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Reduction of parameters in Finite Unified Theories and the MSSM

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Abstract

The method of reduction of couplings developed by W. Zimmermann, combined with supersymmetry, can lead to realistic quantum field theories, where the gauge and Yukawa sectors are related. It is the basis to find all-loop Finite Unified Theories, where the β-function vanishes to all-loops in perturbation theory. It can also be applied to the Minimal Supersymmetric Standard Model, leading to a drastic reduction in the number of parameters. Both Finite Unified Theories and the reduced MSSM lead to successful predictions for the masses of the third generation of quarks and the Higgs boson, and also predict a heavy supersymmetric spectrum, consistent with the non-observation of supersymmetry so far.

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1. Introduction

Although the Standard Model (SM) has been very successful in describing elementary particles and its interactions, it has been known for some time that it must be the low energy limit of a more fundamental theory. This quest for a theory beyond the Standard Model (BSM) has expanded in various directions. The usual, and very efficient, way of reducing the number of free parameters of a theory to render it more predictive, is to introduce a symmetry. Grand Unified Theories (GUTs) are very good examples of such a procedure. First by unifying (approximately) the gauge couplings in a larger symmetry group, the case of minimal $SU(5)$, it was possible to reduce the gauge couplings by one and give a prediction for one of them. By adding a further symmetry, namely $N = 1$ global supersymmetry it was possible to make the prediction viable. Unfortunately, requiring more gauge symmetry does not seem to help, since additional complications are introduced due to new degrees of freedom, for instance in the ways and channels of breaking the symmetry.

A possible way to look for relations among unrelated parameters is the method of *reduction of couplings* [1]. This method, as its name proclaims, reduces the number of couplings in a theory by relating either all or a number of couplings to a single coupling denoted as the "primary coupling". This method might help to identify hidden symmetries in a system, but it is also possible to have reduction of couplings in systems where there is no apparent symmetry. The reduction of couplings is based on the assumption that both the original and the reduced theory are renormalizable and that there exist renormalization group invariant (RGI) relations among parameters.

In our studies [2–7] we have used the reduction of couplings method to look for a more fundamental theory, complemented with the assumption of Grand Unification and N = 1 supersymmetry. The method relies on assuming RGI relations that hold below the Planck scale down to the GUT scale. It leads to the unification of the Gauge and Yukawa (GYU) sectors of the theory at a higher scale, sectors which in the SM are unrelated. An impressive aspect of the RGI relations is that one can guarantee their validity to all-orders in perturbation theory by studying the uniqueness of the resulting relations at one-loop, as was proven in the early days of the programme of reduction of couplings. Even more remarkable is the fact that it is possible to find RGI relations among couplings that guarantee finiteness to all-orders in perturbation theory [8–10]. The reduction of couplings applied to $N = 1$ $SU(5)$ SUSY GUTs has proven very successful by predicting correctly, among others, the top quark mass in the finite and minimal cases [5].

The above mentioned principles have only been applied in supersymmetric GUTs for reasons that will be clear in the following sections. The conjecture of Gauge Yukawa Unification through RGI is by no means in conflict with other interesting proposals (see also ref. [11]), but it rather uses all of them, hopefully in a more successful perspective. For instance, the use of SUSY GUTs comprises the demand of the cancellation of quadratic divergences in the SM. Similarly, the very interesting conjectures about the infrared fixed points are generalized in our proposal, since searching for RGI relations among various couplings corresponds to searching for *fixed points* [12] of the coupled differential equations obeyed by the various couplings of a theory.

Although supersymmetry seems to be an essential feature for a successful realization of the above programme, its breaking has to be understood too. The search for RGI relations has been extended to the soft supersymmetry breaking sector (SSB) of these theories [7,13], which involves parameters of dimension one and two. This is indeed possible to be done on the RGI surface which is defined by the solution of the reduction equations.

Applying the reduction of couplings method to $N = 1$ SUSY theories has led to very interesting phenomenological developments. Previously an appealing "universal" set of soft scalar masses was assumed in the SSB sector of supersymmetric theories, given that apart from economy and simplicity (1) they are part of the constraints that preserve finiteness up to two-loops [14,15], (2) they are RGI up to two-loops in more general supersymmetric gauge theories, subject to the condition known as $P = 1/3\ Q$ [13] and (3) they appear in the attractive dilaton dominated supersymmetry breaking superstring scenarios [16]. However, further studies exhibited problems all due to the restrictive nature of the "universality" assumption for the soft scalar masses. For instance, (a) in Finite Unified Theories (FUTs) the universality predicts that the lightest supersymmetric particle is a charged particle, namely the superpartner of the τ lepton $\tilde{\tau}$, (b) the MSSM with universal soft scalar masses is inconsistent with the attractive radiative electroweak symmetry breaking, and worst of all, (c) the universal soft scalar masses lead to charge and/or color breaking minima deeper than the standard vacuum [17]. Therefore, there have been attempts to relax this constraint without loosing its attractive features. First an interesting observation was made that in $N = 1$ Gauge-Yukawa unified theories there exists a RGI sum rule for the soft scalar masses at lower orders; at one-loop for the non-finite case [18] and at two-loops for the finite case [19]. The sum rule manages to overcome the above unpleasant phenomenological consequences. Moreover it was proven [20] that the sum rule for the soft scalar masses is RGI to all-orders for both the general as well as for the finite case. Finally, the exact β-function for the soft scalar masses in the Novikov–Shifman–Vainstein–Zakharov (NSVZ) scheme [21] for the softly broken supersymmetric QCD has been obtained [20]. The use of RGI both in the dimensionful and dimensionless sector, together with the above mentioned sum rule, allows for the construction of realistic and predictive $N = 1$ all-loop finite $SU(5)$ SUSY GUTS, as well as a reduced version of the MSSM, also with interesting predictions, as we will show [5,6,2,22–28].

