A geometric approach to duality covariant higher-derivative corrections in gravity

Work in progress by Daniel Butter, AG and Falk Hassler

Previously…

- Generators of $O(d+n, d+n)$: $\tilde{R}^{\alpha\beta}, \tilde{R}^{\alpha}_{A}, K^{\alpha}_{\beta}, K_{AB}, R^{A}_{\alpha}, R_{\alpha\beta}$

K^{α}_{β}: $GL(n)$

K_{AB}: $O(d,d)$

$K_{AB}, R^{A}_{\alpha}, R_{\alpha\beta}$: G_{PS}

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- Mega-space frame: $\hat{E} = \tilde{M}(\mathcal{A}E)\tilde{V}$ with $\mathcal{A} \in G_{PS}/O(d,d)$

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- Identify components of $\mathcal{A}$ in terms of fluxes of the physical frame

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- Fix the structure group to be a series of extensions of the double Lorentz group $G_S^{(p,q)} = \underline{\hat{O}^{(p)}(1,d-1)} \times \overline{\hat{O}^{(q)}(d-1,1)} = \underline{G_S}^{(p)} \times \overline{G_S}^{(q)}$

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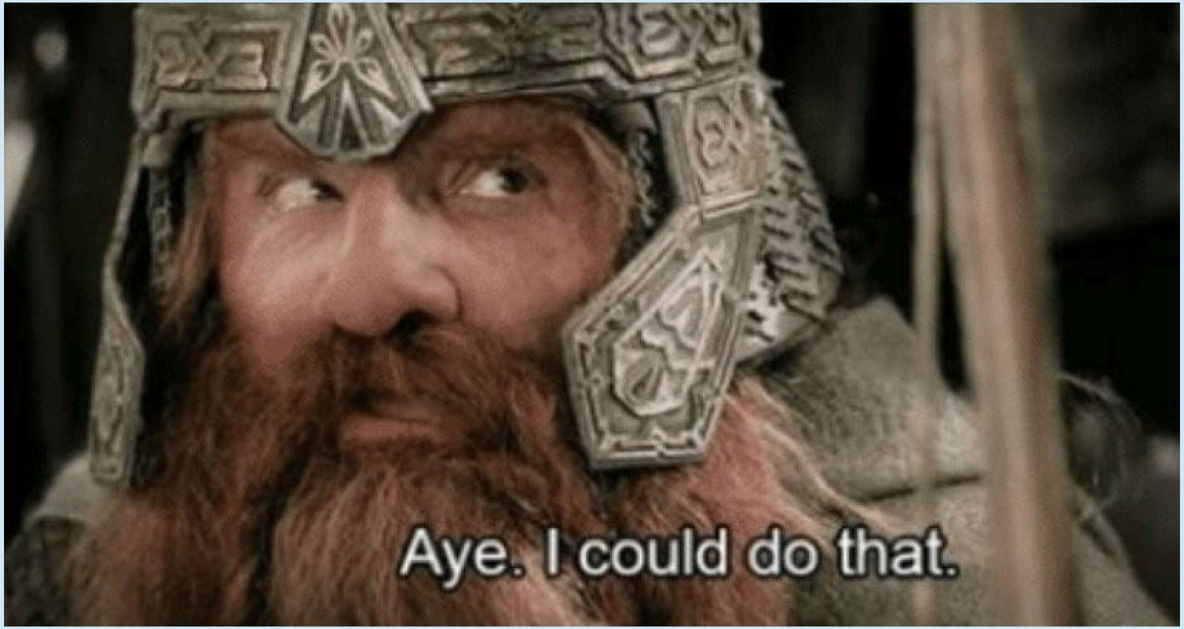

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- $\mathcal{A} = \exp\left(A'\right) = \exp\left(A + c_1 A^3 + c_2 A^5 + \ldots\right)$, A being the connection

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$$A = A_- + A_+$$

Mixed chiral component, will be calculated through torsion constraints

(Anti-)chiral components, set to zero through partial gauge fixing

Construction of the structure group

$$\tau_{\underline{\alpha}_1} = \tau^{\underline{ab}} = \frac{\underline{c}}{\sqrt{a}} K^{\underline{ab}}$$

$$\tau_{\underline{\alpha}_{i+1}} = \left(\tau^{\underline{a}}_{\underline{\beta}_i} \qquad \tau_{\underline{\alpha}_1 \underline{\beta}_1} \ldots \tau_{\underline{\alpha}_i \underline{\beta}_i} \right) = \frac{\underline{c}}{2\sqrt{a}} \left(R^{\underline{a}}_{\underline{\beta}_i} \qquad - R_{\underline{\alpha}_1 \underline{\beta}_i} \cdots - R_{\underline{\alpha}_i \underline{\beta}_i} \right)$$

$$[K_{AB}, K_{CD}] = 2\eta_{[A|[C}K_{D]|B]}$$

$$[K_{AB}, R^C_\gamma] = -\delta^C_{[A}\eta_{B]D}R^D_\gamma$$

$$[R^A_\alpha, R^B_\beta] = \eta^{AB}R_{\alpha\beta} - 2K^{AB}\kappa_{\alpha\beta}$$

$$[R_{\alpha\beta}, R_{\gamma\delta}] = -4\kappa_{[\alpha|[\gamma}R_{\delta]|\beta]}$$

