TWO-LOOP Q.C.D. CORRECTIONS TO NON-LEPTONIC WEAK DECAYS

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Θα 'θελα νά ευχαριστήσω πρώτα καί κύρια τούς γονείς μου - καί ιδιαίτερα τό πατέρα μου πού δέν πρόλαβε νά μέ δή νά τελειώνω τίς σπουδές μου- καί τ'αδέλφια μου γιά τήν ηθική συμπαράσταση πού μού 'δωσαν καί τό οικονομικό βάρος πού σήκωσαν τά χρόνια τών σπουδών μου στήν Αγγλία.

Επίσης θά 'θελα νά ευχαριστήσω τή Χρύσα, καί τό Ρωμύλο πού μαζί του ζήσαμε τρία ολόκληρα χρόνια. Μαζί μ'αυτούς ευχαριστώ όλους τούς 'Ελληνες συμφοιτητές μου στό Πανεπιστήμιο τού SUSSEX πού μού συμπαραστάθηκαν καί μ'έκαναν νά ξεχνάω ότι βρίσκομαι μακρυά από τήν Ελλάδα.

Αλλά πάνω απ'όλα θά 'θελα νά ευχαριστήσω τή γυναίκα μου καί τή κόρη μου πού υπόμεναν τή πολύωρη καθημερινή απουσία μου καθώς καί τή σχεδόν παντελή έλλειψη διασκέδασης. Πού μαζί μου μοιράστηκαν τίς χαρές καί τίς πίκρες μου από τίς επιτυχίες καί τίς δυσκολίες τής ερευνητικής μου δουλειάς καί όλη τήν ένταση τών τελευταίων μηνών μέχρι νά ετοιμαστεί η διατριβή αυτή. Καί τέλος γιά τό κουράγιο καί τήν ηθική δύναμη πού μού 'δειναν συνεχώς.

Γι'αυτό καί αφιερώνω αυτή τή διατριβή ...

... στή ΜΑΡΓΑΡΙΤΑ

καί στήν ΑΡΓΥΡΩ

This is to certify that the subsequent thesis has not been submitted in parts or as awhole to any other University.

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Abstract

The $\Delta I=\frac{1}{2}$ rule in non-leptonic decays of hadrons has been a puzle for a long time. In the last 10 years new approaches have appeared which gave some hopes in explaining the above rule.

In Chapter 1 we give a brief review of the phenomenological theory of weak interactions and the consequences of the $\Delta I=\frac{1}{2}$ rule, as well as earlier attempts to explain it.

In Chapter 2 we outline the idea of dimensional regularization as a tool in evaluating momentum integrals.

In Chapter 3 we review the Renormalization Group Equation and its form under dimensional regularization.

In Chapter 4, Wilson's Operator Product Expansion is described, through which, Wilson proposed (1969), the $\Delta I=\frac{1}{2}$ rule could be explained.

In Chapter 5, we reproduce the results of Gaillard and Lee, Altarelli and Maiani. Leading order correction to the effective operators of the Wilson expansion gave a qualitative but not quantitative explanation to $\Delta I=\frac{1}{2}$ rule. We have performed the next to leading order calculation, which shows that these cannot be ignored and that Quantum Chromodynamics can explain the scale of the effect.

Chapter 1 : Introduction

In the last 6 or 7 years, much attention has been given to the problem of weak non-leptonic decays and especially to hyperon decays, since it was realized that in asymptotically free gauge theories of strong interactions the observed $\Delta I = 1/2$ rule for strange particle decays, can be possibly explained (1). Similar explanations can be derived for strangeness conserving, parity violating weak transitions (2) and also for charm changing transitions which may be observed in the future (3). In the past alternative or complementary explanations have been proposed for the $\Delta I = 1/2$ rule(4).

What is the $\Delta I = 1/2$ rule? In the (phenomenological) current-current theory of weak interactions, the Hamiltonian density has the form:

$$\mathcal{H}_\omega = G/\sqrt{2}\,\left(L_\alpha + J_\alpha\right)\left(L^{+\alpha} + J^{+\alpha}\right) \qquad (1.1)$$

where G is a constant (weak coupling constant) having the value $1.03 \times 10^{-5}\, m_p^{-2}$ (m_p proton mass, $\hbar = c = 1$),

$L_\alpha = \bar{\psi}_e(x)\gamma_\alpha(1-\gamma_5)\psi_{\nu_e}(x) + \bar{\psi}_\mu(x)\gamma_\alpha(1-\gamma_5)\psi_{\nu_\mu}(x)$

is the leptonic current and J_α is the hadronic current, a mixture of vector and axial vector currents constructed entirely from hadronic fields.

The Cabibbo hupothesis states that (5)

$$\mathcal{J}_a = J_a \cos\theta_c + S_a \sin\theta_c \tag{1.2a}$$

$$\mathcal{J}_a^{\dagger} = J_a^{\dagger} \cos\theta_c + S_a^{\dagger} \sin\theta_c \tag{1.2b}$$

where

$$J_a = V_a^1 + i V_a^2 + A_a^1 + i A_a^2 \tag{1.2c}$$

$$S_a = V_a^4 + i V_a^5 + A_a^4 + i A_a^5 \tag{1.2d}$$

$$J_a^{\dagger} = V_a^1 - i V_a^2 + A_a^1 - i A_a^2 \tag{1.2e}$$

$$S_a^{\dagger} = V_a^4 - i V_a^5 + A_a^4 - i A_a^5 \tag{1.2f}$$

and θ_c is the Cabibbo angle. V_a's and A_a's are vectors and axial vectors currents respectively, trasforming like the 1, 2, 4 and 5 members of two octet representations of SU(3). By this we mean that:

i) the space integrals of the quantities $V_0^i(t,\underline{x})$ are the generators $\mathbb{F}^i(t)$ of the group:

$$\mathbb{F}^i(t) = V^i(t) = \int d^3(x)\, V_0^i(t,x) \qquad (i = 1,\dots,8) \tag{1.3}$$

ii) $\mathbb{F}^i(t)$'s obey the commutation relation:

$$\left[\mathbb{F}^i(t), \mathbb{F}^j(t)\right] = i f^{ijk}\, \mathbb{F}^k(t) \tag{1.4}$$

where f^{ijk} are the structure constants of the group (see Appendix 1), and

iii) $V^i{}_\mu$ and $A^i{}_\mu$ transform in the following way:

$$\left[\mathbb{F}^i(x^0), V_\lambda^j(x)\right] = i f^{ijk}\, V_\lambda^k(x) \tag{1.5a}$$

$$\left[\mathbb{F}^i(x^0), A_\lambda^j(x)\right] = i f^{ijk}\, A_\lambda^k(x) \tag{1.5b}$$

If we make the identifications:

third component of isospin $\quad I^3 = \mathbb{F}^3 \qquad (1.6a)$

hypercharge $\quad Y = 2/\sqrt{3}\ \mathbb{F}^8 \qquad (1.6b)$

and $\quad I^2 = (\mathbb{F}^1)^2 + (\mathbb{F}^2)^2 + (\mathbb{F}^3)^2 \qquad (1.6c)$

then the hadron electromagnetic current takes the form:

$$j_\lambda = V^3_\lambda + \frac{1}{\sqrt{3}} V^8_\lambda \tag{1.7}$$

and therfore the electric charge operator (in other words the space integral of j_o) is:

$$Q = I^3 + 1/2\, Y \tag{1.8}$$

With the help of the above equation we have the following commutation rules:

$$[Q, \mathcal{J}_a] = \mathcal{J}_a \quad , \quad [Q, \mathcal{J}^\dagger_a] = -\mathcal{J}^\dagger_a \tag{1.9a}$$
$$[Y, \mathcal{J}_a] = 0 \quad , \quad [Y, \mathcal{J}^\dagger_a] = 0 \tag{1.9b}$$

which means that $\mathcal{J}_a$ creates a unit of charge while $\mathcal{J}^\dagger_a$ destroys a unit of charge and both $\mathcal{J}_a$ and $\mathcal{J}^\dagger_a$ do not change hypercharge.

Finally we can write:

$$\mathcal{J}_a \text{ has } (0, 1, +1) \quad , \quad \mathcal{J}^\dagger_a \quad (0, 1, -1) \tag{1.10a}$$
$$S_a \quad (+1, 1/2, +1/2) \quad , \quad S^\dagger_a \quad (-1, 1/2, -1/2) \tag{1.10b}$$

where the three numbers correspond to the eigenvalues of hypercharge, isospin and third component of isospin respectively. This means that in processes which involve the above operators the quantum numbers <u>change</u> by the amount given in these relations.

<u>Purely hadronic processes.</u> From eq(1.1) we can see that the purely hadronic (non-leptonic) interactions comes from the term of $\mathcal{H}_\omega$:

$$\mathcal{H}_\omega = G/\sqrt{2}\, (\mathcal{J}_a \mathcal{J}^{\dagger a}) \tag{1.11}$$

which can be written in a more symmetric form as:

$$\mathcal{H}_\omega = G/2\sqrt{2} \cdot \{\mathcal{J}_a, \mathcal{J}^{\dagger a}\}_+ \tag{1.12}$$

Using eqs(1.2) we can write:

$$\mathcal{H}_\omega = G/2\sqrt{2}\left[\cos^2\theta_c\{J_a, J^{\dagger a}\} + \sin^2\theta_c\{S_a, S^{\dagger a}\}\right] + \\ + G/2\sqrt{2}\ \sin\theta_c\ \cos\theta_c\left[\{J_a, S^{\dagger a}\} + \{S_a, J^{\dagger a}\}\right] \tag{1.13}$$

Bearing in mind eqs(1.10), we can see that the first term in brackets in $\mathcal{H}_\omega$ conservs hypercharge ($\Delta Y=0$) while the second one has $|\Delta Y|=1$. We are going to consider only the latter.

From eq(1.8), and for the case $\Delta Y=\pm 1$ we get that $\Delta I_3 = \mp 1/2$, since charge must be conserved ($\Delta Q=0$). It follows that the Hamiltonian responsible for the purely hadronic, $\Delta Y=\pm 1$,processes could have $I=1/2, 3/2, 5/2, \ldots$. But from relations (1.10) we can easily see that JS is a mixture of $I=1/2$ and $I=3/2$ only. Thus we have the following four types of purely hadronic, $|\Delta Y|=1$, processes: $\Delta I=1/2$ $\Delta I_3=\pm 1/2$ and $\Delta I=3/2$ $\Delta I_3=\pm 1/2$

Finally considering the SU(3) properties of $\mathcal{H}_\omega$ for purely hadronic processes, we can decompose the Hamiltonian in parts which transform like the 8 and 27 representation of SU(3). We briefly explain how this comes about. The direct product of two identical octets can be decomposed into 1 , 8 and 27 (irreducible) representation of SU(3). From the 36 elements of this product we are interested only in the terms $\mathcal{J}^1\mathcal{J}^4$ and $\mathcal{J}^2\mathcal{J}^5$ (see eqs(1.2) and eq(1.13)), where $\mathcal{J}^i$ is the general element of the octet. The singlet does not contribute since it corresponds to $\sum_{j=1}^{8}\mathcal{J}^j\mathcal{J}^j$. The 8 part of $\mathcal{H}_\omega$ has purely $I=1/2$ since 8 can have only $I=0, 1/2$ and 1 ,and $\mathcal{H}_\omega$ is a mixture of $I=1/2$ and $I=3/2$ only. The 27 part of $\mathcal{H}_\omega$ can have both $I=1/2$ and $I=3/2$. So we can write:

$$\mathcal{H}_\omega(\text{for } \Delta Y=\pm 1) = G/2\sqrt{2}\ \sin\Theta_c \cos\Theta_c \cdot \Big[\mathcal{H}(\underline{8},1,1/2,-1/2)+\mathcal{H}(\underline{8},-1,1/2,+1/2)+ + \mathcal{H}(\underline{27},1,1/2,-1/2)+\mathcal{H}(\underline{27},-1,1/2,+1/2)+ +\sqrt{5}\,\mathcal{H}(\underline{27},1,3/2,-1/2)+\sqrt{5}\,\mathcal{H}(\underline{27},-1,3/2,+1/2)\Big] \tag{1.14}$$

The $\sqrt{5}$ factor comes from the SU(3)-Clebsch-Gordon coefficient and it is important since all four last terms in the $\mathcal{H}_\omega$ contribute to the same reduced amplitude (by Wigner-Eckart theorem). It can be seen clearly now that the $\underline{8}$ representation parts of $\mathcal{H}_\omega$ induce processes with pure $\Delta I=1/2$, while $\underline{27}$ representation parts induce processes with a mixture of $\Delta I=1/2$ and $\Delta I=3/2$ (while always $\Delta I_3=\pm 1/2$). So the $\Delta I=1/2$ rule says that amplitudes coming from the last parts of the $\mathcal{H}_\omega$, are, by a mechanism which we are trying to recover, suppressed relative to the other terms.

Consequences of $\Delta I=1/2$ rule

Hyperon decays. The effective Lagrangian for non leptonic hyperon decays $B_1 \to B_2+\pi$ can be written (6):

$$\mathcal{L}_{eff} = G\mu_c^2\,[\bar{\psi}_2\,(A+B\gamma^5)\,\psi_1]\,\phi_\pi \tag{1.15}$$

where μ_c is the charged pion mass, and A and B are dimensionless complex numbers giving the relative amplitudes of the parity-violating and parity-conserving decays respectively.

The invariant amplitude for the decay is:

$$\mathcal{M} = G\mu_c^2\,[\bar{u}(P)\,(A+B\gamma^5)\,u(p)] \tag{1.16}$$

where P is the 4-momentum of the hyperon of mass M and p is the 4-momentum of the baryon decay product of mass m. The probability $d\Gamma$ for the decay of the hyperon B_1 at rest with its spin polarized in the direction $\underline{s}$ into a pion and a baryon B_2 emitted in the direction $\underline{n}$ at an angle Θ with $\underline{s}$ is:

$$d\Gamma = \frac{G^2\mu_c^4}{8\pi} p \left\{ |A|^2 \frac{(M+m)^2-\mu^2}{M^2} + |B|^2 \frac{(M-m)^2-\mu^2}{M^2} + 2\,\mathrm{Re}\, A^*B \;\; Mp\, \underline{s}\cdot\underline{n} \right\} \frac{1}{2} \sin\Theta \, d\Theta \quad (1.17)$$

Thus $d\Gamma$ can be put in the form:

$$d\Gamma = \Gamma(1 + \alpha\, \underline{s}\cdot\underline{n}) \frac{1}{2} \sin\Theta \, d\Theta$$

where Γ is the total decay rate and α is the asymmetry parameter. Thus measurement of Γ and α suffices to determine the magnitudes of A and B and their relative sign (assuming T-invariance, A and B are relatively real). Table 1 summarizes the amplitudes A and B for the non leptonic decays of Λ, Σ and Ξ.(7)

Table 1

$M \to m + \mu$	A	B
$\Lambda^0_- \to p + \pi^-$	1.47±0.01	9.98±0.24
$\Lambda^0_0 \to n + \pi^0$	-1.07±0.02	-7.14±0.56
$\Sigma^+_+ \to n + \pi^+$	0.07±0.02	19.04±0.16
$\Sigma^+_0 \to p + \pi^0$	1.48±0.05	-11.99±0.58
$\Sigma^-_- \to n + \pi^-$	1.93±0.01	-0.65±0.08
$\Xi^0_0 \to \Lambda + \pi^0$	1.55±0.03	-5.96±1.12
$\Xi^-_- \to \Lambda + \pi^-$	2.04±0.02	-6.70±0.38

Let us now see what the predictions of the $\Delta I = 1/2$ rule are. Suppose that this rule holds exactly. Then it is helpful in visualizing the effects to imagine that the initial hyperon absorbs a spurion (a ficticious particle with $I=1/2, I_3=-1/2$) in making the transition to final particles. With spurion included, the hypothetical process is isospin-invariant. Consider the decay of Λ^0.

	Λ^0 (I,I_3)	+ Spurion (I,I_3)	→ Nucleon (I,I_3)	+ Meson (I,I_3)
$\Lambda^0 \to p\pi^-$	(0,0)	(1/2,-1/2)	(1/2,1/2)	(1,-1)
$\Lambda^0 \to n\pi^0$	(0,0)	(1/2,-1/2)	(1/2,-1/2)	(1,0)

So we get:

$$\frac{\mathcal{M}(\Lambda^0 \to p\pi^-)}{\mathcal{M}(\Lambda^0 \to n\pi^0)} = \frac{\langle \frac{1}{2},\frac{1}{2};1,-1 | \frac{1}{2},-\frac{1}{2}\rangle}{\langle \frac{1}{2},-\frac{1}{2};1,0 | \frac{1}{2},-\frac{1}{2}\rangle} = -\sqrt{2} \qquad (1.18)$$

thus the branching ratio is:

$$R = \frac{\Gamma(\Lambda^0 \to p\pi^-)}{\Gamma(\Lambda^0 \to n\pi^0)} = 2 \qquad (1.19)$$

Experiments give R the value: 2.104 ± 0.028 (7)

Next consider the Σ non leptonic decay

		Σ (I,I_3)	+ Spurion (I,I_3)	$\to$ Nucleon (I,I_3)	+ Meson (I,I_3)
Σ^+_+	$\Sigma^+ \to n\pi^+$	$(1,1)$	$(\frac{1}{2},-\frac{1}{2})$	$(\frac{1}{2},-\frac{1}{2})$	$(1,1)$
Σ^+_0	$\Sigma^+ \to p\pi^0$	$(1,1)$	$(\frac{1}{2},-\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(1,0)$
Σ^-_-	$\Sigma^- \to n\pi^-$	$(1,-1)$	$(\frac{1}{2},-\frac{1}{2})$	$(\frac{1}{2},-\frac{1}{2})$	$(1,-1)$

So we get

$$\Sigma^+_+ \quad \text{Amplitude} \equiv A^+_+ = k\left\langle \frac{-\psi_{3/2,1/2}+\sqrt{2}\,\psi_{1/2,1/2}}{\sqrt{3}} \middle| H_{eff} \middle| \frac{-\psi_{3/2,1/2}+\sqrt{2}\,\psi_{1/2,1/2}}{\sqrt{3}} \right\rangle$$

$$\Sigma^+_0 \quad \text{Amplitude} \equiv A^+_0 = k\left\langle \frac{\sqrt{2}\,\psi_{3/2,1/2}+\psi_{1/2,1/2}}{\sqrt{3}} \middle| H_{eff} \middle| \frac{-\psi_{3/2,1/2}+\sqrt{2}\,\psi_{1/2,1/2}}{\sqrt{3}} \right\rangle$$

and

$$\Sigma^-_- \quad \text{Amplitude} \equiv A^-_- = k\langle \psi_{3/2,-3/2} | H_{eff} | \psi_{3/2,-3/2} \rangle$$

where H_{eff} is isospin invariant and k is a common constant. Thus we find:

$$A^+_+ = 1/3\,x + 2/3\,y \qquad A^+_0 = -\sqrt{2}/3\,x + \sqrt{2}/3\,y \qquad A^-_- = x$$

where x and y are two numbers. From these three relations we get:

$$A^+_+ - A^-_- = \sqrt{2}\,A^+_0 \qquad (1.20)$$

Finally consider the Ξ decay

	Ξ +	Spurion $\rightarrow$	Λ	+ Pion
	(I, I_3)	(I, I_3)	(I, I_3)	(I, I_3)
$\Xi^0 \rightarrow \Lambda \pi^0$	$(1/2, 1/2)$	$(1/2, -1/2)$	$(0, 0)$	$(1, 0)$
$\Xi^- \rightarrow \Lambda^0 \pi^-$	$(1/2, -1/2)$	$(1/2, -1/2)$	$(0, 0)$	$(1, -1)$

Thus:

$$\frac{\mathcal{M}(\Xi^0 \rightarrow \Lambda^0 \pi^0)}{\mathcal{M}(\Xi^- \rightarrow \Lambda^0 \pi^-)} = \frac{\langle 1/2, 1/2; 1/2, -1/2 | 1, 0 \rangle}{\langle 1/2, -1/2; 1/2, -1/2 | 1, -1 \rangle} = -1/\sqrt{2} \qquad (1.21)$$

Experiments give the value 0.548 ± 0.036 to the ratio of the decay rates(7). Therefore we see that exact $\Delta I = 1/2$ are in fair but not perfect agreement with the data.

(It should be noted that relations (1.18)-(1.21) hold separately for s- and p-wave amplitudes).

$\Delta I = 3/2$ amplitudes relative to $\Delta I = 1/2$ amplitudes. It is possible to evaluate from experiments the ratio of $\Delta I = 3/2$-amplitudes to $\Delta I = 1/2$ - amplitudes for s- and p-wave. In the case of Λ^0 decay we get(7):

for s-wave $\qquad A_3/A_1 = 0.027 \pm 0.008 \qquad (1.22)$

for p-wave $\qquad B_3/B_1 = 0.030 \pm 0.037 \qquad (1.23)$

We see again here a violation of the $\Delta I = 1/2$ rule of the order 3%.

For Σ decay the corresponding experimental values are (neglecting $\Delta I = 5/2$ amplitudes) (7) :

$$A_3/A_- = -0.063 \pm 0.025 \qquad (1.24)$$

$$B_3/B_+ = -0.076 \pm 0.029 \qquad (1.25)$$

where + and − correspond to Σ^+_+ and Σ^-_- decays

Finally we get for Ξ decay:

$$A_3/A_1 = -0.043 \pm 0.015 \qquad (1.26)$$

$$B_3/B_1 = -0.13 \pm 0.15 \qquad (1.27)$$

Thus for hyperon decay, present experimental data limit $\Delta I=3/2$ amplitudes to less than about 5%.

Non leptonic Kaon decays. The assumption that the $\Delta I=1/2$ rule holds exactly forbids the $K^+\to\pi^+\pi^0$. And the reason is that the $\pi^+\pi^0$ system must have zero angular momentum and since they are bosons they must have isospin $I=0$ or 2 . However $K^+\to\pi^+\pi^0$ has been observed showing a small violation of the $\Delta I=1/2$ rule:

$$\frac{\Gamma(K^+\to\pi^+\pi^0)}{\Gamma(K^0_S\to\pi^0\pi^0)+\Gamma(K^0_S\to\pi^+\pi^-)} = 1.48\times10^{-3} \qquad (1.28)$$

Furthermore the $\Delta I=1/2$ rule implies:

$$\frac{\Gamma(K^0_S\to\pi^+\pi^-)}{\Gamma(K^0_S\to\pi^0\pi^0)} = 2 \qquad (1.29)$$

Unfortunately, the experimental measurements of the above ratio are not in complete agreement:

$$R=2.285\pm0.055 \quad (8) \quad \text{and} \quad R=2.10\pm0.06 \quad (9) \qquad (1.30)$$

Nevertheless, it seems unlikely that the $\Delta I=1/2$ rule will be exact.

