



A geometric approach to duality covariant higher- derivative corrections in gravity

**Work in progress by Daniel Butter, AG and Falk
Hassler**

Previously...

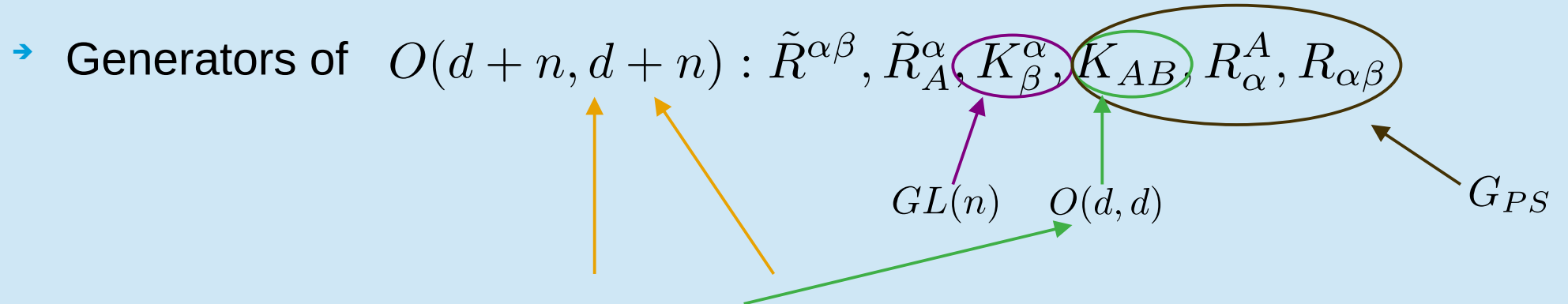
→ Generators of $O(d + n, d + n)$: $\tilde{R}^{\alpha\beta}, \tilde{R}_A^\alpha, K_\beta^\alpha, K_{AB}, R_\alpha^A, R_{\alpha\beta}$


$GL(n)$ $O(d, d)$ G_{PS}

Previously...

- Generators of $O(d+n, d+n)$: $\tilde{R}^{\alpha\beta}, \tilde{R}_A^\alpha, K_\beta^\alpha, K_{AB}, R_\alpha^A, R_{\alpha\beta}$
- Mega-space frame: $\hat{E} = \tilde{M}(\mathcal{A}E)\tilde{V}$ with $\mathcal{A} \in G_{PS}/O(d, d)$
-

Previously...

- Generators of $O(d+n, d+n)$: $\tilde{R}^{\alpha\beta}, \tilde{R}_A^\alpha, K_\beta^\alpha, K_{AB}, R_\alpha^A, R_{\alpha\beta}$ 

The diagram shows the generators of $O(d+n, d+n)$ as a set: $\tilde{R}^{\alpha\beta}, \tilde{R}_A^\alpha, K_\beta^\alpha, K_{AB}, R_\alpha^A, R_{\alpha\beta}$. A purple oval encloses K_β^α and is labeled $GL(n)$. A green oval encloses K_{AB} and R_α^A and is labeled $O(d, d)$. A larger brown oval encloses the entire set of generators. A green arrow points from the $O(d, d)$ label to the brown oval. A purple arrow points from the $GL(n)$ label to the purple oval. Two orange arrows point from the $O(d, d)$ label to $\tilde{R}^{\alpha\beta}$ and \tilde{R}_A^α . A brown arrow points from the G_{PS} label to the brown oval.
- Mega-space frame: $\hat{E} = \tilde{M}(\mathcal{A}E)\tilde{V}$ with $\mathcal{A} \in G_{PS}/O(d, d)$ 

The diagram shows the equation $\hat{E} = \tilde{M}(\mathcal{A}E)\tilde{V}$. The terms \tilde{M} , \mathcal{A} , E , and \tilde{V} are circled in orange. The term \mathcal{A} is also circled in green.
- Identify components of \mathcal{A} in terms of fluxes of the physical frame

Premise

- Reproduction of generalized Bergshoeff-de Roo (gBdR) results by imposing a certain symmetry to an extended space.

Premise

- Reproduction of generalized Bergshoeff-de Roo (gBdR) results by imposing a certain symmetry to an extended space.
- Fix the structure group to be the double Lorentz group $\underline{O(1, d-1)} \times \overline{O(d-1, 1)}$

Premise

→ Reproduction of generalized Bergshoeff-de Roo (gBdR) results by imposing a certain symmetry to an extended space.

~~→ Fix the structure group to be the double Lorentz group
 $\underline{O(1, d - 1)} \times \overline{O(d - 1, 1)}$~~

Premise

- Reproduction of generalized Bergshoeff-de Roo (gBdR) results by imposing a certain symmetry to an extended space.
- ~~→ Fix the structure group to be the double Lorentz group $\underline{O(1, d-1)} \times \overline{O(d-1, 1)}$~~
- Fix the structure group to be a series of extensions of the double Lorentz group $G_S^{(p,q)} = \underline{\hat{O}^{(p)}(1, d-1)} \times \overline{\hat{O}^{(q)}(d-1, 1)} = \underline{G_S^{(p)}} \times \overline{G_S^{(q)}}$

Premise

- Reproduction of generalized Bergshoeff-de Roo (gBdR) results by imposing a certain symmetry to an extended space.

- ~~→ Fix the structure group to be the double Lorentz group $\underline{O(1, d-1)} \times \overline{O(d-1, 1)}$~~

- Fix the structure group to be a series of extensions of the double Lorentz group $G_S^{(p,q)} = \underline{\hat{O}^{(p)}(1, d-1)} \times \overline{\hat{O}^{(q)}(d-1, 1)} = \underline{G_S^{(p)}} \times \overline{G_S^{(q)}}$



Long story short

“So, if I give you the symmetry of the structure group, can you calculate for me all the α' corrections?”

Long story short

“So, if I give you the symmetry of the structure group, can you calculate for me all the α' corrections?”