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$$\underbrace{\hspace{6cm}}$$

$$[\tau_{\underline{\alpha}_I\underline{\beta}_J}, \tau_{\underline{\gamma}_K\underline{\delta}_L}] = \frac{2\underline{c}}{\sqrt{a}}\eta_{[\underline{\alpha}_I|[\underline{\gamma}_K}\tau_{\underline{\delta}_L]|\underline{\beta}_J]} = -f_{\underline{\alpha}_I\underline{\beta}_J\underline{\gamma}_K\underline{\delta}_L}{}^{\underline{\epsilon}_M\underline{\zeta}_N}\tau_{\underline{\epsilon}_M\underline{\zeta}_N}$$

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$$\frac{a}{\underline{c}^2}\langle\langle\tau_{\underline{\alpha}_I\underline{\beta}_J}, \tau_{\underline{\gamma}_K\underline{\delta}_L}\rangle\rangle = \eta_{[\underline{\alpha}_I|[\underline{\gamma}_K}\eta_{\underline{\delta}_L]|\underline{\beta}_J} = \kappa_{\underline{\alpha}_I\underline{\beta}_J\underline{\gamma}_K\underline{\delta}_L}$$

Change of basis

$$t_{\underline{\alpha}} = S_{\underline{\alpha}}{}^{\underline{\beta}} \tau_{\underline{\beta}}$$

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$$S_{\underline{\alpha}}{}^{\underline{\beta}} = \delta_{\alpha}^{\beta} - \frac{\sqrt{a}}{\underline{c}} f_{\underline{a}}{}^{\underline{\beta}'\underline{\beta}''} = \delta_{\underline{\alpha}}^{\underline{\beta}} - f_{\underline{\alpha}}{}^{\underline{\beta}}, \qquad (S^{-1})_{\underline{\alpha}}{}^{\underline{\beta}} = \sum_{n=1}^{p-1} (f^n)_{\underline{\alpha}}{}^{\underline{\beta}}$$

Collapsing the towers

- Infinite contributions when contracting greek indices

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$$\underline{\chi}_m = 1 + \sum_{n=1}^{p-m} \prod_{l=1}^{n} \left[d + dim\underline{G_S}^{(p-l-1)} - 2 \right]$$

Identification

- Vanishing torsion $\Rightarrow \begin{cases} \mathcal{A}_{\underline{a}\overline{\delta}}(\tau^{\overline{\delta}})_{\overline{\mathcal{BC}}} = -\mathcal{F}_{\underline{a}\overline{\mathcal{BC}}}, \\ \mathcal{A}^{\underline{\alpha}}{}_{\overline{\delta}}(\tau^{\overline{\delta}})_{\overline{\mathcal{BC}}} = -\mathcal{F}^{\overline{\alpha}}{}_{\overline{\mathcal{BC}}} \end{cases}$

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$$\mathcal{F}_{\mathcal{A}} = \mathcal{L}_{\mathcal{E}_{\mathcal{A}}}\mathcal{E}\mathcal{E}^{-1}, \qquad \mathcal{E} = \mathcal{A}E$$

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$$A^{(l)\beta}_{\mathcal{A}} t_\beta = \hat{A}^{(l)\beta}_{\mathcal{A}} \tau_\beta \Rightarrow A^{(l)\beta}_{\mathcal{A}} = \hat{A}^{(l)\gamma}_{\mathcal{A}} (S^{-1})_\gamma{}^\beta$$

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$$\mathcal{G}_{\mathcal{A}}^{(l)\beta}[A^{(<l)}] = \mathcal{A}_{\mathcal{A}}^{(l)\beta} - A_{\mathcal{A}}^{(l)\beta}$$

Finally

$$\hat{A}^{(l)}_{\overline{\mathcal{A}}\underline{\beta}} = -\frac{a}{\underline{c}^2}\langle\langle \mathcal{F}^{(l)}_{\overline{\mathcal{A}}}[A^{(<l)}], \tau_{\underline{\beta}}\rangle\rangle - \hat{\mathcal{G}}^{(l)}_{\overline{\mathcal{A}}\underline{\beta}}[A^{(<l)}],$$

$$\hat{A}^{(l)}_{\underline{\mathcal{A}}\overline{\beta}} = -\frac{b}{\overline{c}^2}\langle\langle \mathcal{F}^{(l)}_{\underline{\mathcal{A}}}[A^{(<l)}], \tau_{\overline{\beta}}\rangle\rangle - \hat{\mathcal{G}}^{(l)}_{\underline{\mathcal{A}}\overline{\beta}}[A^{(<l)}]$$

