The Gross–Neveu chiral condensate in finite volume

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The two-dimensional Gross–Neveu model exhibits spontaneous breaking of its discrete chiral symmetry. In finite volume the zero momentum fermionic modes condense and a perturbative condensate is formed. Previous calculations of the chiral condensate, are extended to two-loop order and an estimate for the infinite volume limit is given.

The Gross–Neveu two-dimensional fermionic model has a rich structure, as an asymptotically free field theory, exhibiting dynamical mass generation through the breaking of its discrete chiral symmetry [1]. Its spectrum contains elementary fermions, and tensor multiplets of the $SU(N)$ symmetry of the model [2]. It has also been shown to be classically integrable [3] and 2–2 particle scattering matrices for different sectors of its spectrum have been determined [4–7].

A perturbative treatment of the dynamical mechanisms of this model has been shown to be valid, if we restrict space in a finite periodic box, since then, the zero-momentum modes condense and their density is calculable order by order in powers of the renormalized coupling constant [8,9]. In this letter we extend the calculation of ref. [8] to two-loop order and an estimate of the infinite volume limit is given using the recently calculated beta function of the model to three-loop order [10].

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The calculation is done along the lines of ref. [8]. The lagrangian of the model is

$$\mathscr{L}_{\mathrm{m}}=\bar{\psi}(\mathrm{i}\partial\!\!\!/-m)\psi-\frac{f}{2N}(\bar{\psi}\psi)^2\,, \tag{1}$$

where the fermionic fields carry an internal flavour index ψ^i, $i=1, 2, ..., N$. We restrict the space to be a circle of length L times a, $-\varepsilon$-dimensional real line, $\mathrm{R}^{-\varepsilon}$, for UV regularization reasons. The time is assumed to be infinite with plane wave boundary conditions (b.c.).

These b.c. determine the fermion propagator to be

$$S_{\mathrm{F}}(x,y;m)=\frac{1}{L}\int\frac{\mathrm{d}^Dk}{(2\pi)^D}\sum_{n=-\infty}^{+\infty}\frac{\exp[\mathrm{i}k(x-y)]}{k\!\!\!/+m}\,, \tag{2}$$

where $D=1-\varepsilon$ and the space momenta k_1 are discrete

$$k_1=\frac{2\pi n}{L}\,,\quad n=0,\,1,\,2,\,...\,. \tag{3}$$

For $m=0$ the lagrangian has the discrete chiral symmetry

$$\psi\to\gamma^5\psi\,,\quad\bar{\psi}\to\bar{\psi}\gamma^5\,, \tag{4}$$

which is known to be broken in the $1/N$ approximation [1].

We shall use the two-dimensional euclidean Clifford algebra for the γ-matrices (dimensional reduction),

$$\{\gamma^\mu, \gamma^\nu\} = -2\delta^{\mu\nu} . \tag{5}$$

We define the condensate for $m \neq 0$ as

$$\int d^{2-\varepsilon}x \, \langle \bar\psi(x)\psi(x) \rangle_m \equiv \frac{\partial}{\partial m} \log Z(m) , \tag{6}$$

where $Z(m)$ is the partition function,

$$Z(m) = \int D\psi \, D\bar\psi \exp\left(- \int \mathscr{L} \, d^{2-\varepsilon}x \right) . \tag{7}$$

Finally, sending $m \to 0$, from positive values, in eq. (6), we get the desired result:

$$\int d^{2-\varepsilon}x \, \langle \bar\psi\psi(x) \rangle_0 = \lim_{m\to 0+} \int d^{2-\varepsilon} \langle \bar\psi\psi(x) \rangle_m . \tag{8}$$

The chiral condensate satisfies the RGE (L is the length of the space):