Partial gauge fixing

→ $\mathcal{A} = \exp(A') = \exp(A + c_1 A^3 + c_2 A^5 + \dots)$, A being the connection

Partial gauge fixing

→ $\mathcal{A} = \exp(A') = \exp(A + c_1 A^3 + c_2 A^5 + \dots)$, A being the connection

$$A = A_- + A_+$$

Mixed chiral component, will be calculated through torsion constraints

(Anti-)chiral components, set to zero through partial gauge fixing

Construction of the structure group

$$\tau_{\underline{\alpha}_1} = \tau^{\underline{ab}} = \frac{\underline{c}}{\sqrt{a}} K^{\underline{ab}}$$

$$\tau_{\underline{\alpha}_{i+1}} = \left(\tau_{\underline{\beta}_i}^{\underline{a}} \quad \tau_{\underline{\alpha}_1 \underline{\beta}_1} \cdots \tau_{\underline{\alpha}_i \underline{\beta}_i} \right) = \frac{\underline{c}}{2\sqrt{a}} \left(R_{\underline{\beta}_i}^{\underline{a}} \quad - R_{\underline{\alpha}_1 \underline{\beta}_i} \cdots - R_{\underline{\alpha}_i \underline{\beta}_i} \right)$$

$$[K_{AB}, K_{CD}] = 2\eta_{[A|[C}K_{D]|B]}$$

$$[R_{\alpha}^A, R_{\beta}^B] = \eta^{AB}R_{\alpha\beta} - 2K^{AB}\kappa_{\alpha\beta}$$

$$[K_{AB}, R_{\gamma}^C] = -\delta_{[A}^C\eta_{B]D}R_{\gamma}^D$$

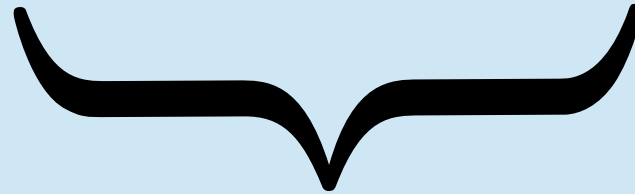
$$[R_{\alpha\beta}, R_{\gamma\delta}] = -4\kappa_{[\alpha|[}\gamma R_{\delta]|}\beta]$$

$$[K_{AB}, K_{CD}] = 2\eta_{[A|[C}K_{D]B]}$$

$$[K_{AB}, R_{\gamma}^C] = -\delta_{[A}^C\eta_{B]D}R_{\gamma}^D$$

$$[R_{\alpha}^A, R_{\beta}^B] = \eta^{AB}R_{\alpha\beta} - 2K^{AB}\kappa_{\alpha\beta}$$

$$[R_{\alpha\beta}, R_{\gamma\delta}] = -4\kappa_{[\alpha|[}\gamma R_{\delta]|\beta]}$$



$$[\tau_{\underline{\alpha}_I\underline{\beta}_J}, \tau_{\underline{\gamma}_K\underline{\delta}_L}] = \frac{2\underline{c}}{\sqrt{a}}\eta_{[\underline{\alpha}_I|[\underline{\gamma}_K} \tau_{\underline{\delta}_L]|\underline{\beta}_J]} = -f_{\underline{\alpha}_I\underline{\beta}_J\underline{\gamma}_K\underline{\delta}_L} \underline{\epsilon}_M \underline{\zeta}_N \tau_{\underline{\epsilon}_M \underline{\zeta}_N}$$

Killing form

- Generators τ_α introduce a new invariant pairing $\langle\langle \cdot, \cdot \rangle\rangle$ on G_S

Killing form

→ Generators τ_α introduce a new invariant pairing $\langle\langle \cdot, \cdot \rangle\rangle$ on G_S

$$\langle\langle R_\alpha^A, R_\beta^B \rangle\rangle = 2\eta^{AB} \kappa_{\alpha\beta}$$

$$\langle\langle K_{AB}, K_{CD} \rangle\rangle = \eta_{[A|[C\eta_{D]}|B]}$$

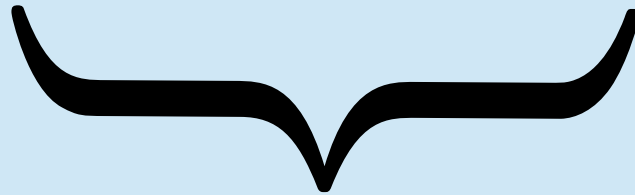
$$\langle\langle R_{\alpha\beta}, R_{\gamma\delta} \rangle\rangle = 4\kappa_{[\alpha|[\gamma\kappa_{\delta]}|\beta]}$$

Killing form

- Generators τ_α introduce a new invariant pairing $\langle\langle \cdot, \cdot \rangle\rangle$ on G_S

$$\langle\langle R_\alpha^A, R_\beta^B \rangle\rangle = 2\eta^{AB} \kappa_{\alpha\beta}$$

$$\langle\langle K_{AB}, K_{CD} \rangle\rangle = \eta_{[A|[C\eta_{D]}|B]} \qquad \langle\langle R_{\alpha\beta}, R_{\gamma\delta} \rangle\rangle = 4\kappa_{[\alpha|[\gamma\kappa_{\delta]}|\beta]}$$



$$\frac{a}{c^2} \langle\langle \tau_{\underline{\alpha}_I \underline{\beta}_J}, \tau_{\underline{\gamma}_K \underline{\delta}_L} \rangle\rangle = \eta_{[\underline{\alpha}_I | [\underline{\gamma}_K \eta_{\underline{\delta}_L}] | \underline{\beta}_J]} = \kappa_{\underline{\alpha}_I \underline{\beta}_J \underline{\gamma}_K \underline{\delta}_L}$$

Change of basis

$$t_{\underline{\alpha}} = S_{\underline{\alpha}}^{\beta} \tau_{\underline{\beta}}$$

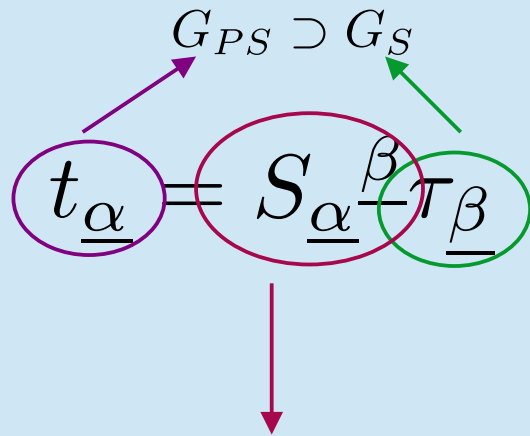
Change of basis

$$G_{PS} \supset G_S$$

The diagram shows the equation $t_{\underline{\alpha}} = S_{\underline{\alpha}}^{\beta} \tau_{\underline{\beta}}$. A purple arrow points from the circled $t_{\underline{\alpha}}$ to the G_{PS} term in the expression above. A green arrow points from the circled $\tau_{\underline{\beta}}$ to the G_S term in the expression above.