$K_{\pi 3}$ decays. Analysis of the three pion isospin states gives:

$$|+-0\rangle=\sqrt{2/5}\,|1,0\rangle+\sqrt{3/5}\,|3,0\rangle \qquad |000\rangle=-\sqrt{3/5}\,|1,0\rangle+\sqrt{2/5}\,|3,0\rangle$$
$$|++-\rangle=\sqrt{4/5}\,|1,1\rangle+\sqrt{1/5}\,|3,1\rangle \qquad |+00\rangle=-\sqrt{1/5}\,|1,1\rangle+\sqrt{4/5}\,|3,1\rangle \qquad (1.31)$$

where $|+-0\rangle\equiv|\pi_1^+\pi_2^-\pi_3^0\rangle$ and $|\alpha,\beta\rangle=|I,I_3\rangle$

Assuming that only $I=1$ and $I=3$ symmetric states contribute we get*:

* In general three pion system could be in a completely symmetric $I=3$ state, in a completely antisymmetric $I=0$ state, and in a $I=2$ and $I=1$ state which have mixed symmetry. However, an appropriate linear combination of $I=1$ states (corresponding to $I_{ab}=0$ and 2 ,where I_{ab} is the isospin of the two pion system), yields a complete symmetric $I=1$ state.

$$\begin{aligned}
\langle +{+}- | H_w | K^+ \rangle &= \tfrac{2}{\sqrt{5}} \langle 1,1 | H_w | K^+ \rangle + \tfrac{1}{\sqrt{5}} \langle 3,1 | H_w | K^+ \rangle \\
\langle +00 | H_w | K^+ \rangle &= -\tfrac{1}{\sqrt{5}} \langle 1,1 | H_w | K^+ \rangle + \tfrac{2}{\sqrt{5}} \langle 3,1 | H_w | K^+ \rangle \\
\langle 000 | H_w | K^0 \rangle &= -\sqrt{\tfrac{3}{5}} \langle 1,1 | H_w | K^0 \rangle + \sqrt{\tfrac{2}{5}} \langle 3,1 | H_w | K^0 \rangle \\
\langle +{-}0 | H_w | K^0 \rangle &= \sqrt{\tfrac{2}{5}} \langle 1,1 | H_w | K^0 \rangle + \sqrt{\tfrac{3}{5}} \langle 3,1 | H_w | K^0 \rangle
\end{aligned} \qquad (1.32)$$

Assuming $\Delta I \geqslant 5/2$ amplitudes to be zero we get:

$$\frac{\gamma(++-)}{\gamma(+00)} = 4 \quad \text{and} \quad \frac{\gamma(K_L \to +-0)}{\gamma(K_L \to 000)} = \frac{2}{3} \qquad (1.33)$$

where γ is the decay rate divided by the phase-space factor for the appropriate decay.

Furthermore if we suppose that the $\Delta I = 1/2$ rule is exact we get (using Wigner-Eckhart theorem):

$$\begin{aligned}
\langle +{+}- | H_w | K^+ \rangle &= \langle \tfrac{1}{2}, \tfrac{1}{2} ; \tfrac{1}{2}, \tfrac{1}{2} | 1,1 \rangle \mathcal{M}_0 = \tfrac{2}{\sqrt{5}} \mathcal{M}_0 \\
\langle +00 | H_w | K^+ \rangle &= \langle \tfrac{1}{2}, \tfrac{1}{2} ; \tfrac{1}{2}, \tfrac{1}{2} | 1,1 \rangle \mathcal{M}_0 = -\tfrac{1}{\sqrt{5}} \mathcal{M}_0 \\
\langle +0- | H_w | K^0 \rangle &= \langle \tfrac{1}{2}, -\tfrac{1}{2} ; \tfrac{1}{2}, \tfrac{1}{2} | 1,0 \rangle \mathcal{M}_0 = -\tfrac{1}{\sqrt{5}} \mathcal{M}_0 \\
\langle 000 | H_w | K^0 \rangle &= \langle \tfrac{1}{2}, -\tfrac{1}{2} ; \tfrac{1}{2}, \tfrac{1}{2} | 1,0 \rangle \mathcal{M}_0 = \sqrt{\tfrac{3}{10}} \mathcal{M}_0
\end{aligned} \qquad (1.34)$$

where $\mathcal{M}_0$ is a reduced matrix element. Then taking into account that $K^0 \approx \frac{1}{\sqrt{2}} | K_L^0 + K_S^0 \rangle$ we predict:

$$\frac{\gamma(K_L \to +-0)}{\gamma(K^+ \to 00+)} = 2 \qquad (1.35)$$

$$\frac{\gamma(K_L \to 000)}{\gamma(K^+ \to ++-) - \gamma(K^+ \to 00+)} = 1 \qquad (1.36)$$

It is also possible to show that the Dalitz plot slope parameters σ are related by the $\Delta I = 1/2$ rule (10):

$$\frac{\sigma(K^+ \to 00+)}{\sigma(K^+ \to ++-)} = 2 \qquad (1.37)$$

$$\frac{\sigma(K^+ \to 00+)}{\sigma(K_L^0 \to +-0)} = +1 \qquad (1.38)$$

Table II shows that the observations are consistent with absence of $\Delta I > 3/2$ amplitudes, but shows a violation of the $\Delta I = 1/2$ rule. Although predictions (1.35) and (1.36) are ambiguous to a certain extend because of electromagnetic effects, one tends to believe that K_{π_3}, as in $K^{\pm} \to \pi^{\pm}\pi^{0}$ decay, the ratio of $I = 3/2$ to $1/2$ amplitudes is about:

$$\left| A_{3/2} / A_{1/2} \right| \approx 5\% \tag{1.39}$$

Table II

Tests of the $\Delta I = 1/2$ rule in $K_{\pi 3}$ decays

Test	Prediction ($\Delta I = 1/2$)	Observation	Reference
$\dfrac{\gamma(K_L \to +-0)}{2\gamma(K^+ \to +00)}$	1	0.85 ± 0.04	(11)
$\dfrac{\gamma(K_L \to 000)}{\gamma(K^+ \to ++-) - \gamma(K^+ \to +00)}$	1	0.95 ± 0.05	(11)
$\dfrac{\sigma(K^+ \to +00)}{\sigma(K^+ \to ++-)}$	-2	-2.63 ± 0.18	(12)
$\dfrac{\sigma(K_L \to +-0)}{\sigma(K^+ \to +00)}$	1	0.99 ± 0.09	(11)
	($\Delta I \leq 3/2$)		
$\dfrac{\gamma(K^+ \to ++-)}{4\gamma(K^+ \to +00)}$	1	1.02 ± 0.03	(11)
$\dfrac{\gamma(K_L \to 000)}{\frac{3}{2}\gamma(K_L \to +-0)}$	1	1.15 ± 0.11	(11)

Finally it is easy to show that the use of the soft pion theorem forbids the $\Delta I = 3/2$ part of the weak Lagrangian.

Consider the commutator of F_i^5, the charges corresponding to the axial currents, with the Lagrangian $\mathcal{L}(I, I_3)$. Then:

$$\left[F_i^5, \mathcal{L}(I,I_3)\right] = \left[F_i, \mathcal{L}(I,I_3)\right] \qquad (1.40)$$

where we have assumed that $\mathcal{L}$ is constructed from $V\text{-}A \times V\text{-}A$ currents and that $\mathcal{L}$ can be written as :

$$\mathcal{L} = \mathcal{L}(1/2, 1/2) + \mathcal{L}(1/2, -1/2) + \mathcal{L}(3/2, 1/2) + \mathcal{L}(3/2, -1/2)$$

Then, using the soft pion theorem we get:

$$\lim_{q_i, q_j, q_k \to 0} \left(\frac{f_\pi}{\sqrt{2}}\right)^3 \langle \pi_i, \pi_j, \pi_k | \mathcal{L}(3/2, \pm 1/2) | K \rangle = -i \cos^2\theta \langle [F_i, [F_j, [F_k, \mathcal{L}(3/2, \pm 1/2)]]] | K \rangle \qquad (1.41)$$

Now the three succesive commutators can be written:

$$\left[F_i, \left[F_j, \left[F_k, \mathcal{L}(3/2, \pm 1/2)\right]\right]\right] = \sum_{m_\alpha} a_\pm(m_\alpha) \mathcal{L}(3/2, m_\alpha) \qquad (1.42)$$

where $m_\alpha = 3/2, \ldots, -3/2$ and $a_\pm$ are numerical coefficients and we have used that F_i are isospin generators. But since K belongs to $I=1/2$ multiplet, the right hand side of eq(1.41) is zero. Therfore the $\Delta I = 1/2$ for $K \to 3\pi$ (and $K \to 2\pi$)decays is seen to be an automatic cansequence of the current algebra assumptions in the unphysical limit of zero pion four momenta.

Octet dominance, Lee-Sugawara relation. We have already seen that the effective Hamiltonian $\mathcal{H}_W$ (for $\Delta S \neq 0$) can be decomposed in parts which transform like the 8 and 27 representation of SU(3) (since it comes from a product of two identical octets). If we assume that $\mathcal{H}_W$ itself transform as an octet under SU(3), then the $\Delta I = 1/2$ rule is automaticaly guaranteed (see eq(1.14)).

At this point two questions can be asked:

i)what octet component must the effective weak Hamiltonian be proportional to?

ii)what are the further experimental consequences of the octet dominance assumption?

In the first place $\mathcal{H}_\omega$ must satisfy $\Delta Q=0$ and $|\Delta S|=1$. Of the eight octet components only $\lambda_3, \lambda_8, \lambda_6$ and λ_7 correspond to $\Delta Q=0$. Among these, only λ_6 and λ_7 correspond to $|\Delta S|=1$. Now we know that K° meson transforms like $\lambda_6+i\lambda_7$ and $\bar{K}^\circ$ transforms like $\lambda_6-i\lambda_7$. Also we know that $|\bar{K}^\circ\rangle=-CP|K^\circ\rangle$, and that both $\bar{K}^\circ$ and K° have no definite lifetime for weak decay. But the states:

$$|K^\circ_S\rangle=\frac{1}{\sqrt{2}}\left[|K^\circ\rangle-|\bar{K}^\circ\rangle\right] \quad , \quad |K^\circ_L\rangle=\frac{1}{\sqrt{2}}\left[|K^\circ\rangle+|\bar{K}^\circ\rangle\right] \qquad (1.43)$$

do have definite lifetimes and are eigenvalues of CP operation ($CP|K^\circ_S\rangle=+|K^\circ_S\rangle$ and $CP|K^\circ_L\rangle=-|K^\circ_L\rangle$). Moreover K°_S must transform like λ_7 and K°_L like λ_6 under SU(3) transformation. If we assume that $\mathcal{H}_\omega$ is CP-invariant then it follows that $\mathcal{H}_\omega$ can transform like λ_6 or λ_7 , but not as a linear combination of the two.

Let us assume that $\mathcal{H}_\omega$ transforms like λ_6 *. As before the weak transition may be examined by pretending the absorption of a spurion h by the decaying huperon. The spurion is endowed with the quantum numbers of the hamiltonian and transforms like λ_6 . In this case the interaction is SU(3)-invariant:

$$h + \text{hyperon} \longrightarrow \text{baryon} + \text{meson}$$

Then the most general invariant amplitude we can write must be a linear combination of the traces of products of the SU(3) matrices which contain creation and destruction operators for the particles.

The initial baryon destuction operators and the final baryon and meson creation operators are represented by the matrices:

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* Actually $\mathcal{H}_w$ could only transform like λ_6 ,and not like λ_7 ,since it is constructed from two identical octets of currents.

$$B = \begin{bmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda^0}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda^0}{\sqrt{6}} & n \\ \Xi^- & \Xi^0 & -\frac{2\Lambda^0}{\sqrt{6}} \end{bmatrix} \qquad (1.44)$$

$$\bar{B} = \begin{bmatrix} \frac{\bar{\Sigma}^0}{\sqrt{2}} + \frac{\bar{\Lambda}^0}{\sqrt{6}} & \bar{\Sigma}^- & \bar{\Xi}^- \\ \bar{\Sigma}^+ & -\frac{\bar{\Sigma}^0}{\sqrt{2}} + \frac{\bar{\Lambda}^0}{\sqrt{6}} & \bar{\Xi}^0 \\ \bar{p} & \bar{n} & -\frac{2\bar{\Lambda}^0}{\sqrt{6}} \end{bmatrix} \qquad (1.45)$$

$$\bar{M} = \begin{bmatrix} \frac{\bar{\pi}^0}{\sqrt{2}} + \frac{\bar{\eta}^0}{\sqrt{6}} & \bar{\pi}^- & \bar{K}^- \\ \bar{\pi}^+ & -\frac{\bar{\pi}^0}{\sqrt{2}} + \frac{\bar{\eta}^0}{\sqrt{6}} & \bar{K}^0 \\ \bar{K}^+ & K^0 & -\frac{2\bar{\eta}^0}{\sqrt{6}} \end{bmatrix} \qquad (1.46)$$

while the spurion annihilation operator h transforms like λ_6 :

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad (1.47)$$

Then the amplitude $\mathcal{M}$ must be:

$$\mathcal{M} = \mathcal{M}_s + \mathcal{M}_p \qquad (1.48)$$

where

$$\mathcal{M}_s = \sum_{i=1}^{9} s_i I_i \quad \text{and} \quad \mathcal{M}_p = \sum_{i=1}^{9} p_i I_i \qquad (1.49)$$

are the s- and p-wave amplitudes and

$$I_1 = tr(h\bar{B}\bar{M}B) \quad I_4 = tr(hB\bar{B}\bar{M}) \quad I_7 = tr(h\bar{M}\bar{B}B)$$
$$I_2 = tr(hB\bar{M}\bar{B}) \quad I_5 = tr(h\bar{B})tr(B\bar{M}) \quad I_8 = tr(h\bar{M}B\bar{B}) \qquad (1.50)$$
$$I_3 = tr(h\bar{B}BM) \quad I_6 = tr(hB)tr(\bar{B}\bar{M}) \quad I_9 = tr(h\bar{M})tr(\bar{B}B)$$

(where we have suppressed γ_5 or I depending on p- or s-wave)

Now the s-wave amplitude is odd under P operation, while p-wave is even. Also we require the amplitude as a whole to be CP-invariant. Therefore the s-wave amplitude must be odd under C operation while the p-wave amplitude must be even under C. However under C:

$$B \to \tilde{\bar{B}}, \quad \bar{B} \to \tilde{B}, \quad \bar{M} \to \tilde{\bar{M}} \quad \text{and} \quad h \to h$$

Hence we find that:

$$I_1 \leftrightarrow I_1, \quad I_2 \leftrightarrow I_2, \quad I_9 \leftrightarrow I_9, \quad I_3 \leftrightarrow I_7, \quad I_4 \leftrightarrow I_8, \quad I_5 \leftrightarrow I_6$$

Thus it follows that $s_1 = s_2 = s_9 = 0$ and eq(1.49) for $\mathcal{M}_5$ gives:

$$\mathcal{M}_5 = s_3(I_3 - I_7) + s_4(I_4 - I_8) + s_5(I_5 - I_6) \qquad (1.51)$$

With the aid of eqs(1.44)-(1.47) eq(1.51) becomes:

$$\mathcal{M}_5 = s_3\left[\frac{\bar{p}\pi^-\Lambda^0}{\sqrt{6}} - \frac{\bar{p}\bar{\pi}^0\Sigma^+}{\sqrt{2}} + \bar{n}\bar{\pi}^-\Sigma^- - \frac{\bar{n}\bar{\pi}^0\Lambda^0}{\sqrt{12}} - \frac{2}{\sqrt{6}}\bar{\Lambda}^0\bar{\pi}^-\Xi^- + \right.$$
$$\left. + \frac{2}{\sqrt{12}}\bar{\Lambda}^0\bar{\pi}^0\Xi^0\right] + s_4\left[\frac{\bar{\Lambda}^0\bar{\pi}^-\Xi^-}{\sqrt{6}} - \frac{\bar{\Lambda}^0\bar{\pi}^0\Xi^0}{\sqrt{12}} - \frac{2}{\sqrt{6}}\bar{p}\bar{\pi}^-\Lambda^0 + \right.$$
$$\left. + \frac{2}{\sqrt{12}}\bar{n}\bar{\pi}^0\Lambda^0\right] + s_5\left[\bar{\pi}^+\bar{n}\Sigma^+ + \bar{\pi}^-\bar{n}\Sigma^-\right] \qquad (1.52)$$

In other words, the s-wave amplitudes, give the following relations:

(Λ^0_-)	$\Lambda^0 \to p\pi^-$	$1/\sqrt{6}\,(s_3 - 2s_4)$	
(Λ^0_0)	$\Lambda^0 \to n\pi^0$	$-1/\sqrt{12}\,(s_3 - 2s_4)$	
(Σ^+_+)	$\Sigma^+ \to n\pi^+$	s_5	
(Σ^+_0)	$\Sigma^+ \to p\pi^0$	$-1/\sqrt{2}\quad s_3$	(1.53)
(Σ^-_-)	$\Sigma^- \to n\pi^-$	$s_5 + s_3$	
(Ξ^0_0)	$\Xi^0 \to \Lambda^0\pi^0$	$1/\sqrt{12}\,(2s_3 - s_4)$	
(Ξ^-_-)	$\Xi^- \to \Lambda^0\pi^-$	$-1/\sqrt{6}\,(2s_3 - s_4)$	

The Lee-Sugawara (13) is a new relation (besides the anticipated $\Delta I = 1/2$ relations), which appears when we combine all seven of the above relations:

$$s(\Lambda^0_-) + 2s(\Xi^-) = \sqrt{3}\, s(\Sigma^+_0) \qquad (11.54)$$

The Lee-Sugawara relation requires the assumption that the effective Hamiltonian transforms like λ_6 , which is a stronger condition than the $\Delta I = 1/2$ rule (see also footnote in page 13).

No further restrictions on the p-wave amplitudes can be derived if the λ_6 assumption is made. On the other hand, if we assume that the weak Hamiltonian transforms like λ_7 (not expected in current-current picture), then the p-wave amplitudes satisfy the Lee-Sugawara relation and the s-wave amplitudes are not restricted. In actual fact, the Lee-Sugawara relation appears to be quite well satisfied by both sets of amplitudes (soft pion techniques give further understanding on this subject).

Chapter 2 : Dimensional Regularization

Ultraviolet infinities and their treatment were one of the most challenging problems in relativistic quantum field theory. The simple idea of subtracting the infinities and absorbing them into the bare quantities (mass, charge etc), must be a process that preserves the gauge symmetry of the underlying Lagrangian. The best method, so far, for regularizing these infinities was proposed by t' Hooft and Veltman (1) and has the name of "dimensional regularization". By regularization we mean a technique, a mathematical prescription, which renders infinite Feynman amplitudes finite, by a specific cut-off procedure.

First we will outline the concept of analytic continuation by a Theorem: Let $g_1(z)$ be an analytic function defined in a region $\mathscr{D}_1$ and let $\mathscr{D}_2$ be another region such that $\mathscr{D}_1 \cap \mathscr{D}_2 \equiv R \neq \emptyset$. Then if a function $g_2(z)$ exists, is analytic in the region $\mathscr{D}_2$ and $g_1(z) = g_2(z)$ for $z \in R$, then there can only be one such function. We call $g_1(z)$ and $g_2(z)$ analytic continuation of each other (Fig. 1).

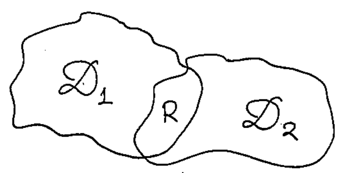

Fig. 1

What is the basic idea in dimensional regularization? Consider the integral in four dimensions (in Euclidean space):

$$I(4) = \int \frac{d^4k}{(k^2+m^2)^2} \tag{2.1}$$

The above integral diverges because of contribution from large k.

But in three dimensions:

$$I(3) = \int \frac{d^3k}{(k^2+m^2)^2} \qquad (2.2)$$

is a convergent integral. The reduction in dimensions makes the integral convergent. So the idea is to define the integral over n-dimensional space ($n=0,1,2,\ldots$), and then, making a further step, for n complex. $I(n)$, where n now is complex, can be defined as an analytic function of n. Using standard techniques, like symmetric integrations, shift of integrations variables, integration by parts, we may compute the integral and finally, by analytic continuation, we return to $n=4$ dimensions. In order for this method to be correct, it must possess two properties:

a) for finite diagrams, the limit $n\to 4$ must give the conventional result, and

b) for infinite diagrams, the method must give a function of n which has poles at $n=4$. As an example we are going to evaluate the infinite part of the vacuum polarization diagram in Quantum Electrodynamics (Feynman rules for Q.E.D. in Appendix B).

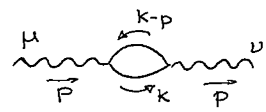

Fig. 2

Fig. 2 gives:

$$I^{\mu\nu} = -\int \frac{d^4k}{(2\pi^4)} \, Tr\left[-ie\gamma^\mu \frac{i}{\not k - m}(-ie)\gamma^\nu \frac{i}{\not k - \not p - m}\right] \qquad (2.3)$$

Going to n dimensions:

$$I^{\mu\nu}(n) = -\int \frac{d^nk}{(2\pi)^n} \, Tr\left[-ie\gamma^\mu \frac{i}{\not k - m}(-ie)\gamma^\nu \frac{i}{\not k - \not p - m}\right] \qquad (2.4)$$

$$= -e^2 \int \frac{d^nk}{(2\pi)^n} \frac{Tr[\gamma^\mu(\not k + m)\gamma^\nu(\not k - \not p + m)]}{(k^2-m^2)((k-p)^2-m^2)}$$

Using traces formulae (see Appendix C) in n dimensions we get:

$$I^{\mu\nu}(n) = -\int \frac{d^nk}{(2\pi)^n} \left\{ \begin{array}{c} 4\left(g^{\mu\phi}g^{\nu\omega} + g^{\mu\omega}g^{\nu\phi} - g^{\mu\nu}g^{\phi\omega}\right)k_\phi k_\omega + \\ + 4g^{\mu\nu}m^2 \end{array} \right\} \cdot \frac{1}{(k^2-m^2)((k-p)^2-m^2)} \qquad (2.5)$$

Now we use the formolae of Appendix C, and since we want the infinite part only we may expand everything in $\epsilon \equiv n-4$ and select the $1/\epsilon$ term only. We know that (see again Appendix C) :

$$\Gamma(2-n/2) = -1/\epsilon + (\text{finite terms for } \epsilon \to 0)$$

$$1/(2\pi)^n = 1/(2\pi)^4 \left(1-\epsilon \ln 2\pi\right) + O(\epsilon^2) \tag{2.6}$$

$$\int_0^1 dx \, \frac{1}{[k^2x(1-x)-m^2]^{2-n/2}} = 1 + \frac{\epsilon}{2} \int dx \, \ln\left(k^2x(1-x)-m^2\right) + O(\epsilon^2) \tag{2.7}$$

$$\int_0^1 dx \, \frac{1}{[k^2x(1-x)-m^2]^{1-n/2}} = \int_0^1 dx \left[[k^2x(1-x)-m^2] \cdot \left(1+\frac{\epsilon}{2} \ln\left(k^2x(1-x)-m^2\right)\right)\right] + O(\epsilon^2) \tag{2.8}$$

So the infinite part is:

$$\boxed{I.P.(I^{\mu\nu}) = \frac{1}{\epsilon}\left(-4/3\right)\frac{i\pi^2}{(2\pi)^4}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)} \tag{2.9}$$

As it was expected * the terms proportional to the mass of the electron m cancel out and eq(2.9) has the correct gauge invariant form.

Dimensional Regularization and Massless Fields (2). In order to cope with the infrared infinities, in the case of massless fields the method must be modified. The trick of introducing a finite mass does not solve the problem at all because :

1)spoils the gauge symmetry of the original theory and

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* $I_{\mu\nu}$ is going to be contracted with the polarization vectors ϵ^μ and ϵ^ν of the external photons. Gauge invariance of Q.E.D. tells us that this quantity stays invariant under the transformation $\epsilon^\mu \to \epsilon^\mu + k^\mu$. Therefore:

$$\epsilon^\mu I_{\mu\nu}\epsilon^\nu = (\epsilon^\mu + k^\mu) I_{\mu\nu}(\epsilon^\nu + k^\nu) = \epsilon^\mu I_{\mu\nu}\epsilon^\nu + k^\mu I_{\mu\nu}\epsilon^\nu + \epsilon^\mu I_{\mu\nu}k^\nu + k^\mu I_{\mu\nu}k^\nu$$

$$\Rightarrow \quad k^\mu I_{\mu\nu} = 0 \;, \quad I_{\mu\nu}k^\nu = 0 \;, \quad k^\mu I_{\mu\nu}k^\nu = 0$$

So the only form $I_{\mu\nu}$ could have is the one in eq(2.9). Clearly, terms like $m^2 g_{\mu\nu}$ would spoil the invariance.