2. Unification of couplings by the RGI method

In this section we will briefly outline the method of reduction of couplings. Any RGI relation among couplings (which does not depend on the renormalization scale μ explicitly) can be expressed in the implicit form $\Phi(g_1, \cdots, g_A) = \text{const.}$, which has to satisfy the partial differential equation (PDE)

$$\mu\, \frac{d\Phi}{d\mu} = \vec{\nabla} \cdot \vec{\beta} = \sum_{a=1}^{A} \beta_a\, \frac{\partial \Phi}{\partial g_a} = 0\,, \tag{1}$$

where β_a is the β-function of g_a. This PDE is equivalent to a set of ordinary differential equations, the so-called reduction equations (REs) [1],

$$\beta_g\, \frac{d g_a}{d g} = \beta_a\ ,\ a = 1, \cdots, A\ , \tag{2}$$

where g and β_g are the primary coupling and its β-function, and the counting on a does not include g. Since maximally $(A-1)$ independent RGI "constraints" in the A-dimensional space of couplings can be imposed by the Φ_a's, one could in principle express all the couplings in terms of a single coupling g. The strongest requirement is to demand power series solutions to the REs,

$$g_a = \sum_n \rho_a^{(n)}\, g^{2n+1}\ , \tag{3}$$

which formally preserve perturbative renormalizability. Remarkably, the uniqueness of such power series solutions can be decided already at the one-loop level [1]. To illustrate this, let us assume that the β-functions have the form

$$\beta_a = \frac{1}{16\pi^2}[\sum_{b,c,d\neq g} \beta_a^{(1)\,bcd} g_b g_c g_d + \sum_{b\neq g} \beta_a^{(1)\,b} g_b g^2] + \cdots ,$$

$$\beta_g = \frac{1}{16\pi^2}\beta_g^{(1)} g^3 + \cdots , \tag{4}$$

where $\cdots$ stands for higher order terms, and $\beta_a^{(1)\,bcd}$'s are symmetric in b, c, d. We then assume that the $\rho_a^{(n)}$'s with $n \leq r$ have been uniquely determined. To obtain $\rho_a^{(r+1)}$'s, we insert the power series (3) into the REs (2) and collect terms of $\mathcal{O}(g^{2r+3})$ and find

$$\sum_{d\neq g} M(r)_a^d \, \rho_d^{(r+1)} = \text{lower order quantities} ,$$

where the r.h.s. is known by assumption, and

$$M(r)_a^d = 3 \sum_{b,c\neq g} \beta_a^{(1)\,bcd} \rho_b^{(1)} \rho_c^{(1)} + \beta_a^{(1)\,d} - (2r+1)\,\beta_g^{(1)}\,\delta_a^d , \tag{5}$$

$$0 = \sum_{b,c,d\neq g} \beta_a^{(1)\,bcd} \rho_b^{(1)} \rho_c^{(1)} \rho_d^{(1)} + \sum_{d\neq g} \beta_a^{(1)\,d} \rho_d^{(1)} - \beta_g^{(1)} \rho_a^{(1)} . \tag{6}$$

Therefore, the $\rho_a^{(n)}$'s for all $n > 1$ for a given set of $\rho_a^{(1)}$'s can be uniquely determined if $\det M(n)_a^d \neq 0$ for all $n \geq 0$.

As it will be clear later by examining specific examples, the various couplings in supersymmetric theories have easily the same asymptotic behavior. Therefore searching for a power series solution of the form (3) to the REs (2) is justified. This is not the case in non-supersymmetric theories, although the deeper reason for this fact is not fully understood.

The possibility of coupling unification described in this section is without any doubt attractive because the "completely reduced" theory contains only one independent coupling, but it can be unrealistic. Therefore, one often would like to impose fewer RGI constraints, and this is the idea of partial reduction [29].

3. Reduction of dimensionful parameters

The reduction of couplings was originally formulated for massless theories on the basis of the Callan–Symanzik equation [1]. The extension to theories with massive parameters is not straightforward if one wants to keep the generality and the rigor on the same level as for the massless case; one has to fulfill a set of requirements coming from the renormalization group equations, the Callan–Symanzik equations, etc. along with the normalization conditions imposed on irreducible Green's functions [30]. See [31] for interesting results in this direction. Here, to simplify the situation, we would like to assume that a mass-independent renormalization scheme has been employed so that all the RG functions have only trivial dependencies of dimensional parameters [7].

To be general, we consider a renormalizable theory which contains a set of $(N+1)$ dimension-zero couplings, $\{\hat{g}_0, \hat{g}_1, \ldots, \hat{g}_N\}$, a set of L parameters with dimension one, $\{\hat{h}_1, \ldots, \hat{h}_L\}$, and a set of M parameters with dimension two, $\{\hat{m}_1^2, \ldots, \hat{m}_M^2\}$. The renormalized irreducible vertex function satisfies the RG equation

$$0 = \mathcal{D}\Gamma[\,\mathbf{\Phi}'s; \hat{g}_0, \hat{g}_1, \ldots, \hat{g}_N; \hat{h}_1, \ldots, \hat{h}_L; \hat{m}_1^2, \ldots, \hat{m}_M^2; \mu\,]\,, \tag{7}$$

$$\mathcal{D} = \mu\frac{\partial}{\partial\mu} + \sum_{i=0}^{N}\beta_i\frac{\partial}{\partial\hat{g}_i} + \sum_{a=1}^{L}\gamma_a^h\frac{\partial}{\partial\hat{h}_a} + \sum_{\alpha=1}^{M}\gamma_\alpha^{m^2}\frac{\partial}{\partial\hat{m}_\alpha^2} + \sum_J \Phi_I\gamma^{\phi I}{}_J\frac{\delta}{\delta\Phi_J}\,.$$

Since we assume a mass-independent renormalization scheme, the γ's have the form

$$\gamma_a^h = \sum_{b=1}^{L}\gamma_a^{h,b}(g_0, \ldots, g_N)\hat{h}_b\,,$$

$$\gamma_\alpha^{m^2} = \sum_{\beta=1}^{M}\gamma_\alpha^{m^2,\beta}(g_0, \ldots, g_N)\hat{m}_\beta^2 + \sum_{a,b=1}^{L}\gamma_\alpha^{m^2,ab}(g_0, \ldots, g_N)\hat{h}_a\hat{h}_b\,, \tag{8}$$

where $\gamma_a^{h,b}$, $\gamma_\alpha^{m^2,\beta}$ and $\gamma_a^{m^2,ab}$ are power series of the dimension-zero couplings g's in perturbation theory.

As in the massless case, we then look for conditions under which the reduction of parameters,

$$\hat{g}_i = \hat{g}_i(g)\,,\ (i = 1, \ldots, N)\,, \tag{9}$$

$$\hat{h}_a = \sum_{b=1}^{P} f_a^b(g)h_b\,,\ (a = P+1, \ldots, L)\,, \tag{10}$$

$$\hat{m}_\alpha^2 = \sum_{\beta=1}^{Q} e_\alpha^\beta(g)m_\beta^2 + \sum_{a,b=1}^{P} k_\alpha^{ab}(g)h_a h_b\,,\ (\alpha = Q+1, \ldots, M)\,, \tag{11}$$

is consistent with the RG equation (1), where we assume that $g \equiv g_0$, $h_a \equiv \hat{h}_a$ $(1 \le a \le P)$ and $m_\alpha^2 \equiv \hat{m}_\alpha^2$ $(1 \le \alpha \le Q)$ are independent parameters of the reduced theory. We find that the following set of equations has to be satisfied:

$$\beta_g\frac{\partial\hat{g}_i}{\partial g} = \beta_i\,,\ (i = 1, \ldots, N)\,, \tag{12}$$

$$\beta_g\frac{\partial\hat{h}_a}{\partial g} + \sum_{b=1}^{P}\gamma_b^h\frac{\partial\hat{h}_a}{\partial h_b} = \gamma_a^h\,,\ (a = P+1, \ldots, L)\,, \tag{13}$$

$$\beta_g\frac{\partial\hat{m}_\alpha^2}{\partial g} + \sum_{a=1}^{P}\gamma_a^h\frac{\partial\hat{m}_\alpha^2}{\partial h_a} + \sum_{\beta=1}^{Q}\gamma_\beta^{m^2}\frac{\partial\hat{m}_\alpha^2}{\partial m_\beta^2} = \gamma_\alpha^{m^2},\ (\alpha = Q+1, \ldots, M)\,. \tag{14}$$