$$t_{\underline{\alpha}} = S_{\underline{\alpha}}^{\beta} \tau_{\underline{\beta}}$$

Change of basis



$$S_\alpha^\beta = \delta_\alpha^\beta - \frac{\sqrt{a}}{\underline{c}} f_\alpha^{\beta' \beta''} = \delta_\alpha^\beta - f_\alpha^\beta,$$

$$(S^{-1})_\alpha^\beta = \sum_{n=1}^{p-1} (f^n)_\alpha^\beta$$

Collapsing the towers

- Infinite contributions when contracting greek indices

Collapsing the towers

- ~~→ Infinite contributions when contracting greek indices~~

Collapsing the towers

- ~~→ Infinite contributions when contracting greek indices~~
- Infinite constant when contracting greek indices

Collapsing the towers

~~→ Infinite contributions when contracting greek indices~~

→ Infinite constant when contracting greek indices



Collapsing the towers

~~→ Infinite contributions when contracting greek indices~~

→ Infinite constant when contracting greek indices



$$\underline{\chi}_m = 1 + \sum_{n=1}^{p-m} \prod_{l=1}^n \left[d + \dim \underline{G}_S^{(p-l-1)} - 2 \right]$$

Identification

→ Vanishing torsion \Rightarrow
$$\begin{cases} \mathcal{A}_{\underline{a}\bar{\delta}}(\tau^{\bar{\delta}})_{\overline{BC}} = -\mathcal{F}_{\underline{a}\overline{BC}}, \\ \mathcal{A}^{\alpha}_{\bar{\delta}}(\tau^{\bar{\delta}})_{\overline{BC}} = -\mathcal{F}^{\alpha}_{\overline{BC}} \end{cases}$$

Identification

→ Vanishing torsion \Rightarrow
$$\begin{cases} \mathcal{A}_{\underline{a}\bar{\delta}}(\tau^{\bar{\delta}})_{\overline{BC}} = -\mathcal{F}_{\underline{a}\overline{BC}}, \\ \mathcal{A}^{\alpha}_{\bar{\delta}}(\tau^{\bar{\delta}})_{\overline{BC}} = -\mathcal{F}^{\alpha}_{\overline{BC}} \end{cases}$$

$$\mathcal{F}_A = \mathcal{L}_{\varepsilon_A} \mathcal{E} \mathcal{E}^{-1}, \quad \mathcal{E} = \mathcal{A}E$$

Change of basis (again)

- t_α are the natural choice for the identification but we want to do it in terms of τ_α

Change of basis (again)

- t_α are the natural choice for the identification but we want to do it in terms of τ_α

$$A_{\mathcal{A}}^{(l)\beta} t_\beta = \hat{A}_{\mathcal{A}}^{(l)\beta} \tau_\beta \Rightarrow A_{\mathcal{A}}^{(l)\beta} = \hat{A}_{\mathcal{A}}^{(l)\gamma} (S^{-1})_\gamma^\beta$$

Change of basis (again)

- t_α are the natural choice for the identification but we want to do it in terms of τ_α

$$A_{\mathcal{A}}^{(l)\beta} t_\beta = \hat{A}_{\mathcal{A}}^{(l)\beta} \tau_\beta \Rightarrow A_{\mathcal{A}}^{(l)\beta} = \hat{A}_{\mathcal{A}}^{(l)\gamma} (S^{-1})_{\gamma}{}^{\beta}$$

$$\mathcal{G}_{\mathcal{A}}^{(l)\beta} t_\beta = \hat{\mathcal{G}}_{\mathcal{A}}^{(l)\beta} \tau_\beta \Rightarrow \mathcal{G}_{\mathcal{A}}^{(l)\beta} = \hat{\mathcal{G}}_{\mathcal{A}}^{(l)\gamma} (S^{-1})_{\gamma}{}^{\beta}$$

$$\mathcal{G}_{\mathcal{A}}^{(l)\beta} [A^{(<l)}] = \mathcal{A}_{\mathcal{A}}^{(l)\beta} - A_{\mathcal{A}}^{(l)\beta}$$

Finally

$$\hat{A}_{\underline{\mathcal{A}}\underline{\beta}}^{(l)} = -\frac{a}{\underline{c}^2} \langle \langle \mathcal{F}_{\underline{\mathcal{A}}}^{(l)} [A^{(<l)}], \tau_{\underline{\beta}} \rangle \rangle - \hat{\mathcal{G}}_{\underline{\mathcal{A}}\underline{\beta}}^{(l)} [A^{(<l)}],$$

$$\hat{A}_{\underline{\mathcal{A}}\overline{\beta}}^{(l)} = -\frac{b}{\overline{c}^2} \langle \langle \mathcal{F}_{\underline{\mathcal{A}}}^{(l)} [A^{(<l)}], \tau_{\overline{\beta}} \rangle \rangle - \hat{\mathcal{G}}_{\underline{\mathcal{A}}\overline{\beta}}^{(l)} [A^{(<l)}]$$