2) when one is going to take the limit $m^2 \to 0$ and $n \to 4$ then $\lim_{n\to 4} \lim_{m^2\to 0} \neq \lim_{m^2\to 0} \lim_{n\to 4}$. So, for example the integral:

$$1/(2\pi)^n \int d^n k \quad 1/(k^2)^a$$

cannot be defined unambiguously (for $n \to 4$ dimensions)

Leibbrand and Capper (1974) proposed a redefinition of the generalized Gaussian integral in n-dimensions. The net effect of this redefinition is an introduction of a mass which is a function of dimensions: $m = m(\frac{n}{2})$, where n are the dimensions of the space, with the following properties:

a) $m(\frac{n}{2})$ is a non-zero analytic function of the complex varable $\frac{n}{2}$

b) $m(\frac{n}{2})=0$ for $\frac{n}{2} = \pm\frac{1}{2}\lambda$ where $\lambda = 0,1,2,\ldots$.

c) $\frac{d^\ell m(n/2)}{d\, n/2} = 0$ for $\frac{n}{2} = \pm\frac{\lambda}{2}$, $\lambda = 0,1,2,\ldots$ and $\ell \leq \ell_0$ where ℓ_0 is finite.

d) $\mathrm{Re}[m(\frac{n}{2})] > 0$ for any $\mathrm{Re}[m] \neq \frac{\lambda}{2}$, $\lambda = 0,1,2,\ldots$ and for some $\mathrm{Im}[m]$

A function which satisfies the above properties is, for example:

$$m(n/2) = 1 - \cos(2\pi\cos(2\pi(\ldots(\cos 2\pi \tfrac{n}{2})\ldots)))$$

with m nested cosine functions, where m is finite integer.

The above function has the additional properties:

e) $\frac{d^\ell m(\frac{n}{2})}{d\, n/2} = 0$ for $\frac{n}{2} = \pm\frac{\lambda}{2}$, $\lambda = 0,1,2,\ldots$ and $\ell < 2^m - 1$

f) $\frac{d^\ell m(\frac{n}{2})}{d\, n/2} \neq 0$ for $\frac{n}{2} = \pm\frac{\lambda}{2}$, $\lambda = 0,1,2,\ldots$ and $\ell \geq 2^m - 1$

The intoduction of $m(\frac{n}{2})$ is not a gauge invariant process for $n \neq 4$, but property e) can make it gauge invariant to any finite order.

<u>Explicit construction of the analytic continuation of an integral</u> (3).

Consider the one-loop integral:

$$I = \int d^n p \; \frac{p_a^{\lambda_1}\, p_b^{\lambda_2} \cdots p_c^{\lambda_j}}{((p+k_1)^2 - m_1^2)^{\alpha_1} \cdots ((p+k_\ell)^2 - m_\ell^2)^{\alpha_\ell}} \tag{2.10}$$

where $\lambda_1, \lambda_2 \ldots \lambda_j$ are not necessarily integers, $k_1, \ldots, k_\ell$ are external momenta, and $p_a, \ldots, p_c$ are components $a, \ldots, c$ of p . The above integral will be convergent if:

$$\lambda_1 > -1 \, , \; \lambda_2 > -1 \, , \; \cdots \, , \; \lambda_j > -1$$

and $$n+\lambda_1+\lambda_2+\dots+\lambda_j < 2(\alpha_1+\alpha_2+\dots+\alpha_\ell) \tag{2.11}$$

(the first relation ensures us that there are no infrared singularities while the second that there are no ultraviolet singularities).

Now we insert in eq(2.10) the expression:

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\partial p_i}{\partial p_i} = 1$$

We may perform now partial integration, within the region defined by (2.11). We obtain trivially:

$$I = \frac{1}{(n+\lambda_1+\lambda_2+\dots+\lambda_j-2\alpha_1-2\alpha_2-\dots-2\alpha_\ell)}\, I' \tag{2.12}$$

where

$$I' = \int d^n p\, p_a^{\lambda_1}\dots p_c^{\lambda_j}\Bigg[\frac{2\alpha_1(m_1^2+k_1^2+(p\cdot k_1))}{((p+k_1)^2-m_1^2)^{\alpha_1+1}(\ )^{\alpha_2}\dots(\ )^{\alpha_\ell}} + \frac{2\alpha_2(m_2^2+k_2^2+(p\cdot k_2))}{(\ )^{\alpha_1}((p+k_2)^2-m_2^2)^{\alpha_2+1}\dots(\ )^{\alpha_\ell}} + \dots + \frac{2\alpha_\ell(m_\ell^2+k_\ell^2+(p\cdot k_\ell))}{(\ )^{\alpha_1}(\ )^{\alpha_2}\dots((p+k_\ell)^2-m_\ell^2)^{\alpha_\ell+1}}\Bigg] \tag{2.13}$$

The integral I' converges if:

$\lambda_1>-1,\ \lambda_2>-1,\ \dots,\ \lambda_j>-1$

and $$n+\lambda_1+\lambda_2+\dots+\lambda_j+1<2(\alpha_1+\alpha_2+\dots+\alpha_\ell+1) \quad \text{or} \tag{2.14}$$

$$n+\lambda_1+\lambda_2+\dots+\lambda_j < 2(\alpha_1+\alpha_2+\dots+\alpha_\ell)+1$$

This is a larger domain than (2.11) and the right-hand side of eq(2.13) is the explicit representation of the analytic continuation of I into this new domain

The above operation is called <u>partial p</u>.

In the case of vector fields, the prescription applies in a similar way. But we must be careful in doing the vector algebra since all internal lines are now n-component vectors.

In two loops, there are four partial operations as we shall see.

Renormalization. In order to obtain a consistent theory, it must be shown that the poles for $n=4$ can be removed order by order in pertubation theory. In any order, the infinite parts, which must be subtracted may not have an imaginary part (optical therem). This means that the new subtraction terms must be finite polynomials in the external momenta.

Let us prove this in the one loop case. Consider the integral I of eq(2.10). If the integral is convergent in 4-dimensional space, it has no singularity for $n=4$. If it is divergent, it is defined only in a region to the left of $n=4$. Analytic continuation, by partial integration, shows that the integral has a single pole at $n=4$. Using Feynman parameters to combine the propagators and the rules of Appendix C, we obtain:

$$I = \Gamma\left(j-\frac{n}{2}\right)\int dx_1 \ldots \int dx_i \quad \frac{P(x_i, m, k)}{(M^2)^{j-\frac{n}{2}}} \tag{2.15}$$

where j is some integer, x_i are the Feynman parameters and P and M polynomials in the masses m ,external momenta k and Feynman parameters x_i . The pole for $n=4$ is in the $\Gamma(j-\frac{n}{2})$ function. There is no trouble hidden in the Feynman parameter integrals, at least in the one loop case.

The residue of the pole is proportional to:

$$\int dx_1 \ldots \int dx_i \quad (M^2)^{\frac{n}{2}-j}\, P(x_i, m, k) \tag{2.16}$$

Obviously (2.16) is a finite polynomial (there are no terms like $\ln k^2$)

Two-loop diagrams. Next, we consider the two-loop case. First we suppose that we have introduced counter-terms of the form

$$\frac{P(m,k)}{n-4}$$

and made the one-loop diagrams finite. Consider now the general two-loop diagram:

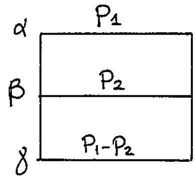


Fig. 3

The corresponding expression is:

$$\int d^n p_1 \, d^n p_2 \, \frac{1}{(p_1^2-m_1^2)^\alpha((p_2+k)^2-m_2^2)^\beta((p_1-p_2)^2-m_3^2)^\gamma} \qquad (2.17)$$

We call this an $(\alpha\beta\gamma)$ diagram. In writing the expression (2.17) we have supposed that all propagators depending on p_1 have been combined by means of Feynman parameters. Similarly for p_2 and p_1-p_2. Also we have suppressed all numerators taking care, of course, of the powers of the loop momenta, i.e. a term $\frac{p_{1\mu}\, p_{1\nu}}{(p_1^2+m_1^2)^4}$ is written as $\frac{1}{(p_1^2+m_1^2)^3}$. k is some external momentum (or momenta).

There are three one -loop diagrams contained in the above two-loop diagram

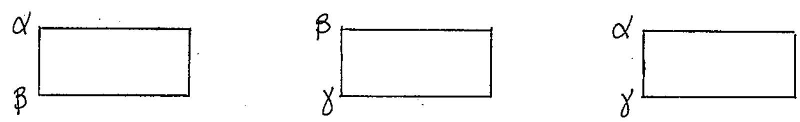


Fig 4

In the case that these diagrams diverge, we have counter-term contributions:

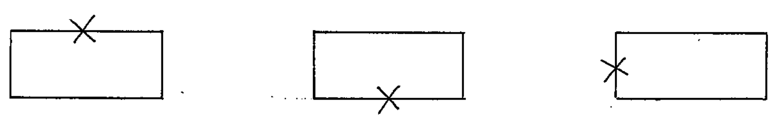


Fig 5

where $\times$ vertex means the pole-part of the one-loop diagrams. Of course these counter-terms have double-poles also: one pole from the $\times$ vertex and one from the loop integration itself. But there are also single pole terms coming from the pole of the vertex multiplying the finite part of the the loop integration. These terms have the form:

$$\frac{A}{\epsilon}\ln K^2$$

These terms must cancel against similar terms coming from the two-loop diagram ($\alpha\beta\gamma$ diagram), since being momentum-dependent they cannot be renormalized away.

Let us see now the four partial integrations, we promised, for the two-loop diagrams:

a) one may insert the expression:

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\partial p_{1i}}{\partial p_{1i}}$$

in eq(2.17) and perform the partial integration. The result shows a pole for $n=2(\alpha+\gamma)$ (since only the terms in the exponents α and γ involve p_1). This is the partial ($\alpha\gamma$) operation. Similarly we may define the partial ($\beta\gamma$) operation, with respect to p_2 (showing a pole for $n=2(\beta+\gamma)$) and partial ($\alpha\beta$), with respect to p_1, <u>after the substitution</u> $p_2'=p_1-p_2$ (showing a pole for $n=2(\alpha+\beta)$). Finally we define the partial ($\alpha\beta\gamma$) showing a pole for $2n=2(\alpha+\beta+\gamma)$. In this case we insert the expression:

$$\frac{1}{2n}\sum_{i=1}^{n}\left(\frac{\partial p_{1i}}{\partial p_{1i}}+\frac{\partial p_{2i}}{\partial p_{2i}}\right)$$

Let us write down what we get after applying partial ($\alpha\beta\gamma$) to eq(2.17):

$$\frac{1}{2n-2\alpha-2\beta-2\gamma}\,I'$$

$$I'=\int d^n p_1\, d^n p_2\left[\frac{2\alpha m_1^2}{(\ )^{\alpha+1}(\ )^{\beta}(\ )^{\gamma}}+\frac{2\beta(m_2^2+k^2+p_2\cdot k)}{(\ )^{\alpha}(\ )^{\beta+1}(\ )^{\gamma}}+\frac{2\gamma m_3^2}{(\ )^{\alpha}(\ )^{\beta}(\ )^{\gamma+1}}\right] \qquad (2.18)$$

Let us find now the form of the counter-terms. Consider the simple case when only one of the sub-diagrams diverges. For instance $\alpha+\gamma=2$

and $\beta=2$ (then, of course $\alpha+\beta>2$, $\beta+\gamma>2$)(*). Consider the one-loop diagram corresponding to the divergent sub-integral ($\alpha\gamma$). We have the expression:

$$\int d^n p_1 \frac{1}{(p_1^2+m_1^2)^{\alpha}((p_1-p_2)^2-m_3^2)^{\gamma}} \qquad (2.19)$$

Combining the propagators by means of Feynman parameters and performing the integration over p_1 we get (ignoring irrelevant factors):

$$\Gamma\left(\alpha+\gamma-\frac{n}{2}\right) \quad \frac{1}{(p_2^2-M^2)^{\alpha+\gamma-\frac{n}{2}}} \qquad (2.20)$$

where M is function of m_1 and the Feynman parameter. The pole part of expression (2.20) is (see Appendix C):

$$\frac{2}{\alpha+\gamma-\frac{n}{2}}$$

So the contributon from the counter-term has the form:

$$\int d^n p_2 \frac{2}{(\alpha+\gamma-\frac{n}{2})} \quad \frac{1}{((p_2+K)^2-m_2^2)^{\beta}} \qquad (2.21)$$

This must be subtructed from the two-loop integral which, from expression (2.17) and (2.20) has the form:

$$\int d^n p_2 \frac{\Gamma(\alpha+\gamma-\frac{n}{2})}{(p_2^2-M^2)^{\alpha+\gamma-\frac{n}{2}}} \quad \frac{1}{((p_2+K)^2-m_2^2)^{\beta}} \qquad (2.22)$$

We are just quoting the follwing theorems (see ref (1),1972):

Theorem 1. The difference of (2.21) and (2.22) for $\alpha+\gamma=2$ and $\beta\geq 1$ contains poles which have as residues polynomials of finite order

---

(*) The sub-integral ($\alpha\gamma$) is said to be divergent if $\alpha+\gamma\leq 2$ and convergent if $\alpha+\gamma>2$. More specifically if $\alpha+\gamma=2$ the diagram is said to be logarithmically divergent, if $\alpha+\gamma=\frac{3}{2}$ linearly divergent etc. The same applies to the other sub-integrals. Similarly if $\alpha+\beta+\gamma=2,\frac{3}{2},\cdots$ the integral is said to have a logarithmic, linear etc overall divergence.

in the external momentum.

Theorem 2. If the integral (2.17) is overall logarithmically divergent and contains no divergent sub-integrals then it contains a "harmless"(*) single pole at $n=4$.

Theorem 3. If the integral (2.17) is overall divergent and contains no divergent sub-integrals then it contains a "harmless" single pole at $n=4$.

Theorem 4. If the integral (2.17) is overall convergent or logarithmically divergent then it contains at most one divergent sub-integral. The denominator not involved in the sub-integral has exponent ≥ 2 .

Theorem 5. If the integral (2.17) is overall convergent or logarithmically divergent and contains one divergent sub-integral then the difference with the subtruction diagram containing the pole subtruction term corresponding to the divergent sub-integral has only harmless poles.

(*) By "harmless" we mean that the residue over the pole is a polynomial of finite order in the external momenta.

Chapter 3: Renormalization Group equation (1) and Dimensional Regulatization

Consider a theory with one dimensionless (in four dimensions) coupling constant g. The lagrangian has bare parameters m_B and g_B. The dimensions of space-time is n, and unrenormalized Green's functions are obtained in pertubation theory by using the rules for evaluating n-dimensional integrals.

The Green's functions will have singularities for $n=4$. By giving g_B and m_B a suitable n-dependence and by multiplying the <u>unrenormalized</u> Green's functions by suitable wave function renormalizations, we obtain <u>renormalized</u> Green's functions which are analytic at $n=4$. The value at $n=4$ is of course finite.

In order to follow the above procedure we must parametrize the renormalized Green's functions by a renormalized mass and renormalized coupling constant. The unrenormalized parameters are defined in terms of : the renormalized ones, n and possibly of an extra mass μ, according to some prescription. Of course, any prescription will have bare and renormalized parameters equal in the lowest order of the expansion in powers of g_R.

The renormalized mass m_R and the renormalized coupling constant g_R can be any function of n, anlytic at $n=4$.

If $\Gamma_u^{(N)}$ is an unrenormalized connected and amputated Green's function with N external legs and $Z^{1/2}$ is the renormalization constant for each of its external legs, then the renormalized Green's function $\tilde{\Gamma}_R^{(N)}$ is defined:

$$\boxed{\tilde{\Gamma}_R^{(N)} \equiv Z^{N/2}\, \Gamma_u^{(N)}} \qquad (3.1)$$

$\tilde{\Gamma}_R^{(N)}$ is analytic at $n=4$ and so:

$$\boxed{\tilde{\Gamma}_R^{(N)} \equiv \lim_{n\to 4} \tilde{\Gamma}_R^{(N)}}$$

is finite. Since for each $n \neq 4$, $\tilde{\Gamma}_R^{(N)}$ is computed as a power series in g_R and as a finite function of m_R, we see that $\Gamma_R^{(N)}$ depends only on g_R and m_R, at $n=4$, and not on dg_R/dn.

In the n-dimensional procedure, the bare coupling constant g_B has dimensions of mass to the power $(4-n)\rho$, for some constant ρ (it is easy to show that for ϕ^4 theory $\rho=1$, for Q.E.D. and Q.C.D. $\rho=1/2$).* The bare mass will have dimensions of mass. We need an extra mass parameter μ, which could for example determine the substruction point†

The wave function renormalization constant Z, the renormalized coupling constant g_R, and the mass renormalization $Z_m (\equiv m_B/m_R)$, which are all dimensionless, are functions of $g_B \mu^{(4-n)\rho}$ and n only. In other words, we assume (and we are going to prove it later, (3))

---

* If we assign to a derivative ∂_μ dimension 1, the Lagrangian has dimension n. A boson field has therfore dimension $(n-2)/2$ (from the kinetic-energy term for example). For ϕ^4 theory the coupling constant has dimension $4-n$, because of the term $\lambda\phi^4/4$, while in Q.C.D. has dimension $(4-n)/2$ because of the term g^2GGGG in the Lagrangian.

† 't Hooft defines μ to be a "unit of mass", having dimensions of mass, and its only use is to absorb the n-dependent part of the dimensions of the bare parameters in order that <u>for any n</u> the renormalized parameters has the same dimensions as the bare one for $n=4$ (2).

some renormalization prescription that allows Z , g_R and Z_m not to depend on m_R . Then by dimensional analysis μ and g_B can only appear in the combination $g_B\mu^{(4-n)P}$.

<u>Scaling behaviour of Green's functions and Renormalization Group Equation</u>. Consider a renormalized, amputated and connected Green's function $\Gamma_R^{(N)}(p, g_R, m_R, \mu)$,where p is the set of external momenta. Γ_R is defined from the unrenormalized $\Gamma_u^{(N)}(p, g_B(n), m_B(n,m))$ through a mass-independent renormalization prescription defined in the previous section:

$$\Gamma_R^{(N)}(p, g_R, m_R, \mu) = \lim_{n\to 4} \tilde{\Gamma}_R^{(N)}(p, g_R(n), m_R(n,m))$$

where

$$\tilde{\Gamma}_R^{(N)}(p, g_R(n), m_R(n), \mu, n) =$$

$$= \tilde{\Gamma}_R^{(N)}(p, g_R(g_B(n)\mu^{(4-n)P}, n), m_B(n), Z_m^{-1}(g_B(n)\mu^{(4-n)P}, n), \mu, n)$$

$$= Z^{N/2}(g_B(n)\mu^{(4-n)P}, n) \quad \Gamma_u^{(N)}(p, g_B(n), m_B(n), n) \tag{3.2}$$

Differentiating with respect to μ , keeping m_B, g_B , p and n fixed (and multiplying afterwards by μ) we get:

$$\left[\mu\frac{\partial}{\partial\mu} + \mu\frac{\partial g_R}{\partial\mu}\cdot\frac{\partial}{\partial g_R} + Z_m m_R \mu \frac{\partial Z_m^{-1}}{\partial\mu}\frac{\partial}{\partial m_R} - \frac{N}{2}\mu\frac{\partial Z}{\partial\mu}Z^{-1}\right]\tilde{\Gamma}_R = 0 \tag{3.3}$$

In the limit $n\to 4$ we get:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g_R)\frac{\partial}{\partial g_R} - \gamma_m(g_R)m_R\frac{\partial}{\partial m_R} - \frac{N}{2}\gamma(g_R)\right]\cdot\Gamma_R = 0 \tag{3.4}$$

where

$$\beta(g_R) = \lim_{n\to 4}\mu\frac{\partial}{\partial\mu}g_R(g_B(n)\mu^{(4-n)P}, n) \tag{3.5}$$

$$\gamma_m(g_R) = \lim_{n\to 4}\mu\frac{\partial}{\partial\mu}\ln Z_m(g_B(n)\mu^{(4-n)P}, n) \tag{3.6}$$

$$\gamma(g_R) = \lim_{n\to 4}\mu\frac{\partial}{\partial\mu}\ln Z(g_B(n)\mu^{(4-n)P}, n) \tag{3.7}$$

All the coefficients β , γ_m and γ are finite as $n \to 4$ since they appear in a differential equation ((3.4)) for a renormalized amplitude.

Suppose now we scale the momenta p with a scale variable α . By dimensional analysis we get:

$$\left[\mu\frac{\partial}{\partial\mu} + \alpha\frac{\partial}{\partial\alpha} + m_R\frac{\partial}{\partial m_R} - D_\Gamma\right]\Gamma_R(\alpha p_0, g_R, m_R, \mu) = 0 \qquad (3.8)$$

where p_0 is a set of fixed momenta, and D_Γ are the mass dimensions of Γ_R . Combining eqs(3.4),(3.8) we get:

$$\left[-\alpha\frac{\partial}{\partial\alpha} + \beta(g_R)\frac{\partial}{\partial g_R} - m_R(1+\gamma_m(g_R))\frac{\partial}{\partial m_R} + D_\Gamma - \frac{N}{2}\gamma(g_R)\right]\cdot \qquad (3.9)$$
$$\cdot\Gamma_R(\alpha p_0, g_R, m_R, \mu) = 0$$

The solution of this equation is given in terms of the effective coupling constant $\bar{g}(g_R,\alpha)$:

$$\alpha\frac{\partial\bar{g}}{\partial\alpha} = \beta(\bar{g}) \qquad \text{and} \qquad \bar{g}(g_R, 1) = g_R \qquad (3.10)$$

A similar relation defines the effective mass $\bar{m}(m_R,\alpha)$. The solution of eq(3.9) is given by:

$$\Gamma_R(\alpha p_0, g_R, m_R, \mu) = \alpha^{D_\Gamma}\exp\left[-\frac{N}{2}\int_0^{\ln\alpha} dt\, \gamma[\bar{g}(g_R, e^t)]\right]\cdot\Gamma(p_0, \bar{g}, \bar{m}, \mu) \qquad (3.11)$$

where t is defined:

$$\alpha = e^t \qquad (3.12)$$

Let us write down now the equations which give the bare parameters g_B, m_B and Z as a power series of inverse powers of $(n-4)$.* The residues are functions of the renormalized parameters and μ :

---

* While 't Hooft uses μ to absorb even the dimension of m_R , rendering m_R dimensionless, we are following reference (3) where m_R has dimension 1 (of mass).

$$g_B \mu^{(n-4)\rho} = g_R + \sum_{\nu=1}^{\infty} \frac{a_\nu(g_R, m_R, \mu)}{(n-4)^\nu} \qquad (3.13a)$$

$$m_B = m_R + \sum_{\nu=1}^{\infty} \frac{b_\nu(g_R, m_R, \mu)}{(n-4)^\nu} \equiv m_R Z_m \qquad (3.13b)$$

$$Z = 1 + \sum_{\nu=1}^{\infty} \frac{c_\nu(g_R, m_R, \mu)}{(n-4)^\nu} \qquad (3.13c)$$

So these quantities are just the renormalized quantities (g_R , m_R and in the case of Z , 1) plus whatever poles are needed to cancel the poles in the Feynman integrals. Given g_R , m_R and μ the coefficient a_ν , b_ν and c_ν are unique.

Now we shall prove that : a_ν , b_ν and c_ν do not depend on μ, and that a_ν and c_ν do not depend on m_R (g_R is dimensionless). From that it follows that Z_m does not depend on m_R and μ . Proof: when we expand the Feynman integrals, in order to find the poles and subtruct them, logarithms of μ will appear, since μ appears only in the combination μ^{4-n} (multiplying the coupling constant). Also, in the residues of the poles m_R only appears in polynomials (2). So we have positive powers of m_R only, and these are independent of n (no logarithms of m_R). But a_ν and c_ν are dimensionless. They cannot therefore depend on μ at all, since if they did, μ must enter in the combination $(\mu/m_R)^{n-4}$,i.e. with powers of m_R depending on n . Logarithmic dependence on μ is also forbidden since it will appear again as $\ln \mu/m_R$ and $\ln m_R$ are forbidden also. Non-dependence on μ ,means that a_ν and c_ν do not depend on m_R either, since they are dimensionless. Now b_ν has dimension of mass (as m_R and μ have). With the same arguments we can easily show that b_ν is proportional to m_R and by writting $m_B = m_R Z_m$, we deduce that Z_m is independent of both μ and m_R .

