Quantum Theory, Gravity and Higher Order Geometry

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Quantum Gravity, Strings and the Swampland

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Summary & Contents

Summarizing statements

- Our understanding of Classical Gravity, encoded in General Relativity, relies on Riemannian Geometry.
- Quantum Theory and (first order) Riemannian Geometry are incompatible.
- Second order geometry is a **minimal extension** that makes Riemannian geometry compatible with quantum theory.
- Extensions to kth-order geometry with $k \in \mathbb{N}$ are possible.

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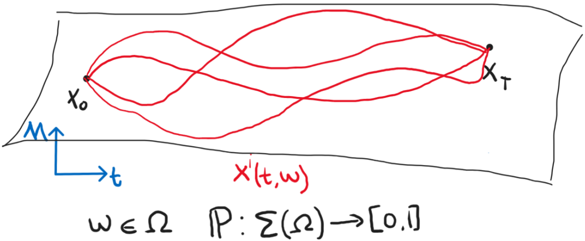
Contents

Incompatibility between Geometry and Quantum Theory

A path integral analysis

Quantum Mechanics

Replace single path by (uncountably) many paths weighted by $\mathbb{P}$



$\mathbb{P}$ on Ω induces $\mu = \mathbb{P} \circ X^{-1}$ on $L^2(\Omega)$, such that $\mathrm{d}\mu_X = e^{\frac{\mathrm{i}\, S(X)}{\hbar}} DX$

Note: Picture can be generalized to (relativistic) field theory.

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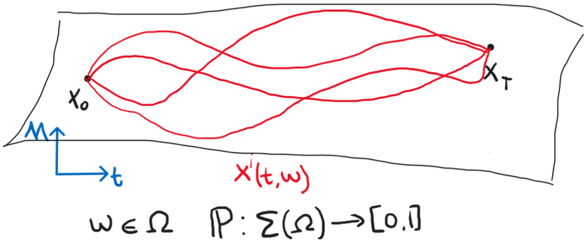

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Problem: Existence[1] of probability measure $\mathbb{P}$

Question: What do the paths look like?

[1] Albeverio, Høegh-Krohn, Mazzucchi, Lecture Notes in Math. 523, Springer (2008).

Existence of $\mathbb{P}$

Euclidean theory

Wick rotation:

1. Change of signature: Lorentzian $\rightarrow$ Euclidean
2. Change of diffusion: Quantum $\rightarrow$ Statistical

Wick rotation $\rightarrow$ Wiener integral[2]

$\Rightarrow$ X becomes a Wiener process (a.k.a. Brownian Motion)

$\Rightarrow$ $\mathbb{P}$ exists in the Euclidean Theory

Lorentzian Theory

$\mathbb{P}$ does not exist (in a minimal formulation).[3]

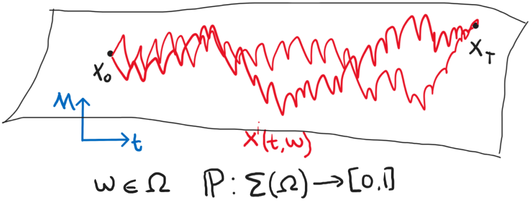
Existence of $\mathbb{P}$ can be achieved by complexifying the tangent bundle:[4]

$$T\mathcal{M} = \bigsqcup_{x\in\mathcal{M}} T_x\mathcal{M} \rightarrow \bigsqcup_{x\in\mathcal{M}} T_x\mathcal{M}^{\mathbb{C}}$$

[2] M. Kac, Trans. Amer. Math. Soc. 65 (1949).

[3] R. Cameron, J. Math. and Phys. (1960); Yu. Daletskii, Russ.Math.Surv. (1962).

[4] M. Pavon, J. Math. Phys. 41 (2000); FK, Eur. Phys. J. Plus 138 (2023).



$X(\omega) \in \mathcal{C}^0$ (continuous everywhere), $X(\omega) \notin \mathcal{C}^1$ (nowhere differentiable)
More precisely,[5] $X(\omega)$ is α-Hölder continuous only for $\alpha < 1/2$:

$$X(\omega) \in \mathcal{C}^{\alpha}(\mathbb{R}, \mathcal{M}), \qquad \alpha \in [0, 1/2).$$

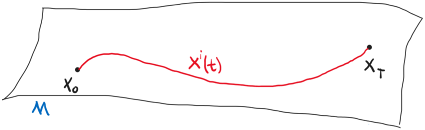
Path integral formulation | **Canonical Quantization**

Roughness of paths $\overset{\text{time-ordering}^6}{\longleftrightarrow}$ Non-commutativity

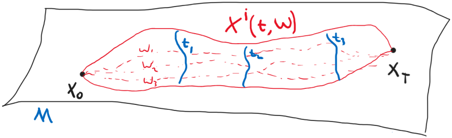
[5] c.f. e.g. Mörters & Peres, Cambridge UP (2012).
[6] R.P. Feynman, Rev. Mod. Phys. 20 (1948).