Results

$$\hat{A}_{\underline{a}\underline{\beta}_1}^{(1)} = \hat{A}_{\underline{a}}^{(1)\underline{b}_1\underline{b}_2} = -\frac{\sqrt{a}}{\underline{c}} F_{\underline{a}}^{\underline{b}_1\underline{b}_2}$$

$$\hat{A}_{\underline{a}\underline{\beta}_2}^{(2)} = \hat{A}_{\underline{a}}^{(2)\underline{b}_1}{}_{\underline{\beta}_1} = \frac{\sqrt{a}}{\underline{c}} \left(F_{\underline{a}}^{\underline{b}_1\bar{c}} \hat{A}_{\bar{c}\underline{\beta}_1}^{(1)} + D^{\underline{b}_1} \hat{A}_{\underline{a}\underline{\beta}_1}^{(1)} \right)$$

$$\hat{A}_{\underline{a}\underline{\beta}_1}^{(3)} = \hat{A}_{\underline{a}}^{(3)\underline{b}_1\underline{b}_2} = \frac{\sqrt{a}}{\underline{c}} \bar{\chi}_1 \left(\hat{A}^{(1)[\underline{b}_1\bar{\gamma}_1} D_{\underline{a}} \hat{A}^{(1)\underline{b}_2]\bar{\gamma}_1} - \hat{A}^{[\underline{b}_1\bar{\gamma}_1} F_{\underline{a}}^{\underline{b}_2]\underline{c}} \hat{A}_{\underline{c}}^{(1)\bar{\gamma}_1} \right)$$

$$\begin{aligned} \hat{A}_{\underline{a}\underline{\beta}_2}^{(3)} = \hat{A}_{\underline{a}\underline{\beta}_1\underline{\gamma}_1}^{(3)} = \frac{\sqrt{a}}{\underline{c}} \left(\hat{A}_{\bar{b}[\underline{\beta}_1]}^{(1)} D_{\underline{a}} \hat{A}^{(1)\bar{b}}{}_{[\underline{\gamma}_1]} - \hat{A}^{(1)\bar{b}}{}_{[\underline{\beta}_1} \hat{A}^{(1)\bar{c}}{}_{\underline{\gamma}_1]} F_{\underline{a}\bar{b}\bar{c}} + \right. \\ \left. + 2D_{\bar{b}} \hat{A}_{\underline{a}[\underline{\beta}_1}^{(1)} \hat{A}^{(1)\bar{b}}{}_{\underline{\gamma}_1]} + \hat{A}_{\underline{a}}^{(1)\underline{\alpha}_1} f_{\underline{\delta}_1[\underline{\beta}_1|\underline{\alpha}_1} \hat{A}_{\bar{b}[\underline{\gamma}_1]}^{(1)} \hat{A}^{(1)\bar{b}\underline{\delta}_1} \right) \end{aligned}$$

Gauge transformations

$$\mathcal{A}^{-1}\delta\mathcal{A} + \delta E^{(l)} E^{-1} = -D_A \xi^\beta R_\beta^A + \frac{1}{2} \xi^\alpha f_{\alpha\beta\gamma} R^{\beta\gamma} + \xi^\alpha \mathcal{A}^{-1} t_\alpha \mathcal{A}$$

Gauge transformations

$$\mathcal{A}^{-1}\delta\mathcal{A} + \delta E^{(l)} E^{-1} = -D_A \xi^\beta R_\beta^A + \frac{1}{2} \xi^\alpha f_{\alpha\beta\gamma} R^{\beta\gamma} + \xi^\alpha \mathcal{A}^{-1} t_\alpha \mathcal{A}$$

\Downarrow

$$\delta A^{(l)} + \delta E^{(l)} E^{-1} - \xi^{(l)\alpha} t_\alpha = \mathcal{X}^{(l)}[\xi^{(<l)}] - \delta G^{(l)}[\xi^{(<l)}]$$

Gauge transformations

$$\mathcal{A}^{-1} \delta \mathcal{A} + \delta E^{(l)} E^{-1} = -D_A \xi^\beta R_\beta^A + \frac{1}{2} \xi^\alpha f_{\alpha\beta\gamma} R^{\beta\gamma} + \xi^\alpha \mathcal{A}^{-1} t_\alpha \mathcal{A}$$

⇓

$$\delta A^{(l)} + \delta E^{(l)} E^{-1} - \xi^{(l)\alpha} t_\alpha = \mathcal{X}^{(l)}[\xi^{(<l)}] - \delta G^{(l)}[\xi^{(<l)}]$$

Mixed chir.
 $R_\alpha^A, R_{\alpha\beta}$

Mixed chir.
 K_{AB}

(Anti-)chir.
 $K_{AB}, R_\alpha^A, R_{\alpha\beta}$

$\delta A'^{(l)} - \delta A^{(l)}$

Results (again)

$$\hat{\xi}^{(0)\underline{\alpha}_1} = \hat{\xi}_{\underline{ab}}^{(0)} = -\frac{\sqrt{a}}{\underline{c}} \Lambda_{\underline{ab}}$$

$$\hat{\xi}^{(1)\underline{\alpha}_2} = \hat{\xi}_{\underline{a}}^{(1)\underline{\alpha}_1} = \frac{2\sqrt{a}}{\underline{c}} D_{\underline{a}} \hat{\xi}^{(0)\underline{\alpha}_1}$$

$$\hat{\xi}^{(2)\underline{\alpha}_1} = \hat{\xi}_{\underline{ab}}^{(2)} = -\frac{\sqrt{a}}{\underline{c}} \bar{\chi}_1 \hat{A}_{[\underline{a}_1|\bar{\beta}_1]}^{(1)} D_{|\underline{b}]}\hat{\xi}^{(0)\bar{\beta}_1}$$

$$\hat{\xi}^{(2)\underline{\alpha}_2} = \hat{\xi}^{(2)\underline{\alpha}_1\underline{\beta}_1} = \frac{\sqrt{a}}{\underline{c}} \hat{A}^{\bar{c}[\underline{\alpha}_1} D_{\bar{c}} \hat{\xi}^{(0)\underline{\beta}_1]}$$

GS transformations (0th order)

$$\delta E^{(0)} E^{-1} = \xi^{(0)}$$

GS transformations (0th order)

$$\delta E^{(0)} E^{-1} = \xi^{(0)}$$

⇓

$$\delta E_{\underline{ab}} = -\Lambda_{\underline{ab}}$$

GS transformations (2nd order)

$$\delta E^{(2)} E^{-1} \cong [A^{(1)}, D\xi^{(0)}]$$

GS transformations (2nd order)

$$\delta E^{(2)} E^{-1} \cong [A^{(1)}, D\xi^{(0)}]$$



$$\delta E_{\underline{a}\bar{b}}^{(2)} = A_{\underline{a}\bar{\alpha}}^{(1)} D_{\bar{b}} \xi^{(0)\bar{\alpha}} - c.c. = \bar{\chi}_1 \hat{A}_{\underline{a}\bar{\alpha}_1}^{(1)} D_{\bar{b}} \hat{\xi}^{(0)\bar{\alpha}_1} - c.c.$$