Let us find now how $\beta(g_R)$, $\gamma_m(g_R)$ and $\gamma(g_R)$ are derived from a_ν, b_ν and c_ν. Differentiation (and succesive multiplication by μ) with respect to μ of eq(3.13α)gives:

$$\rho\left[(n-4)g_R + a_1 + \frac{a_2}{n-4}\cdots\right] = \left(\mu\frac{\partial g_R}{\partial\mu}\right)_{g_B}\left[1 + \frac{da_1/dg_R}{n-4} + \frac{da_2/dg_R}{(n-4)^2} + \cdots\right] \tag{3.14}$$

We know that $\left(\mu\frac{\partial g_R}{\partial\mu}\right)_{g_B}$ is analytic (from its position in the eq(3.4) for $\tilde{\Gamma}_R$) at $n=4$. So we can write:

$$\left(\mu\frac{\partial}{\partial\mu}g_R\right)_{g_B} = X_0 + X_1(n-4) + X_2(n-4)^2 + \cdots \tag{3.15}$$

Applying eq(3.15) in eq(3.14) we get:

$$\begin{aligned} X_0 &= \left(a_1 - g_R\frac{\partial a_1}{\partial g_R}\right)\rho \\ X_1 &= g_R\rho \\ X_\nu &= 0 \qquad \text{for} \quad \nu \geqslant 2 \end{aligned} \tag{3.16}$$

and the identities:

$$\frac{\partial a_{\nu-1}}{\partial g_R}X_0 + \frac{\partial a_\nu}{\partial g_R}X_1 = \rho a_\nu \qquad \nu = 2,3,\ldots \tag{3.17}$$

Therefore by using the definition of $\beta(g_R)$ (eq(3.5)) we get:

$$\beta(g_R) = \left(a_1 - g_R\frac{\partial a_1}{\partial g_R}\right)\rho \tag{3.18}$$

Similarly we get:

$$\begin{aligned} \gamma_m(g_R) &= \rho g_R\frac{\partial b_1}{\partial g_R}\cdot\frac{1}{m_R} \\ \frac{\partial b_{\nu-1}}{\partial g_R}X_0 + \frac{\partial b_\nu}{\partial g_R}X_1 &= \rho g_R b_{\nu-1}\frac{\partial b_1}{\partial g_R}\,\frac{1}{m_R} \qquad \nu = 2,3,\ldots \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} \gamma(g_R) &= \rho g_R\frac{\partial c_1}{\partial g_R} \\ \frac{\partial c_{\nu-1}}{\partial g_R}X_0 + \frac{\partial c_\nu}{\partial g_R}X_1 &= \rho g_R c_{\nu-1}\frac{\partial c_1}{\partial g_R} \qquad \nu = 2,3,\ldots \end{aligned} \tag{3.20}$$

Eq(3.20) shows clearly that $\gamma(g_R)$ depends only on the residue c_1 of the single pole.

Dependence of γ function on the renormalization scheme(4). Consider two renormalization schemes with the coupling constant g defined at the point where all external momenta equal μ. If $\Gamma_R^{o\,\prime}$ and Γ_R^o are renormalized inserted Green's functions (with the insertion of the operator O) then*:

$$\Gamma_R^{o\,\prime} = Z'\,\Gamma_B^o \qquad \text{and} \qquad \Gamma_R^o = Z\,\Gamma_B^o \tag{3.21}$$

where Z' and Z are the renormalization constants in the two schemes. Thus:

$$\Gamma_R^{o\,\prime}/\Gamma_R^o = Z'/Z \tag{3.22}$$

Now it is easy to see that:

$$Z' = Z\;F(g) = Z\;\left(1+g^2\alpha+O(g^4)\right) \tag{3.23}$$

Consider now the γ - function. From its definition (see eq(3.7)) we get:

$$\begin{aligned}\gamma' &= \mu\frac{\partial}{\partial\mu}\ln Z' = \frac{1}{Z'}\mu\frac{\partial}{\partial\mu}Z' = \frac{1}{Z'}\left(\mu\frac{\partial}{\partial\mu}Z\right)F + \\ &\qquad + \frac{1}{Z'}\left(\mu\frac{\partial}{\partial\mu}F\right)Z = \\ &= \mu\frac{\partial}{\partial\mu}\ln Z + \mu\frac{\partial}{\partial\mu}\ln F = \gamma + \mu\frac{\partial}{\partial\mu}\ln F = \\ &= \gamma + \frac{\partial}{\partial g}\ln F\cdot\beta(g)\end{aligned} \tag{3.24}$$

where $\beta(g)$ is the β-function (see eq(3.5)). Now $\beta(g)$ can be written:

$$\beta(g) = \beta_0 g^3 + \beta_1 g^5 + O(g^7) \tag{3.25}$$

Using eq(3.23) and eq(3.25), eq(3.24) becomes:

* Although we have for the two schemes the same g defined at the same point μ, we could have different Z. This is not quite true in the case where minimal substraction procedure is used.

$$\gamma' = \gamma + \frac{1}{1+\alpha g^2}\frac{\partial}{\partial g}(1+\alpha g^2)\,\beta(g) =$$

$$= \gamma + (1-\alpha g^2)\,2\alpha g\,\left(\beta_1 g^3 + \beta_2 g^5 + O(g^7)\right) =$$

$$= \gamma + 2\alpha\beta_0 g^4 + \dots \qquad (3.26)$$

Therefore we note that the coefficient of g^4 differs in the two schemes.

Chapter 4 : The Operator Product Expansion (1)

The operator product expansion expresses the product of local operators $A(y)$ and $B(x)$ in terms of operators O_i in the limit $x-y \rightarrow 0$:

$$A(y)\,B(x) \xrightarrow{x-y\rightarrow 0} \sum_i C_i(x-y)\,O_i(y) \tag{4.1}$$

where the C's are c-number functions, generally singular. By saying that this is an operator expansion we mean that the singularities are independent of the matrix elements:

$$\langle\alpha|\,AB\,|\beta\rangle \longrightarrow \sum_i C_i \langle\alpha|\,O_i\,|\beta\rangle \tag{4.2}$$

where the C's are independent of α and β .

If d^A, d^B, d^{C_i} and d^{O_i} are the (mass) dimensions of A, B, C_i and O_i, then the follwing equation must hold:

$$C_i(x-y) \xrightarrow{x\rightarrow y} \left(\frac{1}{x-y}\right)^{d^{C_i}} = \left(\frac{1}{x-y}\right)^{d^A+d^B-d^{O_i}} \tag{4.3}$$

From eq(4.3) we can easily see that the operator O_i with the smallest dimension d^{O_i} dominates for $x\rightarrow y$, since the corresponding is the most singular ($d^A+d^B-d^{O_i}$ maximum)

<u>Operator Product Expansion and the $\Delta I=\frac{1}{2}$ rule</u> . Wilson (1) first proposed a mechanism for explaining the $\Delta I=\frac{1}{2}$ rule through the Operator Product Expansion (O.P.E.). To lowest order in the weak interaction, the matrix element for a weak non leptonic transition is expected to be of the form:

$$\langle f | H_{NL} | i \rangle \sim \int d^4x \, D_{\mu\nu}(x^2, M_W^2) \langle f | T(J^\mu(x) J^{+\nu}(0)) | i \rangle \quad (4.4)$$

where $|f\rangle$ and $|i\rangle$ are final and initial hadronic states, J is the weak hadronic current (*) and $D_{\mu\nu}(x^2, M_W^2)$ is the W-boson propagator. The matrix element of $T(J\,J)$ is completely determined by strong interactions.

Now for x being space-like ($x^2 < 0$) and for $|x^2| M_W^2 \simeq 1$ and assuming that $M_W \gg 1\,GeV$ (1 GeV is consider to be a typical Hadronic mass), then the contribution to the integral of eq(4.4) comes from small x and in this region we can use the O.P.E. for the T-product of the two weak currents:

$$T(J(x) J^+(0)) = \sum_i C_i(x) O_i(0) \quad (4.5)$$

The mass dimension of a spinor field is $3/2$. So the product has dimension 6 (since each current involves two spinor fields). If d^{O_i} is the dimension of the operator O_i then we have:

$$C_i(x) \propto (M)^{6-d^{O_i}} \quad \text{or} \quad C_i(x) \propto (1/x)^{6-d^{O_i}} \quad (4.6)$$

Having eq(4.6) in mind we can rewrite eq(4.5) in the form:

$$T(J(x) J^+(0)) = \sum_i \frac{c_i(mx)}{x^2} O_i^{(4)}(0) + \sum_j b_j(mx) O_j^{(6)}(0) + \cdots \quad (4.7)$$

where $O_i^{(4)}$ and $O_j^{(6)}$ are operators with dimensions 4 and 6 respectively and m is a hadronic mass parameter. In the free field theory, the C_i, b_i's are dimensionless. In general they show deviation from naive scaling and they are singular when $x \to 0$. But in asymptotically free gauge theories, this deviation is only logarithmic.

(*) We consider here only charged weak currents since the neutral current is strangeness conserving and therefore does not contribute to the processes.

So Wilson proposed : if we can show that the operator which have $|\Delta S|=1$, $\Delta I=1/2$ requires a sufficiently stronger singularity than the operator with $|\Delta S|=1$, $\Delta I=1/2, 3/2$, the $\Delta I=1/2$ rule is demonstated. Let us see how this comes about. If

$$C_{1/2}(mx) \sim \ln(mx)^{d_{1/2}} \quad , \quad C_{3/2}(mx) \sim \ln(mx)^{d_{3/2}}$$

where $C_{1/2}$ and $C_{3/2}$ are the coefficients in the Wilson expansion corresponding to the operators with $|\Delta S|=1$, $\Delta I=1/2$ and $\Delta I=1/2, 3/2$ respectively, then from eq(4.4) we get:

$$\frac{\langle f|H_{NL}|i\rangle_{\Delta I=1/2}}{\langle f|H_{NL}|i\rangle_{\Delta I=1/2,3/2}} \sim \ln\left(\frac{M_W^2}{m^2}\right)^{d_{1/2}-d_{3/2}} \frac{\langle f|O_{1/2}|i\rangle}{\langle f|O_{3/2}|i\rangle}$$

(the integration in eq(4.4) which leads to the above equation will be made explicitly in Chapter 5). Since $M \sim 100$GeV and $m \sim 1$GeV (typical hadronic mass), then $\ln \frac{M_W^2}{m^2} \sim 10$ and reasonable values of $d_{1/2}$ and $d_{3/2}$ could give the desired value of the enhancement. For example , if $d_{1/2}-d_{3/2} \sim 1.3$,then $\ln\left(\frac{M_W^2}{m^2}\right)^{1.3} \sim 20$ confirming the experimental results.

O.P.E. and Renormalization Group Equation. The renormalization group equation for the Wilson coefficients are easily derived. One simply considers the short distance expansion of Green's function of products of operators, and explores the consequences of the renormalization group equation satisfied by these Green's functions.

Consider the short distance expansion of the operators A and B :

$$A(x)B(0) \xrightarrow{x\to 0} \sum C_i(x, g, m, \mu)\, O_i(0)$$

where g, m and μ are defined as in Chapter 3 and O_i are operators with the appropriate quantum numbers.

The above expansion means that the inserted Green's function Γ_{AB}^n

$$\Gamma_{AB}^n = \langle 0|T\{A(x)B(0)\prod_{k=1}^{n}\phi_u(y_k)\}|0\rangle \qquad (4.8)$$

can be expressed, when $x \to 0$, as an asymptotic expansion:

$$\sum_i C_i(x,g,m,\mu)\, G^n_{O_i} \quad \text{where} \qquad (4.9)$$

$$G^n_{O_i} = \langle 0 | T\{ O_i(0) \prod_{k=1}^{n} \phi_k(y_k)\} | 0 \rangle$$

Now the n-particle Green's function with the insertion of A and B satisfies the Renormalization Group Equation :

$$\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} + \gamma_m m\frac{\partial}{\partial m} - \sum \gamma_k(g) + \gamma_A(g) + \gamma_B(g)\right] \Gamma^n_{AB} = 0 \qquad (4.10)$$

where γ_A and γ_B are the anomalous dimensions of A and B. In our case A and B are conserved or partially conserved currents which have zero anomalous dimensions. (*). Also we can replase the sum over $k=1,\ldots,n$ by $n/2$ times the anomalous dimension γ ,since all external particles are the same.

Bearing in mind that the Green's function with the insertion of O_i satisfy the Renormalization Group Equation, we conclude:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} + \gamma_m m\frac{\partial}{\partial m} - \gamma_i(g)\right] C_i(x,g,m,\mu) = 0 \qquad (4.11)$$

The Wilson coefficients behave as if it were Green's function of A, B and O (irreducible with respect to O).

The above consideration applies in the case where no mixing of O 's arises. In this case, the operators O_i are not multiplicatively renormalized and therefore the anomalous dimension γ takes the form of a matrix γ_{ij}. So, eq(4.11) becomes :

$$\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} + \gamma_m m\frac{\partial}{\partial m}\right] C_i(x,g,m,\mu) = \gamma_{ij}\, C_j(x,g,m,\mu) \qquad (4.12)$$

Now the solution to eq(4.11) and eq(4.12) are given :

(*) The electromagnetic current, being a conserved current, is not renormalized and therefore has zero anomalous dimension, since anomalous dimension arises from renormalization itself. By current algebra all currents, including partially conserved currents, have zero anomalous dimension.

$$\tilde{C}_i(q^2/\mu^2, g, m) = \tilde{C}_i(1, \bar{g}, \bar{m}) \cdot \exp\left[-\int_0^t dt' \, \gamma_i(\bar{g}(g,t'))\right]$$

and

$$\tilde{C}_i(q^2/\mu^2, g, m) = \tilde{C}_j(1, \bar{g}, m) \; \exp\left[T\left\{-\int_0^t dt' \, \gamma_{ij}(\bar{g}(g,t'))\right\}\right]$$

where $\tilde{C}$ is the Fourier transform of C, $\bar{g}$ and $\bar{m}$ are the effective coupling constant and the effective mass, and T refers to t-ordering.

Note. In general the R.G.E. which the Γ_0 obeys must have a term which depends on the gauge parameter :

$$\delta(g,\alpha)\frac{\partial}{\partial\alpha} = \left(\mu\frac{\partial}{\partial\mu}\alpha\right)\frac{\partial}{\partial\alpha}$$

But it can be shown (2), in the case where O is a gauge invariant operator (current etc) or S-matrix element this term can be eliminated from the R.G.E.

(*) Since $\gamma(t')$ is a matrix, in general $\gamma_{ij}(t')$ does not commute with $\gamma_{ij}(t'')$. Consider the third term of the expansion of $\exp[-\int\ ]$:

$$\frac{1}{3!}\int_0^t dt' \, \gamma_{ik}(t') \int_0^{t'} dt'' \gamma_{kl}(t'') \int_0^{t''} dt''' \gamma_{lj}(t''')$$

where we must have $t > t' > t''$. So t-ordering takes care of that situation.

Chapter 5 : Weak Non-Leptonic Amplitudes

Consider a gauge theory based on a group structure of the form:

$$G = G_S \otimes G_W$$

where G_S is the non-abelian gauge group of the strong interaction and G_W is the gauge group of weak and electromagnetic interactions. The two groups are assumed to commute with each other. This means that the gluons of the strong interaction are neutral under weak and electromagnetic interactions and that the weak gauge bosons have no strong interactions. G_S is specified to be the colour SU(3) group, while G_W can be specified to be the SU(2) $\otimes$ U(1) group of Weinberg-Salam (1) extended to hadrons. The colour symmetry is unbroken. The coupling constant for the strong interaction is g and for the weak interaction is g_W(actually in the SU(2)$\otimes$U(1) model there are two coupling constants g_W and g' related by the tangent of the Weinberg angle Θ_W: $g_W \tan\Theta_W = g'$. The electromagnetic coupling constant e is $e = g_W \sin\Theta_W = g_W g'/(g_W^2+g'^2)^{1/2}$. The Fermi weak coupling constant G is related to g_W by $\frac{g_W^2}{8M_W^2} = \frac{G}{\sqrt{2}}$ where M_W is the mass of W meson).

We are interested in an arbitrary transition between hadronic states to order g_W^2 and to all orders in g. Then the non-leptonic effective hamiltonian can be written (2):

$$H_{NL} = \frac{G}{\sqrt{2}} M_W^2 \int d^4x \; D(x^2, M_W^2) \; T(J_\alpha(x) J^{+\alpha}(0)) + \frac{G}{\sqrt{2}} M_Z^2 \int d^4x \; D(x^2, M_Z^2) \; T(J_\alpha^0(x) J^{0+\alpha}(0)) \qquad (5.1)$$

+ (Higgs exchange)
+ (tadpoles exchange) .

where $D(x^2, M_W^2)$ is the propagator for a scalar particle of mass M, M_Z^2 is the mass of the neutral gauge boson and J^0 is the neutral weak hadronic current.

Let us consider the first term in eq(5.1). We can expand the time ordered product of weak currents in terms of local operators of given canonical dimensions (see eq(4.7)):

$$T\left(J_\alpha(x) J^{+\alpha}(0)\right) = \sum_i \frac{C_4^i(mx)}{x^2} \, O_i^{(4)}(0) + \sum_\ell C_6^\ell(mx) \, O_\ell^{(6)}(0) + \ldots \tag{5.2}$$

$O^{(4)}$ and $O^{(6)}$ are local operators of canonical dimension 4 and 6 respectively (the dots stand for operators of higher canonical dimensions). C_4^i and C_6^ℓ are singular functions of x, and m is a mass parameter. Dimension 2 operators do not appear in the Wilson expansion (2,3). Odd dimension operators, dimension 3 or 5, could in principle appear in eq(5.2). However since fractional powers of x^2 are not allowed, these operators enter the expansion multiplied by one power of mass, So that the corresponding coefficient $C(mx)$ have behaviour similar to that of dimension 4 and 6 operators.

We have to recall at this point that since the theory we are discussing is asymptotically free in the strong sector the C-coefficients have at most logarithmic singularities for $x \to 0$:

$$C(mx) \sim k \left(\ln(m^2x^2)\right)^d \tag{5.3}$$

d being a computable number in the pertubation theory.

Now, operators of dimension 4, or less, are responsible for terms of order α $(\sim G M_W^2)$. But these operators can be reabsorbed into counterterms of $\mathcal{L}_{STRONG}(2)$, (see also Appendix F).

Operators of dimension 6 give leading contribution to $\mathcal{H}_{NL}$:

$$G \sum_{\alpha} k_{\alpha} \left(\ln \frac{M_W^2}{m^2} \right)^{d_\alpha} O_{\alpha}^{(6)}(0) \qquad (5.4)$$

(d_α is the exponent in eq(5.3)).

Operators of dimension larger than 6 give leading contribution to $\mathcal{H}_{NL}$ which is smaller with respect to those given in eq(5.4), by at least one power of M_W^2 and therefore completely negligible.

Thus the leading contribution to $\mathcal{H}_{NL}$ from the first term in eq(5.1) is given by eq(5.4), i.e. by dimension-6 and -5 operators. Notice that these operators contribute to non-leptonic amplitudes terms of order :

$$G \left[\ln \left(\frac{M_W^2}{m^2} \right) \right]^d \sim G \left[\ln \left(\frac{\alpha}{G m^2} \right) \right]^d \qquad (5.5)$$

which may be considerably enhanced or suppressed (depending upon the sign of d) with respect to the naive estimate $\mathcal{H}_{NL} \sim G$. In fact what Gallaird and Lee found in their one-loop calculations (4) was $d>0$ for $\Delta I = \frac{1}{2}$ and $d<0$ for $\Delta I = \frac{1}{2}, \frac{3}{2}$ operators leading to $\Delta I = \frac{1}{2}$ enhancement in $\mathcal{H}_{NL}$ ($\Delta S \neq 0$).

The same discussion applies to the second term in eq(5.1) but in our case ($\Delta S \neq 0$) the neutral current piece is excluded since by construction (in the G.I.M. model (5)) J_α^0 is strangeness conserving ($\Delta S = 0$).

Consider now the third term in eq(5.1), namely the case of physical and unphysical Higgs scalar meson exchange. The coupling constant of Higgs scalars to the quarks is of the order:

$$g_s^2 \approx G m^2 \qquad (5.6)$$

m being a hadronic mass scale (e.e. the quark mass in aquark model). Thus Higgs exchange terms are explicity of order G . But here we have the problem that M_S (the mass of the Higgs scalar) is not restricted by the theory. So we consider two extreme alternatives:

i) $M_S \gtrsim M_W >> m$

ii) $M_S \approx m$

In case i) short distance behaviour is relevant. But eq(5.6) shows that only operators of dimension 4 or less can give rise to terms comparable to those in eq(5.4). Again such operators can be dropped out from $\mathcal{H}_{NL}$. In case ii) the short distance expansion is useless. In our case, where $\Delta S \neq 0$, Higgs exchange terms certainly do not contribute since they are hypercharge conserving.

Finally tadpole exchange terms do not matter since they only contribute to the mass matrix of the quarks and can be reabsorbed like all dimension-4 or less operators, into counterterms of $\mathcal{L}_{STRONG}$.

The net conclusion of the foregoing discussion is that we must consider all operators of dimension 5 or 6 appearing in the short distance expansion of $T(J_\alpha(x) J^{+\alpha}(0))$ coming from the first term of eq(5.1). These operators must possess the correct quantum numbers i.e. $|\Delta S| = 1$, $CP = +1$, $\Delta C = 0$ and be colour singlets. We can classify them as follows according to the number of fermion fields:

i)Operators with no fermion fields. These can be disregarded, since they can only constucted out of gluons and thus have $\Delta S = 0$.

ii)Operators with two fermion fields. These operators cannot contain $\Delta I = \frac{3}{2}$ pieces, since the maximum isospin of a single quark is $I = \frac{1}{2}$. In this class, those containing gluons through covariants derivatives only, can be reduced by the equation of motion $((i\not{D} - m)\psi = 0)$ to operators of lower dimension and therefore disregarded. Operators where gluon fields appear explicity trough $G^a_{\mu\nu}$ cannot be present, so long as we are concerned with left-handed weak currents, as we are. The reason is simple. The operators appearing in the short distance expansion of $T(J_\alpha(x) J^{+\alpha}(0))$ are symmetric in the exchange of the two currents. Therefore they must be proportional to $\{L^+, L^-\}_+$, where L^+ is defined through the weak current J_μ:

$$J_\mu = \bar{\psi}\gamma_\mu L^+(1-\gamma_5)\psi \tag{5.7}$$

where ψ is the quark field. This anticommutator must be purely $\Delta S=0$ in order to avoid $\Delta S\neq 0$ neutral transition at order $G\alpha$.

The only remaining two-fermion operators are those of dimension 5 containing a mass matrix. These operators do not mix with operators of dimension 6 and in fact it can be shown (4) that they only matter in theories where both V-A and V+A currents exist.

iii) Operators with four fermionfields. There are two operators in this class: O^1 and O^2 (ther possibilities can be expressed as linear combinations of them) :

$$O^1 = (\bar{\psi}\gamma_\mu L^+(1-\gamma_5)\,I\,\psi)(\bar{\psi}\gamma^\mu L^-(1-\gamma_5)\,I\,\psi) \tag{5.8}$$

$$O^2 = (\bar{\psi}\gamma_\mu L^+(1-\gamma_5)\,t^a\psi)(\bar{\psi}\gamma^\mu L^-(1-\gamma_5)\,t^a\psi) \tag{5.9}$$

where t^a is the SU(3) matrices (see Appendix A).

The above discussion implies that we have to restrict ourselves to the operators O^1 and O^2. As we shall see O^1 and O^2 are not multiplicatively renormalized. In other words, strong interactions to the above effective operators mix them. All these will be shown explicitly in the calculation of the one-loop diagrams.

