Towards spectral evolution

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ALGEBRAIC APPROACH

- algebras (generators and relations): nice, but?
- depends on the choice of the (polynomial) basis (!)

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FUNCTIONAL ANALYSIS

- analysis (mostly) on operator algebras
- the core of (spectral) noncommutative approach
- unifies algebraic and topological approach

GEOMETRY AND SMOOTHNESS

Classical geometry is differential

- an orientable manifold M, smooth functions, $C^\infty(M)$,
- differential algebra $\Omega(M)$, metric $g^{\mu\nu}$, Laplace operator Δ,
- spin^c structure(s), real spin structure, Dirac operator

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Differential operators - as **operators on a Hilbert space**

- came with the dawn of quantum mechanics
- the core of noncommutative (spectral) approach

Geometry and the Hilbert spaces.

The **significance** of (differential) operators

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6. scalar of curvature, metric tensor, Einstein tensor - via spectral computations

   Spectral Metric and Einstein Functionals,
   L.Dabrowski, A.Sitarz, P.Zalecki,
   Advances in Mathematics, Volume 427, 2023, 109128

GEOMETRY ENCODED AS OPERATORS

Connes' spectral triple approach

Algebra $\mathscr{A}$, its faithful representation π on a Hilbert space $\mathscr{H}$, a selfadjoint, unbounded operator D, satisfying several conditions:

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4. assure that D is a first order differential operator
5. ...+ a lot of conditions assuring smoothness

Theorem [Connes]

$\mathscr{A} = C^\infty(M)$, M spin Riemannian compact manifold, $\mathscr{H} = L^2(S)$, (sections of spinor bundle) and D the Dirac operator on M then $(\mathscr{A}, \mathscr{H}, D)$ is a spectral triple (with a real structure).

Accomplishments and failures

The Standard Model

A finite spectral triple describes the Standard Model based on an algebra $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. A **nonproduct** geometry removes fermion doubling and explains CP-violation in geometrical terms

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A. Bochniak, A. Sitarz, A spectral geometry for the Standard Model without the fermion doubling Phys. Rev. D 101, 075038

Quantum spaces

The spectral triple approach allows to study geometry of quantum spheres (including Podles sphere and $SU_q(2)$.)

But..

The spectral approach is **limited** to Euclidean metrics. A version of a spectral triple formalism for indefinite metric is possible (using the Krein space formalism) - but there is no generic spectral approach (noncommutative residue).

Instead: a model of $N+1$ NC geometries?

The **classical** simplest $N+1$ geometry.

$$ds^2 = \eta dt^2 + \gamma_{ij} dx^i dx^j,$$

where η determines the signature of the metric, $\eta = \pm 1$.
The extrinsic curvature reduces to:

$$K_{ij} = -\frac{1}{2}\frac{d\gamma_{ij}(t)}{dt}, \qquad \frac{d\gamma^{ij}(t)}{dt} = 2\gamma^{im}\gamma^{jk}K_{mk}$$

Ricci tensor

$$R_{ik} = r_{ik} + \frac{1}{\eta}\dot{K}_{ik} + \frac{2}{\eta}\gamma^{pm}K_{mk}K_{ip} - \frac{1}{\eta}KK_{ik},$$

and

$$R_{00} = \partial_t(\gamma^{mi}K_{mi}) - \gamma^{pm}\gamma^{rt}K_{mr}K_{pt}.$$

A model of $N+1$ NC geometries.

The curvature of the **classical** simplest $N+1$ geometry.

$$R = r - \frac{1}{\eta}\left(K^2 + K_{ij}K^{ij} - 2\dot{K}\right)$$

FRLW geometries

Assuming $\gamma_{ij}(x,t) = a(t)\zeta_{ij}(x)$ we obtain:

$$R_{ik} = r_{ik} + \frac{1}{\eta}\left(-\frac{1}{2}\ddot{a} + \frac{2-N}{4}\frac{\dot{a}^2}{a}\right)\zeta_{ik},$$

$$R_{00} = N\left(\frac{d}{dt}\left(-\frac{1}{2}\frac{\dot{a}}{a}\right) - \frac{1}{4}\frac{\dot{a}^2}{a^2}\right), \qquad ,$$

$$R = \frac{1}{a}r - \frac{1}{\eta}\left(\frac{N^2+N}{4}\left(\frac{\dot{a}^2}{a^2}\right) + N\frac{d}{dt}\left(\frac{\dot{a}}{a}\right)\right).$$

Can the curvature be recovered spectrally?

