Diagrammatic approach to weak quark decays

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We calculate the leading logarithms of the four-point Green's function $G^{(4)}$ responsible for hadronic weak decays to order g_s^6. The result shows factorization of the two types of logarithms, each factor appearing in exponential form.

I. INTRODUCTION

The use of the operator-product expansion (OPE) has proved to be a powerful tool in the study of phenomena which are dominated by large invariants. Properties depending on these invariants can be absorbed into multiplicative factors (Wilson coefficients), which can be calculated in several cases. Asymptotic freedom of the underlying dynamical theory is usually a sufficient condition for the feasibility of such a calculation. There are cases, however, where the OPE cannot be applied directly, due to kinematic restrictions.

For this reason, a diagrammatic approach even in cases where the OPE has been found to work, is useful, not only for providing an independent test, but also for helping an intuitive understanding of the underlying physics.

In this work we make a diagrammatic analysis of a supposedly short-distance-dominated process, namely, the $\Delta S=1$ weak quark decays through charged-W-boson exchange. This process is rather well understood within the OPE framework where the Wilson coefficients are found to be of the form $(\ln M_W{}^2/\mu^2)^d$, where M_W is the W mass and μ is a typical QCD momentum scale.[1]

Our analysis extends up to three-loop graphs, an order which is sufficient for providing us with all the necessary information needed.

The quark multiplets will be considered to be elements of an unbroken SU(4) flavor group (in order to avoid complications arising from penguin graphs), where the underlying color group will be SU(3) as usual.

The lowest-order graph is shown in Fig. 1 and corresponds to a color-singlet amplitude

$$M_0=-\frac{1}{M_W{}^2}\bar{s}\gamma_\mu(1-\gamma_5)u\;\bar{u}\gamma^\mu(1-\gamma_5)d\;. \tag{1.1}$$

Addition of one gluon will produce the graphs of Fig. 2, which by means of standard methods, are found to give the leading-logarithm-approximation (LLA) result:

$$M=-\frac{6}{M_W{}^2}\bar{s}\gamma_\mu(1-\gamma_5)t^a u\;\bar{u}\gamma^\mu(1-\gamma_5)t^a d\,\ln\frac{M_W{}^2}{p^2}\;, \tag{1.2}$$

where p^2 is the quark virtual mass squared and t^a the usual SU(3) color matrices.

It is interesting to point out that in the Landau gauge the graphs (b) in Fig. 2 do not contribute and Eq. (1.2) will come out from graph (a) alone.

The same result is obtained by shrinking the W line to a point.[2] The resulting reduced graph is then superficially UV divergent and has to be calculated by means of an effective cutoff M_W.

Addition of extra gluons will not modify the chiral structure $(V-A)\times(V-A)$ of Eq. (1.2) and the general form of the amplitude will be a linear combination of the color singlets available,

$$\Gamma_{O_1}=\bar{s}\gamma_\mu(1-\gamma_5)u\;\bar{u}\gamma^\mu(1-\gamma_5)d\;, \tag{1.3}$$

$$\Gamma_{O_2}=\bar{s}\gamma_\mu(1-\gamma_5)t^a u\;\bar{u}\gamma_\mu(1-\gamma_5)t^a d\;, \tag{1.4}$$

namely,

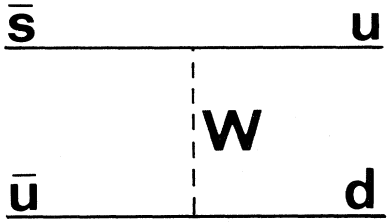


FIG. 1. Tree graph.

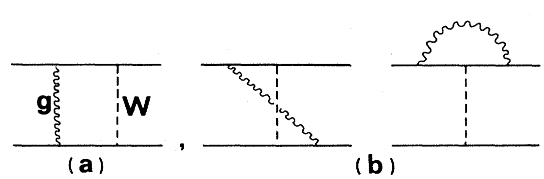
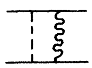
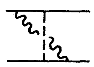
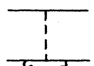


FIG. 2. (a) First-order corrections. (b) shows the two graphs which do not have leading contributions in the Landau gauge.