GS transformations (2nd order)

$$\delta E^{(2)} E^{-1} \cong [A^{(1)}, D\xi^{(0)}]$$

↓

$$\delta E_{\underline{a}\bar{b}}^{(2)} = A_{\underline{a}\bar{\alpha}}^{(1)} D_{\bar{b}} \xi^{(0)\bar{\alpha}} - c.c. = \bar{\chi}_1 \hat{A}_{\underline{a}\bar{\alpha}_1}^{(1)} D_{\bar{b}} \hat{\xi}^{(0)\bar{\alpha}_1} - c.c. =$$

$$= -\frac{a}{\underline{c}^2} \underline{\chi}_1 D_{\underline{a}} \Lambda^{\underline{c}_1 \underline{c}_2} F_{\bar{b}\underline{c}_1 \underline{c}_2} + \frac{b}{\bar{c}^2} \bar{\chi}_1 D_{\bar{b}} \Lambda^{\bar{c}_1 \bar{c}_2} F_{\underline{a}\bar{c}_1 \bar{c}_2} = \frac{a}{2} D_{\underline{a}} \Lambda^{\underline{c}_1 \underline{c}_2} F_{\bar{b}\underline{c}_1 \underline{c}_2} + \frac{b}{2} D_{\bar{b}} \Lambda^{\bar{c}_1 \bar{c}_2} F_{\underline{a}\bar{c}_1 \bar{c}_2}$$

GS transformations (2nd order)

$$\delta E^{(2)} E^{-1} \cong [A^{(1)}, D\xi^{(0)}]$$

↓

$$\delta E_{\underline{a}\bar{b}}^{(2)} = A_{\underline{a}\bar{\alpha}}^{(1)} D_{\bar{b}} \xi^{(0)\bar{\alpha}} - c.c. = \bar{\chi}_1 \hat{A}_{\underline{a}\bar{\alpha}_1}^{(1)} D_{\bar{b}} \hat{\xi}^{(0)\bar{\alpha}_1} - c.c. =$$

$$= -\frac{a}{\underline{c}^2} \underline{\chi}_1 D_{\underline{a}} \Lambda^{\underline{c}_1 \underline{c}_2} F_{\bar{b}\underline{c}_1 \underline{c}_2} + \frac{b}{\bar{c}^2} \bar{\chi}_1 D_{\bar{b}} \Lambda^{\bar{c}_1 \bar{c}_2} F_{\underline{a}\bar{c}_1 \bar{c}_2} = \frac{a}{2} D_{\underline{a}} \Lambda^{\underline{c}_1 \underline{c}_2} F_{\bar{b}\underline{c}_1 \underline{c}_2} + \frac{b}{2} D_{\bar{b}} \Lambda^{\bar{c}_1 \bar{c}_2} F_{\underline{a}\bar{c}_1 \bar{c}_2}$$

$$\underline{c}^2 = -2\underline{\chi}_1, \quad \bar{c}^2 = 2\bar{\chi}_1$$

GS transformations (4th order)

$$\delta E^{(4)} E^{-1} \cong -[A^{(1)}, D\xi^{(2)}] - [A^{(2)}, D\xi^{(1)}] - [A^{(3)}, D\xi^{(0)}]$$

GS transformations (4th order)

$$\delta E^{(4)} E^{-1} \cong -[A^{(1)}, D\xi^{(2)}] - [A^{(2)}, D\xi^{(1)}] - [A^{(3)}, D\xi^{(0)}]$$

\Downarrow

$$\delta E_{\underline{a}\bar{b}}^{(4)} = A_{\underline{a}\bar{\alpha}}^{(1)} D_{\bar{b}} \xi^{(2)\bar{\alpha}} + A_{\underline{a}\bar{\alpha}}^{(2)} D_{\bar{b}} \xi^{(1)\bar{\alpha}} + A_{\underline{a}\bar{\alpha}}^{(3)} D_{\bar{b}} \xi^{(0)\bar{\alpha}} - c.c.$$

GS transformations (4th order)

$$\delta E^{(4)} E^{-1} \cong -[A^{(1)}, D\xi^{(2)}] - [A^{(2)}, D\xi^{(1)}] - [A^{(3)}, D\xi^{(0)}]$$

↓

$$\delta E_{\underline{a}\bar{b}}^{(4)} = A_{\underline{a}\bar{\alpha}}^{(1)} D_{\bar{b}} \xi^{(2)\bar{\alpha}} + A_{\underline{a}\bar{\alpha}}^{(2)} D_{\bar{b}} \xi^{(1)\bar{\alpha}} + A_{\underline{a}\bar{\alpha}}^{(3)} D_{\bar{b}} \xi^{(0)\bar{\alpha}} - c.c.$$