Results

$$\hat{A}^{(1)}_{\overline{a}\underline{\beta}_1} = \hat{A}^{(1)\underline{b}_1\underline{b}_2}_{\overline{a}} = -\frac{\sqrt{a}}{\underline{c}} F_{\overline{a}}{}^{\underline{b}_1\underline{b}_2}$$

$$\hat{A}^{(2)}_{\overline{a}\underline{\beta}_2} = \hat{A}^{(2)\underline{b}_1}_{\overline{a}}{}_{\underline{\beta}_1} = \frac{\sqrt{a}}{\underline{c}}\left(F_{\overline{a}}{}^{\underline{b}_1\overline{c}}\hat{A}^{(1)}_{\overline{c}\underline{\beta}_1} + D^{\underline{b}_1}\hat{A}^{(1)}_{\overline{a}\underline{\beta}_1}\right)$$

$$\hat{A}^{(3)}_{\overline{a}\underline{\beta}_1} = \hat{A}^{(3)\underline{b}_1\underline{b}_2}_{\overline{a}} = \frac{\sqrt{a}}{\underline{c}}\overline{\chi}_1\left(\hat{A}^{(1)[\underline{b}_1}{}_{\overline{\gamma}_1} D_{\overline{a}}\hat{A}^{(1)\underline{b}_2]\overline{\gamma}_1} - \hat{A}^{[\underline{b}_1}{}_{\overline{\gamma}_1}F_{\overline{a}}{}^{\underline{b}_2]\underline{c}}\hat{A}^{(1)\overline{\gamma}_1}_{\underline{c}}\right)$$

$$\begin{aligned}\hat{A}^{(3)}_{\overline{a}\underline{\beta}_2} = \hat{A}^{(3)}_{\overline{a}\underline{\beta}_1\underline{\gamma}_1} = \frac{\sqrt{a}}{\underline{c}}\Big(&\hat{A}^{(1)}_{\overline{b}[\underline{\beta}_1|}D_{\overline{a}}\hat{A}^{(1)\overline{b}}{}_{|\underline{\gamma}_1]} - \hat{A}^{(1)\overline{b}}{}_{[\underline{\beta}_1}\hat{A}^{(1)\overline{c}}{}_{\underline{\gamma}_1]}F_{\overline{a}\overline{b}\overline{c}} + \\ &+2D_{\overline{b}}\hat{A}^{(1)}_{\overline{a}[\underline{\beta}_1}\hat{A}^{(1)\overline{b}}{}_{\underline{\gamma}_1]} + \hat{A}^{(1)\underline{\alpha}_1}_{\overline{a}} f_{\underline{\delta}_1[\underline{\beta}_1|\underline{\alpha}_1}\hat{A}^{(1)}_{\overline{b}|\underline{\gamma}_1]}\hat{A}^{(1)\overline{b}\underline{\delta}_1}\Big)\end{aligned}$$

Gauge transformations

$$\mathcal{A}^{-1}\delta\mathcal{A} + \delta E^{(l)}E^{-1} = -D_A\xi^\beta R^A_\beta + \frac{1}{2}\xi^\alpha f_{\alpha\beta\gamma}R^{\beta\gamma} + \xi^\alpha\mathcal{A}^{-1}t_\alpha\mathcal{A}$$

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$$\Downarrow$$

$$\delta A^{(l)} + \delta E^{(l)}E^{-1} - \xi^{(l)\alpha}t_\alpha = \mathcal{X}^{(l)}[\xi^{(<l)}] - \delta G^{(l)}[\xi^{(<l)}]$$

Gauge transformations

$$\mathcal{A}^{-1}\delta\mathcal{A} + \delta E^{(l)} E^{-1} = -D_A \xi^\beta R_\beta^A + \frac{1}{2}\xi^\alpha f_{\alpha\beta\gamma} R^{\beta\gamma} + \xi^\alpha \mathcal{A}^{-1} t_\alpha \mathcal{A}$$

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Mixed chir. $R_\alpha^A, R_{\alpha\beta}$

