Higher order calculations and renormalization scheme dependence in the 2D Gross–Neveu model

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We present a systematic algorithm for calculating certain classes of p-loop integrals. The results are used for calculating the highest-N term of the β-function for the Gross–Neveu model in four and five loops respectively. The scheme dependence of the results is also being discussed.

The recent evaluation of the β-function for the Gross–Neveu (G–N) model [1] in three loops [3] raised the question whether the observed vanishing of the renormalization scheme independent part (i.e. the term proportional to N^2) persists in the next orders. In this letter we develop a systematic procedure for calculating the highest-N term of the β-function to any loop order. All contributing graphs can be generated by means of two basic ones, while the (infinite part of the) momentum integrals is calculated through a recurrent relation.

The lagrangian of the model is

$$L=\bar{\Psi}(\mathrm{i}\partial\!\!\!/)\Psi+\tfrac{1}{2}\lambda(\bar{\Psi}\Psi)(\bar{\Psi}\Psi)\,, \qquad (1)$$

or in the so-called σ-formulation which is more suitable for our calculations

$$L_\sigma=\bar{\Psi}(\mathrm{i}\partial\!\!\!/)\Psi-g(\bar{\Psi}\Psi)\sigma+\tfrac{1}{2}\sigma^2\,, \qquad (2)$$

where $g^2=\lambda$ and in both eq. (1) and eq. (2) we have suppressed the summation over the flavour index N of the fermion field Ψ. The field σ serves as an auxiliary one. The β-function of the model has been calculated up to three loops [2,3] and equals

$$\beta(\lambda)=(N-1)\left(-\frac{4\lambda^2}{4\pi}+\frac{8\lambda^3}{(4\pi)^2}-\frac{420\lambda^4}{(4\pi)^3}\right).$$

The vanishing of the β-function for $N=1$ shows the equivalence to the massless Thirring model. It is useful to note that to any j-loop ($j\geqslant 2$) order, the corresponding β-function term is a polynomial in N of order $j-1$. Evaluation of the third order term revealed that the coefficient of the N^2 term is zero, as eq. (3) shows. Let us now examine the renormalization scheme dependence of the β-function. Under an (N-independent) analytic rescaling of the coupling

$$\lambda'=\lambda+c_2\lambda^2+c_3\lambda^3+c_4\lambda^4 \qquad (3)$$

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we get

$$\beta_1'=\beta_1\,,\quad \beta_2'=\beta_2\,,\quad \beta_3'=\beta_3-c_2\beta_2+(c_3-c_2^2)\beta_2\,, \tag{4a,b,c}$$

$$\beta_4'=\beta_4+2\beta_1(c_4+2c_2^3-3c_3c_2)+\beta_2c_2^2-2\beta_3c_2\,, \tag{4d}$$

with an obvious definition of the β_i. Since β_1 and β_2 are proportional to $(N-1)$, an N^2-proportional term in β_3 would be scheme independent, as eq. (4c) shows. The lack of this term shows that by a suitable rescaling of the coupling, β_3 can be made to vanish. Finally eq. (4d) shows that possible N^3- and N^2-proportional terms of β_4 are both scheme independent.

All the terms of the β-function can be extracted from the infinite parts of the four-and two-point functions calculated to the appropriate order. Although the number of contributing diagrams gets large very quickly and the corresponding momentum integrals become more and more complicated, graphs contributing to the highest power of N (namely $j-1$ for the jth term of the β-function) are represented by the three diagrams of fig. 1, where the blobs represent jth loop corrections containing $(j-1)$ fermion loops. As far as the two-point function is concerned, the only such possible diagram is the one shown in fig. 2a. The correction to the vertex can be easily obtained by differentiating the two-point function with respect to m, a fermion mass which plays the role of infrared regulator. Finally, the correction to the σ-propagator can be obtained in a similar way by differentiating the vacuum-to-vacuum diagram, shown in fig. 2b, twice with respect to m (for more details on the methods used see ref. [2]).

The building block for all calculations is the one-loop correction to the σ-propagator, shown in fig. 3, which we denote by $\Sigma(k)$. Separating out the infinite part we can write

$$\Sigma(k)=-N\omega[2I-k^2\zeta(k)+4m^2\zeta(k)]g^2\,, \tag{5a}$$

where

$$I=\int\frac{\mathrm{d}^{2\omega}p}{(2\pi)^{2\omega}}\frac{1}{p^2-m^2}\,,\quad \zeta(k)=\int\frac{\mathrm{d}^{2\omega}p}{(2\pi)^{2\omega}}\frac{1}{(p^2-m^2)[(p+k)^2-m^2]}\,, \tag{5b,c}$$

and 2ω is the dimension of space–time. The integral I contains the UV divergence for $\omega=1$ while $\zeta(k)$ is finite. Now the vacuum-to-vacuum diagram shown in fig. 2b can be written as

$$V(m)=\mathrm{i}^{(j-1)}\int\frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}}\,[\Sigma(k)]^{(j-1)}\,, \tag{6}$$

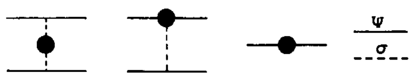

Fig. 1. Highest N generic diagrams.