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(f) \frac{\partial}{\partial f} + \gamma_{\bar\psi\psi}(f) \right) \langle \bar\psi\psi \rangle_0 = 0 , \tag{9}$$

where the $\beta(f)$ and $\gamma_{\bar\psi\psi}$ functions are normalized as

$$\beta(f) = -f^2(\beta_0 + \beta_1 f + \beta_2 f^2 + ...) , \tag{10}$$

$$\gamma_{\bar\psi\psi}(f) = -f(\gamma_1 + \gamma_2 f + ...) . \tag{11}$$

So from the RGE (9) we expect that the dependence of $\langle \bar\psi\psi \rangle_L$ on L will be logarithmic (apart from the canonical dimensions),

$$\langle \bar\psi\psi \rangle_L = \frac{1}{L} [\alpha_{00} + f(\alpha_{11} \ln \mu L + \alpha_{10}) + ...] , \tag{12}$$

and the coefficients α_{11}, α_{21}, α_{22} should be given as

$$\alpha_{11} = \alpha_{00}\gamma_1 , \quad \alpha_{21} = \alpha_{10}(\beta_0 + \gamma_1) + \alpha_{00}\gamma_2 , \tag{13a}$$

$$\alpha_{22} = \tfrac{1}{2}\alpha_{11}(\beta_0 + \gamma_1) , \tag{13b}$$

if the RGE is to be satisfied up to two loops.

The following coefficients are known [1,11,10]:

$$\beta_0 = \frac{N-1}{2\pi N} , \quad \beta_1 = -\frac{N-1}{(2\pi N)^2} , \quad \beta_2 = -\frac{105}{4} \frac{N-1}{(2\pi N)^3} \tag{14}$$

and [1,8]

$$\gamma_1 = \frac{2N-1}{2\pi N} , \tag{15}$$

as also [8] ($\gamma_E = 0.5772$)

$$\alpha_{00} = N , \quad \alpha_{10} = -\frac{2N-1}{4\pi} (\ln 4\pi - \gamma_E) . \tag{16}$$

It remains thus to calculate the coefficients α_{20} as well as γ_2.

The diagrams appearing for $Z(m)$ up to two loops and their contributions to the coefficient α_{20} are presented in fig. 1. The final result for α_{20} is

$$\alpha_{20} = \frac{N-\frac{1}{2}}{4\pi^2 N} [(N-\tfrac{3}{4})a + b] ,$$

$$a = -\ln 4\pi + \gamma_E , \quad b = \ln \pi - \gamma_E , \tag{17}$$

and the coefficient γ_2 turns out to be [10]

$$\gamma_2 = 0 . \tag{18}$$

After presenting the perturbative calculations up to two loops, we shall extract from these the non-perturbative and renormalization scheme independent information about the chiral condensate. What we are after is to give an estimate of the constant c relating

$$\widehat{\langle \bar\psi\psi \rangle} = c\Lambda_{\overline{MS}} , \tag{19}$$

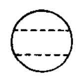
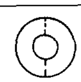
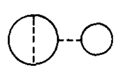
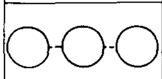
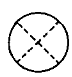
	Nr.	$(2\pi)^2 a_{20}$
	1α	$\frac{1}{4N} \cdot a$
	1β	$b - \frac{a^2}{4}$
	1γ	$-a^2$
	1δ	$N . a^2$
	1ε	$\frac{1}{8N}(a^2 - 4b)$

Fig. 1. The two-loop diagrams and their contribution to the coefficient α_{20}: $a = \gamma_E - \ln 4\pi$, $b = \ln \pi - \gamma_E$.

where Λ_{MS} is the MS renormalization group invariant (RGI) scale of the model,

$$\Lambda_{\mathrm{MS}}=\mu(\beta_0 f)^{-\beta_1/\beta_0^2} \times\exp\left(-\frac{1}{\beta_0 f}\right)\left(1+\frac{\beta_1^2-\beta_0\beta_2}{\beta_0^3}f+\ldots\right) \tag{20}$$

and $\widehat{\langle\bar\psi\psi\rangle}$ an appropriately defined RGI chiral condensate, which satisfies RGE (9) without the anomalous dimension $\gamma_{\bar\psi\psi}$. This defines $\widehat{\langle\bar\psi\psi\rangle}$ uniquely up to an overall constant [8],