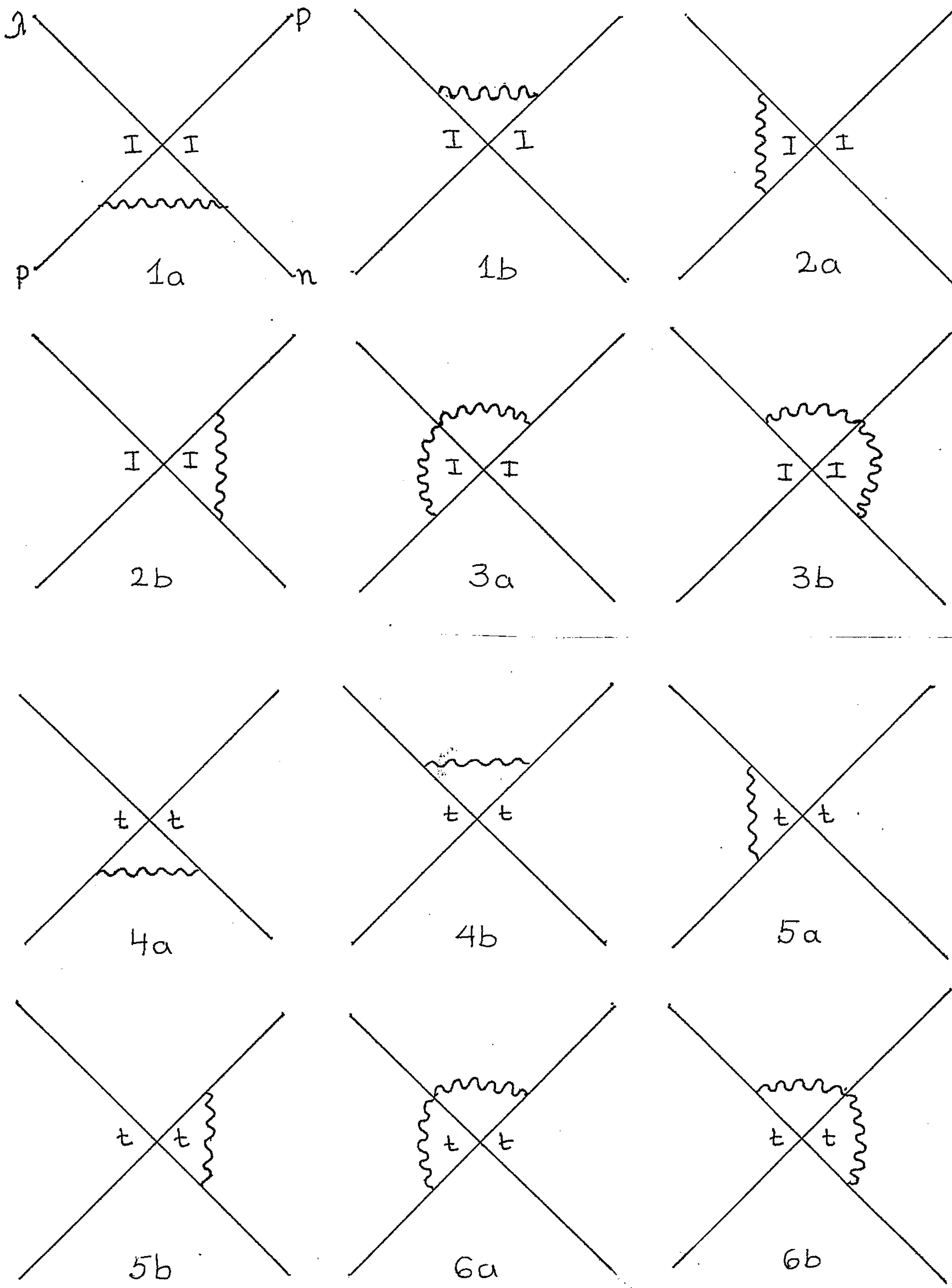
The only terms in eq(5.8) and eq(5.9) with $\Delta S=1$, $\Delta C=0$, and a $\Delta T=3/2$ component are of the form $(\bar{\lambda}p)(\bar{p}n)$.

One-loop correction. We will now reproduce the results of Gaillard and Lee, Altarelli and Maiani, using dimensional regularization. As we have just said, there are two operators O^1 and O^2 appearing in the Wilson expansion, namely:

$$O^1 = (\bar{\lambda}\gamma^\mu(1-\gamma_5)\,I\,p)(\bar{p}\gamma_\mu(1-\gamma_5)\,I\,n) \tag{5.10}$$

$$O^2 = (\bar{\lambda}\gamma^\mu(1-\gamma_5)\,t^a p)\,(\bar{p}\gamma_\mu(1-\gamma_5)\,t^a n) \qquad (5.11)$$

where I is the unit (colour) matrix and t^a are the SU(3) matrices. There are 12 one-loop diagrams, as shown in fig(5.1):



fig(5.1)

Of course the divergent parts of 1a and 1b are the same. This applies to 2a and 2b and so on. Moreover the difference between 1a and 4a is in the t-algebra only (the same for 2 and 5, 3 and 6).

Since all the diagrams have logarithmic singularities, the infinite part, in which we are interested, does not depend on external momenta. Thus we perform the calculation with zero external momenta, we give to the gluon the Leibbrand-Capper mass $m=m(n)$, and, to simplify the calculation, we give the same mass to all fermions. Now in general (and this is especially true in the second-loop order), the γ-algebra must be done in n-dimensions. So we can write:

$$\Gamma_{algebra} = A_0' + A_1' n + A_2' n + O(n^3) \qquad (5.12)$$

or if $\epsilon = n-4$

$$\Gamma_{algebra} = A_0 + A_1 \epsilon + A_2 \epsilon^2 + O(\epsilon^3) \qquad (5.13)$$

where of course

$$\left(\Gamma_{algebra} \right)_{n \to 4} = A_0 \qquad (5.14)$$

Now the integration with respect to the internal momentum will have a divergent part, proportional to $\frac{1}{n-4} = \frac{1}{\epsilon}$ and a finite part (for $n=4$) :

$$\text{integral } I = \frac{a}{\epsilon} + b + c e + O(\epsilon^2) \qquad (5.15)$$

So, the diagram D is:

$$\begin{aligned} D &= \left[\frac{a}{\epsilon} + b + c e + O(\epsilon^2)\right] \left[A_0 + A_1 \epsilon + A_2 \epsilon^2 + O(\epsilon^3) \right] = \\ &= \frac{a A_0}{\epsilon} + a A_1 + b A_0 + O(\epsilon) \end{aligned} \qquad (5.16)$$

From this last relation we can see that the residue (the coefficient of $1/\epsilon$), depends only on A_0, as far as γ-algebra is conserned, which means that in the one-loop order we can perform the γ-algebra in 4-dimensions, without fear of losing anything.

We evaluate one of the diagrams for demonstration. The others can be evaluated in the same way.

Diagram 4b. Applying Feynman rules for Q.C.D. (see Appendix D) we get:

$$-ig^2\int\frac{d^nk}{(2\pi)^n}\left(\bar{\lambda}\gamma_\mu(1-\gamma_5)t^b\frac{\not{k}+m}{k^2-m^2}\gamma^\rho t^a p\right)\left(\bar{p}\gamma^\mu(1-\gamma_5)t^b\frac{-\not{k}+m}{k^2-m^2}\gamma_\rho t^a n\right)\frac{1}{k^2-m^2}$$

The above can be written, isolating the momentum integral:

$$-ig^2\Big[\int\frac{d^nk}{(2\pi)^n}\frac{-k_\nu k_\lambda}{(k^2-m^2)^3}\left(\bar{\lambda}\gamma_\mu(1-\gamma_5)t^b\gamma^\nu\gamma^\rho t^a p\right)\left(\bar{\lambda}\gamma^\mu(1-\gamma_5)t^b\gamma^\lambda\gamma_\rho t^a n\right)+$$

$$\int\frac{d^nk}{(2\pi)^n}\frac{m^2}{(k^2-m^2)^3}\left(\bar{\lambda}\gamma_\mu(1-\gamma_5)t^b\gamma^\rho t^a p\right)\left(\bar{p}\gamma^\mu(1-\gamma_5)t^b\gamma_\rho t^a n\right)\Big]$$

(where the terms proportional to k_λ or k_ν vanish because of symmetric integration). Now using Appendix C we get:

$$\int\frac{d^nk}{(2\pi)^n}\frac{-k_\nu k_\lambda}{(k^2-m^2)^3}=\frac{-i\pi^{n/2}}{(2\pi)^n}\frac{1}{\Gamma(3)}\left\{\Gamma(2-\tfrac{n}{2})\tfrac{1}{2}(-m^2)^{n/2-2}g_{\nu\lambda}\right\}$$

$$\int\frac{d^nk}{(2\pi)^n}\frac{m^2}{(k^2-m^2)^3}=\frac{i\pi^{n/2}}{(2\pi)^n}\frac{1}{\Gamma(3)}\left\{\Gamma(3-\tfrac{n}{2})\;(-m^2)^{n/2-3}(m^2)\right\}$$

The first one has an infinite part I.P. (see Appendix C)

$$I.P.=\frac{-i\pi^2}{(2\pi)^4}\frac{1}{2}\left(-\frac{2}{\epsilon}\right)\frac{1}{2}g_{\nu\lambda}$$

while the second one has only a finite part.

Now the γ-algebra. As we said we can work in 4-dimensions. Using the formula:

$$\gamma^\mu\gamma^\nu\gamma^\rho=g^{\mu\nu}\gamma^\rho+g^{\nu\rho}\gamma^\mu-g^{\mu\rho}\gamma^\nu+i\epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\sigma$$

we can easily prove that:

$$\left(\gamma^\mu(1-\gamma_5)\gamma^\nu\gamma^\rho\right)\left(\gamma_\mu(1-\gamma_5)\gamma_\nu\gamma_\rho\right)=16\left(\gamma^\mu(1-\gamma_5)\right)\left(\gamma_\mu(1-\gamma_5)\right)$$

The t-algebra reads (see Appendix A):

$$(t^bt^a)(t^bt^a)=2/9\,(I)(I)-1/3\,(t^a)(t^a)$$

So the final result for this diagram is (not writing explicity the quarks spinors):

$$(\text{Diagram 4b}) = -g^2 \frac{\pi^2}{(2\pi)^4} \frac{1}{2}\left(-\frac{2}{\epsilon}\right) \frac{1}{2}\, 16 \begin{Bmatrix} 2/9\ (\gamma^\mu(1-\gamma_5)I)(\gamma_\mu(1-\gamma_5)I) \\ -1/3\ (\gamma^\mu(1-\gamma_5)t^a)(\gamma_\mu(1-\gamma_5)t^a) \end{Bmatrix}$$

$$= -g^2 \frac{\pi^2}{(2\pi)^4} \left(-\frac{1}{\epsilon}\right) 8 \begin{Bmatrix} 2/9\ (I)(I) \\ -1/3\ (t)(t) \end{Bmatrix}$$

This example shows the "phenomenon" of mixing in the operators: diagram 4b has the insertion of the operator O_2 while the result shows the appearence of O_1 operator too. By calculating all the one-loop diagrams,we get the follwing divergent part (Appendix E):

$$\text{Diagrams } (I)(I) = g^2 \frac{\pi^2}{(2\pi)^4} \frac{1}{\epsilon} \begin{Bmatrix} -\frac{16}{3}\ (\gamma^\mu(1-\gamma_5)I)(\gamma_\mu(1-\gamma_5)I) \\ 12\ (\gamma_\mu(1-\gamma_5)t^a)(\gamma^\mu(1-\gamma_5)t^a) \end{Bmatrix}$$

$$\text{Diagrams } (t)(t) = g^2 \frac{\pi^2}{(2\pi)^4} \frac{1}{\epsilon} \begin{Bmatrix} \frac{8}{3}\ (\gamma^\mu(1-\gamma_5)I)(\gamma_\mu(1-\gamma_5)I) \\ -\frac{28}{3}\ (\gamma^\mu(1-\gamma_5)t^a)(\gamma_\mu(1-\gamma_5)t^a) \end{Bmatrix}$$

or in matrix notation:

$$\begin{pmatrix} (I)(I)_{\text{insertion}} \\ \\ (t)(t)_{\text{insertion}} \end{pmatrix} = g^2 \frac{\pi^2}{(2\pi)^4} \frac{1}{\epsilon} \begin{pmatrix} -\frac{16}{3} & 12 \\ \\ \frac{8}{3} & -\frac{28}{3} \end{pmatrix} \begin{pmatrix} (I)(I) \\ \\ (t)(t) \end{pmatrix} \qquad (5.17)$$

So we see that the renormalization constants for the inserted Green's functions have become a matrix Z_{ij}, which to order $\frac{1}{\epsilon}$ is of course:

$$Z_{ij} = -\frac{g^2}{16\pi^2} \frac{1}{\epsilon} \begin{pmatrix} -\frac{16}{3} & 12 \\ \\ \frac{8}{3} & -\frac{28}{3} \end{pmatrix}$$

where g is the renormalized coupling constant.

Therefore the anomalous dimension matrix for the inserted Green's function is (see Chapter 3) :

$$\gamma = \frac{1}{2} g \frac{\partial Z_{ij}}{\partial g} = -\frac{g^2}{16\pi^2} \begin{pmatrix} -\frac{16}{3} & 12 \\ 8/3 & -\frac{28}{3} \end{pmatrix} \tag{5.18}$$

Now (see Appendix G) :

$$\gamma = 2\gamma_F - \gamma_O \tag{5.19}$$

where γ_F is the anomalous dimensions of the fermion field and γ_O the anomalous dimensions of the operators O_1 and O_2 we are interested in.

The anomalous dimensions of the fermion field, to order g^2 is (see Appendix E) :

$$\gamma_F = -\frac{g^2}{16\pi^2}\left(-\frac{8}{3}\right) I \tag{5.20}$$

Therefore we have, diaganolizing the γ_{ij} matrix:

$$\gamma_O = 2\gamma_F - \gamma = -\frac{g^2}{16\pi^2}\left[2\begin{pmatrix} -8/3 & 0 \\ 0 & -8/3 \end{pmatrix} - \begin{pmatrix} -40/3 & 0 \\ 0 & -4/3 \end{pmatrix}\right] = $$
$$= -\frac{g^2}{16\pi^2}\begin{pmatrix} 8 & 0 \\ 0 & -4 \end{pmatrix} \tag{5.21}$$

Let us find now the linear combinations of O^1 and O^2 that correspond to the eigenvalues of the matrix in eq(5.18). We are seeking c_i and d_i such that:

$$O^+ = \sum_{i=1,2} c_i O_i \qquad \text{and} \qquad O^- = \sum_{i=1,2} d_i O_i \tag{5.22}$$

where O^+ and O^- are the eigenvectors. Thus we have:

$$O^+_{\text{inserted}} = \sum_{i=1,2} c_i O_{i\,\text{inserted}} = \sum_{i=1,2} c_i \sum_{j=1,2} A_{ij} O_j \tag{5.23}$$

where A_{ij} is the matrix in eq(5.18). Eq(5.23) can be rewritten:

$$O^+_{\text{Inserted}} = \sum_{j=1,2}\left[\sum_{i=1,2} c_i A_{ij}\right] O_j = \lambda O^+ = \lambda \sum_{j=1,2} c_j O_j \tag{5.24}$$

Thus we get:

$$\sum_{i=1,2} c_i A_{ij} = \lambda c_j \tag{5.25}$$

Eq(5.25) tells us that c_i are the eigenvectors of the transpose of the matrix A_{ij} (λ is one of the two eigenvalues of A_{ij}. Note that both A_{ij} and $\tilde{A}_{ij}$ have the same eigenvalues). Thus we get:

$$\text{eigenvalue} = +\frac{40}{3} \qquad \text{eigenvector} \quad O^- = O_1 - 3O_2 \tag{5.26a}$$

eigenvalue $= +\frac{4}{3}$ eigenvector $O^{+} = O_1 + \frac{3}{2} O_2$ (5.26b)

These are the eigenvectors of the matrix γ_{ij} but also of the matrix $(\gamma_0)_{ij}$ eq(5.21).

We shall prove now that O^{+} operator induces transitions which are mixture of $\Delta I = \frac{1}{2}$ and $\frac{3}{2}$ while O^{-} induces transitions with $\Delta I = \frac{1}{2}$. Thus we want O^{-} operator to be enhanced while O^{+} to be suppressed.

Using the Fierz identities (see Appendix A) and writing $O^{1} = (\bar{\lambda} I p)(\bar{p} I n)$, $O^{2} = (\bar{\lambda} t p)(\bar{p} t n)$ (suppressing spinor indices) we get:

$$O^{2} = (\bar{\lambda} t p)(\bar{p} t n) = \frac{1}{2}(\bar{\lambda} I n)(\bar{p} I p) - \frac{1}{6}(\bar{\lambda} I p)(\bar{p} I n) \tag{5.27}$$

Then simple manipulations give:

$$\begin{aligned} O^{-} = O_1 - 3O_2 &= (\bar{\lambda} I p)(\bar{p} I n) - 3(\bar{\lambda} t p)(\bar{p} t n) = \\ &= \frac{3}{2}(\bar{\lambda} I p)(\bar{p} I n) - \frac{3}{2}(\bar{\lambda} I n)(\bar{p} I p) = \frac{3}{2}\hat{O}^{-} \end{aligned} \tag{5.28}$$

$$\begin{aligned} O^{+} = O_1 + \frac{3}{2} O_2 &= (\bar{\lambda} I p)(\bar{p} I n) + \frac{3}{2}(\bar{\lambda} t p)(\bar{p} t n) = \\ &= \frac{3}{4}(\bar{\lambda} I p)(\bar{p} I n) + \frac{3}{4}(\bar{\lambda} I n)(\bar{p} I p) = \frac{3}{4}\hat{O}^{+} \end{aligned} \tag{5.29}$$

where we have defined:

$$\begin{aligned} \hat{O}^{-} &= (\bar{\lambda} I p)(\bar{p} I n) - (\bar{\lambda} I n)(\bar{p} I p) \\ \hat{O}^{+} &= (\bar{\lambda} I p)(\bar{p} I n) + (\bar{\lambda} I n)(\bar{p} I p) \end{aligned} \tag{5.30}$$

It is easy to see that $\hat{O}^{-}$ is antisymmetric in the exchange of p and n, while $\hat{O}^{+}$ is symmetric. Thus the first operator has the p and n in a $I=0$ state while the second operator in a $I=1$ state. Since λ is isosinglet, it follows that $\hat{O}^{-}$ induces transitions with $\Delta I = \frac{1}{2}$ (combination of $T=0$ and $T=\frac{1}{2}$) while $\hat{O}^{+}$ induces transitions with a mixture of $\Delta I = \frac{1}{2}$ and $\frac{3}{2}$ (combination of $T=1$ and $T=\frac{1}{2}$). Thus eq(5.28) and eq(5.29) show that the proof has been completed.

As we have already seen in Chapter 4, the coefficients of Wilson expansion obey a renormalization group equation (eq(4.14)), namely:

$$\tilde{C}(q^2/\mu^2, g) = \tilde{C}(1, \bar{g}) \quad \exp\left\{T\left[-\int_0^t dt' \, \gamma(\bar{g}(g, t'))\right]\right\} \tag{5.31}$$

where we are working in a mass independent renormalization scheme. Bearing in mind the footnote in page 39 and that we only consider the first term in the γ -function:

$$\gamma = \gamma_0 g^2 + O(g^4) \tag{5.32}$$

we can drop the t-ordering. Thus eq(5.31) can be worked out completely independently for the two eigenvalues of the γ -function matrix. So the integral in the exponent in eq(5.31) reads:

$$-\int_0^t dt' \, \gamma_0^i \bar{g}^2 \qquad i = 1,2 \tag{5.33}$$

where $\gamma_0^1 = -8$ and $\gamma_0^2 = 4$.

Now returning to eq(3.10) it is easy to solve the differential equation for $\beta(\bar{g}) = \beta_0' \bar{g}^3 + O(\bar{g}^5)$. The solution is:

$$\bar{g}^2 = g^2 [1 - g^2 2\beta_0' t]^{-1} \tag{5.34}$$

where t is defined in eq(3.12). Thus eq(5.33) gives:

$$-\int_0^t dt' \, \gamma_0^i \bar{g}^2 = -\frac{\gamma_0^i g^2}{16\pi^2} \int_0^t dt' \, [1 - g^2 2\beta_0' t]^{-1} =$$

$$= -\frac{\gamma_0^i g^2}{16\pi^2} \frac{1}{g^2 2\beta_0'} \ln [1 - g^2 2\beta_0' t] = \ln [1 - g^2 2\beta_0' t]^{\gamma_0^i / 2\beta_0} \tag{5.35}$$

where we define $\beta_0 = \beta_0' 16\pi^2$. Now β_0 has the value(4):

$$\beta_0 = -\frac{25}{3} \tag{5.36}$$

Therefore the exponent in eq(5.35) have the values:

$$0.48 \qquad \text{and} \qquad -0.24 \tag{5.37}$$

where the positive corresponds to the pure $\Delta I = \frac{1}{2}$ operator and the negative to the operator having $\Delta I = \frac{1}{2}$ and $\frac{3}{2}$ mixture.

Finally the first term in eq(5.1) can be written:

$$G/\sqrt{2} \quad M_W^2 \int d^4k \frac{1}{k^2 - M_W^2} C(k^2/\mu^2, g) \langle | O^{\pm} | \rangle \tag{5.38}$$

Now $\tilde{C}(k^2/\mu^2)$ can be written :

$$\tilde{C}(k^2/\mu^2, g) = \tilde{C}(1, \bar{g}) \left[1 + \frac{1}{16\pi^2} g^2 \frac{25}{3} \ln k^2/\mu^2\right]^{\alpha}$$

where $\alpha = 0.48$ and -0.24, and $\tilde{C}(1,\bar{g}) = \frac{1}{\sqrt{2}} \frac{1}{k^4}$ and $\sqrt{\frac{2}{5}} \frac{1}{k^4}$ for 0^- and 0^+ respectively(*). The matrix elements of the operators 0 do not depend on k. Thus we have:

$$G/\sqrt{2}\ M_W^2 \int d^4k \frac{1}{k^2 - M_W^2} \left[1 + \frac{1}{16\pi^2} g^2 \frac{25}{3} \ln k^2/\mu^2\right]^{\alpha} \frac{1}{k^4} \propto$$

$$G/\sqrt{2}\ M_W^2 \int dk^2 \frac{1}{k^2} \frac{1}{k^2 - M_W^2} \left[1 + \frac{1}{16\pi^2} g^2 \frac{25}{3} \ln k^2/\mu^2\right]^{\alpha} =$$

$$G/\sqrt{2} \left[1 + \frac{1}{16\pi^2} g^2 \frac{25}{3} \ln M_W^2/\mu^2\right]^{\alpha} \qquad (5.39)$$

since the main contribution to the integral comes from $k^2 \sim M_W^2$.

Thus we see that the enhancement is of the order:

$$\frac{1/\sqrt{2} \left[1 + \frac{1}{16\pi^2} g^2 \frac{25}{3} \ln M_W^2/\mu^2\right]^{0.48}}{\sqrt{2/5} \left[1 + \frac{1}{16\pi^2} g^2 \frac{25}{3} \ln M_W^2/\mu^2\right]^{-0.24}} \simeq 5 \qquad (5.40)$$

taking $M_W \simeq 100$ GeV and $\mu \simeq 1$ GeV (at this subtruction point $g^2/4\pi \simeq 1$). Therefore the relative enhancement is not itself sufficient to account for the observed small (around 5%) violation of the $\Delta I = 1/2$ rule.

(*) While the canonical dimension of $C(x)$ is zero (for operators with dimension 6), the Fourier transform $\tilde{C}(k)$ has canonical dimension (-4) $C(k) \sim \frac{1}{k^4}$.

Second-loop calculations

The next to leading order corrections to eq(5.31) are given by calculating the g^4-order term of eq(5.32) for the γ-function. So we need the g -order term of the C_i, and the residue of the single pole of the wavefunction renormalization constant series (see eq(3.13) and eq(3.20)).