TABLE I. Coefficients of order-g^2 leading terms calculated in the Landau gauge. Symmetry factors are included. A factor of $1/16\pi^2$ is understood. $C'=6$.

	Γ_{O_1}	Γ_{O_2}
	0	C'
	0	0
	0	0

$$M \sim \frac{1}{M_W{}^2}[F_1(p^2,M_W{}^2,\mu^2)\Gamma_{O_1} + F_2(p^2,M_W{}^2,\mu^2)\Gamma_{O_2}] \ . \quad (1.5)$$

The purpose of this paper is the calculation of the dimensionless functions F_1 and F_2 in the LLA, where μ is a scale introduced by renormalization.

The organization of this paper is as follows. In Sec. II, we calculate the μ-independent term of both skeleton graphs and graphs containing renormalization parts. In Sec. III, we calculate the μ-dependent term of graphs containing renormalization parts, and in Sec. IV, we present our final results and conclusions.

II. CALCULATION OF μ-INDEPENDENT TERMS OF $G^{(4)}$

In this section, we calculate the leading contribution, with respect to $\ln p^2/M_W{}^2$, of the diagrams up to order g^6. In Table I, we give the diagrams contributing to g^2 order and the corresponding coefficient of $(1/16\pi^2)\ \ln p^2/M_W{}^2$. Obviously they contribute only to Γ_{O_2} [pure $(t)(t)$]. In Table II, we show the skeleton graphs, which contribute a nonzero leading term to order g^4. All other skeleton graphs giving nonleading contributions, as shown in the Appendix, are given in Fig. 3 for completeness. In Table III, we show the graphs with a renormalization part contributing a nonzero leading term to order g^4. Note that to this order these are the only graphs with a leading contribution. All other graphs are given in Fig. 4. Table IV gives the order-g^6 skeleton graphs and the corresponding coefficient of $[1/(16\pi^2)^3]\ln^3 p^2/M_W{}^2$. In order to evaluate the order-g^6 graphs having renormalization part(s), consider first the element of the ladder shown in Fig. 5. The blobs A, D, and Γ, which represent the finite part of the fermion propagator, gluon propagator, and vertex corrections, respectively, will generate logarithms (UV logarithms) from the region where the internal momentum flow is large. In Landau gauge, A is zero to leading order, therefore we neglect it. Renormalization will introduce a momentum scale μ^2 so that D and Γ will be functions of $\ln p^2/\mu^2$ where p^2 is the square of the largest momentum involved. We have

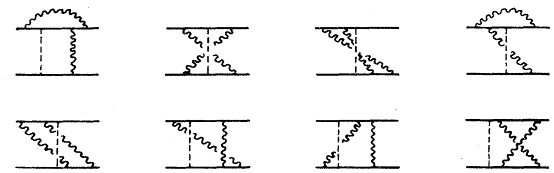


FIG. 3. Skeleton graphs which do not give leading contributions to order g^2.

$$D_{\mu\nu}(k^2,\mu^2)=i\left[\frac{g_{\mu\nu}}{k^2}-k_\mu k_\nu\right]\Big/[1+D(k^2,\mu^2)] \ , \quad (2.1a)$$

$$\Gamma_\mu(p,p',k^2,\mu^2)=-i\gamma_\mu\Gamma(k^2,\mu^2) \ . \quad (2.1b)$$

Note that in leading order there is a momentum hierarchy $p'^2 \simeq k^2 \gg p^2$.

The ladder element is proportional to

$$G=\Gamma^2(k^2,\mu^2)/[1+D(k^2,\mu^2)] \ , \quad (2.2)$$

where factors involving Lorentz indices have been omitted. The functions Γ and $(1+D)$ satisfy the renormalization-group equations (RGE's)

$$\mathscr{D}\Gamma-\tfrac{1}{2}\gamma_G\Gamma=0, \quad \mathscr{D}(1+D)-\gamma_G(1+D)=0 \ , \quad (2.3)$$

TABLE II. Coefficients of order-g^4 skeleton graphs calculated in the Landau gauge. Only the nonzero contributing graphs are shown. Symmetry factors are included. A factor of $1/(16\pi^2)^2$ is understood.