Mixed chir. K_{AB}

(Anti-)chir. $K_{AB}, R_\alpha^A, R_{\alpha\beta}$

$\delta A'^{(l)} - \delta A^{(l)}$

Results (again)

$$\hat{\xi}^{(0)\underline{\alpha}_1} = \hat{\xi}^{(0)}_{\underline{ab}} = -\frac{\sqrt{a}}{\underline{c}}\Lambda_{\underline{ab}}$$

$$\hat{\xi}^{(1)\underline{\alpha}_2} = \hat{\xi}^{(1)\underline{\alpha}_1}_{\underline{a}} = \frac{2\sqrt{a}}{\underline{c}} D_{\underline{a}}\hat{\xi}^{(0)\underline{\alpha}_1}$$

$$\hat{\xi}^{(2)\underline{\alpha}_1} = \hat{\xi}^{(2)}_{\underline{ab}} = -\frac{\sqrt{a}}{\underline{c}}\overline{\chi}_1 \hat{A}^{(1)}_{[\underline{a}_1|\overline{\beta}_1} D_{|\underline{b}]}\hat{\xi}^{(0)\overline{\beta}_1}$$

$$\hat{\xi}^{(2)\underline{\alpha}_2} = \hat{\xi}^{(2)\underline{\alpha}_1\underline{\beta}_1} = \frac{\sqrt{a}}{\underline{c}} \hat{A}^{\overline{c}[\underline{\alpha}_1} D_{\overline{c}}\hat{\xi}^{(0)\underline{\beta}_1]}$$

GS transformations (0th order)

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$$\delta E_{\underline{ab}} = -\Lambda_{\underline{ab}}$$

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$$\delta E^{(2)}_{\underline{a}\bar{b}} = A^{(1)}_{\underline{a}\overline{\alpha}} D_{\overline{b}} \xi^{(0)\overline{\alpha}} - c.c. = \overline{\chi}_1 \hat{A}^{(1)}_{\underline{a}\overline{\alpha}_1} D_{\overline{b}} \hat{\xi}^{(0)\overline{\alpha}_1} - c.c.$$

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$$= -\frac{a}{\underline{c}^2} \underline{\chi}_1 D_{\underline{a}} \Lambda^{\underline{c}_1\underline{c}_2} F_{\overline{b}\underline{c}_1\underline{c}_2} + \frac{b}{\overline{c}^2} \overline{\chi}_1 D_{\overline{b}} \Lambda^{\overline{c}_1\overline{c}_2} F_{\underline{a}\overline{c}_1\overline{c}_2} = \frac{a}{2} D_{\underline{a}} \Lambda^{\underline{c}_1\underline{c}_2} F_{\overline{b}\underline{c}_1\underline{c}_2} + \frac{b}{2} D_{\overline{b}} \Lambda^{\overline{c}_1\overline{c}_2} F_{\underline{a}\overline{c}_1\overline{c}_2}$$

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$$\boxed{\underline{c}^2 = -2\underline{\chi}_1, \qquad \overline{c}^2 = 2\overline{\chi}_1}$$

GS transformations (4th order)

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$$\begin{aligned}
\delta E^{(4)}_{\underline{a}\bar{b}} = -\frac{a^2\underline{\chi}_2}{2\underline{\chi}_1^2} \Big[& D_{\underline{a}} D_{\underline{c}} \Lambda_{\underline{de}} \left(F^{\underline{c}}{}_{\underline{f}\bar{b}} F^{\bar{f}\underline{de}} + D^{\underline{c}} F_{\bar{b}}{}^{\underline{de}} \right) - F_{\bar{b}\underline{f}}{}^{\underline{g}} F^{\bar{c}}{}_{\underline{dg}} \left(F_{\bar{c}}{}^{\underline{ed}} D_{\underline{a}} \Lambda_{\underline{e}}{}^{\underline{f}} - F_{\bar{c}}{}^{\underline{ef}} D_{\underline{a}} \Lambda_{\underline{e}}{}^{\underline{d}} \right) \\
& + D_{\underline{a}} \Lambda_{\underline{ef}} F^{\bar{c}\underline{e}}{}_{\underline{d}} \left(F_{\bar{b}\bar{c}\bar{g}} F^{\bar{g}\underline{fd}} - D_{\bar{b}} F_{\bar{c}}{}^{\underline{fd}} + 2 D_{\bar{c}} F_{\bar{b}}{}^{\underline{fd}} \right) + F_{\bar{b}}{}^{\underline{ed}} D_{\underline{a}} \left(D^{\bar{c}} \Lambda_{\underline{e}}{}^{\underline{f}} F_{\bar{c}\underline{fd}} \right) \Big] \\
-\frac{ab}{4} \Big[& D_{\underline{a}} \Lambda^{\underline{cd}} \left(F_{\bar{b}\underline{cg}} F^{\underline{g}\bar{e}\bar{f}} F_{\underline{d}\bar{e}\bar{f}} - D_{\bar{b}} F_{\underline{c}}{}^{\bar{e}\bar{f}} F_{\underline{d}\bar{e}\bar{f}} \right) + F_{\bar{b}\underline{cd}} D_{\underline{a}} \left(D^{\underline{c}} \Lambda_{\bar{e}\bar{f}} F^{\underline{d}\bar{e}\bar{f}} \right) \\
& - D_{\bar{b}} \Lambda^{\bar{c}\bar{d}} \left(F_{\underline{a}\bar{c}\bar{g}} F^{\bar{g}\underline{ef}} F_{\bar{d}\underline{ef}} - D_{\underline{a}} F_{\bar{c}}{}^{\underline{ef}} F_{\bar{d}\underline{ef}} \right) - F_{\underline{a}\bar{c}\bar{d}} D_{\bar{b}} \left(D^{\bar{c}} \Lambda_{\underline{ef}} F^{\bar{d}\underline{ef}} \right) \Big] \\
+\frac{b^2\overline{\chi}_2}{2\overline{\chi}_1^2} \Big[& D_{\bar{b}} D_{\bar{c}} \Lambda_{\underline{de}} \left(F^{\bar{c}}{}_{\underline{fa}} F^{\underline{f}\bar{d}\bar{e}} + D^{\bar{c}} F_{\underline{a}}{}^{\bar{d}\bar{e}} \right) - F_{\underline{a}\bar{f}}{}^{\bar{g}} F^{\underline{c}}{}_{\bar{d}\bar{g}} \left(F_{\underline{c}}{}^{\bar{e}\bar{d}} D_{\bar{b}} \Lambda_{\bar{e}}{}^{\bar{f}} - F_{\underline{c}}{}^{\bar{e}\bar{f}} D_{\bar{b}} \Lambda_{\bar{e}}{}^{\bar{d}} \right) \\
& + D_{\bar{b}} \Lambda_{\bar{e}\bar{f}} F^{\underline{c}\bar{e}}{}_{\bar{d}} \left(F_{\underline{acg}} F^{\underline{g}\bar{f}\bar{d}} - D_{\underline{a}} F_{\underline{c}}{}^{\bar{f}\bar{d}} + 2 D_{\underline{c}} F_{\underline{a}}{}^{\bar{f}\bar{d}} \right) + F_{\underline{a}}{}^{\bar{e}\bar{d}} D_{\bar{b}} \left(D^{\underline{c}} \Lambda_{\bar{e}}{}^{\bar{f}} F_{\underline{c}\bar{f}\bar{d}} \right) \Big]
\end{aligned}$$