The procedure is the same as in the one-loop. Using dimensional regularization we are trying to find the infinite part of the diagrams. Nevertheless noumerous difficulties arise in the second-loop calculations. Namely a) over a hundred diagrams must be considered, which fortunately are reduced to 34 different types only(!)

b) momentum integration must be done with great care. We must avoid combining propagators belonging to different loops, because in this case a possible transfer of the ultraviolet divergence to the Feynman-parameter integral could appear.

c) γ-algebra gave us a big trouble, which arose from the need of working in n dimensions.

d) the Feynman-parameter integral was calculated by computer. The accuracy of this calculation was tested with success, in several ways (for example see the case of "bad currents" later in this chapter) and

e) t-algebra was just a little bit more tedious than in the first-loop case. But let us be more specific about the above five points.

a) All the diagrams contributing to the second-loop order (two gluon exchange) correction to the operator O_1 are listed below (Fig(5.2)). The same diagrams must be considered with the insertion of the operator O_2. But as we shall see later on the final calculation will be done in a slightly different way from the one-loop calculations.

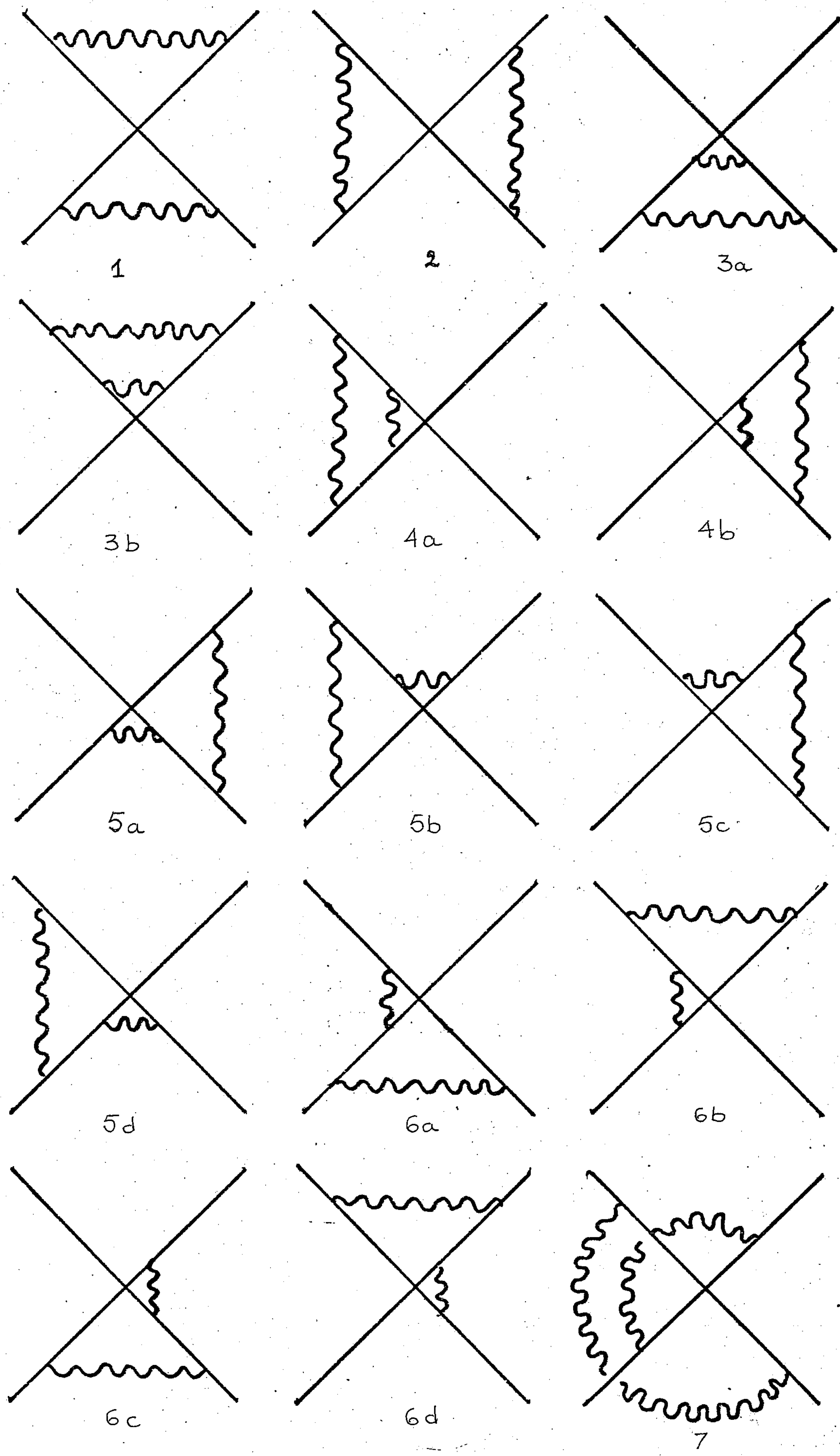
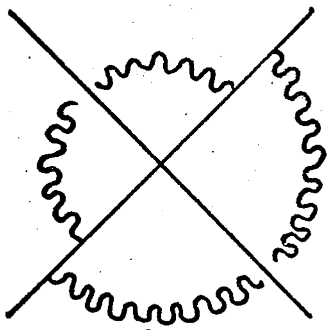


Fig (5.2)



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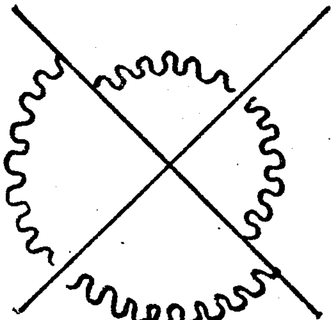

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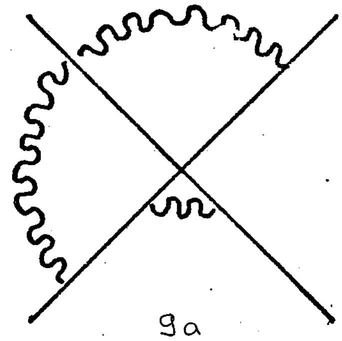

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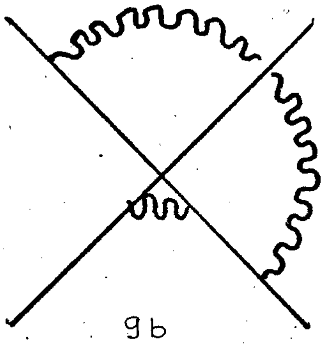

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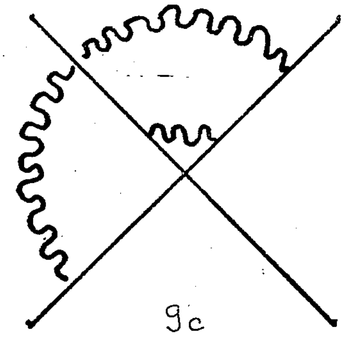

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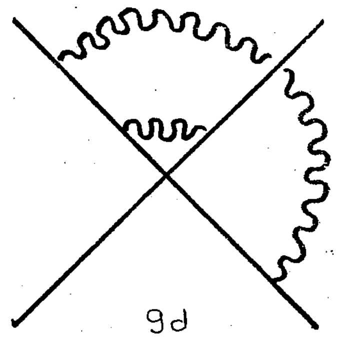

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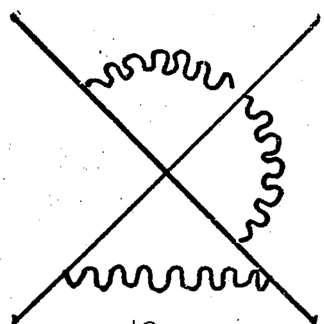

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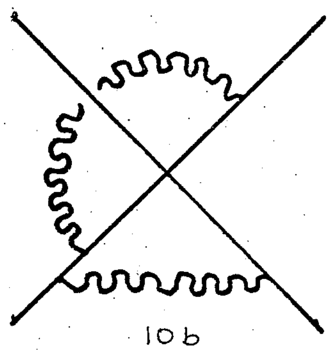

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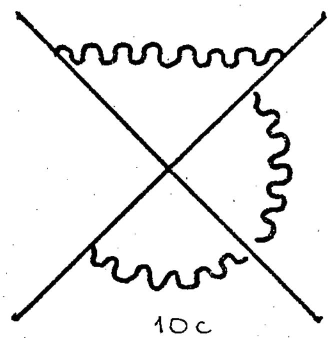

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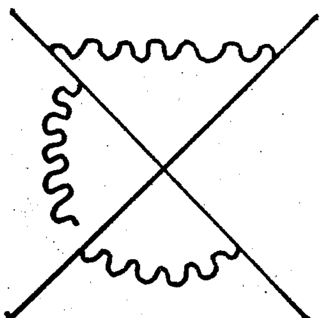

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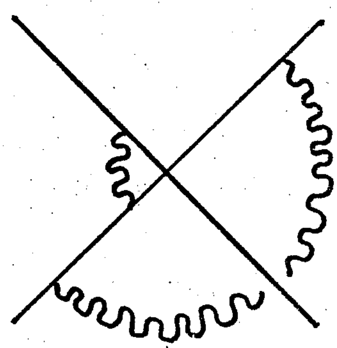

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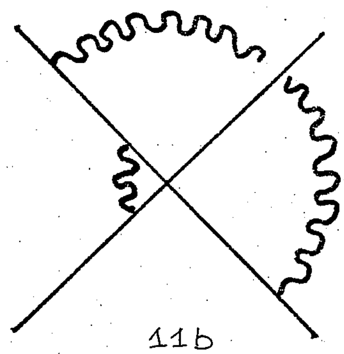

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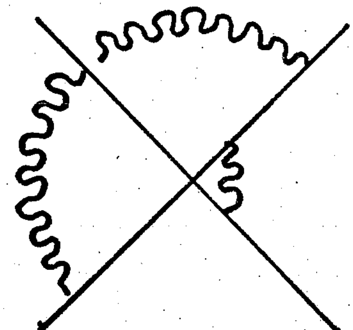

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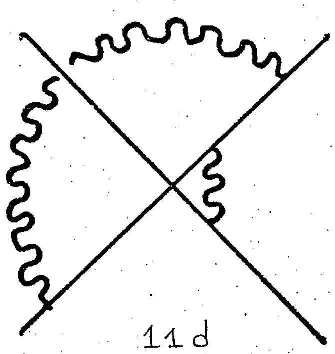

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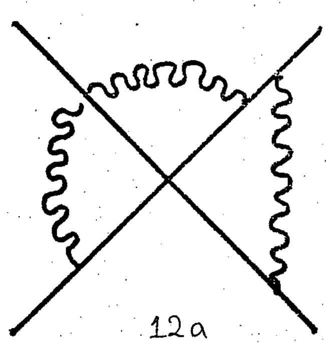

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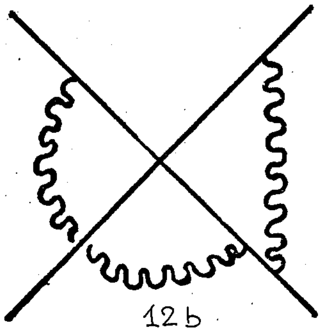

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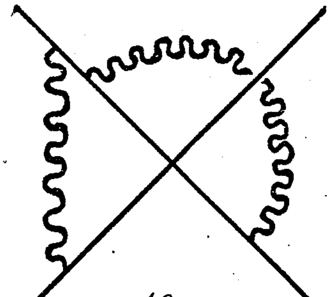

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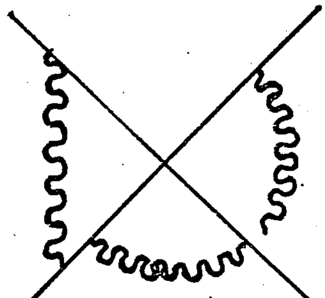

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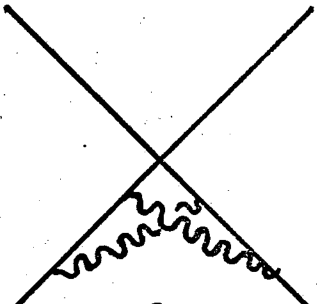

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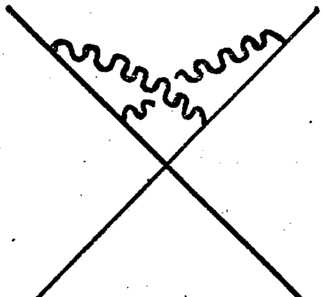

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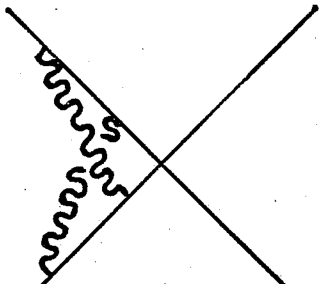

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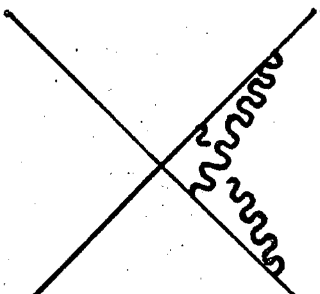

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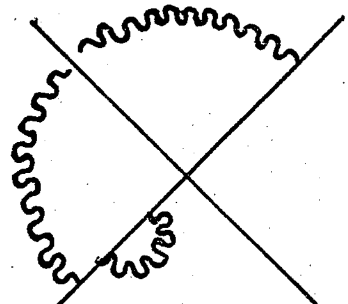

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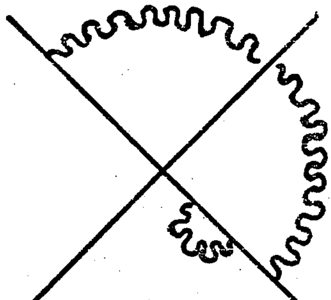

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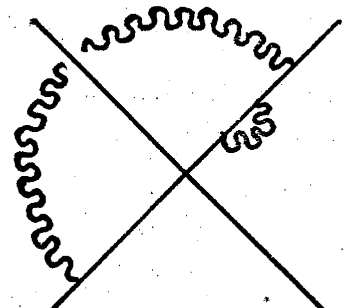

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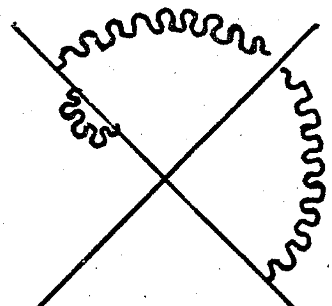

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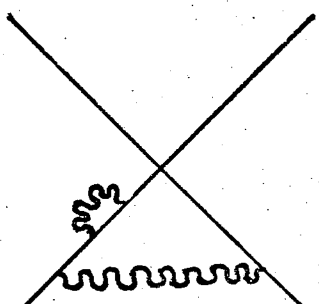

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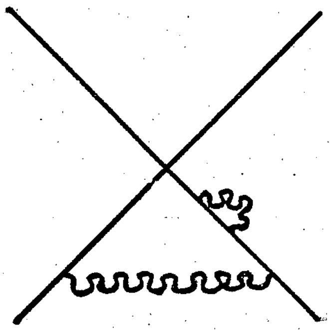

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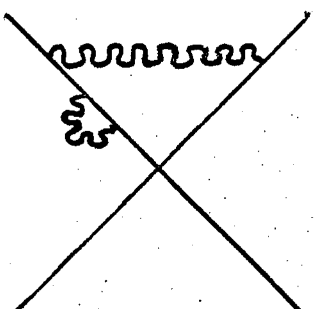

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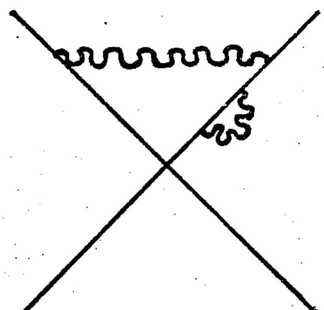

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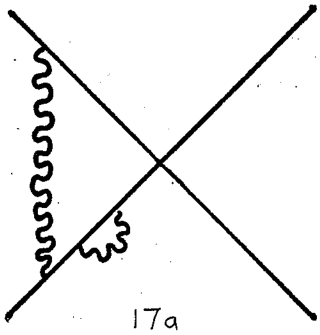
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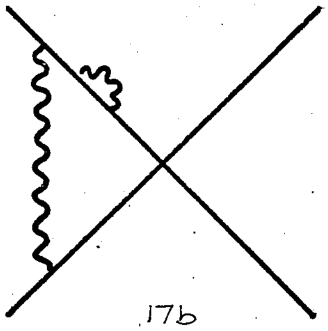
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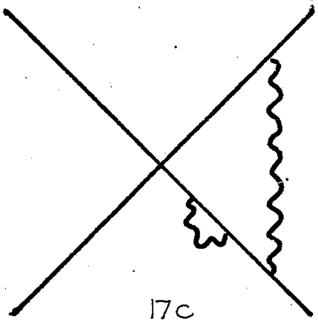
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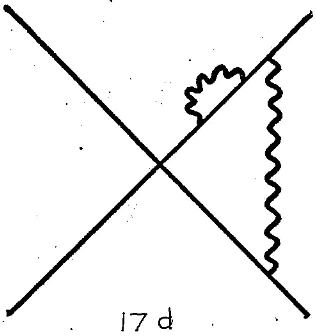
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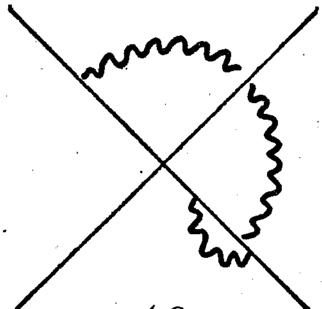
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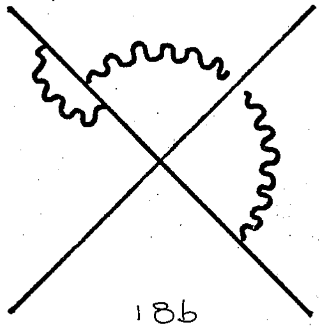
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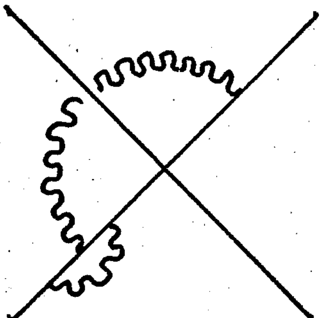
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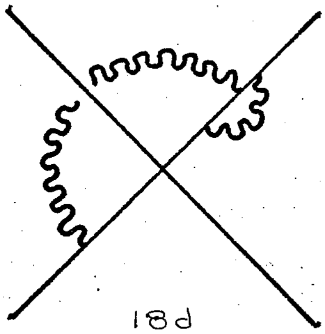
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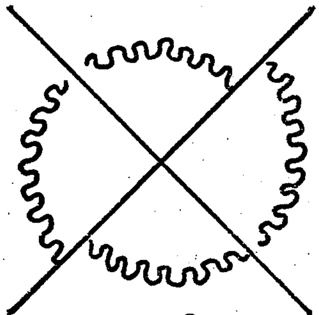
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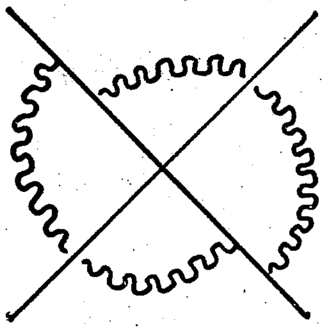
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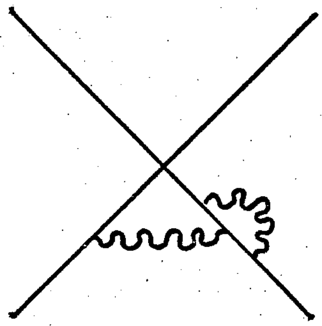
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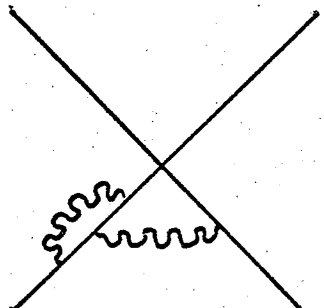
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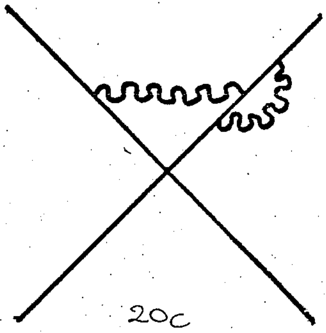
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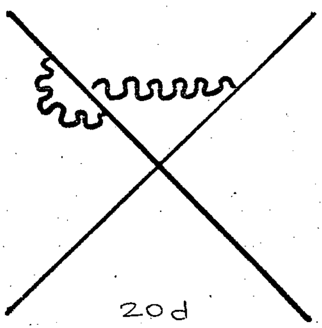
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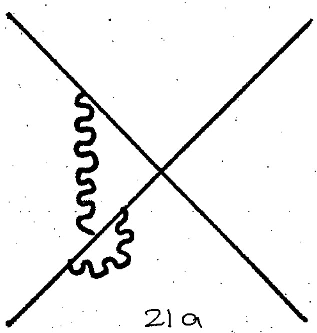
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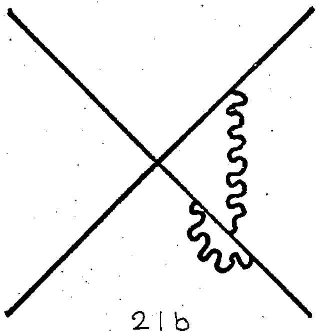

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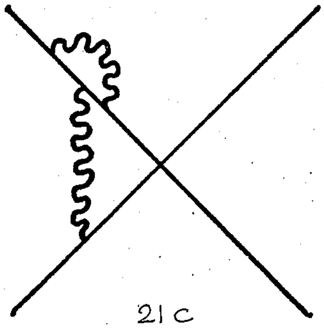

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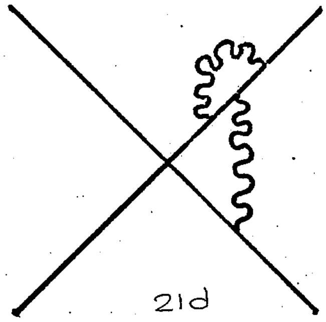

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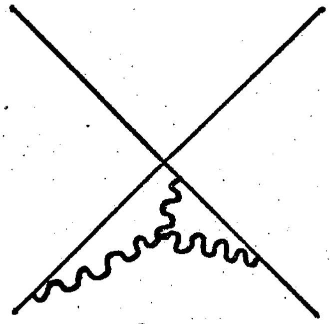

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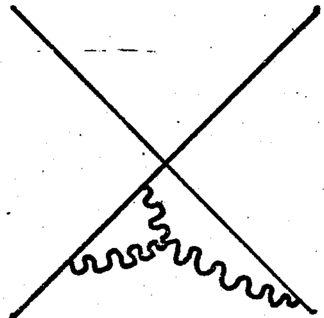

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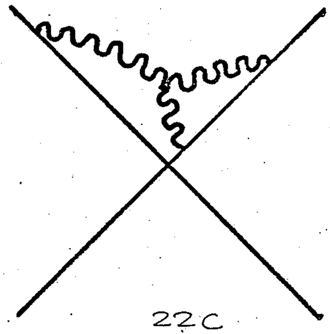

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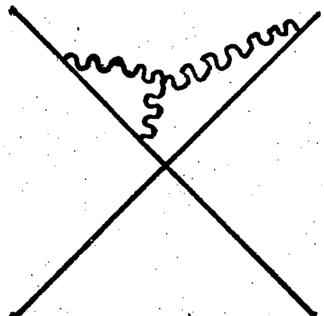

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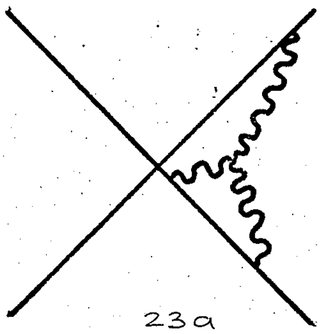

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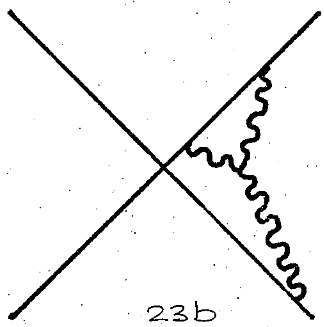