Using eq. (7) for γ's, one finds that eqs. (12)–(14) reduce to

$$\beta_g\frac{df_a^b}{dg} + \sum_{c=1}^{P} f_a^c[\gamma_c^{h,b} + \sum_{d=P+1}^{L}\gamma_c^{h,d}f_d^b] - \gamma_a^{h,b} - \sum_{d=P+1}^{L}\gamma_a^{h,d}f_d^b = 0\,, \tag{15}$$
$$(a = P+1, \ldots, L; b = 1, \ldots, P)\,,$$

$$\beta_g\frac{de_\alpha^\beta}{dg} + \sum_{\gamma=1}^{Q} e_\alpha^\gamma[\gamma_\gamma^{m^2,\beta} + \sum_{\delta=Q+1}^{M}\gamma_\gamma^{m^2,\delta}e_\delta^\beta] - \gamma_\alpha^{m^2,\beta} - \sum_{\delta=Q+1}^{M}\gamma_\alpha^{m^2,\delta}e_\delta^\beta = 0\,, \tag{16}$$
$$(\alpha = Q+1, \ldots, M; \beta = 1, \ldots, Q)\,,$$

$$
\begin{aligned}
&\beta_g \frac{dk_\alpha^{ab}}{dg} + 2\sum_{c=1}^{P}(\gamma_c^{h,a} + \sum_{d=P+1}^{L} \gamma_c^{h,d} f_d^a) k_\alpha^{cb} + \sum_{\beta=1}^{Q} e_\alpha^\beta [\gamma_\beta^{m^2,ab} + \sum_{c,d=P+1}^{L} \gamma_\beta^{m^2,cd} f_c^a f_d^b \\
&+2 \sum_{c=P+1}^{L} \gamma_\beta^{m^2,cb} f_c^a + \sum_{\delta=Q+1}^{M} \gamma_\beta^{m^2,\delta} k_\delta^{ab}] - [\gamma_\alpha^{m^2,ab} + \sum_{c,d=P+1}^{L} \gamma_\alpha^{m^2,cd} f_c^a f_d^b \\
&+2 \sum_{c=P+1}^{L} \gamma_\alpha^{m^2,cb} f_c^a + \sum_{\delta=Q+1}^{M} \gamma_\alpha^{m^2,\delta} k_\delta^{ab}] = 0 \,, \qquad (17) \\
&(\alpha = Q+1, \ldots, M; a, b = 1, \ldots, P) \,.
\end{aligned}
$$

If these equations are satisfied, the irreducible vertex function of the reduced theory

$$
\begin{aligned}
&\Gamma_R[\, \mathbf{\Phi}'s; g; h_1, \ldots, h_P; m_1^2, \ldots, \hat{m}_Q^2; \mu \,] \\
\equiv &\Gamma[\, \mathbf{\Phi}'s; g, \hat{g}_1(g), \ldots, \hat{g}_N(g); h_1, \ldots, h_P, \hat{h}_{P+1}(g,h), \ldots, \hat{h}_L(g,h); \\
&m_1^2, \ldots, \hat{m}_Q^2, \hat{m}_{Q+1}^2(g,h,m^2), \ldots, \hat{m}_M^2(g,h,m^2); \mu \,] \qquad (18)
\end{aligned}
$$

has the same renormalization group flow as the original one.

The requirement for the reduced theory to be perturbative renormalizable means that the functions $\hat{g}_i$, f_a^b, e_α^β and k_α^{ab}, defined in eqs. (9)–(11), should have a power series expansion in the primary coupling g:

$$
\begin{aligned}
&\hat{g}_i = g \sum_{n=0}^{\infty} \rho_i^{(n)} g^n \,, \quad f_a^b = g \sum_{n=0}^{\infty} \eta_a^{b\,(n)} g^n \,, \\
&e_\alpha^\beta = \sum_{n=0}^{\infty} \xi_\alpha^{\beta\,(n)} g^n \,, \quad k_\alpha^{ab} = \sum_{n=0}^{\infty} \chi_\alpha^{ab\,(n)} g^n \,. \qquad (19)
\end{aligned}
$$

To obtain the expansion coefficients, we insert the power series ansatz above into eqs. (12), (15)–(17) and require that the equations are satisfied at each order in g. Note that the existence of a unique power series solution is a non-trivial matter: It depends on the theory as well as on the choice of the set of independent parameters.

4. Finiteness in N = 1 supersymmetric gauge theories

Let us consider a chiral, anomaly free, $N = 1$ globally supersymmetric gauge theory based on a group G with gauge coupling constant g. The superpotential of the theory is given by

$$
W = \frac{1}{2} m_{ij}\, \phi_i\, \phi_j + \frac{1}{6} C_{ijk}\, \phi_i\, \phi_j\, \phi_k \,, \qquad (20)
$$

where m_{ij} and C_{ijk} are gauge invariant tensors and the matter field ϕ_i transforms according to the irreducible representation R_i of the gauge group G. The renormalization constants associated with the superpotential (20), assuming that supersymmetry is preserved, are

$$
\phi_i^0 = (Z_i^j)^{(1/2)}\, \phi_j \,, \qquad (21)
$$

$$
m_{ij}^0 = Z_{ij}^{i'j'}\, m_{i'j'} \,, \qquad (22)
$$

$$
C_{ijk}^0 = Z_{ijk}^{i'j'k'}\, C_{i'j'k'} \,. \qquad (23)
$$

The $N = 1$ non-renormalization theorem [32] ensures that there are no mass and cubic-interaction-term infinities and therefore

$$Z_{ijk}^{i'j'k'}\, Z_{i'}^{1/2\,i''}\, Z_{j'}^{1/2\,j''}\, Z_{k'}^{1/2\,k''} = \delta_{(i}^{i''}\, \delta_{j}^{j''}\, \delta_{k)}^{k''} \,,$$
$$Z_{ij}^{i'j'}\, Z_{i'}^{1/2\,i''}\, Z_{j'}^{1/2\,j''} = \delta_{(i}^{i''}\, \delta_{j)}^{j''} \,. \tag{24}$$