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$$\delta E^{(4)} E^{-1} \cong -[A^{(1)}, D\xi^{(2)}] - [A^{(2)}, D\xi^{(1)}] - [A^{(3)}, D\xi^{(0)}]$$

$$\Downarrow$$

$$\delta E^{(4)}_{\underline{a}\bar{b}} = A^{(1)}_{\underline{a}\bar{\alpha}} D_{\bar{b}} \xi^{(2)\bar{\alpha}} + A^{(2)}_{\underline{a}\bar{\alpha}} D_{\bar{b}} \xi^{(1)\bar{\alpha}} + A^{(3)}_{\underline{a}\bar{\alpha}} D_{\bar{b}} \xi^{(0)\bar{\alpha}} - c.c.$$

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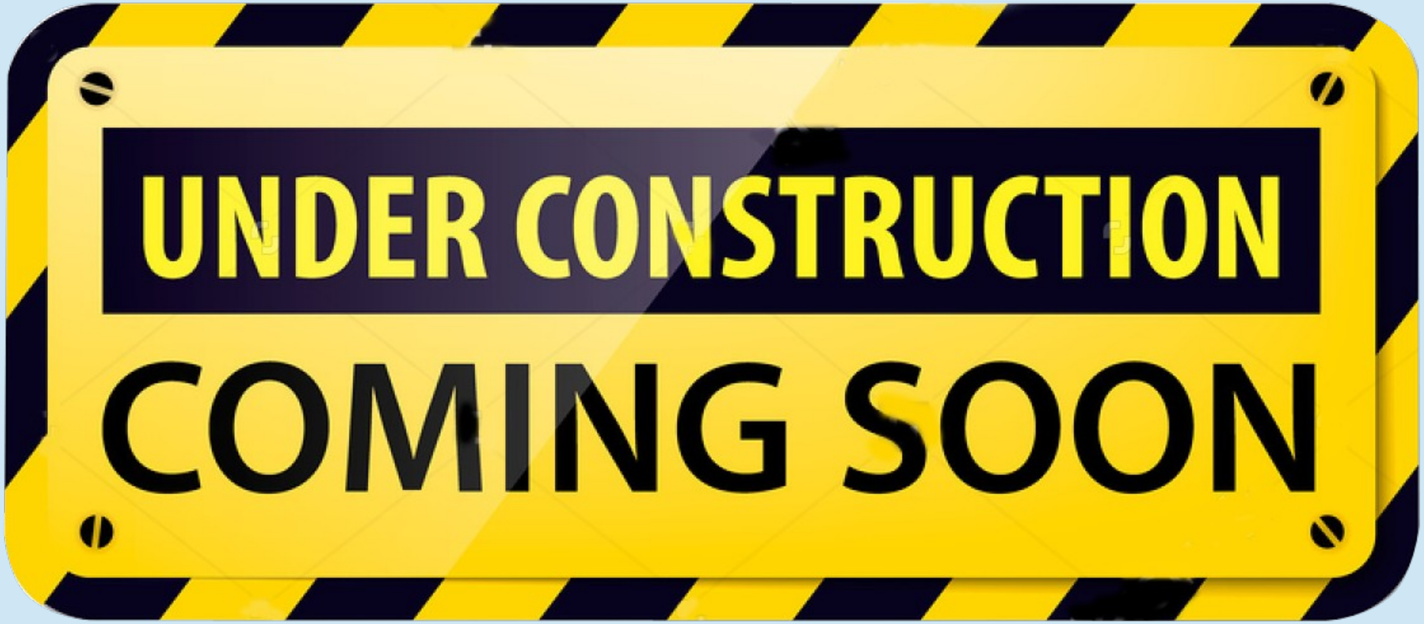
& + D_{\underline{a}} \Lambda_{\underline{ef}} F^{\bar{c}\underline{e}}{}_{\underline{d}} \left(F_{\bar{b}\bar{c}\underline{g}} F^{\bar{g}\underline{fd}} - D_{\bar{b}} F_{\bar{c}}{}^{\underline{fd}} + 2 D_{\bar{c}} F_{\bar{b}}{}^{\underline{fd}} \right) + F_{\bar{b}}{}^{\underline{ed}} D_{\underline{a}} \left(D^{\bar{c}} \Lambda_{\underline{e}}{}^{\underline{f}} F_{\bar{c}\underline{fd}} \right) \Big] \\
& - \frac{ab}{4} \Big[D_{\underline{a}} \Lambda^{\underline{cd}} \left(F_{\bar{b}\underline{cg}} F^{\underline{g}\bar{e}\bar{f}} F_{\underline{d}\bar{e}\bar{f}} - D_{\bar{b}} F_{\underline{c}}{}^{\bar{e}\bar{f}} F_{\underline{d}\bar{e}\bar{f}} \right) + F_{\bar{b}\underline{cd}} D_{\underline{a}} \left(D^{\underline{c}} \Lambda_{\bar{e}\bar{f}} F^{\underline{d}\bar{e}\bar{f}} \right) \\