$$\widehat{\langle\bar\psi\psi\rangle}=\exp\left(\int_{f_0}^{f}\frac{\gamma_{\bar\psi\psi}(\xi)}{\beta(\xi)}\,\mathrm{d}\xi\right)\langle\bar\psi\psi\rangle_L \tag{21}$$

$$=(\beta_0 f)^{\gamma_1/\beta_1}\left(1+\frac{\gamma_2\beta_0-\gamma_1\beta_1}{\beta_0^2}f+\ldots\right)\langle\bar\psi\psi\rangle_L\,. \tag{22}$$

In eq. (22) we have fixed the overall constant in an arbitrary way.

Now define the RGI variable Z, which measures how much bigger the condensate $\widehat{\langle\bar\psi\psi\rangle}$ is compared to L, the size of the box:

$$Z\equiv L\widehat{\langle\bar\psi\psi\rangle}_L\,. \tag{23}$$

Remember that there is a relation between the condensate and the mass gap of the elementary fermions [9,12]:

$$m(L)\underset{L\to 0}{=}c_{\bar\psi\psi}(f)\langle\bar\psi\psi\rangle_L+\text{power corrections}\,,$$

$$c_{\bar\psi\psi}\sim\mathrm{O}(f)\,. \tag{24}$$

So if we knew, by some independent calculation, the short distance coefficient function $c_{\bar\psi\psi}$ up to two-loop order, we could extend the calculation of ref. [9] for the mass gap of elementary fermions to two loops using the presented results for $\langle\bar\psi\psi\rangle_L$.

The variable Z thus qualitatively measures whether our box is bigger or smaller than the size of the elementary fermions. The perturbative regime is for Z small, but we should not squeeze the fermions too much, so moderate values of $Z\sim 2$–3 are reasonable [13,8,9]. Finally we introduce the function $F(Z)$:

$$F(Z)=\frac{\widehat{\langle\bar\psi\psi\rangle}_L}{\Lambda_{\mathrm{MS}}}\,. \tag{25}$$

Following the philosophy of refs. [13,8,9] we should determine the minimum of this function, that is, its value which is less dependent on L, in order to get a rough estimate of the value at infinite volume. This is what seems to be the correct procedure judging from the behaviour of this function in the leading $1/N$ expansion [8]. More careful study though of the finite size effects should be done, using techniques developed in ref. [14] and applied to the mass-gap of the O(N) two-dimensional non-linear sigma model [15]. With the results we have obtained for $\langle\bar\psi\psi\rangle_L$ up to two-loop order and the knowledge of the β-function up to three loops we calculate the function $F(Z)$, as a function of the variable Z:

$$F(Z)=F_0 Z^{\mu}\exp(A/Z^{\nu})\,[1+F_1 Z^{\nu}+\mathrm{O}(Z^{2\nu})]\,, \tag{26a}$$

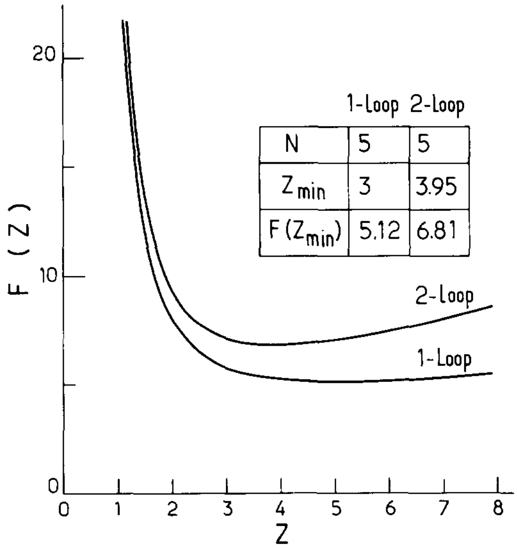
$$F_0=\alpha_{00}^{-\beta_1/\beta_0\gamma_1}\,\frac{\exp(\alpha_{10}/\alpha_{00}+\gamma_2/\beta_0-\gamma_1\beta_1/\beta_0^2)}{\gamma_1}\,, \tag{26b}$$