$$\begin{aligned} \delta E_{\underline{a}\bar{b}}^{(4)} = & -\frac{a^2 \chi_2}{2\chi_1^2} \left[D_{\underline{a}} D_{\underline{c}} \Lambda_{\underline{d}\underline{e}} \left(F_{\underline{f}\underline{b}}^{\underline{c}} F^{\underline{f}\underline{d}\underline{e}} + D^{\underline{c}} F_{\underline{b}}^{\underline{d}\underline{e}} \right) - F_{\underline{b}\underline{f}}^{\underline{g}} F^{\underline{c}\underline{d}\underline{g}} \left(F_{\underline{c}}^{\underline{e}\underline{d}} D_{\underline{a}} \Lambda_{\underline{e}}^{\underline{f}} - F_{\underline{c}}^{\underline{e}\underline{f}} D_{\underline{a}} \Lambda_{\underline{e}}^{\underline{d}} \right) \right. \\ & \left. + D_{\underline{a}} \Lambda_{\underline{e}\underline{f}} F^{\underline{c}\underline{e}}_{\underline{d}} \left(F_{\underline{b}\underline{c}\underline{g}}^{\underline{g}} F^{\underline{g}\underline{f}\underline{d}} - D_{\underline{b}} F_{\underline{c}}^{\underline{f}\underline{d}} + 2D_{\underline{c}} F_{\underline{b}}^{\underline{f}\underline{d}} \right) + F_{\underline{b}}^{\underline{e}\underline{d}} D_{\underline{a}} \left(D^{\underline{c}} \Lambda_{\underline{e}\underline{f}}^{\underline{f}} F_{\underline{c}\underline{f}\underline{d}} \right) \right] \\ & - \frac{ab}{4} \left[D_{\underline{a}} \Lambda^{\underline{c}\underline{d}} \left(F_{\underline{b}\underline{c}\underline{g}}^{\underline{g}} F^{\underline{g}\underline{e}\underline{f}} F_{\underline{d}\underline{e}\underline{f}} - D_{\underline{b}} F_{\underline{c}}^{\underline{e}\underline{f}} F_{\underline{d}\underline{e}\underline{f}} \right) + F_{\underline{b}\underline{c}\underline{d}} D_{\underline{a}} \left(D^{\underline{c}} \Lambda_{\underline{e}\underline{f}}^{\underline{f}} F^{\underline{d}\underline{e}\underline{f}} \right) \right. \\ & \left. - D_{\underline{b}} \Lambda^{\underline{c}\underline{d}} \left(F_{\underline{a}\underline{c}\underline{g}}^{\underline{g}} F^{\underline{g}\underline{e}\underline{f}} F_{\underline{d}\underline{e}\underline{f}} - D_{\underline{a}} F_{\underline{c}}^{\underline{e}\underline{f}} F_{\underline{d}\underline{e}\underline{f}} \right) - F_{\underline{a}\underline{c}\underline{d}} D_{\underline{b}} \left(D^{\underline{c}} \Lambda_{\underline{e}\underline{f}}^{\underline{f}} F^{\underline{d}\underline{e}\underline{f}} \right) \right] \\ & + \frac{b^2 \bar{\chi}_2}{2\bar{\chi}_1^2} \left[D_{\bar{b}} D_{\bar{c}} \Lambda_{\bar{d}\bar{e}} \left(F_{\bar{f}\bar{a}}^{\bar{c}} F^{\bar{f}\bar{d}\bar{e}} + D^{\bar{c}} F_{\bar{a}}^{\bar{d}\bar{e}} \right) - F_{\bar{a}\bar{f}}^{\bar{g}} F^{\bar{c}\bar{d}\bar{g}} \left(F_{\bar{c}}^{\bar{e}\bar{d}} D_{\bar{b}} \Lambda_{\bar{e}}^{\bar{f}} - F_{\bar{c}}^{\bar{e}\bar{f}} D_{\bar{b}} \Lambda_{\bar{e}}^{\bar{d}} \right) \right. \\ & \left. + D_{\bar{b}} \Lambda_{\bar{e}\bar{f}} F^{\bar{c}\bar{e}}_{\bar{d}} \left(F_{\bar{a}\bar{c}\bar{g}}^{\bar{g}} F^{\bar{g}\bar{f}\bar{d}} - D_{\bar{a}} F_{\bar{c}}^{\bar{f}\bar{d}} + 2D_{\bar{c}} F_{\bar{a}}^{\bar{f}\bar{d}} \right) + F_{\bar{a}}^{\bar{e}\bar{d}} D_{\bar{b}} \left(D^{\bar{c}} \Lambda_{\bar{e}\bar{f}}^{\bar{f}} F_{\bar{c}\bar{f}\bar{d}} \right) \right] \end{aligned}$$

GS transformations (4th order)

$$\delta E^{(4)} E^{-1} \cong -[A^{(1)}, D\xi^{(2)}] - [A^{(2)}, D\xi^{(1)}] - [A^{(3)}, D\xi^{(0)}]$$



$$\delta E_{\underline{a}\bar{b}}^{(4)} = A_{\underline{a}\bar{\alpha}}^{(1)} D_{\bar{b}} \xi^{(2)\bar{\alpha}} + A_{\underline{a}\bar{\alpha}}^{(2)} D_{\bar{b}} \xi^{(1)\bar{\alpha}} + A_{\underline{a}\bar{\alpha}}^{(3)} D_{\bar{b}} \xi^{(0)\bar{\alpha}} - c.c.$$