Property II: Worldlines → Worldsheets

Classical theory: $\dim\{X(\tau) : \tau \in [0, T]\} = 1$



Quantum Theory:[7] $\dim_{\text{Hausdorff}}\{X(\tau, \omega) : \tau \in [0, T], \omega \in \Omega\} = 2.$



Note: Upper bound is set by α-Hölder regularity:

$$\dim_{\text{Hausdorff}}\{X(\tau, \omega) : \tau \in [0, T], \omega \in \Omega\} \leq \alpha_{\text{sup}}^{-1}$$

[7] c.f. e.g. Mörters & Peres, Cambridge UP (2012).

Implications of roughness

Paths are non-differentiable:

$$\lim_{d\tau\to 0} \frac{X^\mu(\tau + d\tau) - X^\mu(\tau)}{\mathrm{d}\tau} = \pm\infty\,.$$

By taking an expectation value, we obtain well-defined limits

$$v_+^\mu = \lim_{d\tau\to 0} \left\langle \frac{X^\mu(\tau + d\tau) - X^\mu(\tau)}{d\tau} \right\rangle,$$
$$v_-^\mu = \lim_{d\tau\to 0} \left\langle \frac{X^\mu(\tau) - X^\mu(\tau - d\tau)}{d\tau} \right\rangle$$

defining independent velocities along the path.
Similarly, one can define the second order object

$$v_2^{\mu\nu} = \lim_{d\tau\to 0} \left\langle \frac{[X^\mu(\tau + d\tau) - X^\mu(\tau)][X^\nu(\tau + d\tau) - X^\nu(\tau)]}{d\tau} \right\rangle,$$

which is the velocity of the variance.

Given $\omega \in \Omega$, $X(\omega) : \mathbb{R} \to \mathcal{M}$ and $f, g \in \mathcal{C}^2(\mathcal{M}, \mathbb{C})$, $h \in \mathcal{C}^2(\mathbb{C}, \mathbb{C})$
We define the increment

$$\mathrm{d}_+ X(\tau) := X(\tau + d\tau) - X(\tau)\,,$$
$$\mathrm{d}_- X(\tau) := X(\tau) - X(\tau - d\tau)\,.$$

Then, since $X(\omega) \in \mathcal{C}^{1/2-\epsilon}(\mathbb{R}, \mathcal{M})$, one has

$$\begin{aligned}
\mathrm{d}_\pm X^\mu &= b_\pm^\mu\, d\tau^{1/2} + v_\pm^\mu\, d\tau + o(d\tau)\,,\\
\mathrm{d}_\pm X^\mu \mathrm{d}_\pm X^\nu &= b_\pm^\mu b_\pm^\nu\, d\tau + o(d\tau)\,,\\
\mathrm{d}_\pm f &= \partial_\mu f\, \mathrm{d}_\pm X^\mu \pm \frac{1}{2}\, \partial_\nu \partial_\mu f\, \mathrm{d}_\pm X^\mu \mathrm{d}_\pm X^\nu + o(d\tau)\,,\\
\mathrm{d}_\pm f \mathrm{d}_\pm g &= \partial_\mu f\, \partial_\nu g\, \mathrm{d}_\pm X^\mu \mathrm{d}_\pm X^\nu + o(d\tau)\,.
\end{aligned}$$

$\Rightarrow$ modification of Leibniz rule and chain rule:

$$\mathrm{d}_\pm (f\, g) = f\, \mathrm{d}_\pm g + g\, \mathrm{d}_\pm f \pm \mathrm{d}_\pm f \mathrm{d}_\pm g\,,$$
$$\mathrm{d}_\pm (h \circ f) = (h' \circ f)\, \mathrm{d}_\pm f \pm \frac{1}{2} (h'' \circ f)\, \mathrm{d}_\pm f \mathrm{d}_\pm f\,.$$