As a result the only surviving possible infinities are the wave-function renormalization constants Z_i^j, i.e., one infinity for each field. The one-loop β-function of the gauge coupling g is given by [33]

$$\beta_g^{(1)} = \frac{dg}{dt} = \frac{g^3}{16\pi^2}\,[\sum_i l(R_i) - 3\,C_2(G)\,]\,, \tag{25}$$

where $l(R_i)$ is the Dynkin index of R_i and $C_2(G)$ is the quadratic Casimir of the adjoint representation of the gauge group G. The β-functions of C_{ijk}, by virtue of the non-renormalization theorem, are related to the anomalous dimension matrix γ_{ij} of the matter fields ϕ_i as:

$$\beta_{ijk} = \frac{dC_{ijk}}{dt} \;=\; C_{ijl}\,\gamma_k^l + C_{ikl}\,\gamma_j^l + C_{jkl}\,\gamma_i^l \,. \tag{26}$$

At one-loop level γ_{ij} is [33]

$$\gamma_j^{i(1)} = \frac{1}{32\pi^2}\,[\,C^{ikl}\,C_{jkl} - 2\,g^2\,C_2(R_i)\delta_j^1\,], \tag{27}$$

where $C_2(R_i)$ is the quadratic Casimir of the representation R_i, and $C^{ijk} = C_{ijk}^*$. Since dimensional coupling parameters such as masses and couplings of cubic scalar field terms do not influence the asymptotic properties of a theory on which we are interested here, it is sufficient to take into account only the dimensionless supersymmetric couplings such as g and C_{ijk}. So we neglect the existence of dimensional parameters, and assume furthermore that C_{ijk} are real so that C_{ijk}^2 always are positive numbers.

As one can see from Eqs. (25) and (27), all the one-loop β-functions of the theory vanish if $\beta_g^{(1)}$ and $\gamma_{ij}^{(1)}$ vanish, i.e.

$$\sum_i \ell(R_i) = 3C_2(G)\,, \tag{28}$$

$$C^{ikl}C_{jkl} = 2\delta_j^i g^2 C_2(R_i)\,, \tag{29}$$

The conditions for finiteness for $N = 1$ field theories with $SU(N)$ gauge symmetry and the analysis of the anomaly-free and no-charge renormalization requirements for these theories can be found in [34]. A very interesting result is that the conditions (28), (29) are necessary and sufficient for finiteness at the two-loop level [33,35].

In case supersymmetry is broken by soft terms, the requirement of finiteness in the one-loop soft breaking terms imposes further constraints among themselves [14]. In addition, the same set of conditions that are sufficient for one-loop finiteness of the soft breaking terms render the soft sector of the theory two-loop finite [14].

The one- and two-loop finiteness conditions (28), (29) restrict considerably the possible choices of the irreps. R_i for a given group G as well as the Yukawa couplings in the superpotential (20). Note in particular that the finiteness conditions cannot be applied to the minimal supersymmetric standard model (MSSM), since the presence of a $U(1)$ gauge group is incompatible with the condition (28), due to $C_2[U(1)] = 0$. This naturally leads to the expectation that finiteness should be attained at the grand unified level only, the MSSM being just the corresponding, low-energy, effective theory.

Another important consequence of one- and two-loop finiteness is that supersymmetry (most probably) can only be broken due to the soft breaking terms. Indeed, due to the unacceptability of gauge singlets, F-type spontaneous symmetry breaking [36] terms are incompatible with finiteness, as well as D-type [37] spontaneous breaking which requires the existence of a $U(1)$ gauge group.

A natural question to ask is what happens at higher loop orders. The answer is contained in a theorem [8] which states the necessary and sufficient conditions to achieve finiteness at all orders. Before we discuss the theorem let us make some introductory remarks. The finiteness conditions impose relations between gauge and Yukawa couplings. To require such relations which render the couplings mutually dependent at a given renormalization point is trivial. What is not trivial is to guarantee that relations leading to a reduction of the couplings hold at any renormalization point. As we have seen, the necessary and also sufficient, condition for this to happen is to require that such relations are solutions to the REs

$$\beta_g \frac{dC_{ijk}}{dg} = \beta_{ijk} \tag{30}$$

and hold at all orders. Remarkably, the existence of all-order power series solutions to (30) can be decided at one-loop level, as already mentioned.

Let us now turn to the all-order finiteness theorem [8], which states that if an $N = 1$ supersymmetric gauge theory can become finite to all orders in the sense of vanishing β-functions, that is of physical scale invariance. It is based on (a) the structure of the supercurrent in $N = 1$ supersymmetric gauge theory [38], and on (b) the non-renormalization properties of $N = 1$ chiral anomalies [8,39,40]. Details on the proof and further discussion can be found in refs. [8,9].

Theorem. *Consider an $N = 1$ supersymmetric Yang–Mills theory, with simple gauge group. If the following conditions are satisfied*

1. *There is no gauge anomaly.*
2. *The gauge β-function vanishes at one-loop*

$$\beta_g^{(1)} = 0 = \sum_i l(R_i) - 3\, C_2(G). \tag{31}$$

3. *There exist solutions of the form*

$$C_{ijk} = \rho_{ijk} g, \qquad \rho_{ijk} \in \mathbb{C} \tag{32}$$

to the conditions of vanishing one-loop matter fields anomalous dimensions

$$\begin{aligned} \gamma_j^{i\,(1)} &= 0 \\ &= \frac{1}{32\pi^2}\, [\, C^{ikl}\, C_{jkl} - 2\, g^2\, C_2(R_i)\delta_{ij}]. \end{aligned} \tag{33}$$

4. *These solutions are isolated and non-degenerate when considered as solutions of vanishing one-loop Yukawa β-functions:*

$$\beta_{ijk} = 0. \tag{34}$$

Then, each of the solutions (32) can be uniquely extended to a formal power series in g, and the associated super Yang–Mills models depend on the single coupling constant g with a β function which vanishes at all-orders.

It is important to note a few things: The requirement of isolated and non-degenerate solutions guarantees the existence of a unique formal power series solution to the reduction equations. The vanishing of the gauge β-function at one-loop, $\beta_g^{(1)}$, is equivalent to the vanishing of the R current anomaly. The vanishing of the anomalous dimensions at one-loop implies the vanishing of the Yukawa couplings β-functions at that order. It also implies the vanishing of the chiral anomaly coefficients r^A. This last property is a necessary condition for having β functions vanishing at all orders.[1]

Thus, finiteness and reduction of couplings are intimately related.