$$A=\alpha_{00}^{\beta_0/\gamma_1}\,,\quad \mu=1+\beta_1/\beta_0\gamma_1\,,\quad \nu=\beta_0/\gamma_1\,, \tag{26c}$$

$$F_1=\frac{(\beta_0\beta_2-\beta_1^2)B_1^2+3\beta_0^2B_2^2-\beta_0\beta_1B_1B_2-\beta_0^2B_1B_3}{\beta_0^3B_1^3}\,, \tag{26d}$$

where

$$B_1=\beta_0\alpha_{00}^{\beta_0/\gamma_1}\,, \tag{27a}$$



	1-Loop	2-Loop
N	5	5
Z min	3	3.95
F (Z min)	5.12	6.81

Fig. 2. The plot of the function $F(Z)$, against Z, $N=2$, for the one- and two-loop results.

$$B_2 = \frac{\beta_0}{\gamma_1 \alpha_{00}} (\alpha_{10} + K\alpha_{00}) B_1 , \qquad (27b)$$

$$B_3 = \frac{\beta_0}{\gamma_1 \alpha_{00}} (\alpha_{20} + K\alpha_{10}) + \frac{\beta_0 - \gamma_1}{2\beta_0} B_2^2 , \qquad (27c)$$

$$K = (\gamma_2 \beta_0 - \gamma_1 \beta_1) / \beta_0^2 . \qquad (27d)$$

In fig. 2 we plot for $N=5$ the one-loop and two-loop functions $F(Z)$ as functions of Z. Note that the variables Z, in one and two loops are different.

Concluding this work we notice that our method though reliable in the perturbative regime (small volumes) could be trusted for large volumes only after the complete resummation of all orders (at least in every order in the $1/N$ expansion).

Recently very interesting work of the Bern group provided the exact mass gap of the $O(N)$ two-dimensional model [16]. It could be the case that the exact mass gap for the elementary fermions of the Gross–Neveu model is exactly calculable, since the S-matrix in this case is also known [4].

References

[1] D. Gross and A. Neveu, Phys. Rev. D 10 (1974) 3235.
[2] R. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. D 12 (1975) 2443.
[3] A. Neveu and N, Papanicolaou, Commun. Math. Phys. 58 (1978) 31.
[4] A.B. Zamolodchikov and Al.B. Zamolodchikov, Ann. Phys. 120 (1979) 253; Nucl. Phys. B 133 (1978) 525.
[5] B. Berg, M. Karowski, V. Kurak and P. Weiz, Nucl. Phys. B 134 (1978) 125.
[6] R. Shankar and E. Witten, Nucl. Phys. B 141 (1978) 349.
[7] E. Witten, Nucl. Phys. B 142 (1978) 285.
[8] E.G. Floratos, Phys. Lett. B 154 (1985) 173.
[9] E.G. Floratos and N. Vlachos, Phys. Lett. B 189 (1987) 137.
[10] N.D. Tracas and N.D. Vlachos, Phys. Lett. B 236 (1990) 333; Analytic properties for the Gross–Neveu model b-function, in three loops, National Technical University of Athens preprint NTUA-10/1990, THES-TP 90/02, Nucl. Phys. B, to be published.
[11] W. Wetzel, Phys. Lett. B 153 (1985) 297.
[12] H.D. Politzer, Nucl. Phys. B 117 (1976) 397.
[13] M. Luscher, Phys. Lett. B 118 (1982) 391.
[14] M. Luscher, DESY preprint, 83-116 (1983).
[15] E.G. Florators and D. Petcher, Phys. Lett. B 160 (1985) 271.
[16] P. Hasenfratz, M. Maggiore and F. Niedermayer, Phys. Lett. B 245 (1990) 522;
P. Hasenfratz and F. Niedermayer, Phys. Lett. B 245 (1990) 529.