Change of basis

We may also define the increments

$$\mathrm{d}_0 X(\tau) := X(\tau + d\tau/2) - X(\tau - d\tau/2)\,,$$
$$\mathrm{d}_0^2 X(\tau) := X(\tau + d\tau) - 2\,X(\tau) + X(\tau - d\tau)\,.$$

Then

$$\mathrm{d}_0 X(\tau) = \frac{b_+ + b_-}{\sqrt{2}}\,d\tau^{1/2} + v_\circ\,d\tau + o(d\tau)\,,$$
$$\mathrm{d}_0^2 X(\tau) = (b_+ - b_-)\,d\tau^{1/2} + 2\,v_\perp\,d\tau + o(d\tau)\,,$$

where

$$v_\circ = \frac{v_+ + v_-}{2}\,, \qquad v_\perp = \frac{v_+ - v_-}{2}\,.$$

Remark: $b_\pm$ can be related to creation/annihilation operators using the Wiener-Itô chaos expansion[8]

[8] P. Biane, Stoch. Process. Their Appl. 120 (2010).

First order geometry: differential forms are given by

$$\mathrm{d}f(x) = \partial_\mu f \, \mathrm{d}x^\mu \, .$$

Riemannian Geometry: the line element is assumed[9] to be given by

$$\mathrm{d}s^2 = g_{\mu\nu} \, \mathrm{d}x^\mu \mathrm{d}x^\nu \, .$$

For rough paths, this does not encode all necessary information.
Regularity requirement: $\alpha = 1$, while $\alpha < 1/2$.
$\Rightarrow$ **Incompatibility between quantum theory and geometry**

[9] B. Riemann (1868).

Rough paths and Geometry

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Solution: Second order geometry

$$\begin{aligned} \mathrm{d}_2 f &= \partial_\mu f \, \mathrm{d}_2 x^\mu + \frac{1}{2} \partial_\nu \partial_\mu f \, \mathrm{d}x^\mu \mathrm{d}x^\nu \, , \\ \mathrm{d}s^2 &= g_{\mu\nu} \mathrm{d}_2 x^\mu \mathrm{d}_2 x^\nu + g_{\mu(\kappa\lambda)} \mathrm{d}_2 x^\mu \mathrm{d}x^\kappa \mathrm{d}x^\lambda + g_{(\rho\sigma)\nu} \mathrm{d}x^\rho \mathrm{d}x^\sigma \mathrm{d}_2 x^\nu \\ &\quad + g_{(\rho\sigma)(\kappa\lambda)} \mathrm{d}x^\rho \mathrm{d}x^\sigma \mathrm{d}x^\kappa \mathrm{d}x^\lambda \, . \end{aligned}$$

[9] B. Riemann (1868).

Introduction to Second Order Geometry[10]

[10] P.A. Meyer, Springer Berlin (1981); L. Schwartz, Montreal University Press (1984); M. Emery, Springer Berlin (1989); Q. Huang, J.C. Zambrini, J. Nonlinear Sci. (2023).

Given a manifold $\mathcal{M}$ and $x \in \mathcal{M}$, the (co)tangent spaces are extended:

$$T_x\mathcal{M} \to T_{2,x}\mathcal{M} \cong T_x\mathcal{M} \oplus \mathrm{Sym}(T_x\mathcal{M} \otimes T_x\mathcal{M})$$

Vectors $v \in T_{2,x}\mathcal{M}$ and forms $\alpha \in T^*_{2,x}\mathcal{M}$ are represented as

$$v = v^\mu \, \partial_\mu + \frac{1}{2} \, v_2^{\mu\nu} \partial_\mu \partial_\nu \, ,$$
$$\alpha = \alpha_\mu \, \mathrm{d}_2 x^\mu + \frac{1}{2} \, \alpha_{\mu\nu} \, \mathrm{d}x^\mu \mathrm{d}x^\nu \, .$$

The duality pairing is given by

$$\langle \alpha, v \rangle = \alpha_\mu v^\mu + \frac{1}{2} \, \alpha_{\mu\nu} v_2^{\mu\nu} \, .$$

Dimensions

$$\dim(\mathcal{M}) = n = 4 \, ,$$
$$\dim(T_x\mathcal{M}) = n = 4 \, ,$$
$$\dim(T_{2,x}\mathcal{M}) = \frac{n(n+3)}{2} = 14 \, .$$