$$\begin{aligned} \delta E_{\underline{a}\bar{b}}^{(4)} = & -\frac{a^2 \chi_2}{2\chi_1^2} \left[D_{\underline{a}} D_{\underline{c}} \Lambda_{\underline{d}\underline{e}} \left(F_{\underline{f}\underline{b}}^{\underline{c}} F^{\underline{f}\underline{d}\underline{e}} + D^{\underline{c}} F_{\underline{b}}^{\underline{d}\underline{e}} \right) - F_{\underline{b}\underline{f}}^{\underline{g}} F_{\underline{d}\underline{g}}^{\underline{c}} \left(F_{\underline{c}}^{\underline{e}\underline{d}} D_{\underline{a}} \Lambda_{\underline{e}}^{\underline{f}} - F_{\underline{c}}^{\underline{e}\underline{f}} D_{\underline{a}} \Lambda_{\underline{e}}^{\underline{d}} \right) \right. \\ & \left. + D_{\underline{a}} \Lambda_{\underline{e}\underline{f}} F_{\underline{d}}^{\underline{c}\underline{e}} \left(F_{\underline{b}\underline{c}\underline{g}} F^{\underline{g}\underline{f}\underline{d}} - D_{\underline{b}} F_{\underline{c}}^{\underline{f}\underline{d}} + 2D_{\underline{c}} F_{\underline{b}}^{\underline{f}\underline{d}} \right) + F_{\underline{b}}^{\underline{e}\underline{d}} D_{\underline{a}} \left(D^{\underline{c}} \Lambda_{\underline{e}}^{\underline{f}} F_{\underline{c}\underline{f}\underline{d}} \right) \right] \\ & - \frac{ab}{4} \left[D_{\underline{a}} \Lambda^{\underline{c}\underline{d}} \left(F_{\underline{b}\underline{c}\underline{g}} F^{\underline{g}\underline{e}\underline{f}} F_{\underline{d}\underline{e}\underline{f}} - D_{\underline{b}} F_{\underline{c}}^{\underline{e}\underline{f}} F_{\underline{d}\underline{e}\underline{f}} \right) + F_{\underline{b}\underline{c}\underline{d}} D_{\underline{a}} \left(D^{\underline{c}} \Lambda_{\underline{e}\underline{f}} F^{\underline{d}\underline{e}\underline{f}} \right) \right. \\ & \left. - D_{\underline{b}} \Lambda^{\underline{c}\underline{d}} \left(F_{\underline{a}\underline{c}\underline{g}} F^{\underline{g}\underline{e}\underline{f}} F_{\underline{d}\underline{e}\underline{f}} - D_{\underline{a}} F_{\underline{c}}^{\underline{e}\underline{f}} F_{\underline{d}\underline{e}\underline{f}} \right) - F_{\underline{a}\underline{c}\underline{d}} D_{\underline{b}} \left(D^{\underline{c}} \Lambda_{\underline{e}\underline{f}} F^{\underline{d}\underline{e}\underline{f}} \right) \right] \\ & + \frac{b^2 \chi_2}{2\chi_1^2} \left[D_{\bar{b}} D_{\bar{c}} \Lambda_{\underline{d}\underline{e}} \left(F_{\underline{f}\underline{a}}^{\bar{c}} F^{\underline{f}\bar{d}\bar{e}} + D^{\bar{c}} F_{\underline{a}}^{\bar{d}\bar{e}} \right) - F_{\underline{a}\underline{f}}^{\bar{g}} F_{\underline{d}\underline{g}}^{\bar{c}} \left(F_{\underline{c}}^{\bar{e}\bar{d}} D_{\bar{b}} \Lambda_{\bar{e}}^{\bar{f}} - F_{\underline{c}}^{\bar{e}\bar{f}} D_{\bar{b}} \Lambda_{\bar{e}}^{\bar{d}} \right) \right. \\ & \left. + D_{\bar{b}} \Lambda_{\bar{e}\bar{f}} F_{\bar{d}}^{\bar{c}\bar{e}} \left(F_{\underline{a}\underline{c}\underline{g}} F^{\underline{g}\bar{f}\bar{d}} - D_{\underline{a}} F_{\underline{c}}^{\bar{f}\bar{d}} + 2D_{\underline{c}} F_{\underline{a}}^{\bar{f}\bar{d}} \right) + F_{\underline{a}}^{\bar{e}\bar{d}} D_{\bar{b}} \left(D^{\bar{c}} \Lambda_{\bar{e}}^{\bar{f}} F_{\underline{c}\bar{f}\bar{d}} \right) \right] \end{aligned}$$

Need futher study

GS transformations (6th order)



A pattern (?)

$$\delta E_{\underline{a}\bar{b}}^{(2m)} = \sum_{n=1}^{2m-1} A_{\underline{a}\bar{\alpha}}^{(n)} D_{\bar{b}\bar{\xi}}^{(2m-n-1)\bar{\alpha}}$$

A pattern (?)

$$\delta E_{\underline{a}\bar{b}}^{(2m)} = \sum_{n=1}^{2m-1} A_{\underline{a}\bar{\alpha}}^{(n)} D_{\bar{b}\xi}^{(2m-n-1)\bar{\alpha}}$$

Depends on the coefficients of $A' = A + c_1 A^3 + c_2 A^5 + c_3 A^7 + \dots$

gbdRi vs deformed PS

$$\begin{aligned}
 \delta E_{\underline{a}\bar{b}}^{(4)} = & -\frac{a^2 \underline{\chi}_2}{2\underline{\chi}_1^2} \left[D_{\underline{a}} D_{\underline{c}} \Lambda_{\underline{de}} \left(F_{\underline{fb}}^{\underline{c}} F^{\bar{f}\underline{de}} + D_{\underline{b}}^{\underline{c}} F_{\underline{de}} \right) - F_{\underline{bf}}^{\underline{g}} F^{\bar{c}}_{\underline{dg}} \left(F_{\underline{c}}^{\underline{ed}} D_{\underline{a}} \Lambda_{\underline{e}}^{\underline{f}} - F_{\underline{c}}^{\underline{ef}} D_{\underline{a}} \Lambda_{\underline{e}}^{\underline{d}} \right) \right. \\
 & \left. + D_{\underline{a}} \Lambda_{\underline{ef}} F^{\bar{c}\underline{e}}_{\underline{d}} \left(F_{\underline{bcg}}^{\bar{c}} F^{\underline{g}\underline{fd}} - D_{\underline{b}} F_{\underline{c}}^{\underline{fd}} + 2D_{\underline{c}} F_{\underline{b}}^{\underline{fd}} \right) + F_{\underline{b}}^{\underline{ed}} D_{\underline{a}} \left(D^{\bar{c}} \Lambda_{\underline{e}}^{\underline{f}} F_{\underline{c}}^{\underline{fd}} \right) \right] \\
 & -\frac{ab}{4} \left[D_{\underline{a}} \Lambda^{\underline{cd}} \left(F_{\underline{bcg}}^{\underline{g}\bar{e}\bar{f}} F_{\underline{de}\bar{f}} - D_{\underline{b}} F_{\underline{c}}^{\bar{e}\bar{f}} F_{\underline{de}\bar{f}} \right) + F_{\underline{bcd}} D_{\underline{a}} \left(D^{\underline{c}} \Lambda_{\underline{e}\bar{f}} F^{\underline{d}\bar{e}\bar{f}} \right) \right. \\
 & \left. - D_{\underline{b}} \Lambda^{\bar{c}\underline{d}} \left(F_{\underline{acg}}^{\bar{g}\underline{e}\underline{f}} F_{\underline{de}\bar{f}} - D_{\underline{a}} F_{\underline{c}}^{\underline{ef}} F_{\underline{de}\bar{f}} \right) - F_{\underline{acd}} D_{\underline{b}} \left(D^{\bar{c}} \Lambda_{\underline{e}\bar{f}} F^{\underline{d}\bar{e}\underline{f}} \right) \right] \\
 & +\frac{b^2 \bar{\chi}_2}{2\bar{\chi}_1^2} \left[D_{\bar{b}} D_{\bar{c}} \Lambda_{\underline{de}} \left(F_{\underline{fa}}^{\bar{c}} F^{\underline{f}\bar{de}} + D_{\bar{a}}^{\bar{c}} F_{\underline{de}} \right) - F_{\underline{af}}^{\bar{g}} F_{\underline{dg}}^{\underline{c}} \left(F_{\underline{c}}^{\bar{e}\bar{d}} D_{\bar{b}} \Lambda_{\bar{e}}^{\bar{f}} - F_{\underline{c}}^{\bar{e}\bar{f}} D_{\bar{b}} \Lambda_{\bar{e}}^{\bar{d}} \right) \right. \\
 & \left. + D_{\bar{b}} \Lambda_{\bar{e}\bar{f}} F^{\bar{c}\underline{e}}_{\bar{d}} \left(F_{\underline{acg}}^{\bar{g}\bar{f}\bar{d}} - D_{\underline{a}} F_{\underline{c}}^{\bar{f}\bar{d}} + 2D_{\underline{c}} F_{\underline{a}}^{\bar{f}\bar{d}} \right) + F_{\underline{a}}^{\bar{e}\bar{d}} D_{\bar{b}} \left(D^{\underline{c}} \Lambda_{\bar{e}}^{\bar{f}} F_{\underline{c}}^{\bar{f}\bar{d}} \right) \right]
 \end{aligned}$$