5. Sum rule for SB terms in N = 1 supersymmetric and finite theories: all-loop results

As we have seen in section 3, the method of reducing the dimensionless couplings can be extended [7] to the soft supersymmetry breaking (SSB) dimensionful parameters of $N = 1$ supersymmetric theories. In addition it was found [18] that RGI SSB scalar masses in Gauge-Yukawa unified models satisfy a universal sum rule. Here we will describe first how the use of the available two-loop RG functions and the requirement of finiteness of the SSB parameters up to this order leads to the soft scalar-mass sum rule [19].

Consider the superpotential given by (20) along with the Lagrangian for SSB terms

$$
\begin{aligned}
-\mathcal{L}_{\rm SB} &= \frac{1}{6}\, h^{ijk}\, \phi_i\phi_j\phi_k + \frac{1}{2}\, b^{ij}\, \phi_i\phi_j \\
&+ \frac{1}{2}\, (m^2)^{j}_{i}\, \phi^{*\,i}\phi_j + \frac{1}{2}\, M\, \lambda\lambda + \text{h.c.},
\end{aligned}
\tag{35}
$$

where the ϕ_i are the scalar parts of the chiral superfields Φ_i, λ are the gauginos and M their unified mass. Since we would like to consider only finite theories here, we assume that the gauge group is a simple group and the one-loop β-function of the gauge coupling g vanishes. We also assume that the reduction equations admit power series solutions of the form

$$
C^{ijk} = g \sum_{n} \rho^{ijk}_{(n)}\, g^{2n}\ .
\tag{36}
$$

According to the finiteness theorem of ref. [8], the theory is then finite to all orders in perturbation theory, if, among others, the one-loop anomalous dimensions $\gamma_i^{j\,(1)}$ vanish. The one- and two-loop finiteness for h^{ijk} can be achieved by [15]

$$
h^{ijk} = -MC^{ijk} + \cdots = -M\rho^{ijk}_{(0)}\, g + O(g^5)\ ,
\tag{37}
$$

where ... stand for higher order terms.

Now, to obtain the two-loop sum rule for soft scalar masses, we assume that the lowest order coefficients $\rho^{ijk}_{(0)}$ and also $(m^2)^{i}_{j}$ satisfy the diagonality relations

$$
\rho_{ipq(0)}\rho^{jpq}_{(0)} \propto \delta_i^j \ \text{for all}\ p\ \text{and}\ q\ \ \text{and}\ \ (m^2)^{i}_{j} = m^2_j\delta^i_j\ ,
\tag{38}
$$

[1] There is an alternative way to find finite theories [41].

respectively. Then we find the following soft scalar-mass sum rule [19,4,42]

$$(m_i^2 + m_j^2 + m_k^2)/MM^\dagger = 1 + \frac{g^2}{16\pi^2}\,\Delta^{(2)} + O(g^4) \tag{39}$$

for i, j, k with $\rho_{(0)}^{ijk} \neq 0$, where $\Delta^{(2)}$ is the two-loop correction

$$\Delta^{(2)} = -2\sum_l [(m_l^2/MM^\dagger) - (1/3)]\; T(R_l), \tag{40}$$

which vanishes for the universal choice in accordance with the previous findings of ref. [15].

If we know higher-loop β-functions explicitly, we can follow the same procedure and find higher-loop RGI relations among SSB terms. However, the β-functions of the soft scalar masses are explicitly known only up to two loops. In order to obtain higher-loop results some relations among β-functions are needed.

Making use of the spurion technique [43], it is possible to find the following all-loop relations among SSB β-functions, [44–46]

$$\beta_M = 2\mathcal{O}\left(\frac{\beta_g}{g}\right) , \tag{41}$$

$$\begin{aligned}\beta_h^{ijk} &= \gamma^i{}_l h^{ljk} + \gamma^j{}_l h^{ilk} + \gamma^k{}_l h^{ijl} \\ &\quad -2\gamma_1^i{}_l C^{ljk} - 2\gamma_1^j{}_l C^{ilk} - 2\gamma_1^k{}_l C^{ijl} ,\end{aligned} \tag{42}$$

$$(\beta_{m^2})^i{}_j = \left[\Delta + X\frac{\partial}{\partial g}\right]\gamma^i{}_j , \tag{43}$$

$$\mathcal{O} = \left(Mg^2\frac{\partial}{\partial g^2} - h^{lmn}\frac{\partial}{\partial C^{lmn}}\right) , \tag{44}$$

$$\Delta = 2\mathcal{O}\mathcal{O}^* + 2|M|^2 g^2\frac{\partial}{\partial g^2} + \tilde{C}_{lmn}\frac{\partial}{\partial C_{lmn}} + \tilde{C}^{lmn}\frac{\partial}{\partial C^{lmn}} , \tag{45}$$

where $(\gamma_1)^i{}_j = \mathcal{O}\gamma^i{}_j$, $C_{lmn} = (C^{lmn})^*$, and

$$\tilde{C}^{ijk} = (m^2)^i{}_l C^{ljk} + (m^2)^j{}_l C^{ilk} + (m^2)^k{}_l C^{ijl} . \tag{46}$$

The assumption, following [45], that the relation among couplings

$$h^{ijk} = -M(C^{ijk})' \equiv -M\frac{dC^{ijk}(g)}{d\ln g} , \tag{47}$$

is RGI, and furthermore, the use of the all-loop β-function of Novikov *et al.* [21] given by

$$\beta_g^{\rm NSVZ} = \frac{g^3}{16\pi^2}\left[\frac{\sum_l T(R_l)(1-\gamma_l/2) - 3C(G)}{1 - g^2C(G)/8\pi^2}\right] , \tag{48}$$

leads to the all-loop RGI sum rule [20],

$$\begin{aligned} m_i^2 + m_j^2 + m_k^2 = |M|^2\{ &\frac{1}{1 - g^2C(G)/(8\pi^2)}\frac{d\ln C^{ijk}}{d\ln g} + \frac{1}{2}\frac{d^2\ln C^{ijk}}{d(\ln g)^2} \} \\ &+ \sum_l \frac{m_l^2 T(R_l)}{C(G) - 8\pi^2/g^2}\frac{d\ln C^{ijk}}{d\ln g} . \end{aligned} \tag{49}$$

In addition the exact-β-function for m^2 in the NSVZ scheme has been obtained [20] for the first time and is given by

$$\beta_{m_i^2}^{\rm NSVZ} = \left[|M|^2 \{ \frac{1}{1 - g^2 C(G)/(8\pi^2)} \frac{d}{d \ln g} + \frac{1}{2} \frac{d^2}{d (\ln g)^2} \} + \sum_l \frac{m_l^2 T(R_l)}{C(G) - 8\pi^2/g^2} \frac{d}{d \ln g} \right] \gamma_i^{\rm NSVZ} . \tag{50}$$

Surprisingly enough, the all-loop result (49) coincides with the superstring result for the finite case in a certain class of orbifold models [19] if $d \ln C^{ijk}/d \ln g = 1$.