Coordinate transformations

Structure group of $T_2\mathcal{M}$ is the Itô group:

$$G_n^I = \mathrm{GL}(n,\mathbb{R}) \ltimes \mathrm{Lin}(\mathbb{R}^n \otimes \mathbb{R}^n, \mathbb{R}^n)\,,$$

such that $(v, v_2) \in T_2\mathcal{M}$ transforms under $x^\mu \to \tilde{x}^\mu$ as

$$\begin{pmatrix} \tilde{v}^\mu \\ \tilde{v}_2^{\rho\sigma} \end{pmatrix} = \begin{pmatrix} \frac{\partial \tilde{x}^\mu}{\partial x^\nu} & \frac{1}{2}\frac{\partial^2 \tilde{x}^\mu}{\partial x^\kappa \partial x^\lambda} \\ 0 & \frac{\partial \tilde{x}^\rho}{\partial x^\kappa}\frac{\partial \tilde{x}^\sigma}{\partial x^\lambda} \end{pmatrix} \begin{pmatrix} v^\nu \\ v_2^{\kappa\lambda} \end{pmatrix}.$$

$\Rightarrow$ v^μ does not transform covariantly.
There exist covariant representations:

$$\hat{v}^\mu = v^\mu + \frac{1}{2}\,\Gamma^\mu_{\rho\sigma} v_2^{\rho\sigma}\,,$$
$$\hat{v}_2^{\mu\nu} = v_2^{\mu\nu}\,.$$

where Γ is the first order Christoffel symbol.

A similar analysis applies to forms.

Lorentz symmetry

Under the Lorentz symmetry, second order vectors transform as

$$\begin{pmatrix} \tilde{v}^a \\ \tilde{v}_2^{bc} \end{pmatrix} = \begin{pmatrix} \Lambda^a{}_d & e^\mu_e \partial_\mu \Lambda^a{}_f \\ 0 & \Lambda^b{}_e \Lambda^c{}_f \end{pmatrix} \begin{pmatrix} v^d \\ v_2^{ef} \end{pmatrix}$$

where $\Lambda \in SO^+(3,1)$ and

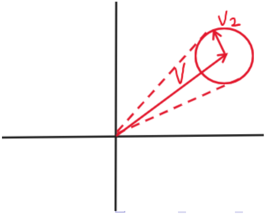
$$\begin{aligned} v^a &= \hat{v}^\mu e_\mu{}^a - v_2^{\mu b} \omega_\mu{}^a{}_b \\ v_2^{bc} &= v_2^{\mu\nu} e_\mu{}^b e_\nu{}^c \end{aligned}$$

with e the polyad (vielbein) and ω the spin connection.

$\Rightarrow$ Lorentz symmetry is deformed by the off-diagonal term.

Deformations vanish in two limits:

- No fluctuations: $\hbar \to 0 \Rightarrow v_2 \to 0$;
- Flat space(time): $G \to 0 \Rightarrow \partial\Lambda \to 0$.



Inner product on the tangent spaces:

$$g_{\mu\nu} v^\mu v^\nu \to G_{\mu\nu} \hat{v}^\mu \hat{v}^\nu + \frac{1}{2} G_{\mu(\kappa\lambda)} \hat{v}^\mu \hat{v}_2^{\kappa\lambda} + \frac{1}{2} G_{(\rho\sigma)\nu} \hat{v}_2^{\rho\sigma} \hat{v}_2^{\nu} + \frac{1}{4} G_{(\rho\sigma)(\kappa\lambda)} \hat{v}_2^{\rho\sigma} \hat{v}_2^{\kappa\lambda}$$

where

$$\begin{pmatrix} G_{\mu\nu} & G_{\mu(\kappa\lambda)} \\ G_{(\rho\sigma)\nu} & G_{(\rho\sigma)(\kappa\lambda)} \end{pmatrix} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & \begin{matrix} \ell_s^{-2} \left(g_{\rho\kappa} g_{\sigma\lambda} + g_{\rho\lambda} g_{\sigma\kappa} - \frac{2(1-a)}{n} g_{\rho\sigma} g_{\kappa\lambda} \right) \\ + b \left(\mathcal{R}_{\rho\kappa\sigma\lambda} + \mathcal{R}_{\rho\lambda\sigma\kappa} \right) \end{matrix} \end{pmatrix}$$

- ℓ_s a small length scale;
- $a \in \mathbb{R}$;
- $n = \dim(\mathcal{M})$;
- b is related to the Pauli-DeWitt term:
  - B.S. DeWitt, Rev. Mod. Phys. 29 (1957) $\Rightarrow b = \frac{1}{3}$,
  - B.S. DeWitt, Int. Ser. Monogr. Phys. 114 (2003) $\Rightarrow b = \frac{1}{2}$.