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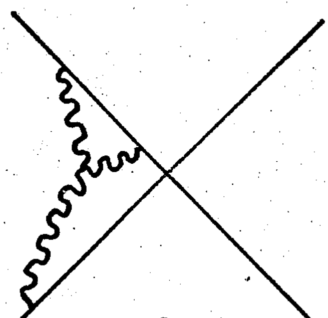

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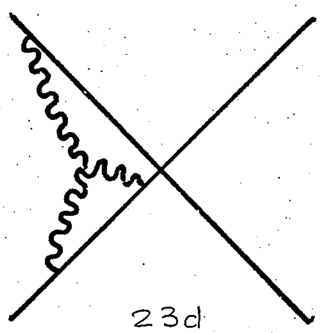

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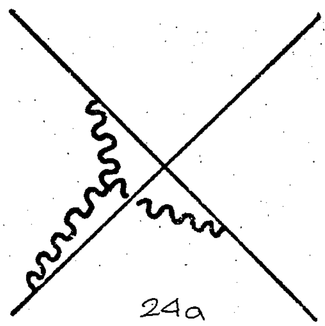

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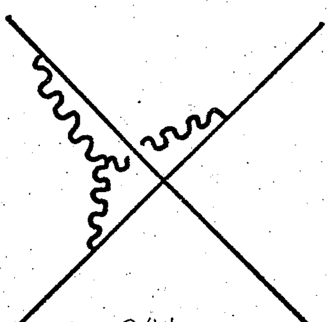

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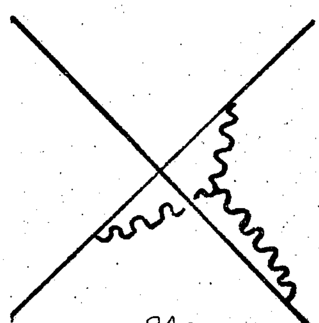

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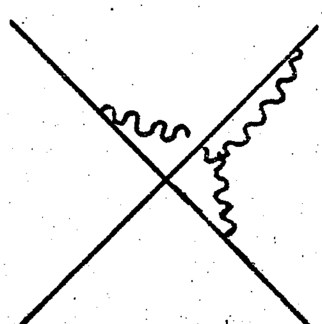

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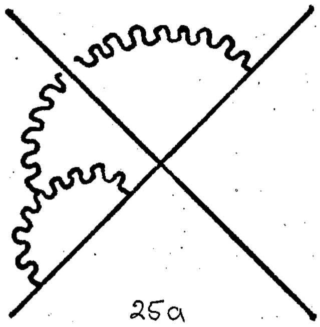
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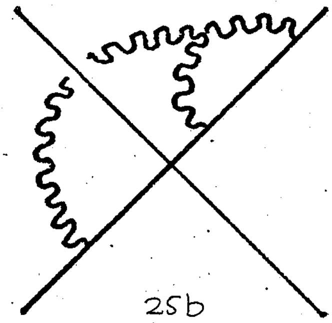
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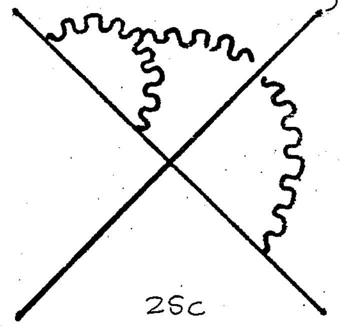
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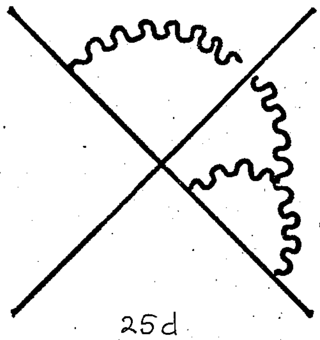
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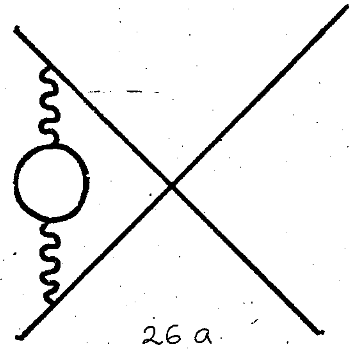
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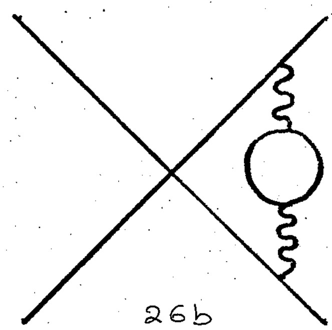
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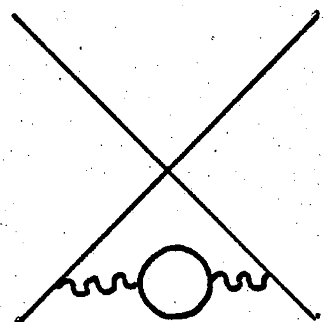
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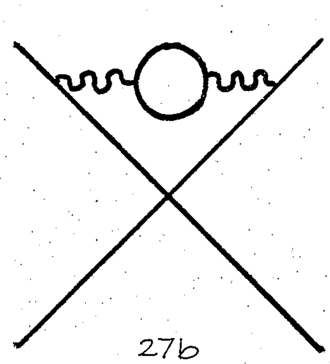
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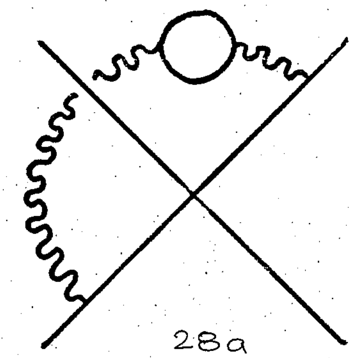
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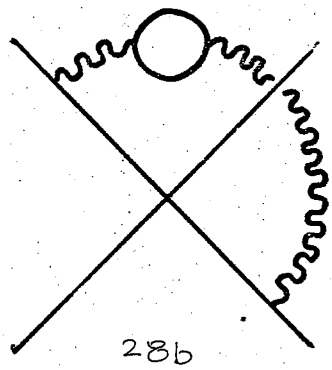
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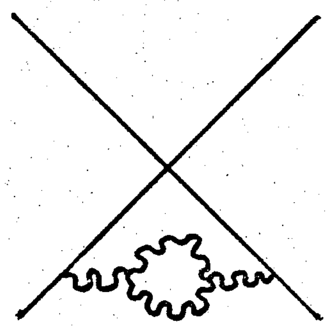
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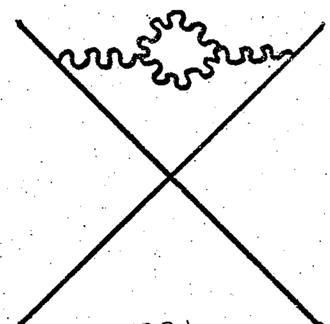
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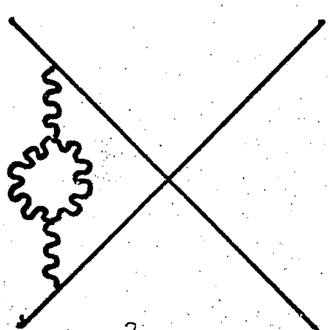
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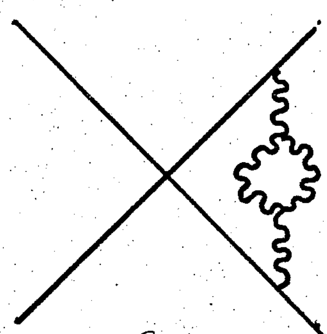
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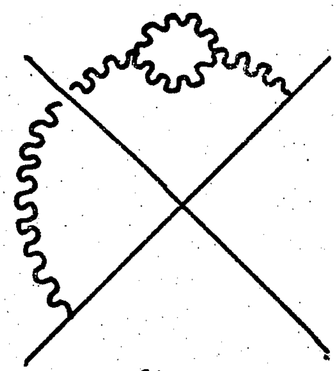
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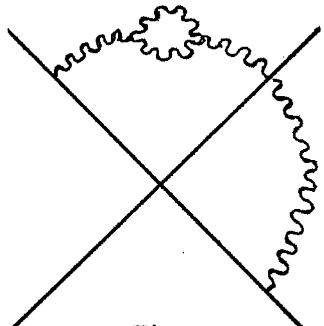

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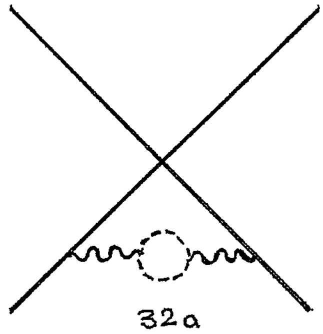

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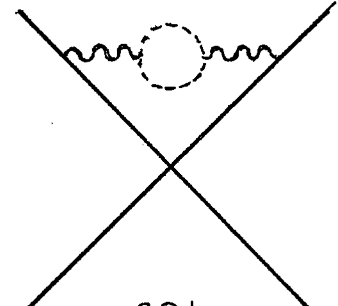

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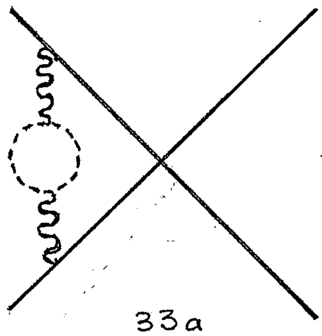

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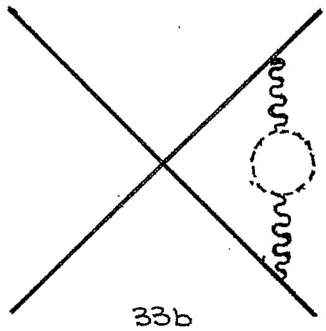

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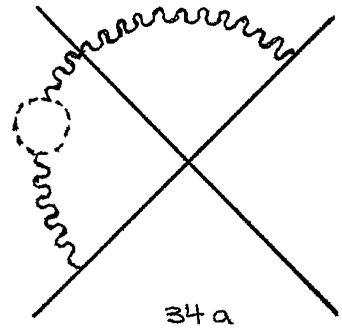

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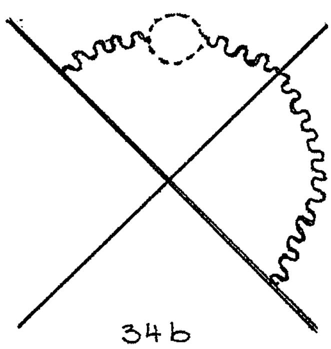


34b

b) As in the one-loop calculations we are considering gluons and fermions having the Leibbrand mass depending on the dimensions of the space. This simplify greatly the calculations since we need at most two Feynman parameters to combine all propagators. In a second-loop calculation a $\frac{1}{\epsilon^2}$ -term is expected as well as a $\frac{1}{\epsilon}$ -term. Bearing in mind that all the diagrams are overall logerithmically divergent, $\frac{1}{\epsilon^2}$ -terms are coming from diagrams having a divergent subintegral. $\frac{1}{\epsilon}$ -terms always come from the expansion of $\Gamma(4-n)$ ($\epsilon \equiv n-4$), while $\frac{1}{\epsilon^2}$ -terms come from the expansion of $\Gamma(4-n)\,\Gamma(2-n/2)$. Diagrams giving $\frac{1}{\epsilon^2}$ -terms have also "bad" terms of the form $(\ln m/\epsilon)$ which, as we noted in the Chapter on Dimensional Regularization, cannot be renormalized by adding counter terms in the Lagrangian. But it is shown diagram by diagram that these terms vanish when we subtract the one-loop counter terms (one-loop diagrams with the insertion of the infinite part from the one-loop calculation. See again the Chapter on Dimensional Regularization) The counter terms are listed in Appendix J. The last thing we want to mention on the subject of momentum integration is the following. Consider a double-loop integral with respect to momenta k and p and suppose that one of the subintegrals is logarithmically divergent. By performing the integration which contains this divergence we are left with an integral of the form (neglecting overall factors) :

$$\int_0^1 d\alpha_1 \int d^n k \, F(k,m,\alpha_1) \, \frac{\Gamma(2-n/2)}{[k^2 - D(m,\alpha_1)]^{2-n/2}} \qquad (5.41)$$

where D is a function of m and the Feynman parameter α_1 while F is a function of m, k ,and α_1. It should be noted that in order to perform the first integration we only need one Feynman parameter. This is not true in general but a special feature of our way of calculation. Now we have to combine the propagators which are in the function $F(k,m,\alpha_1)$ with the $[k^2 - D(m,\alpha_1)]^{2-n/2}$ using Feynman parameter. The general formula:

$$\frac{1}{\alpha^a \beta^b} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 dx \frac{x^{a-1}(1-x)^{b-1}}{[\alpha x+(1-x)\beta]^{a+b}} \qquad (5.42)$$

is valid when $a,b>0$. So in order to apply the above formula we must write eq(5.41):

$$\int_0^1 d\alpha_1 \int d^n K \; F(K,m,\alpha_1) \; \frac{\Gamma(2-n/2)\,[K^2-D(m,\alpha_1)]}{[K^2-D(m,\alpha_1)]^{3-n/2}} \qquad (5.43)$$

and then combine the propagators in $F(K,m,\alpha_1)$ with the $[K^2-D(m,\alpha_1)]^{3-n/2}$.

c) We saw in eqs(5.12)-(5.16) that as far as the one-loop calculations are concerned we can work out the gammology in 4 dimensions. In the tow-loop calculation this is not true any more. The momentum integral has the form, after integration :

$$\text{integral} \quad I = \frac{a}{\epsilon^2} + \frac{b}{\epsilon} + \gamma + O(\epsilon)$$

So the diagram D becomes, using eq(5.13)

$$D = \left[\frac{a}{\epsilon^2} + \frac{b}{\epsilon} + \gamma + O(\epsilon)\right]\left[A_0 + A_1\epsilon + A_2\epsilon^2 + O(\epsilon^3)\right] = \qquad (5.44)$$

$$= \frac{A_0 a}{\epsilon^2} + \frac{bA_0 + aA_1}{\epsilon} + (\gamma A_0 + A_1 b + A_2 a) + O(\epsilon)$$

Thus we see that the residue over the single pole depends on A_1 as well as on A_0. The above consideration tells us that the γ-algebra must be done in n-dimensions and at the end take the limit $n\to 4$ ($\epsilon\equiv n-4\to 0$)

Now in n dimensions the expression:

$$\gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_K} \qquad K\leqslant n-2 \qquad (5.45)$$

is an element of the basis of the Clifford algebra, while expressions of the form :

$$\gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_\ell} \qquad \ell \geqslant n-1$$

can be written as a linear combination of expression in eq(5.45). So two-gluon-exchange correction to the operators O_1 and O_2, which the following Lorentz structure (colour structure and external spinors suppressed) :

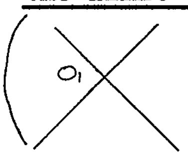
$$(\gamma^\mu)_{\alpha\beta}(\gamma_\mu)_{\gamma\delta}$$

give a linear combination of new effective operators of the general form :

$$(\gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_K})_{\alpha\beta}\,(\gamma_{\alpha_1}\gamma_{\alpha_2}\cdots\gamma_{\alpha_K})_{\gamma\delta}\qquad K\leq n-2 \qquad (5.46)$$

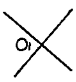
This procedure stops, in principle, when correction to one of the new effective operators gives an expression of the form (5.46) with $K > n-2$. Of course the above procedure is not practical to use.

The way we used to overcome(but not at all sure to solve) the above difficulty is the following. Suppose the calculations had been made and the limit $n\to 4$ had been taken. Then we get :

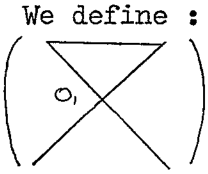
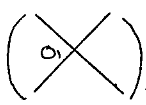


$$\Big(\quad\Big)_{\substack{\alpha\beta\,\gamma\delta\\ ab\;cd}} = A\left((\gamma_\mu)_{\alpha\beta}\,I_{ab}\right)\left((\gamma^\mu)_{\gamma\delta}\,I_{cd}\right) + B\left((\gamma_\mu)_{\alpha\beta}\,t^i_{ab}\right)\left((\gamma^\mu)_{\gamma\delta}\,t^i_{cd}\right) \qquad (5.47)$$

$$[\gamma^\mu \equiv \gamma^\mu(1-\gamma_5)]$$

where  is the two-gluon-exchange correction diagram with the insertion of the operator O_1 ,$\alpha\beta$, $\gamma\delta$ are the spinor indices and ab ,cd the colour indices. A and B are constants which could be infinite.

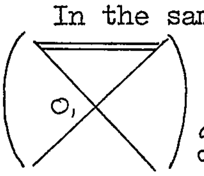
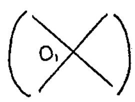
We define :

$$\Big(\quad\Big)_{\substack{\alpha\delta\\ ad}} \equiv \Big(\quad\Big)_{\substack{\alpha\beta\,\gamma\delta\\ ab\;cd}}\,\gamma^\nu_{\beta\gamma}\,I_{bc} = A(-2)\gamma^\nu_{\alpha\delta}I_{ab} + B(-2)\,\gamma^\nu_{\alpha\delta}\tfrac{4}{3}\,I_{ad} \qquad (5.48)$$

where we have used that $\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$ and that $t^i t^i = \frac{4}{3}I$.

In the same way we can define :

$$\Big(\quad\Big)_{\substack{\alpha\delta\\ ab}} \equiv \Big(\quad\Big)_{\substack{\alpha\beta\,\gamma\delta\\ ab\;cd}}\,\gamma^\nu_{\beta\gamma}\,t^j_{bc} = A(-2)\gamma^\nu_{\alpha\delta}\,t^j_{ad} + B(-2)\gamma^\nu_{\alpha\delta}\left(-\tfrac{1}{6}\right)t^j_{ad} \qquad (5.49)$$

where we have used that $t^i t^j t^i = (-\frac{1}{6})t^j$.

Eq(5.48) and eq(5.49) can be solved for A and B which will be expressed in terms of  and  . But the γ-algebra in these diagrams can be done in n-dimensions since all γ-matrices are in "the same line". Thus A and B may be defined in n-dimensions using eq(5.48) and eq(5.49). A computer program has been used to contract γ-matrices in n-dimensions and give the first and second term in the expansion with respect to $\epsilon = n-4$ (the program is called MACSYMA -MIT-. I would like to thank A.D.Kennedy for helping me with this program).

Another feature of the way we did the second-loop calculation is the "bad currents". Let us explain what we mean by this. Terms which are proportional to the mass m in the momentum integral, give rise (after the γ-algebra manipulations) to terms which have "wrong" chiral properties:

$$[\gamma_\mu\gamma_\nu(1\pm\gamma_5)]_{\alpha\beta}\,[\gamma^\mu\gamma^\nu(1\pm\gamma_5)]_{\gamma\delta}$$

These terms are residues over single pole ($1/\epsilon$)only and they vanish diagram by diagram when we subtract the counter terms. This fact has been checked in every diagram considered in our calculation.

d) There seem to be a limited number of Feynman parameter integrals needed. The table of integrals and their numerical values are given in Appendix I. Feynman parameter integrals coming from n-dimensional momentum integration depend also on n. Therefore in diagrams having divergent subintegrals, the Feynman parameter integral must be expanded with respect to ϵ . These Feynman parameter integrals (having a $\ln$) have also been listed in Appendix I.

e) Formulae for t-algebra can be found in Appendix A.

We will follow here, as we have already noted, a slightly different method from the one used in the one-loop diagrams. Insertion of the operator O_1 in every diagram considered in the two-gluon correction has the form (as far as colour algebra is concerned) :

$$D_{O_1} = \alpha(I)(I) + \beta(t)(t) \qquad (5.50)$$

The procedure we followed in the one-loop calculation (diagonalization of the matrix Z_{ij} in order to find the eigenvectors which are proportional to $\hat{O}^{\pm}=(I)(I)\pm(I)\times(I)$ (*))tells us that if we introduce the operators $\hat{O}^{\pm}$ then these operators will be multiplicatively renormalized (the mixing will disappear) :

$$D_{\hat{O}^{\pm}} = \alpha^{\pm}\hat{O}^{\pm} = \alpha^{\pm}\left[(I)(I)\pm(I)\times(I)\right] \qquad (5.51)$$

This is very easy to prove since (see Apendix A) :

$$(t)(t)+(t)\times(t)=\tfrac{1}{3}\hat{O}^{+} \quad , \quad (t)(t)-(t)\times(t)=-\tfrac{2}{3}\hat{O}^{-} \qquad (5.52)$$

So using eqs(5.50,51,52) we get :

$$\begin{aligned} D_{\hat{O}^{+}} &= D_{(I)(I)} + D_{(I)\times(I)} = \\ &= \left[\alpha(I)(I)+\beta(t)(t)\right]+\left[\alpha(I)\times(I)+\beta(t)\times(t)\right] = \\ &= \alpha\left[(I)(I)+(I)\times(I)\right]+\beta\left[(t)(t)+(t)\times(t)\right] \\ &= \alpha\,\hat{O}^{+}+\beta\,\tfrac{1}{3}\,\hat{O}^{+} = \\ &= (\alpha+\beta/3)\,\hat{O}^{+} \equiv \alpha^{+}\hat{O}^{+} \end{aligned} \qquad (5.53)$$

Similarly we get :

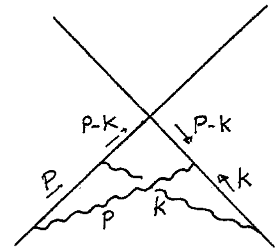
$$D_{\hat{O}^{-}} = (\alpha-\tfrac{2\beta}{3})\,\hat{O}^{-} \equiv \alpha^{-}\hat{O}^{-} \qquad (5.54)$$

Therefore the procedure we follow here is the following: we evaluate α and β (eq(5.54)) inserting the operator $O_1=(I)(I)$. Then by using the eqs(5.53,54) we calculate α^{+} and α^{-}. Addition of α^{+} for all diagrams concerned gives the desired eigenvalue for the operator $\hat{O}^{+}$ while addition of α^{-} gives the eigenvalue for $\hat{O}^{-}$.

As an example we find the contribution to the anomalous dimension from diagram 13a :

(*) $(I)(I)$ means $(\bar{n}Ip)(\bar{p}In)$ while $(I)\times(I)$ means the Fierz rearanged expression $(\bar{n}In)(\bar{p}Ip)$.