It is important to emphasize that the sum rule holds always, to the extent that there is a reduction of couplings. A consequence from the reduction of dimensionful parameters is that in some cases, for instance in the reduced MSSM, it is possible to have exact relations among the soft scalar masses and a mass-dimension one parameter, which could be the gaugino which corresponds to the primary coupling [28]. This option cannot be applied to the case of Finite Unified Theories, though [18].

6. Finite $SU(5)$ Unified Theories

We shall study an all-loop Finite Unified Theory (FUT) based on the $SU(5)$ gauge group, applying the coupling reduction to quarks and leptons of the third generation. The particle content of the model consists of the following supermultiplets: three $(\overline{\mathbf{5}} + \mathbf{10})$, needed for each of the three generations of quarks and leptons, four $(\overline{\mathbf{5}} + \mathbf{5})$ and one $\mathbf{24}$ considered as Higgs supermultiplets. When the gauge group of the finite GUT is broken the theory is no longer finite, and we will assume that we are left with the MSSM.

A predictive FUT, in addition to the requirements mentioned already, should also posses the following properties

1. One-loop anomalous dimensions are diagonal, i.e., $\gamma_i^{(1)\,j} \propto \delta_i^j$.
2. The three fermion generations, in the irreducible representations $\overline{\mathbf{5}}_i, \mathbf{10}_i$ $(i = 1, 2, 3)$, should not couple to the adjoint $\mathbf{24}$.
3. The two Higgs doublets of the MSSM should mostly be made out of a pair of Higgs quintet and anti-quintet, which couple to the third generation.

Since the gauge symmetry is spontaneously broken below $M_{\rm GUT}$, the finiteness conditions do not restrict the renormalization properties at low energies, and all it remains are boundary conditions on the gauge and Yukawa couplings (29), the $h = -MC$ relation (37), and the soft scalar-mass sum rule (39) at $M_{\rm GUT}$. Thus we examine the evolution of these parameters according to their RGEs up to two-loops for dimensionless and dimensionful parameters with the relevant boundary conditions. Below $M_{\rm GUT}$ their evolution is assumed to be governed by the MSSM. We further assume a unique supersymmetry breaking scale $M_{\rm SUSY}$ (which we define as the geometrical average of the stop masses) and therefore below that scale the effective theory is just the SM. This allows to evaluate observables at the electroweak scale.

We briefly describe now the low-energy observables used in our analysis. As precision observables we first discuss the third generation quark masses that are leading to the strongest

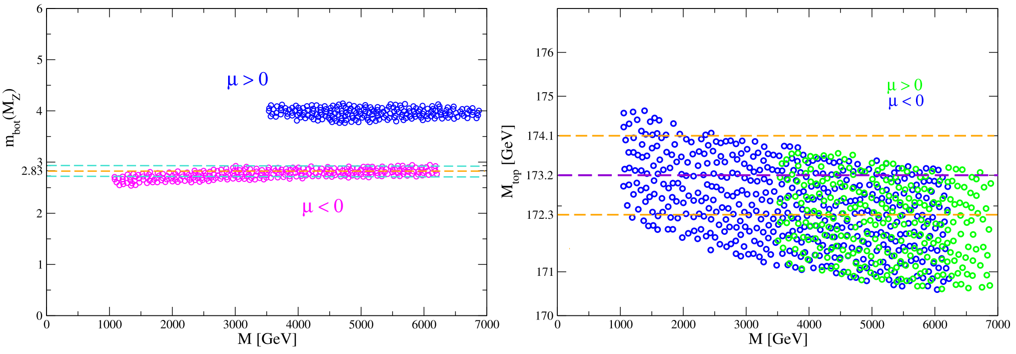


Fig. 1. The bottom quark mass at the Z boson scale (left) and top quark pole mass (right) are shown as function of M, the unified gaugino mass, and the two values of sign μ.

constraints on the models under investigation. Next we apply B physics and Higgs-boson mass constraints.

For the evaluation of the bottom and tau masses the one-loop radiative corrections from the SUSY breaking are incorporated [47] which can provide sizeable corrections to the bottom mass for large $\tan\beta$. We calculate the bottom mass at M_Z in order to avoid running down to the pole mass which induces uncertainties, while we take into account the tau and bottom quark mass SUSY radiative corrections

$$m_b(M_Z) = (2.83 \pm 0.10)\ \mathrm{GeV}\ . \tag{51}$$

We use the experimental value of the top quark pole mass as [48][2]

$$m_t^{\mathrm{exp}} = (173.2 \pm 0.9)\ \mathrm{GeV}\ . \tag{52}$$

The theoretical values for M_{top} may suffer from a correction of $\sim 4\%$ [3,50,42].

The **FUT** predictions are shown in Fig. 1, for the bottom mass $m_b(M_Z)$ and the top mass m_t and as a function of the gaugino mass M, distinguishing the two cases $\mu < 0$ and $\mu > 0$. The bounds on the two quark masses leave only the $\mu < 0$ case as a phenomenologically viable [22, 51]. A small variation of up to 5% of the FUT boundary conditions, due to threshold corrections at the GUT scale, is also included.

As additional constraints we consider the following observables[3]: the rare b decays $\mathrm{BR}(b \to s\gamma)$ and $\mathrm{BR}(B_s \to \mu^+\mu^-)$, as well as the lightest Higgs boson mass. For the branching ratio $\mathrm{BR}(b \to s\gamma)$, we take an experimental value estimated by the Heavy Flavour Averaging Group (HFAG) is [52]

$$\mathrm{BR}(b \to s\gamma)_{\mathrm{SM/MSSM}} = 1.089 \pm 0.27\ . \tag{53}$$

For the branching ratio $\mathrm{BR}(B_s \to \mu^+\mu^-)$ we use a combination of CMS and LHCb data [53]

$$\mathrm{BR}(B_s \to \mu^+\mu^-) = (2.9 \pm 1.4) \times 10^{-9}\ . \tag{54}$$

[2] We did not include the latest LHC/Tevatron combination, leading to $m_t^{\mathrm{exp}} = (173.34 \pm 0.76)$ GeV [49], which would have a negligible impact on our analysis.

[3] We do not employ the very latest experimental data, but this has a minor impact on our analysis.