Lie derivative[11] of a **first order** tensor along a **second order** vector field:

$$\mathcal{L}_{(v,v_2)} T = \mathcal{L}_{\hat{v}} T + v_2^{\mu\nu} \left(\nabla_\mu \nabla_\nu + \mathcal{R}^{\cdot}{}_{\mu\cdot\nu}\right) T$$

$\Rightarrow$ Killing equation[12]

$$\nabla_{(\mu} \hat{k}_{\nu)} = \hat{k}_2^{\rho\sigma} \mathcal{R}_{\mu\rho\nu\sigma}$$

$\rightarrow$ For any diffeormorphism generated by (v, v_2)

$$\left(g_{\nu\rho} \hat{v}^\rho \nabla_\mu + v_2^{\rho\sigma} \mathcal{R}_{\mu\rho\nu\sigma}\right) T^{\mu\nu} = 0\,.$$

$\Rightarrow$ Gravitational anomaly as $\nabla_\mu \langle T^{\mu\nu} \rangle \neq 0$.

$\rightarrow$ For any Killing vector one can define a conserved current

$$\hat{J}^\mu = \hat{k}_\nu T^{\mu\nu} = \left(k^\nu + \frac{1}{2} \Gamma^\nu_{\rho\sigma} k_2^{\rho\sigma}\right) T^\mu_\nu \qquad \Rightarrow \qquad \nabla_\mu \hat{J}^\mu = 0\,.$$

[11] FK, JHEP 05 (2021); Q. Huang, J.C. Zambrini, J. Nonlinear Sci. (2023).
[12] FK, Springer Proc. Math. Stat. (2022).

Implications and Extensions

Path integral measure

Any physical theory must depend on the second order vectors $\{(v_+, v_2), (v_-, v_2)\}$ or $\{(v_\circ, v_2), (v_\perp, v_2)\}$, such that

$$L(x, v) \to L(x, v_\circ, v_\perp, v_2)$$

Thus, the path integral measure is given by

$$\mathrm{d}\mu(X) = e^{\mathrm{i} \int L(x, v_\circ, v_\perp, v_2) d\tau} DX$$

On a flat space(time),

$$\mathrm{d}\mu(X) = e^{\mathrm{i} \int L(x, v_\circ) d\tau} e^{\mathrm{i} \int L(x, v_\perp) d\tau} e^{\mathrm{i} \int L(x, v_2) d\tau} DX \,,$$

Hence, the new parts containing $v_\perp$ and v_2 only affect the normalization.

On curved space(times) one must consider the full measure, as $v_\circ, v_\perp$ couple to v_2.

Hamiltonian

The Hamiltonian is constructed through a second order Legendre transform:[13]

$$H(x, p_\circ, \partial_\perp, p_2) = \langle p_\circ, v_\circ \rangle + \langle p_\perp, v_\perp \rangle - L(x, v_\circ, v_\perp, v_2)$$

Example for free particle:

$$H = \frac{\varepsilon}{2}\left[g^{\mu\nu}\left(\hat{p}^\circ_\mu \hat{p}^\circ_\nu + \hat{p}^\perp_\mu \hat{p}^\perp_\nu \right) + \frac{1}{4} G^{(\mu\nu)(\rho\sigma)} \hat{p}^{(2)}_{\mu\nu} \hat{p}^{(2)}_{\rho\sigma} + m^2 \right]$$

Note:

- One obtains a modified energy-momentum relation.
- The Hamiltonian is quadratic in all momenta $\Rightarrow$ it is bounded from below.
- Ostragradski's theorem does not apply, as we consider $\frac{dx}{d\tau}, \frac{d^2x}{d\tau}, \frac{dxdx}{d\tau}$ instead of $\frac{dx}{d\tau}, \frac{d^2x}{d\tau^2}$.