The question: If we have an N-dimensional Euclidean space and a family of Laplace operators $\Delta(t)$ that are given by the metrics $\gamma_{ij}(t)$, can we compute the scalar of curvature using the spectral properties of $\Delta(t)$?

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The $3+1$ case.

We start with a family $\Delta_3(x,t)$ of 3-dimensional Laplace-type operators set by the metric $g_{ij}(x)$:

$$\Delta_3(x,t) = -\frac{1}{\sqrt{g(t)}}\partial_j\left(\sqrt{g(t)}g^{jk}(t)\partial_k\right).$$

and we extend it to a 4-dimensional operator acting on $M_3 \times \mathbb{R}$:

$$\Delta_4(x,t) = -\partial_t^2 + c(t)\partial_t + \Delta_3(t),$$

where

$$c(t) = -\frac{1}{\sqrt{g}}\frac{d\sqrt{g}}{dt}.$$

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Curvature for $3+1$ geometry.

The scalar of curvature.

We compute the scalar of curvature using purely spectral methods:

$$R \sim \mathrm{Wres}(\Delta_4^{-1}),$$

where the Wodzicki residue depends solely on the spectral properties of the operator Δ.

How to compute it?

The Wodzicki residue is easily computable using the calculus of pseudodifferential operators, if a ΨDO T has an expansion in homogeneous symbols:

$$T = T_p(\xi,x) + T_{p-1}(\xi,x) + T_{p-2}(\xi,x) + \cdots$$

then

$$\mathrm{Wres}(T) = \int_M \int_{|\xi|=1} T_{-n}(\xi,x).$$

Curvature for $3+1$ geometry.

The scalar of curvature.

So using the symbol of homogeneity -4 of $\Delta(t)^{-1}$ we obtain:

$$\mathrm{Wres}\big(f(x,t)\Delta_4^{-1}(t)\big) = \int dt \int_M \sqrt{g(t)} f(x,t) \Big[r(x,t) + \\ + \Big(-\frac{\pi^2}{6}(2K_{ij}K^{ij} + K^2) - \frac{\pi^2}{2} K^2 + \pi^2 \dot{K} \Big) \Big].$$

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Can it be rewritten?

Next, can we rewrite it using only $\Delta_3(t)$?
The 3-dimensional scalar of curvature is

$$\mathrm{Wres}\big(f(x,t)\Delta_3^{-\frac{1}{2}}(t)\big) \sim \int dt \int_M \sqrt{g(t)} f(x,t) r(x,t)$$

Evolution of $3+1$ geometry.

No go theorem

There exist no functional depending on

$$\mathrm{Wres}\big(\dot{\Delta}_3(t)^2\Delta_3^{-\frac{7}{2}}(t)\big)$$

and

$$\mathrm{Wres}\big(\ddot{\Delta}_3(t)\Delta_3^{-\frac{5}{2}}(t)\big)$$

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Covariant derivative

We introduce a covariant derivative (acting on operators):

$$\nabla_t = \partial_t + c,$$

where c is an operator (!) and require that the volume functional is covariantly constant:

$$\mathrm{Wres}\left(f(x,t)(\nabla_t)\Delta_3(t)^{-\frac{3}{2}} \right) = 0$$

Evolution of $3+1$ geometry.

Solution

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But...

- no proof that such covariant derivative always exists
- assumptions on covariantly constant volume (?)

Conclusions & outlook

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- Almost noncommutative geometries ?
- Evolution of fuzzy spaces ?

THANK YOU !