& - D_{\bar{b}} \Lambda^{\bar{c}\bar{d}} \left(F_{\underline{a}\bar{c}\underline{g}} F^{\bar{g}\underline{ef}} F_{\bar{d}\underline{ef}} - D_{\underline{a}} F_{\bar{c}}{}^{\underline{ef}} F_{\bar{d}\underline{ef}} \right) - F_{\underline{a}\bar{c}\bar{d}} D_{\bar{b}} \left(D^{\bar{c}} \Lambda_{\underline{ef}} F^{\bar{d}\underline{ef}} \right) \Big] \\
& + \frac{b^2 \bar{\chi}_2}{2\bar{\chi}_1^2} \Big[D_{\bar{b}} D_{\bar{c}} \Lambda_{\underline{de}} \left(F^{\bar{c}}{}_{\underline{fa}} F^{\underline{f}\bar{d}\bar{e}} + D^{\bar{c}} F_{\underline{a}}{}^{\bar{d}\bar{e}} \right) - F_{\underline{a}\bar{f}}{}^{\bar{g}} F^{\underline{c}}{}_{\bar{d}\bar{g}} \left(F_{\underline{c}}{}^{\bar{e}\bar{d}} D_{\bar{b}} \Lambda_{\bar{e}}{}^{\bar{f}} - F_{\underline{c}}{}^{\bar{e}\bar{f}} D_{\bar{b}} \Lambda_{\bar{e}}{}^{\bar{d}} \right) \\
& + D_{\bar{b}} \Lambda_{\bar{e}\bar{f}} F^{\underline{c}\bar{e}}{}_{\bar{d}} \left(F_{\underline{acg}} F^{\underline{g}\bar{f}\bar{d}} - D_{\underline{a}} F_{\underline{c}}{}^{\bar{f}\bar{d}} + 2 D_{\underline{c}} F_{\underline{a}}{}^{\bar{f}\bar{d}} \right) + F_{\underline{a}}{}^{\bar{e}\bar{d}} D_{\bar{b}} \left(D^{\underline{c}} \Lambda_{\bar{e}}{}^{\bar{f}} F_{\underline{c}\bar{f}\bar{d}} \right) \Big]
\end{aligned}$$

Need futher study

GS transformations (6th order)



A pattern (?)

$$\delta E^{(2m)}_{\underline{a}\overline{b}} = \sum_{n=1}^{2m-1} A^{(n)}_{\underline{a}\overline{\alpha}} D_{\overline{b}} \xi^{(2m-n-1)\overline{\alpha}}$$

A pattern (?)