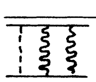
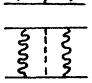
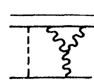
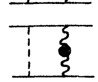
	Γ_{O_1}	Γ_{O_2}
	$\dfrac{C'^2}{2\times 9}$	$-\dfrac{C'^2}{3\times 4}$
	$\dfrac{C'^2}{2\times 9}$	$-\dfrac{C'^2}{3\times 4}$
Sum	$C'^2/9$	$-C'^2/6$

TABLE III. Coefficients of order-g^4 graphs with a renormalization part calculated in the Landau gauge. Only the nonzero contributing graphs are shown. Symmetry factors are included. $\gamma_G{}^1$ is the first coefficient in the γ function of the gluon field. A factor of $1/(16\pi^2)^2$ is understood.

	Γ_{O_1}	Γ_{O_2}
	0	$-\dfrac{\beta_1 C'}{2}+\dfrac{\gamma_G^1 C'}{2}$
	0	$-\dfrac{\gamma_G^1 C'}{2}$
Sum	0	$-\beta_1 C'/2$

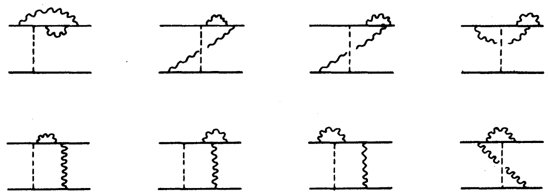


FIG. 4. Graphs with renormalization parts which do not give leading contributions to order g^4.

where as usual γ_G is the anomalous dimension of the gluon field and

$$\mathscr{D}=\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}\ .$$

Since we are interested in leading order only, we neglect, in Landau gauge, the factor $-\gamma_F\Gamma$ in the RGE for Γ. Now G itself satisfies the equation

$$\mathscr{D}G=0 \tag{2.4}$$

with leading-order solution

$$\begin{aligned} G&=g^2\Big/\left[1-g^2\beta_1\ln\frac{p^2}{\mu^2}\right]\\ &=g^2+\beta_1 g^4\ln\frac{p^2}{\mu^2}-\beta_1^2g^6\ln^2\frac{p^2}{\mu^2}+O(g^8)\ , \end{aligned}\tag{2.5}$$

where β_1 is the order-g^3 coefficient of the β function of the RGE. This expansion represents the sum of all the graphs shown in Fig. 5 with insertions up to the relevant order.

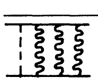
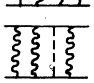
Using Eq. (2.5) to the required order, we calculate the LLA contributions of diagrams shown in Table V, where the coefficient of $[1/(16\pi^2)^3]\ln^3 p^2/M_W^2$ is given.

III. EVALUATION OF THE μ-DEPENDENT TERMS OF $G^{(4)}$

The RGE gives the dependence of a Green's function from the renormalization point μ, up to the order that the β and γ functions are known.[3] Our Green's function $G^{(4)}$ obeys the RGE

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}-2\gamma_F(g)\right]G^4=0\ , \tag{3.1}$$

TABLE IV. Same as in Table II, to order g^6. A factor of $1/(16\pi^2)^3$ is understood.