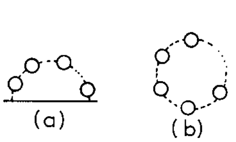

Fig. 2. (a) Diagram contributing to the two-point function. (b) Vacuum diagram.

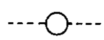

Fig. 3. The one-loop correction to the σ propagator.

while fig. 2a takes the form

$$W(\not{p}, m) = -\mathrm{i}^{(j+1)} g^2 \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}} [\Sigma(k)]^{(j-1)} \frac{(\not{p}-\not{k})+m}{(p-k)^2-m^2} . \tag{7}$$

As expected, $W(\not{p}, m)$ has the form $A\not{p}+Bm$. The infinite parts of A and B are not independent (note that $W(\not{p}, m)$ is not the whole two-point function), but related through the simple equation Inf. Part $(A) = (1-1/\omega)$Inf. Part(B). This can be most easily checked by expanding the fermion propagator in eq. (7) around $p_\mu = 0$

$$\frac{\not{p}-\not{k}+m}{(p-k)^2-m^2} = \frac{-\not{k}+m}{k^2-m^2} + \frac{\partial}{\partial p^\mu} \frac{\not{p}-\not{k}+m}{(p-k)^2-m^2}\bigg|_{p^\mu=0} + ... = \frac{-\not{k}+m}{k^2-m^2} + \frac{\not{p}}{k^2-m^2} + 2(pk)\frac{-\not{k}+m}{(k^2-m^2)^2} + ... \tag{8}$$

After all these simplifications the integrals needed to be evaluated can be put into the form

$$K(a, b) = \int \frac{\mathrm{d}^{2\omega}p}{(2\pi)^{2\omega}} (p^2)^a [\zeta(p)]^b , \quad a \geqslant 0, b \geqslant 1, a \leqslant b , \tag{9}$$

which is a $(b+1)$-loop integral.

Upon employing Feynman parametrization, $\zeta(k)$ becomes

$$\zeta(k) = \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{[k^2+p^2x(1-x)-m^2]^2} .$$

It is now straightforward to show that $\zeta(k)$ satisfies the following differential equation:

$$\zeta(p) = \frac{\partial I}{\partial m^2} - \tfrac{1}{2}p^2 \frac{\partial \zeta}{\partial m^2} - 2p^2 \frac{\partial \zeta}{\partial p^2} . \tag{10}$$

Differentiating eq. (9) with respect to m^2 and using eq. (10) we get

$$\frac{\partial}{\partial m^2} K(a+1, b) = \tfrac{2}{3}(2a+2\omega-b)K(a, b) + \tfrac{2}{3}b \frac{\partial I}{\partial m^2} K(a, b-1) . \tag{11}$$

Taking into account that

$$K(a, b) = (-m^2)^{(b+1)\omega-2b+a} f(\omega) , \tag{12}$$

we finally get the following powerful recurrent formula:

$$K(a+1, b) = \frac{2m^2(2a-b-2\omega)K(a, b) + 2b(\omega-1)IK(a, b-1)}{(b+1)\omega-2b+a+1} , \tag{13a}$$

with the following initial values

$$K(0, 0) = 0 , \quad K(0, 1) = I^2 , \quad K(0, j) = R^j, j > 1 , \tag{13b}$$

where

$$R^j = \int \frac{\mathrm{d}^{2\omega}p}{(2\pi)^{2\omega}} [\zeta(p)]^j , \tag{13c}$$

which are finite for $j > 1$.

Taking care of the counterterms is easy. In the two-point function of fig. 2a the only counterterms arising are those connected with fermion loop(s). Replacing $\Sigma(k)$, in eq. (7), by

$$\Sigma(k) - \text{Inf.Part}\,[\Sigma(k)] ,$$

we automatically take care of all counterterms. The same procedure can also be applied to eq. (6), for the vacuum diagrams. In this case we have to add one more counterterm, not included in the previous procedure, namely the one shown in fig. 4, where the cross stands for the $(j-1)$-loop wave function counterterm (times a factor of $2j$).

Now the desired infinite part of the σ-propagator can be taken as

$$\frac{g^2}{2(j-1)}(\partial_m)^2 V(m)\,, \tag{14a}$$

while the corresponding term for the vertex is

$$g\partial_m W(\not\phi, m)\,. \tag{14b}$$

The recurrent formula of eq. (13a) can be easily manipulated up to any order by means of a standard algebraic computer package (e.g. Mathematica). Defining the wave function renormalization constant by Z_2 and the corresponding four-point function renormalization constant by Z_4, the β-function is defined as usual to be [4]

$$\beta(\lambda)=2\lambda^2\frac{\partial}{\partial\lambda}\mathrm{Res}(Z_4-2Z_2)\,. \tag{15}$$

As an application, we evalute the highest-N term of the β-function in three, four and five loops respectively. To that end we need the infinite part of $K(a, b)$ for $a, b\leqslant 4$.