$$\delta E^{(2m)}_{\underline{a}\overline{b}} = \sum_{n=1}^{2m-1} A^{(n)}_{\underline{a}\overline{\alpha}} D_{\overline{b}} \xi^{(2m-n-1)\overline{\alpha}}$$

Depends on the coefficients of $A' = A + c_1 A^3 + c_2 A^5 + c_3 A^7 + \dots$

gbdRi vs deformed PS

$$
\begin{aligned}
\delta E^{(4)}_{\underline{a}\bar{b}} = & -\frac{a^2\underline{\chi}_2}{2\underline{\chi}_1^2}\Big[D_{\underline{a}} D_{\underline{c}} \Lambda_{\underline{de}} \Big(F^{\underline{c}}{}_{\overline{fb}} F^{\bar{f}\underline{de}} + D^{\underline{c}} F_{\bar{b}}{}^{\underline{de}}\Big) - F_{\bar{b}\underline{f}}{}^{\underline{g}} F^{\bar{c}}{}_{\underline{dg}} \Big(F_{\bar{c}}{}^{\underline{ed}} D_{\underline{a}}\Lambda_{\underline{e}}{}^{\underline{f}} - F_{\bar{c}}{}^{\underline{ef}} D_{\underline{a}} \Lambda_{\underline{e}}{}^{\underline{d}}\Big) \\
& + D_{\underline{a}} \Lambda_{\underline{ef}} F^{\bar{c}\underline{e}}{}_{\underline{d}} \Big(F_{\bar{b}\overline{cg}} F^{\bar{g}\underline{fd}} - D_{\bar{b}} F_{\bar{c}}{}^{\underline{fd}} + 2 D_{\bar{c}} F_{\bar{b}}{}^{\underline{fd}}\Big) + F_{\bar{b}}{}^{\underline{ed}} D_{\underline{a}} \Big(D^{\bar{c}} \Lambda_{\underline{e}}{}^{\underline{f}} F_{\bar{c}\underline{fd}}\Big)\Big] \\
& -\frac{ab}{4}\Big[D_{\underline{a}} \Lambda^{\underline{cd}} \Big(F_{\bar{b}\underline{cg}} F^{\underline{g}\overline{ef}} F_{\underline{d}\overline{ef}} - D_{\bar{b}} F_{\underline{c}}{}^{\overline{ef}} F_{\underline{d}\overline{ef}}\Big) + F_{\bar{b}\underline{cd}} D_{\underline{a}} \Big(D^{\underline{c}} \Lambda_{\overline{ef}} F^{\underline{d}\overline{ef}}\Big) \\
& - D_{\bar{b}} \Lambda^{\overline{cd}} \Big(F_{\underline{a}\overline{cg}} F^{\bar{g}\underline{ef}} F_{\bar{d}\underline{ef}} - D_{\underline{a}} F_{\bar{c}}{}^{\underline{ef}} F_{\bar{d}\underline{ef}}\Big) - F_{\underline{a}\overline{cd}} D_{\bar{b}} \Big(D^{\bar{c}} \Lambda_{\underline{ef}} F^{\bar{d}\underline{ef}}\Big)\Big] \\
& + \frac{b^2\bar{\chi}_2}{2\bar{\chi}_1^2}\Big[D_{\bar{b}} D_{\bar{c}} \Lambda_{\underline{de}} \Big(F^{\bar{c}}{}_{\underline{fa}} F^{\underline{f}\overline{de}} + D^{\bar{c}} F_{\underline{a}}{}^{\overline{de}}\Big) - F_{\underline{a}\bar{f}}{}^{\bar{g}} F^{\underline{c}}{}_{\overline{dg}} \Big(F_{\underline{c}}{}^{\overline{ed}} D_{\bar{b}} \Lambda_{\bar{e}}{}^{\bar{f}} - F_{\underline{c}}{}^{\overline{ef}} D_{\bar{b}} \Lambda_{\bar{e}}{}^{\bar{d}}\Big) \\
& + D_{\bar{b}} \Lambda_{\overline{ef}} F^{\underline{c}\bar{e}}{}_{\bar{d}} \Big(F_{\underline{acg}} F^{\underline{g}\overline{fd}} - D_{\underline{a}} F_{\underline{c}}{}^{\overline{fd}} + 2 D_{\underline{c}} F_{\underline{a}}{}^{\overline{fd}}\Big) + F_{\underline{a}}{}^{\overline{ed}} D_{\bar{b}} \Big(D^{\underline{c}} \Lambda_{\bar{e}}{}^{\bar{f}} F_{\underline{c}\overline{fd}}\Big)\Big]
\end{aligned}
$$