	Γ_{O_1}	Γ_{O_2}
	$-\frac{C'^3}{2\times 81}$	$\frac{C'^3}{2\times 18}$
	$-\frac{C'^3}{2\times 81}$	$\frac{C'^3}{2\times 18}$
Sum	$-C'^3/81$	$C'^3/18$

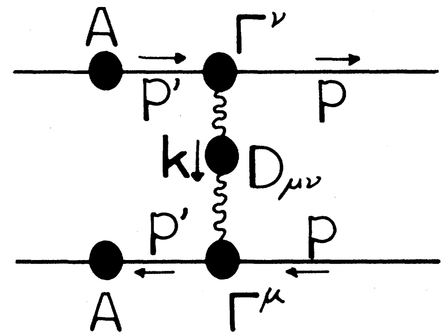


FIG. 5. Fully dressed ladder element.

where

$$\gamma_F(g)=\gamma_1\frac{g^2}{16\pi^2}+\gamma_2\left[\frac{g^2}{16\pi^2}\right]^2+O(g^6) \tag{3.2a}$$

is the anomalous dimension of the fermion field and $\beta(g)$ is given by

$$\beta(g)=g\left[\beta_1\frac{g^2}{16\pi^2}+\beta_2\left[\frac{g^2}{16\pi^2}\right]^2+O(g^6)\right]\ . \tag{3.2b}$$

We write

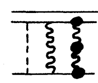
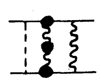
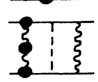
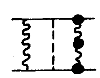
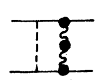
$$G^{(4)}=\Gamma_{O_1}(1+g^2A+g^4B)+\Gamma_{O_2}(g^2C+g^4D) \tag{3.3}$$

(in order to make the calculations as simple as possible we shall proceed to the g^6 order in the next step). Now, applying Eq. (3.1) to (3.3), we get a simple differential equation for A, B, C, and D, in each order of g^2. Since in the Landau gauge $\gamma_1=0$, we get

$$A=0,\ \ C=0\ . \tag{3.4}$$

Since the RGE gives only the μ-dependent terms, we add

TABLE V. Same as in Table III, to order g^6. A factor of $1/(16\pi^2)^3$ is understood.

	Γ_{O_1}	Γ_{O_2}
	$-\frac{\beta_1 C'^2}{3\times 9}$	$\frac{\beta_1 C'^2}{3\times 6}$
	$-\frac{\beta_1 C'^2}{3\times 9}$	$\frac{\beta_1 C'^2}{3\times 6}$
	$-\frac{\beta_1 C'^2}{2\times 3\times 9}$	$\frac{\beta_1 C'^2}{2\times 3\times 6}$
	$-\frac{\beta_1 C'^2}{2\times 3\times 9}$	$\frac{\beta_1 C'^2}{2\times 3\times 6}$
	0	$\frac{\beta_1^2 C}{3}$
Sum	$-\beta_1 C'^2/9$	$\frac{\beta_1^2 C'}{3}+\frac{\beta_1 C'^2}{6}$

to the above results the ones shown in Table I, namely, the μ-independent terms, to get

$$A=0,\quad C=\frac{C'}{16\pi^2}\ln\frac{p^2}{M_W{}^2}\,, \tag{3.5}$$

where $C'=6$.

Solving now for B we get

$$B=\frac{1}{(16\pi^2)^2}\frac{1}{2}2\gamma_2\ln\frac{\mu^2}{p^2}\,, \tag{3.6}$$

which is obviously a subleading contribution. Taking into account the μ-independent leading contribution of Table II we get

$$B=\frac{1}{(16\pi^2)^2}\frac{C'^2}{9}\ln^2\frac{p^2}{M_W{}^2}\,. \tag{3.7}$$

Solving, finally, for D we get

$$D=-\frac{1}{(16\pi^2)^2}\beta_1 C'\ln\frac{p^2}{M_W{}^2}\ln\frac{\mu^2}{p^2} \tag{3.8}$$

and adding the contribution shown in Tables II and III, we find

$$D=\frac{1}{(16\pi^2)^2}\left[-\beta_1 C'\ln\frac{p^2}{M_W{}^2}\ln\frac{\mu^2}{p^2} - \left[\frac{\beta_1 C'}{2}+\frac{C'^2}{6}\right]\ln^2\frac{p^2}{M_W{}^2}\right]. \tag{3.9}$$

We next make a further step and add g^6 terms to the Green's function of Eq. (3.3):

$$G=\Gamma_{O_1}(1+g^2A+g^4B+g^6F) + \Gamma_{O_2}(g^2C+g^4D+g^6E)\,. \tag{3.10}$$