[13] Q. Huang, J.C. Zambrini, J. Nonlinear Sci. (2023)

Field Theory

All analysis so far was for the single particle.
However, we can derive some implications for field theories:

- A QFT on curved spacetime has a deformed Lorentz symmetry $\leftrightarrow$ modified dispersion relations;
- Deformations scale with $\hbar\, G = l_p^2$.
- Field theories are defined on the second order jet bundle $L(\phi, \nabla\phi) \to L(\phi, \nabla\phi, \nabla\nabla\phi)$.
- Ostragradski's theorem is avoided $\Rightarrow$ ghosts may be avoided.
- $\Rightarrow$ Quadratic gravity may be both normalizable and ghost-free.

Unification?

$$T_x\mathcal{M} \to \tilde{T}_x\mathcal{M} \cong T_x\mathcal{M} \oplus T_x\mathcal{M} \oplus \mathrm{Sym}(T_x\mathcal{M} \otimes T_x\mathcal{M})$$

$$\begin{aligned} \dim(T_x\mathcal{M}) &= n &&= 4 \quad = (1,3)\,, \\ \dim(\tilde{T}_x\mathcal{M}) &= N = \frac{n(n+5)}{2} &&= 18 = \begin{cases} (5,13) & \text{if} \quad a > 0\,, \\ (6,12) & \text{if} \quad a < 0\,. \end{cases} \end{aligned}$$

It has been suggested that $n = 4 < N$ allows for unification of forces[14]

- For $N = 14$, scenario with[15] $SO(1,13)$ and[16] $SO(3,11)$.
- For $N = 18$, scenario with[17] $SO(1,17)$ and[18] $SO(2,16)$.
- Unification in kth-order geometry.[19]

[14] Krasnov, Percacci, Class. Quantum Grav. 35 (2018).
[15] Percacci, Phys. Lett. B144 (1984); Chamseddine, Mukhanov, JHEP 03 (2016).
[16] Nesti, Percacci, Phys.Rev.D 81 (2010).
[17] Konitopoulos, Roumelioti, Zoupanos, Forthschr. Phys. 72 (2024).
[18] Roumelioti, Stefas, Zoupanos, Eur.Phys.J.C 84 (2024).
[19] Bies, arXiv:2406.06605 [math.DG] (2024).

Extensions: non-commutative geometry

We have assumed a symmetric second order vector field, since

$$\mathrm{d}_2 f = v^\mu \partial_\mu f + \frac{1}{2} v_2^{\mu\nu} \partial_\nu \partial_\mu f \,.$$

By promoting $\partial_\mu \to D_\mu$ with $[D_\mu, D_\nu] \neq 0$, $v_2 = \frac{dx^\mu dx^\nu}{d\tau}$ is no longer required to be symmetric. This imposes a spacetime non-commutativity relation of the form

$$[x^\mu, x^\nu] = B^{\mu\nu}(x)\,.$$

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We may also extend to infinite order geometry:

$$\mathrm{d}_2 f = v^\mu \partial_\mu f + \frac{1}{2} v_2^{\mu\nu} \partial_\nu \partial_\mu f + \frac{1}{6} v_3^{\mu\nu\rho} \partial_\rho \partial_\nu \partial_\mu + \dots \,.$$

This corresponds to studying non-continuous paths $X \notin \mathcal{C}^0$. This can be related to a space-time non-commutativity relation of the form[20]

$$[x^\mu, x^\nu] = C^{\mu\nu}_\rho(x)\, x^\rho \,.$$

[20] Arzano, FK, arxiv:2409.xxxxx (2024).

Conclusions & Outlook

Conclusions

- Path integrals violate basic assumptions of Riemannian geometry $\Rightarrow$ incompatibility between Quantum theory and Gravity;
- **Solution:** Higher order Geometry.
- **Implication:** $\dim(\tilde{T}_x\mathcal{M}) > \dim(\mathcal{M}) = 4$.

Outlook I: Further development of 2nd-order geometry

- Math: generalize concepts from 1st-order to 2nd-order geometry.
- Physics: investigate consequences such as
  - Dynamical theory of gravity in 2nd-order geometry
  - Unification of gauge forces and gravity
- ...

Outlook II: Generalizations beyond 2nd-order geometry

- Rougher paths: consider $\mathcal{C}^{1/k}$ paths with $k \in \mathbb{N}$
  - requires kth-order geometry.
  - $k \to \infty \Rightarrow [x^\mu, x^\nu] = C_\rho^{\mu\nu} x^\rho$.
- Non-symmetric v_2 fields $\Rightarrow [x^\mu, x^\nu] = B^{\mu\nu}$
- Sector $(v_\circ, v_\perp) \leftrightarrow$ generalized geometry?
- ...