gBdRi vs deformed PS

$$\begin{aligned}
 \delta E_{\underline{a}\bar{b}}^{(4)} = & -\frac{a^2 \chi_2}{2\chi_1^2} \left[D_{\underline{a}} D_{\underline{c}} \Lambda_{\underline{de}} \left(F_{\underline{fb}}^{\underline{c}} F^{\bar{f}de} + D_{\underline{b}}^{\underline{c}} F_{\underline{d}}^{\underline{e}} \right) - F_{\underline{bf}}^{\underline{g}} F^{\bar{c}}_{\underline{dg}} \left(F_{\underline{c}}^{\underline{ed}} D_{\underline{a}} \Lambda_{\underline{e}}^{\underline{f}} - F_{\underline{c}}^{\underline{ef}} D_{\underline{a}} \Lambda_{\underline{e}}^{\underline{d}} \right) \right. \\
 & \left. + D_{\underline{a}} \Lambda_{\underline{ef}} F^{\bar{c}\underline{e}}_{\underline{d}} \left(F_{\underline{bcg}}^{\bar{c}} F^{\bar{g}fd} - D_{\underline{b}} F_{\underline{c}}^{\underline{fd}} + 2D_{\underline{c}} F_{\underline{b}}^{\underline{fd}} \right) + F_{\underline{b}}^{\underline{ed}} D_{\underline{a}} \left(D^{\bar{c}} \Lambda_{\underline{e}}^{\underline{f}} F_{\underline{c}}^{\underline{fd}} \right) \right] \\
 & -\frac{ab}{4} \left[D_{\underline{a}} \Lambda^{\underline{cd}} \left(F_{\underline{bcg}}^{\bar{c}} F^{\bar{g}\underline{e}\underline{f}} F_{\underline{de}\underline{f}} - D_{\underline{b}} F_{\underline{c}}^{\bar{e}\underline{f}} F_{\underline{de}\underline{f}} \right) + F_{\underline{bcd}} D_{\underline{a}} \left(D^{\underline{c}} \Lambda_{\underline{e}\underline{f}} F^{\underline{d}\bar{e}\underline{f}} \right) \right. \\
 & \left. - D_{\underline{b}} \Lambda^{\bar{c}\underline{d}} \left(F_{\underline{acg}}^{\bar{c}} F^{\bar{g}\underline{e}\underline{f}} F_{\underline{de}\underline{f}} - D_{\underline{a}} F_{\underline{c}}^{\underline{ef}} F_{\underline{de}\underline{f}} \right) - F_{\underline{acd}} D_{\underline{b}} \left(D^{\bar{c}} \Lambda_{\underline{e}\underline{f}} F^{\underline{d}\bar{e}\underline{f}} \right) \right] \\
 & +\frac{b^2 \bar{\chi}_2}{2\bar{\chi}_1^2} \left[D_{\bar{b}} D_{\bar{c}} \Lambda_{\underline{de}} \left(F_{\underline{fa}}^{\bar{c}} F^{\underline{f}\bar{d}\bar{e}} + D_{\bar{a}}^{\bar{c}} F_{\underline{d}}^{\bar{e}} \right) - F_{\underline{af}}^{\bar{g}} F_{\underline{dg}}^{\underline{c}} \left(F_{\underline{c}}^{\bar{e}\bar{d}} D_{\bar{b}} \Lambda_{\bar{e}}^{\bar{f}} - F_{\underline{c}}^{\bar{e}\bar{f}} D_{\bar{b}} \Lambda_{\bar{e}}^{\bar{d}} \right) \right. \\
 & \left. + D_{\bar{b}} \Lambda_{\bar{e}\bar{f}} F^{\bar{c}\underline{e}}_{\bar{d}} \left(F_{\underline{acg}}^{\bar{c}} F^{\bar{g}\underline{f}\bar{d}} - D_{\underline{a}} F_{\underline{c}}^{\underline{f}\bar{d}} + 2D_{\underline{c}} F_{\underline{a}}^{\underline{f}\bar{d}} \right) + F_{\underline{a}}^{\bar{e}\bar{d}} D_{\bar{b}} \left(D^{\underline{c}} \Lambda_{\bar{e}}^{\bar{f}} F_{\underline{c}}^{\underline{f}\bar{d}} \right) \right]
 \end{aligned}$$

VS

$$\delta E_{\underline{a}\bar{b}}^{(4)} = A_{\underline{a}\bar{\alpha}}^{(1)} D_{\bar{b}} \xi^{(2)\bar{\alpha}} + A_{\underline{a}\bar{\alpha}}^{(2)} D_{\bar{b}} \xi^{(1)\bar{\alpha}} + A_{\underline{a}\bar{\alpha}}^{(3)} D_{\bar{b}} \xi^{(0)\bar{\alpha}} - c.c.$$

THANK YOU FOR YOUR ATTENTION