Working as before we find

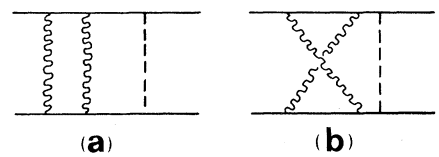


FIG. 6. Two typical order-g^4 graphs. (a) gives leading contribution while (b) does not.

$$F=\frac{1}{(16\pi^2)^3}\left[-2\beta_1\frac{C'^2}{9}\ln^2\frac{p^2}{M_W{}^2}\ln\frac{\mu^2}{p^2}\right], \tag{3.11}$$

$$E=\frac{1}{(16\pi^2)^3}\left[\beta_1{}^2 C'\ln^2\frac{\mu^2}{p^2}\ln\frac{p^2}{M_W{}^2} + \beta_1 C'(\beta_1+\tfrac{1}{3}C')\ln\frac{p^2}{M_W{}^2}\ln\frac{\mu^2}{p^2}\right]. \tag{3.12}$$

Again we have to go back to Tables IV and V and add the corresponding μ-independent contribution to get

$$F=\frac{1}{(16\pi^2)^3}\left[-2\beta_1\frac{C'^2}{9}\ln\frac{\mu^2}{p^2}\ln^2\frac{p^2}{M_W{}^2} - \left[\frac{C'^3}{81}+\frac{\beta_1 C'^2}{9}\right]\ln^3\frac{p^2}{M_W{}^2}\right], \tag{3.13a}$$

$$E=\frac{1}{(16\pi^2)^3}\left[\beta_1{}^2 C'\ln^2\frac{\mu^2}{p^2}\ln\frac{p^2}{M_W{}^2} + \beta_1 C'(\beta_1+\tfrac{1}{3}C')\ln\frac{\mu^2}{p^2}\ln^2\frac{p^2}{M_W{}^2} + \left[\frac{C'^3}{18}+\frac{\beta_1{}^2 C'}{3}+\frac{\beta_1 C'^2}{6}\right]\ln^3\frac{p^2}{M_W{}^2}\right]. \tag{3.13b}$$

We can now write the full $G^{(4)}$ function up to order g^6,

$$\begin{aligned} G^{(4)}=\Gamma_{O_1}&\left\{1+\left[\frac{g^2}{16\pi^2}\right]^2\frac{C'^2}{9}\ln^2\frac{p^2}{M_W{}^2}+\left[\frac{g^2}{16\pi^2}\right]\left[-2\beta_1\frac{C'^2}{9}\ln\frac{\mu^2}{p^2}\ln^2\frac{p^2}{M_W{}^2}-\left[\frac{C'^3}{81}+\frac{\beta_1 C'^2}{9}\right]\ln^3\frac{p^2}{M_W{}^2}\right]\right\} \\ &+\Gamma_{O_2}\left\{\frac{g^2}{16\pi^2}C'\ln\frac{p^2}{M_W{}^2}+\left[\frac{g^2}{16\pi^2}\right]^2\left[-\beta_1 C'\ln\frac{\mu^2}{p^2}\ln\frac{p^2}{M_W{}^2}-\left[\frac{\beta_1 C'}{3}+\frac{C'^2}{6}\right]\ln^2\frac{p^2}{M_W{}^2}\right]\right. \\ &\left.+\left[\frac{g^2}{16\pi^2}\right]^3\left[\beta_1{}^2C'\ln^2\frac{\mu^2}{p^2}\ln\frac{p^2}{M_W{}^2}+\beta_1 C'\left[\beta_1+\frac{C'}{3}\right]\ln\frac{\mu^2}{p^2}\ln^2\frac{p^2}{M_W{}^2}+\left[\frac{C'^3}{18}+\frac{\beta_1{}^2C'}{3}+\frac{\beta_1 C'^2}{6}\right]\ln^3\frac{p^2}{M_W{}^2}\right]\right\}. \end{aligned} \tag{3.14}$$