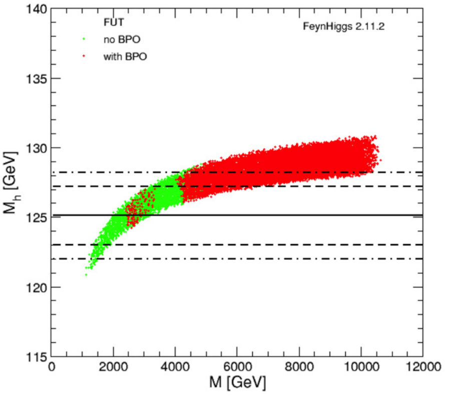


Fig. 2. The lightest Higgs mass, M_h, as function of M for the **FUT** model with $\mu < 0$. The red (darker) points are the ones that satisfy the B physics constraints. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

For the lightest Higgs mass prediction we used the code `FeynHiggs 2.11.2` [55,56,54][4] where the prediction for M_h of **FUT** with $\mu < 0$ is shown in Fig. 2. The red (green) points include (exclude) the B physics constraints. In a range where the unified gaugino mass varies from 1 TeV $\lesssim M \lesssim$ 11 TeV, the lightest Higgs mass varies as

$$M_h \sim 121\text{–}131 \text{ GeV}\,, \tag{55}$$

where the uncertainty comes from variations of the soft scalar masses. To this value one has to add at least ± 2 GeV coming from unknown higher order corrections [56].

Additional to the BPO constraints, we now impose the experimental constraint of the lightest Higgs boson, which is

$$M_h \sim 125.1 \pm 3.1(\pm 2.1) \text{ GeV}\,, \tag{56}$$

where ± 3.1 GeV corresponds to the current theory and experimental uncertainty, and ± 2.1 GeV to a reduced theory uncertainty in the future.

We find that constraining the allowed values of the Higgs mass puts a limit on the allowed values of the unified gaugino mass, as can be seen from Fig. 3. It can be seen from the figure that the lightest observable SUSY particle (LOSP) is the light scalar tau. In the left (right) plot we impose $M_h = 125.1 \pm 3.1$ (2.1) GeV. Including the Higgs mass constraints in general favors the lower part of the SUSY particle mass spectra [24,58], however in particular very heavy colored SUSY particles are favored, in agreement with the non-observation of those particles at the LHC [59]. Going to the anticipated future theory uncertainty of M_h (as shown in the right plot of Fig. 3) still permits SUSY masses which would remain unobservable at the LHC, the ILC or CLIC. On the other hand, large parts of the allowed spectrum of the lighter scalar tau or the lighter neutralinos might be accessible at CLIC with $\sqrt{s} = 3$ TeV.

[4] The M_h evaluation employed here and later does not yet take into account some important refinements that are relevant for high SUSY mass scales (as given in our analyses) [57], and that yield Higgs boson masses which are slightly smaller by $\mathcal{O}(2 \text{ GeV})$.

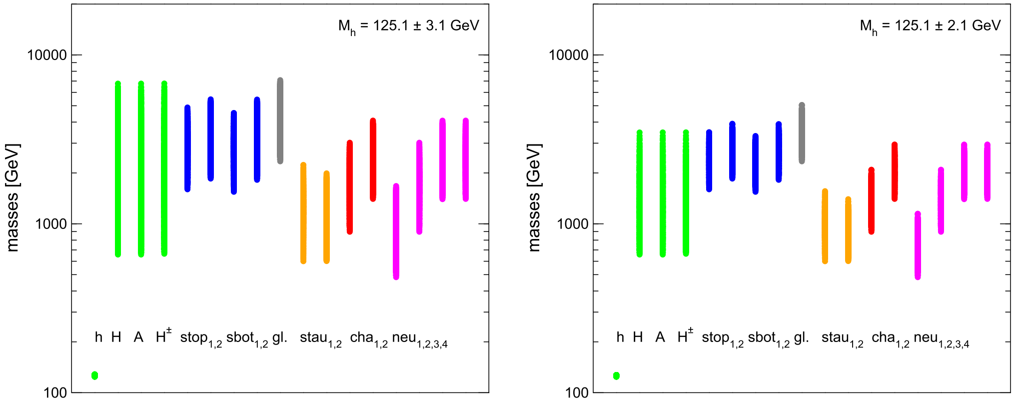


Fig. 3. The left (right) plot shows the spectrum of the **FUT** model after imposing the constraint $M_h = 125.1 \pm 3.1\,(2.1)$ GeV. The light (green) points are the various Higgs boson masses, the dark (blue) points following are the two scalar top and bottom masses, the gray ones are the gluino masses, then come the scalar tau masses in orange (light gray), the darker (red) points to the right are the two chargino masses followed by the lighter shaded (pink) points indicating the neutralino masses. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

7. Reduction of couplings in the MSSM

The method of reduction of couplings can also be applied successfully to the MSSM, not only reducing greatly the number of free parameters, but also giving different allowed parameter regions than the usual CMSSM.

The superpotential of the MSSM is defined by

$$W = Y_t H_2 Q t^c + Y_b H_1 Q b^c + Y_\tau H_1 L \tau^c + \mu H_1 H_2 \tag{57}$$

while the SSB Lagrangian is given by

$$\begin{aligned} -\mathcal{L}_{\text{SSB}} &= \sum_\phi m_\phi^2 \phi^* \phi + \left[m_3^2 H_1 H_2 + \sum_{i=1}^{3} \frac{1}{2} M_i \lambda_i \lambda_i + \text{h.c} \right] \\ &\quad + \left[h_t H_2 Q t^c + h_b H_1 Q b^c + h_\tau H_1 L \tau^c + \text{h.c.} \right], \end{aligned} \tag{58}$$

where in the last four terms we refer to the scalar components of the superfield. The Yukawa $Y_{t,b,\tau}$ and the trilinear $h_{t,b,\tau}$ couplings refer to the third generator only, neglecting the first two generations.

Following the procedure of reduction, in the first stage we keep only the g_3 coupling and treat the two other gauge coupling g_2 and g_1 (which cannot be reduced in favor of g_3) as corrections. The same happens with the tau Yukawa, since assuming that Y_τ is proportional to g_3 leads to an imaginary coefficient. This “reduced” system, holding at any scale, can serve as boundary conditions of the RGE of MSSM at the unification scale [23].