gBdRi vs deformed PS

$$
\begin{aligned}
\delta E^{(4)}_{\underline{a}\bar{b}} = -\frac{a^2\underline{\chi}_2}{2\underline{\chi}_1^2}\Big[& D_{\underline{a}}D_{\underline{c}}\Lambda_{\underline{de}}\Big(F^{\underline{c}}{}_{\bar{f}\bar{b}}F^{\bar{f}\underline{de}} + D^{\underline{c}}F_{\bar{b}}{}^{\underline{de}}\Big) - F_{\bar{b}\underline{f}}{}^{\underline{g}}F^{\bar{c}}{}_{\underline{dg}}\Big(F_{\bar{c}}{}^{\underline{ed}}D_{\underline{a}}\Lambda_{\underline{e}}{}^{\underline{f}} - F_{\bar{c}}{}^{\underline{ef}}D_{\underline{a}}\Lambda_{\underline{e}}{}^{\underline{d}}\Big) \\
& + D_{\underline{a}}\Lambda_{\underline{ef}}F^{\bar{c}\underline{e}}{}_{\underline{d}}\Big(F_{\bar{b}\bar{c}\bar{g}}F^{\bar{g}\underline{fd}} - D_{\bar{b}}F_{\bar{c}}{}^{\underline{fd}} + 2D_{\bar{c}}F_{\bar{b}}{}^{\underline{fd}}\Big) + F_{\bar{b}}{}^{\underline{ed}}D_{\underline{a}}\Big(D^{\bar{c}}\Lambda_{\underline{e}}{}^{\underline{f}}F_{\bar{c}\underline{fd}}\Big)\Big] \\
-\frac{ab}{4}\Big[& D_{\underline{a}}\Lambda^{\underline{cd}}\Big(F_{\bar{b}\underline{cg}}F^{\underline{g}\bar{e}\bar{f}}F_{\underline{d}\bar{e}\bar{f}} - D_{\bar{b}}F_{\underline{c}}{}^{\bar{e}\bar{f}}F_{\underline{d}\bar{e}\bar{f}}\Big) + F_{\bar{b}\underline{cd}}D_{\underline{a}}\Big(D^{\underline{c}}\Lambda_{\bar{e}\bar{f}}F^{\underline{d}\bar{e}\bar{f}}\Big) \\
& - D_{\bar{b}}\Lambda^{\bar{c}\bar{d}}\Big(F_{\underline{a}\bar{c}\bar{g}}F^{\bar{g}\underline{ef}}F_{\bar{d}\underline{ef}} - D_{\underline{a}}F_{\bar{c}}{}^{\underline{ef}}F_{\bar{d}\underline{ef}}\Big) - F_{\underline{a}\bar{c}\bar{d}}D_{\bar{b}}\Big(D^{\bar{c}}\Lambda_{\underline{ef}}F^{\bar{d}\underline{ef}}\Big)\Big] \\
+\frac{b^2\overline{\chi}_2}{2\overline{\chi}_1^2}\Big[& D_{\bar{b}}D_{\bar{c}}\Lambda_{\underline{de}}\Big(F^{\bar{c}}{}_{\underline{fa}}F^{\underline{f}\bar{d}\bar{e}} + D^{\bar{c}}F_{\underline{a}}{}^{\bar{d}\bar{e}}\Big) - F_{\underline{a}\bar{f}}{}^{\bar{g}}F^{\underline{c}}{}_{\bar{d}\bar{g}}\Big(F_{\underline{c}}{}^{\bar{e}\bar{d}}D_{\bar{b}}\Lambda_{\bar{e}}{}^{\bar{f}} - F_{\underline{c}}{}^{\bar{e}\bar{f}}D_{\bar{b}}\Lambda_{\bar{e}}{}^{\bar{d}}\Big) \\
& + D_{\bar{b}}\Lambda_{\bar{e}\bar{f}}F^{\underline{c}\bar{e}}{}_{\bar{d}}\Big(F_{\underline{acg}}F^{\underline{g}\overline{fd}} - D_{\underline{a}}F_{\underline{c}}{}^{\overline{fd}} + 2D_{\underline{c}}F_{\underline{a}}{}^{\overline{fd}}\Big) + F_{\underline{a}}{}^{\bar{e}\bar{d}}D_{\bar{b}}\Big(D^{\underline{c}}\Lambda_{\bar{e}}{}^{\bar{f}}F_{\underline{c}\overline{fd}}\Big)\Big]
\end{aligned}
$$

vs

$$
\delta E^{(4)}_{\underline{a}\bar{b}} = A^{(1)}_{\underline{a}\bar{\alpha}}D_{\bar{b}}\xi^{(2)\bar{\alpha}} + A^{(2)}_{\underline{a}\bar{\alpha}}D_{\bar{b}}\xi^{(1)\bar{\alpha}} + A^{(3)}_{\underline{a}\bar{\alpha}}D_{\bar{b}}\xi^{(0)\bar{\alpha}} - c.c.
$$

THANK YOU FOR YOUR ATTENTION