IV. FINAL RESULTS AND CONCLUSIONS

The form of $G^{(4)}$ in Eq. (3.14) does not exhibit explicit factorization. However, if we reexpress the logarithms in terms of $\ln\mu^2/p^2$ and $\ln\mu^2/M_W{}^2$ and use the linear combination $\frac{2}{3}(\Gamma_{O_1}+\frac{3}{2}\Gamma_{O_2})$, $\frac{1}{3}(\Gamma_{O_1}-3\Gamma_{O_2})$, which are the eigenvectors of the anomalous dimension matrix that were found in Ref. 1, with eigenvalues $2C'/3$ and $-4C'/3$, we actually resurrect the first three terms, in the g^2 expansion, of the exponential form

$$G^{(4)}=\tfrac{2}{3}(\Gamma_{O_1}+\tfrac{3}{2}\Gamma_{O_2})\left[1+\frac{g^2}{16\pi^2}\beta_1\ln\frac{\mu^2}{p^2}\right]^{-a}\left[1+\frac{g^2}{16\pi^2}\beta_1\ln\frac{\mu^2}{M_W{}^2}\right]^{a}$$
$$+\tfrac{1}{3}(\Gamma_{O_1}-3\Gamma_{O_2})\left[1+\frac{g^2}{16\pi^2}\beta_1\ln\frac{\mu^2}{p^2}\right]^{d}\left[1+\frac{g^2}{16\pi^2}\beta_1\ln\frac{\mu^2}{M_W{}^2}\right]^{-d}, \tag{4.1}$$

where

$$a=\frac{2C'/3}{2\beta_1},\quad d=\frac{4C'/3}{2\beta_1}\ . \tag{4.2}$$

This exponentiation shows clearly that results obtained by means of the OPE under the assumption of short-distance dominance, can be reproduced in perturbation theory by summing up selected terms taken from individual Feynman graphs. These terms which are the leading logarithms of the W mass, correspond to the short-distance part of the specific process. The low-momentum (large-distance) dynamics have been oversimplified by assuming a universal off-shell quark momentum p^2, so that an attempt to compare the results found with experiment could be unrealistic.

A final comment has to be made concerning the role of penguin graphs. In a broken SU(4) flavor group, where the c quark is heavy, these graphs may play an important role.[4–6] It has been shown[6] that the sum of all penguin-type graphs still equals a sum of Wilson coefficients times the matrix elements of local four-fermion operators. As the problem now involves three momentum scales (p,M_W,m_c), the resulting OPE effective Hamiltonian is quite complicated. The equivalence between the leading-logarithm summation and the OPE in this case has not, to the best of our knowledge, been checked to all orders so far.

ACKNOWLEDGMENTS

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APPENDIX

In this appendix we will outline why the graphs in Figs. 3 and 4 are not giving leading contribution. It is clear that internal momenta larger than $M_W{}^2$ are not important in this case. We can therefore use $M_W{}^2$ as an UV cutoff for all the graphs. In this region the W propagator can be shrunk to a point and all graphs will become logarithmically divergent by power counting. This trick will identify the dominant graphs as these graphs which contain the maximum number of superficially divergent subintegrals. As an example, consider the two graphs in Fig. 6. The above line of thought would identify graph (a) as dominant while graph (b) should not be dominant because it does not contain a superficially divergent subintegration when the W propagator is shrunk to a point.

In order to check this conjecture we calculated all the graphs (by means of standard methods) and extracted the leading-logarithmic behavior by Mellin transforming the amplitude[7] with respect to $M_W{}^2$. The result proves that the cutoff trick is in fact correct.

As we have already mentioned, the two graphs shown in Fig. 2(b) do not give leading contributions in the Landau gauge. And this is due to the Lorentz and γ-algebra structure of the graph. This structure, in leading order, persists even in the case when we add more gluons, in any way we like, to these two graphs. The same is true for the graphs of Fig. 4 if we recall that γ_1, the coefficient of $g^2/16\pi^2$ in the fermion γ function, is zero in Landau gauge.

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