The reduction of the top and bottom Yukawa couplings in favor of g_3 leads, at the unification scale M_U, to the expansions

$$Y_t^2 = c_1 g_U^2 + c_2 g_U^4/(4\pi), \qquad Y_b^2 = p_1 g_U^2 + p_2 g_U^4/(4\pi) \tag{59}$$

where $g_U = g_3(M_U)$ and

$$
\begin{aligned}
c_1 &= \frac{157}{175} + \frac{1}{35} K_\tau = 0.897 + 0.029 K_\tau \; , \\
p_1 &= \frac{143}{175} - \frac{6}{35} K_\tau = 0.817 - 0.171 K_\tau \; , \\
c_2 &= \frac{1}{4\pi} \, \frac{1457.55 - 84.491 K_\tau - 9.66181 K_\tau^2 - 0.174927 K_\tau^3}{818.943 - 89.2143 K_\tau - 2.14286 K_\tau^2} \; , \\
p_2 &= \frac{1}{4\pi} \, \frac{1402.52 - 223.777 K_\tau - 13.9475 K_\tau^2 - 0.174927 K_\tau^3}{818.943 - 89.2143 K_\tau - 2.14286 K_\tau^2} \; , \\
K_\tau &= Y_\tau^2 / g_3^2 \; .
\end{aligned}
\tag{60}
$$

Keeping only the first term of the perturbative expansion of the Yukawas in favor of g_3 we get also

$$
h_{t,b} = -M(M_U) Y_{t,b} \; , \qquad m_3^2 = -M(M_U)\mu \; , \tag{61}
$$

and finally a set of equations resulting from the application of the sum rule

$$
m_{H_2}^2 + m_Q^2 + m_{t^c}^2 = M(M_U)^2 \; , \quad m_{H_1}^2 + m_Q^2 + m_{b^c}^2 = M(M_U)^2 \; , \tag{62}
$$

where $M(M_U)$ is the unified gluino mass at the GUT scale. A more refined evaluation, including an updated Higgs boson mass prediction, can be found in [28].

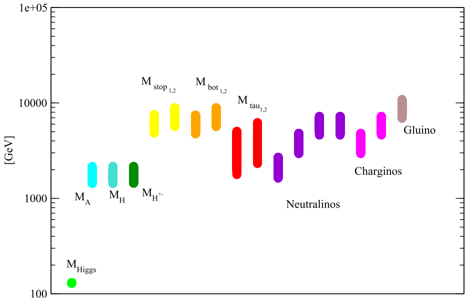
Let us proceed now to our predictions on the reduced MSSM. Starting at the unification scale M_U with the boundary conditions described above, we run the MSSM RGEs down to the SUSY scale and then the SM ones down to the M_Z scale. At that scale we compare our calculated third generation quark masses values with the corresponding experimental ones. The unification scale M_U and $|\mu|$ at M_U are varied in the range $\sim$ 1–11 TeV for both possible signs of μ.

The values of the top and bottom quark masses are taken as in the previous section. The value of the parameter $K_\tau = Y_\tau^2/g_3^2$ (see Eq. (60)), which is fixed by the value of the tau lepton at M_Z, is now constrained in order to get both the mass of the top and bottom quarks within 1σ and 2σ from the central experimental values simultaneously. This requirement is not fulfilled in the case where sign $\mu > 0$ and therefore in what follows we consider only the case where the sign of μ is negative. In that case, the variation of the value of K_τ, demanding 2σ agreement with the top and bottom mass experimental values, is in the range $\sim$ 0.38–0.5.

In Fig. 4 we present the Higgs mass along with the whole sparticle and Higgs mass spectrum calculated according to Eqs. (59), (61) and (62). The “mixed-scale” 1-loop approach was used in order to calculate the Higgs mass. This approach approximates the leading 2-loop corrections given by the full diagrammatic calculations [60]. However, results as the ones in [54,57] (with more refined calculations of the Higgs mass) are not yet included.

In Fig. 4, the left plot presents the mass spectrum of the model. The heavier Higgses mass are above the TeV scale while we note a heavy SUSY spectrum in general, in agreement with the non-observation of colored SUSY particles put by the LHC bounds [59]. As it was mentioned above, we are considering only the case where sign(μ) < 0, which is known not to be compatible with the muon anomalous magnetic moment, but our heavy spectrum provides very small corrections to the predictions of the SM anyway.

Going to the right plot of Fig. 4 we present the mass of the light Higgs as a function of the unified gaugino mass M. The value of K_τ is constrained to give simultaneously the right masses for the top and bottom quarks within 2σ of their experimental value, as explained before. All the points satisfy the BPO constraints. The different colored points correspond to different values of



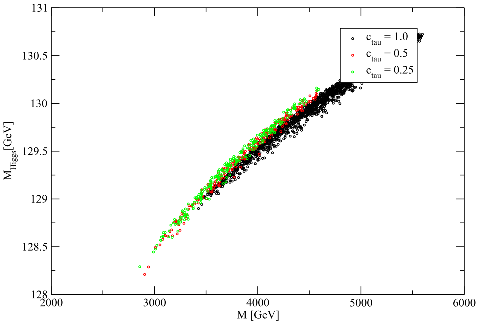


Fig. 4. The left plot shows the SUSY spectrum in the reduced MSSM. From left to right are shown: The lightest Higgs mass, the pseudoscalar one M_A, the heavy neutral one M_H, the two charged Higgses $M_{H^\pm}$; then come the two stops, two sbottoms and two staus, the four neutralinos, the two charginos, and at the end the gluino. The right plot shows the lightest Higgs mass as a function of the unified gaugino mass for three values of the unconstrained parameter c_τ.

c_τ, the constant between h_τ and Y_τ, $h_\tau = c_\tau M Y_\tau$, which is the only unconstrained parameter. The m_3^2 and μ parameters are constrained by the requirement of electroweak symmetry breaking. The value of the Higgs mass varies in the range $128 \sim 130$ GeV, but we expect that using the new version of the code FeynHiggs [55,56,54,57] this value will go down by $\sim \mathcal{O}(2 \text{ GeV})$, as already mentioned. See [28] for further details.

8. Conclusions

The reduction of couplings principle, expressed via RGI relations among couplings, provides a way to search for more fundamental quantum field theories in which a group of couplings are related to a primary one, thus reducing greatly the number of free parameters of the theory. In particular, supplemented with supersymmetry, it leads to theories where the gauge and Yukawa sectors are unified. It is essential to the construction of $N = 1$ Finite Unified Theories described here, in which the β-function vanishes to all-loops. From the theoretical side, FUTs solve the problem of UV divergences in a minimal way. On the phenomenological side, the reduction of couplings principle provides strict selection rules in choosing realistic models which lead to testable predictions. The celebrated success of predicting the top-quark mass in FUTs [5–7] was extended to the correct prediction of the Higgs boson mass, as well a prediction for the supersymmetric spectrum of the MSSM [22,24,25]. It is also possible to apply a reduction of couplings in the MSSM, as we have also shown here, again decreasing greatly the number of free parameters and making the model more predictive [23,27,28]. The two models analyzed (FUT and reduced MSSM) share similar features and are in natural agreement with all LHC measurements and searches. For the reduced MSSM the SUSY and heavy Higgs particles will likely escape the detection at the LHC, as well as at ILC and CLIC. In the FUT case parts of the allowed spectrum of the lighter scalar tau or the lighter neutralinos might be accessible at CLIC. On the other hand, the FCC-hh will be able to test the predicted parameter space for both models.

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