Three loops. The diagram of figs. 2a gives the following $1/\epsilon$-term ($\epsilon=1-\omega$):

$$\mathrm{i}\frac{\lambda^3}{(4\pi)^3}N^2\frac{1}{\epsilon}\left(-\tfrac{8}{3}\not\phi+\tfrac{4}{3}m\right)\,, \tag{16a}$$

while the diagram of fig. 2b gives

$$\mathrm{i}\frac{\lambda^2}{(4\pi)^3}N^2\frac{1}{\epsilon}\frac{16}{3}m^2\,. \tag{16b}$$

Using eqs. (14a) and (14b) we get the following contribution to Z_4:

$$\frac{\lambda^3}{(4\pi)^3}N^2\frac{1}{\epsilon}\frac{16}{3}\,, \tag{17a}$$

while the contribution to Z_2 is

$$\frac{\lambda^3}{(4\pi)^3}N^2\frac{1}{\epsilon}\frac{8}{3}\,. \tag{17b}$$

Using eq. (15) the N^2-proportional contribution to the β-function vanishes.

Four loops. The corresponding results that we get are as follows:

$$(\text{fig. 2a}):\ \mathrm{i}\frac{\lambda^4}{(4\pi)^4}N^3\frac{1}{\epsilon}\left[6\not\phi-m(2-m^2R^2-m^4R^3)\right]\,, \tag{18a}$$

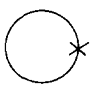

Fig. 4. Counterdiagram to the vacuum diagram of fig. 2b. The cross stands for the $(j-1)$-loop wave function counterterm.

$$(\text{fig. 2b}):\ \mathrm{i}\frac{\lambda^3}{(4\pi)^4}N^3m^2\frac{1}{\epsilon}(-28-6m^2R^2-6m^4R^3)\,, \tag{18b}$$

$$Z_4:\frac{\lambda^4}{(4\pi)^4}N^3\frac{1}{\epsilon}(-\tfrac{40}{3})\quad Z_2:\frac{\lambda^4}{(4\pi)^4}N^3\frac{1}{\epsilon}(-6)\,. \tag{19a,b}$$

Now the N^3-proportional contribution to the β-function turns out to be

$$\frac{\lambda^5}{(4\pi)^4}N^3(-\tfrac{64}{3})\,, \tag{19c}$$

showing that to this order the contribution of the highest N term does not vanish.

Five loops. The corresponding results are

$$(\text{fig. 2a}):\ \mathrm{i}\frac{\lambda^5}{(4\pi)^5}N^4\frac{1}{\epsilon}[\not p(-\tfrac{64}{5}+\tfrac{8}{5}m^2R^2+\tfrac{8}{5}m^4R^3)+m(\tfrac{16}{5}-\tfrac{56}{5}m^2R^2-8m^4R^3)]\,, \tag{20a}$$

$$(\text{fig. 2b}):\ \mathrm{i}\frac{\lambda^5}{(4\pi)^5}N^4\frac{1}{\epsilon}m^2(\tfrac{448}{5}+80m^2R^2+\tfrac{272}{5}m^4R^3)\,, \tag{20b}$$

$$Z_4:\frac{\lambda^5}{(4\pi)^5}N^4\frac{1}{\epsilon}(\tfrac{144}{5}-\tfrac{12}{5}m^2R^2-\tfrac{12}{5}m^4R^3)\,,\quad Z_2:\frac{\lambda^5}{(4\pi)^5}N^4\frac{1}{\epsilon}(\tfrac{64}{5}-\tfrac{8}{5}m^2R^2-\tfrac{8}{5}m^4R^3)\,, \tag{21a,b}$$

and the N^4-proportional term of the β-function is

$$\frac{\lambda^6}{(4\pi)^5}N^4(32+8m^2R^2+8m^4R^3)\,. \tag{21c}$$

The integrals R^2 and R^3 can be calculated numerically. Taking into account that $m^2R^2\simeq 8.4144$ and $m^4R^3\simeq 3.6062$ we conclude that in five loops the highest-N term of the β-function is not zero.

The intriguing dependence on N found in three loops, does not seem to persist in higher orders. The four-loop result is a third order polynomial in N, while the five-loop one is a fourth order polynomial, suggesting that the three-loop result is a fluke, connected most probably to the topological structure of the contributing graphs and not to physics. However, the results of this work are not completely negative. The algorithm presented can actually be used for calculating next to leading N corrections in perturbation theory, and as such might be of some interest.

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