Applying Feynman rules to diagram 13a we get :



$$-g^4 \int \frac{d^n k}{(2\pi)^n}\,\frac{d^n p}{(2\pi)^n} \left(\bar{\lambda}\gamma^\sigma(1-\gamma_5)\frac{(\not p-\not K+m)}{((p-k)^2-m^2)}\gamma^\lambda \frac{\not p+m}{(p^2-m^2)}\gamma^\tau p\right)$$

$$\left(\bar p\, \gamma_\sigma(1-\gamma_5)\frac{\not p-\not K+m}{(p-k)^2-m^2}\gamma_\tau\frac{\not K+m}{(k^2-m^2)}\gamma_\lambda n\right)\frac{1}{(p^2-m^2)(k^2-m^2)} \qquad \text{(Ex.1)}$$

Isolating the momenta integral we have :

$$\int \frac{d^n k}{(2\pi)^n}\,\frac{d^n p}{(2\pi)^n}\;\frac{(P_\mu+m)(P_\phi-k_\phi+m)(-p_\omega+k_\omega+m)(k_\nu+m)}{(p^2-m^2)^2\,(k^2-m^2)^2\,[(p-k)^2-m^2]^2} \qquad \text{(Ex.2)}$$

It is easy to see that this diagram does not have any divergent subintegral. Therefore the only terms in eq(Ex.2)which contribute are:

$$-p_\phi p_\omega p_\mu k_\nu + p_\phi p_\mu k_\omega k_\nu + p_\omega p_\mu k_\phi k_\nu - p_\mu k_\phi k_\omega k_\nu\,. \qquad \text{(Ex.3)}$$

We apply partial $(\alpha\beta\gamma)$ (see Chapter 2) to demonstrate the method:

$$\int \frac{d^n k}{(2\pi)^n}\,\frac{d^n p}{(2\pi)^n}\,(-p_\mu p_\phi p_\omega k_\nu)\,\frac{1}{2n+4-2(2+2+2)}\,.$$

$$\left\{ \frac{2\cdot 2\; m^2}{(p^2-m^2)^3\,(k^2-m^2)^3\,[(p-k)^2-m^2]^2} + \frac{2\cdot 2\; m^2}{(p^2-m^2)^2\,(k^2-m^2)^3\,[(p-k)^2-m^2]} + \qquad \text{(Ex.4)}\right.$$

$$\left. + \frac{2\cdot 2\; m^2}{(p^2-m^2)^2\,(k^2-m^2)^2\,[(p-k)^2-m^2]^3}\right\}$$

<u>First term</u>

$$\frac{1}{(k^2-m^2)^2\,((p-k)^2-m^2)^2} = \frac{\Gamma(4)}{\Gamma(2)\,\Gamma(2)}\int_0^1 dx\,\frac{(1-x)\,x}{[k^2-m^2-k^2x+m^2x+p^2x+k^2x-2pkx-m^2x]^4}$$

$$= \frac{\Gamma(4)}{\Gamma(2)\,\Gamma(2)}\int_0^1 dx\,\frac{(1-x)\,x}{[k^2-2pkx+p^2x-m^2]^4}$$

and $\dfrac{k_\nu}{[k^2-2pkx+p^2x-m^2]^4}$ gives when integrated.

$$\frac{in^{\frac{n}{2}}}{(2\pi)^n} \quad \frac{1}{\Gamma(4)} \quad \frac{\Gamma(4-n/2) \quad p_\nu x}{[p^2x-m^2-p^2x^2]^{4-n/2}}$$

Now

$$\frac{1}{[p^2x-m^2-p^2x^2]^{4-n/2}\ (p^2-m^2)^3} = \frac{\Gamma(7-n/2)}{\Gamma(3)\Gamma(4-n/2)} \int_0^1 dy \ \frac{y^{3-n/2}(1-y)^2}{[p^2(xy-x^2y+1-y)-m^2]^{7-n/2}}$$

and integrating with respect to p we get

$$-\frac{in^{n/2}}{(2\pi)^n}\frac{1}{\Gamma(7-n/2)} \quad \frac{\Gamma(5-n)\ \frac{1}{4}\ (g_{\mu\nu}g_{\phi\omega}+g_{\mu\phi}g_{\nu\omega}+g_{\mu\omega}g_{\phi\nu})}{[(1-x)xy+1-y]^{7-n/2}\left[\frac{-m^2}{x(1-x)y+1-y}\right]^{5-n}}$$

So from the first term in eq(Ex.4) we get :

$$-\left[\frac{in^{n/2}}{(2\pi)^n}\right]^2 \frac{(-m^2)^{n-5}\ 4\,m^2}{\Gamma(2)\Gamma(2)\Gamma(3)} \quad \Gamma(5-n)\ \frac{1}{4}\ (g_{\mu\nu}g_{\phi\omega}+g_{\mu\phi}g_{\nu\omega}+g_{\mu\omega}g_{\phi\nu})\ .$$

$$.\int_0^1 dx dy \ \frac{(1-x)\ x^2\ y^{3-n/2}\ (1-y)^2}{[(1-x)\ xy+1-y]^{2+n/2}}$$

Of course this term is multiplied by $\frac{1}{2n+4-12} = \frac{1}{2(n-4)} = \frac{1}{2\epsilon}$

Second term Similarly we get :

$$-\left[\frac{in^{n/2}}{(2\pi)^n}\right]^2 \quad \frac{(-m^2)^{n-5}\ 4\,m^2}{\Gamma(3)\Gamma(2)\Gamma(2)} \quad \Gamma(5-n)\ \frac{1}{4}\ (g_{\mu\nu}g_{\phi\omega}+g_{\mu\phi}g_{\nu\omega}+g_{\mu\omega}g_{\nu\phi})\ .$$

$$.\int_0^1 dx dy \ \frac{(1-x)^2x^2\ y^{4-n/2}(1-y)}{[(1-x)xy+1-y]^{2+n/2}}$$

Third term

$$-\left[\frac{in^{n/2}}{(2\pi)^n}\right]^2 \quad \frac{(-m^2)^{n-5}\ 4\,m^2}{\Gamma(2)\Gamma(2)\Gamma(3)} \quad \Gamma(5-n)\ \frac{1}{4}\ (g_{\mu\nu}g_{\phi\omega}+g_{\mu\phi}g_{\nu\omega}+g_{\mu\omega}g_{\nu\phi})$$

$$.\int_0^1 dx dy \ \frac{(1-x)\,x^3\,y^{4-n/2}\,(1-y)}{[(1-x)xy+1-y]^{2+n/2}}$$

The last Feynman parameter integral is divergent but, as we shall see, it cancels out.

From the next term in eq(Ex.3), namely $-k_\phi k_\nu k_\omega p_\mu$, we get exactly the same thing (the first two terms are symmetric under $p \leftrightarrow k$)

Now the two remaining terms in eq(Ex.3). They are again symmetric under $p \leftrightarrow k$. Overall partial integration gives a logarithmic divergence as before. The corresponding three terms give :

First term

$$\left[\frac{i\pi^{n/2}}{(2\pi)^n}\right]^2 \frac{1}{\Gamma(2)\Gamma(2)\Gamma(3)} (-m^2)^{n-5}\, 4m^2 \left\{ \begin{aligned} &\Gamma(5-n)\ \tfrac{1}{4}\left(g_{\mu\nu}g_{\phi\omega}+g_{\mu\phi}g_{\nu\omega}+g_{\mu\omega}g_{\nu\phi}\right) \\ &\cdot \int_0^1 dx\,dy \ \frac{x^3(1-x)\, y^{3-n/2}(1-y)^2}{[x(1-x)y+1-y]^{2+n/2}} \\ &+ \Gamma(5-n)\ \tfrac{1}{4}\ g_{\nu\phi}g_{\mu\omega} \\ &\cdot \int_0^1 dx\,dy \ \frac{x(1-x)\, y^{2-n/2}(1-y)^2}{[x(1-x)y+1-y]^{1+n/2}} \end{aligned} \right.$$

Second term

$$\left[\frac{i\pi^{n/2}}{(2\pi)^n}\right]^2 \frac{(-m^2)^{n-5}\, 4m^2}{\Gamma(2)\Gamma(3)\Gamma(2)} \left\{ \begin{aligned} &\Gamma(5-n)\ \tfrac{1}{4}\left(g_{\mu\nu}g_{\phi\omega}+g_{\mu\phi}g_{\nu\omega}+g_{\mu\omega}g_{\nu\phi}\right) \\ &\cdot \int_0^1 dx\,dy \ \frac{x^3(1-x)^2\, y^{4-n/2}(1-y)}{[x(1-x)y+1-y]^{2+n/2}} \\ &+ \Gamma(5-n)\ \tfrac{1}{4}\ g_{\nu\phi}g_{\mu\omega} \\ &\cdot \int_0^1 dx\,dy \ \frac{x(1-x)^2\, y^{3-n/2}(1-y)}{[x(1-x)y+1-y]^{1+n/2}} \end{aligned} \right.$$

<u>Third term</u>

$$\left[\frac{i\pi^{n/2}}{(2\pi)^n}\right]^2 \frac{(-m^2)^{n-5}\,4m^2}{\Gamma(3)\Gamma(2)\Gamma(2)} \left\{ \begin{array}{l} \Gamma(5-n)\ \frac{1}{4}\,(g_{\mu\nu}g_{\phi\omega}+g_{\mu\phi}g_{\nu\omega}+g_{\mu\omega}g_{\nu\phi}) \\ \qquad \cdot \int_0^1 dx\,dy\ \dfrac{x^3(1-x)^2\,y^{4-n/2}(1-y)}{[x(1-x)y+1-y]^{2+n/2}} \\ \Gamma(5-n)\ \frac{1}{4}\,g_{\nu\phi}g_{\mu\omega} \\ \qquad \int_0^1 dx\,dy\ \dfrac{x(1-x)^2\,y^{3-n/2}(1-y)}{[x(1-x)\,y+1-y]^{1+n/2}} \end{array}\right.$$

The last Feynman parameter integral is again divergent.

The final term, $k_\nu k_\omega p_\mu p_\phi$, gives the same thing except that instead of $\frac{1}{4}g_{\nu\phi}g_{\mu\omega}$ we have $\frac{1}{4}g_{\nu\omega}g_{\mu\phi}$.

So the four terms of eq(Ex.3) gives:

$$-g^4\left[\frac{i\pi^2}{(2\pi)^4}\right]^2 4\ \frac{1}{2\epsilon}\ \frac{1}{4}\,(g_{\mu\phi}g_{\omega\nu}+g_{\mu\omega}g_{\phi\nu}+g_{\mu\nu}g_{\phi\omega})\ \frac{\Gamma(1)}{\Gamma(2)\Gamma(2)\Gamma(3)}$$

$$\Bigg[\Bigg\{ -\int_0^1 dx\,dy\ \frac{(1-x)\,x^2 y\,(1-y)^2}{[x(1-x)y+1-y]^4} + \int_0^1 dx\,dy\ \frac{x^3(1-x)\ y(1-y)^2}{[x(1-x)y+1-y]^4}$$

$$-\int_0^1 dx\,dy\ \frac{x^2(1-x)^2\ y^2\,(1-y)}{[x(1-x)y+1-y]^4} + \int_0^1 dx\,dy\ \frac{x^3(1-x)^2\ y^2\,(1-y)}{[(1-x)\,xy+1-y]^4}$$

$$-\int_0^1 dx\,dy\ \frac{x^3(1-x)\ y^2(1-y)}{[x(1-x)y+1-y]^4} + \int_0^1 dx\,dy\ \frac{x^4(1-x)\ y^2(1-y)}{[x(1-x)y+1-y]^4}\Bigg\}\cdot 2$$

$$+\ \frac{1}{4}\,(g_{\nu\phi}g_{\mu\omega}+g_{\nu\omega}g_{\mu\phi})\cdot\Bigg\{\int_0^1 dx\,dy\,\frac{(1-x)\,x\,(1-y)^2}{[x(1-x)y+1-y]^3}+\int_0^1 dx\,dy\,\frac{x(1-x)^2 y\,(1-y)}{[x(1-x)y+1-y]^3}$$

$$+\int_0^1 dx\,dy\ \frac{x^2(1-x)\ y(1-y)}{[x(1-x)y+1-y]^3}\Bigg\}\Bigg]$$

Where we have expanded with respect to ϵ and take only the $\frac{1}{\epsilon}$ -term (therefore everything else will be in 4 dimensions since this diagram gives only $\frac{1}{\epsilon}$).

The fifth and sixth integral in the last expression, although divergent, when combined give a divergent integral.

Now the γ-algebra can be made in 4 dimensions. We write formally:

$$\{ g_{\mu\phi} g_{\nu\omega} + g_{\mu\omega} g_{\phi\nu} + g_{\mu\nu} g_{\phi\omega} \} \; (\gamma^{\sigma}(1-\gamma_5)\gamma^{\phi}\gamma^{\lambda}\gamma^{\mu}\gamma^{\tau})(\gamma_{\sigma}(1-\gamma_5)\gamma^{\omega}\gamma_{\tau}\gamma^{\nu}\gamma_{\lambda}) =$$

$$= (-192) \; (\gamma^{\sigma}(1-\gamma_5))(\gamma_{\sigma}(1-\gamma_5))$$

$$\{ g_{\nu\phi} g_{\mu\omega} + g_{\nu\omega} g_{\mu\phi} \} \; (\gamma^{\sigma}(1-\gamma_5)\gamma^{\phi}\gamma^{\lambda}\gamma^{\mu}\gamma^{\tau})(\gamma_{\sigma}(1-\gamma_5)\gamma^{\omega}\gamma_{\tau}\gamma^{\nu}\gamma_{\lambda}) =$$

$$= (-64) \; (\gamma^{\sigma}(1-\gamma_5))(\gamma_{\sigma}(1-\gamma_5)) .$$

The t-algebra reads :

$$(t^a t^b)(t^b t^a) = \begin{Bmatrix} 2/9 & (I)(I) \\ 7/6 & (t)(t) \end{Bmatrix}$$

Using the tables of Feynman integrals from Appendix I, we find that the contribution of diagram 13a is : $\alpha^{+} = -g^4 \, 2.4444/(16\pi^2)^2$, $\alpha^{-} = +g^4 \, 2.2222/(16\pi^2)^2$.

We give now the contribution from all the diagrams.

Diagram	α^{+} $\frac{-g^4}{(16\pi^2)^2}$	α^{-} $\frac{-g^4}{(16\pi^2)^2}$	Number of eq. diag.
1	+ 2.2222	+ 8.8889	1
2	0	0	1
3	− 5.9063	− 23.6251	2
4	− 4.3507	− 4.3507	2
5	+ 0.6550	− 1.3099	4
6	− 10.1285	+ 20.2570	4

7	-1.2222	1.1111	1
8	+0.8844	-0.8040	2
9	-5.3711	+4.8888	4
10	-10.2600	+9.3273	4
11	+1.1618	-2.3235	4
12	-0.5804	+1.1608	4
13	+2.4444	-2.2222	2
14	-0.8888	-0.8888	2
15	-0.1482	+0.2964	4
16	+2.0446	-4.0893	4
17	+2.6664	+2.6664	4
18	+0.0655	-0.1309	4
19	0	0	2
20	+0.1734	-0.3468	4
21	+0.3266	+0.3266	4
22	-53.1837	+106.3763	4
23	-10.8189	-10.8189	4
24	+3.5000	-7.0000	4
25	-7.2047	+14.4094	2
26	$-1.3333 \cdot N_f$	$-1.3333 \cdot N_f$	2
27	$-1.7899 \cdot N_f$	$+3.5799 \cdot N_f$	2
28	$+1.4045 \cdot N_f$	$-2.8090 \cdot N_f$	2
29+32	+22.5211	-45.0428	2
30+33	-10.5817	-10.5817	2
31+34	-44.2454	+88.4909	2

N_f = number of flavours.

Adding we get, for $N_f=4$, :

$$\sum_{diag} \alpha^+ = \frac{g^4}{(16\pi^2)^2}(426.995) \qquad \sum_{diag} \alpha^- = -\frac{g^4}{(16\pi^2)^2}(513.392) \qquad (5.55)$$

Following the same procedure as in the one-loop calculation we get :

$$\gamma^+ = \frac{1}{2} g \frac{\partial}{\partial g}\alpha^+ = 2g^4\left(\frac{426.995}{(16\pi^2)^2}\right) = g^4\left(\frac{853.99}{(16\pi^2)^2}\right) \qquad (5.56)$$

$$\gamma^- = \frac{1}{2} g \frac{\partial}{\partial g}\alpha^- = 2g^4\left(-\frac{513.392}{(16\pi^2)^2}\right) = g^4\left(-\frac{1026.78}{(16\pi^2)^2}\right)$$

Now the second-loop contribution to the anomalous dimension of the operators $\hat{O}^{\pm}$ is given by:

$$\gamma^{\pm} = 2\gamma_F - \gamma_{\hat{O}^{\pm}} \qquad (5.57)$$

where γ_F is the anomalous dimension of the fermion field. From reference (5) we get that the second-loop contributes a term :

$$\gamma_F = 52 g^4 \qquad (5.58)$$

Thus we get :

$$\hat{O}^+ : \qquad \gamma_{\hat{O}^+} = g^4(-749.99) \qquad (5.59)$$

$$\hat{O}^- : \qquad \gamma_{\hat{O}^-} = g^4(1130.78) \qquad (5.60)$$

But there are still some problems to be solved. As we saw in Chapter 3, the g^4-term of the γ-function depends on the renormalization scheme.

<u>Proof of the independence of physical quantities from the renormalization scheme</u>. As we already know the leading asymptotic form of the solution of the Renormalization Group Equation of the C-coefficient is :

$$\tilde{C}(q^2/\mu^2, g) = \tilde{C}(1,0)\, t^{\gamma_0/2\beta_0} \quad \text{for } \bar{g}\to 0 \text{ when } q^2\to\infty \qquad (5.61)$$

where $t=\frac{1}{2}\ln(-q^2/\mu^2)$ and β_0 and γ_0 are the coefficients of the g^3- and g^2-term in the β- and γ-function respectively.

The next to leading order corrections to eq(5.61) may arise from the term $\tilde{C}(1,\bar{g})$ as well as from the integral of the γ-function (see for example eq(5.31)) :

$$\tilde{C}(1,\bar{g}) = \tilde{C}(1,0) + \bar{g}^2 \frac{\partial \tilde{C}(1,\bar{g})}{\partial \bar{g}^2} \qquad (5.62)$$

We have to find therefore the term $\frac{\partial \tilde{C}(1,\bar{g})}{\partial \bar{g}^2}$. Let us write again the complete solution of the R.G.E. for the C-coefficients :

$$\tilde{C}(q^2/\mu^2, g) = \tilde{C}(1,\bar{g}) \exp\left[-\int_0^t \gamma(\bar{g}(g,t'))\,dt'\right] \qquad (5.63)$$

The left-hand side is directly connected with the physical process. This is clear since by Wilson expansion the Green's function Γ_{JJ} is equal the C-coefficient times the Green's function Γ_0 which is independent of q^2 . Thus the left-hand side of eq(5.63) can be calculated, up to order g^2, from the diagrams :

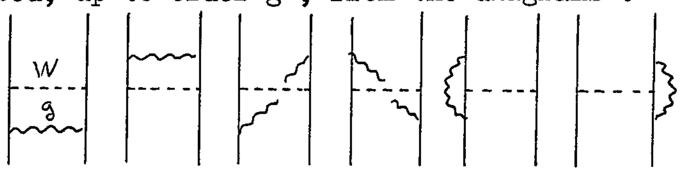


Fig (5.3)

Now for $q^2=\mu^2=M_W^2$ the exponent in the right-hand side vanishes. Therefore $\partial\tilde{C}/\partial\bar{g}^2$ is equal to the contribution from the diagrams in Fig (5.3) (for $q^2=\mu^2=M_W^2$, $g=\bar{g}$). Now from Appendix K we get that :

$$\frac{\partial \tilde{C}}{\partial \bar{g}^2} = \frac{20}{3}(1.4464) \quad \text{for } \hat{O}^- \qquad (5.64)$$

$$\frac{\partial \tilde{C}}{\partial \bar{g}^2} = -\frac{20}{6}(1.4464) \quad \text{for } \hat{O}^+$$

Let us write now eq(5.63) up to the next to leading order correction:

$$\tilde{C}(q^2/\mu^2, g) = \left(\tilde{C}(1,0) + \bar{g}^2 \frac{\partial \tilde{C}}{\partial \bar{g}^2}\right) t^{\gamma_0/2\beta_0} \qquad (5.65)$$

$$\left[1 - \frac{\gamma_0\beta_1 \ln(-2\beta_0 t)}{4\beta_0^3 t} - \frac{\gamma_0\beta_1}{4\beta_0^3 t} + \frac{\gamma_1}{4\beta_0^2 t} + O\left(\frac{1}{t}\right)\right]$$

where we have used that (6) :

$$\gamma = \gamma_0 \bar{g}^2 + \gamma_1 \bar{g}^4 + O(\bar{g}^6) \quad , \quad \beta = \beta_0 \bar{g}^3 + \beta_1 \bar{g}^5 + O(\bar{g}^7) \qquad (5.66\,a,b)$$

$$\bar{g} \xrightarrow{t\to\infty} -\frac{1}{2\beta_0 t} - \frac{\beta_1}{4\beta_0^3} \frac{\ln(-2\beta_0 t)}{t^2} + O\left(\frac{1}{t^2}\right) \qquad (5.66\,c)$$

where $O(1/t)$ for the eq(5.65) and $O(1/t^2)$ for the eq(5.66c) depend on the physical coupling constant and so are uncalculable.

As we have noted in Chapter 4, the evaluation of γ is scheme dependent. We shall demonstrate that eq(5.65) is scheme independent to order $1/t$.

In eq(5.65) there are two sources of scheme dependence

i) in the first paranthesis is the evaluation of diagrams in Fig(5.3) which determines $\partial\tilde{C}/\partial\bar{g}^2$, differs in the two schemes (as defined in Chapter 3) by an amount α (where α is given by $Z' = (1+g_R^2\alpha + O(g_R^4))Z$ see again Chapter 3). So the difference, inthe two schemes, of the term $\bar{g}^2 \frac{\partial\tilde{C}}{\partial\bar{g}^2}$ is :

$$\bar{g}^2\alpha = -\frac{1}{2\beta_0 t}\alpha \tag{5.67}$$

ii) in the second paranthesis γ_1 itself appears and the difference in the two schemes is (see eq(3.26)) :

$$\frac{2\alpha\beta_0}{4\beta_0^2 t} = \frac{\alpha}{2\beta_0 t} \tag{5.68}$$

Eqs(5.65,67,68) show that, to order $\frac{1}{t}$, the scheme dependence cancels.

<u>Results and conclusions</u>. Let us find the contribution of each term in eq(5.65). We know the values of the parameters :

$$\left.\begin{array}{l}\tilde{C}(1,0) = \frac{1}{\sqrt{2}}\\ \gamma_0 = -\frac{8}{16\pi^2}\\ \gamma_1 = \frac{1130}{(16\pi^2)^2}\end{array}\right\}\text{for } \hat{O}^- \qquad \left.\begin{array}{l}\sqrt{\frac{2}{5}}\\ \frac{4}{16\pi^2}\\ -\frac{749}{(16\pi^2)^2}\end{array}\right\}\text{for } \hat{O}^+$$

$$\beta_0 = -\frac{25}{3}\frac{1}{16\pi^2}\ , \quad \beta_1 = \left(102 - \frac{38}{3}\cdot 4\right)\frac{1}{(16\pi^2)^2}$$

For $t=5$ we get :

$$-\frac{\gamma_0\beta_1 \ln(-2\beta_0 t)}{4\beta_0^3 t} = \quad -0.0227 \text{ for } O^- \qquad +0.0113 \text{ for } O^+$$

$$-\frac{\gamma_0\beta_1}{4\beta_0^3 t} = +0.0355 \qquad -0.0177$$

$$\frac{\gamma_1}{4\beta_0^2 t} = +0.8000 \quad \text{for } \hat{O}^- \qquad -0.5300 \quad \text{for } \hat{O}^+$$

$$\bar{g}^2 \frac{\partial \vec{C}}{\partial \bar{g}^2} = +0.11355 \qquad -0.05786$$

Thus to order $1/t$ we have :

$$\tilde{C}(q^2/\mu^2, g) = t^{\gamma_0/2\beta_0} \left\{ \vec{C}(1,0)\left[1 - \frac{\gamma_0\beta_1 \ln(-2\beta_0 t)}{4\beta_0^3 t} - \frac{\gamma_0\beta_1}{4\beta_0^3 t} + \frac{\gamma_1}{4\beta_0^2 t}\right] + \right.$$

$$\left. + \bar{g}^2 \frac{\partial \vec{C}}{\partial \bar{g}^2} \right\} \qquad (5.69)$$

which gives a ratio of the order :

$$R = t^{0.72} \, \frac{1.39}{0.23} \approx 19 \qquad (5.70)$$

In other words, the calculation of the next to leading order contributions, shows that these cannot be ignored. In any case they show that Q.C.D. can explain the scale of the effect, but to prove that it does will acquire understanding of all (?) higher order effects. The result of our calculation does not imply that the third loop calculations will definitely improve the ratio in eq(5.70). In deep inelastic scattering, second loop calculations (5) also show considerable individual contributions for the new terms, which in the end cancel each other. In our calculation it seems that they add up to give the above result.

The last comment to make is about the terms which depends on the physical coupling constant. The function $c(g_{ph})$ of $g_{ph.}$ in eq(5.66c) is given by :

$$\frac{C(g_{ph.})}{t^2} = \left\{ \frac{1}{t^2 4\beta_0^2 g_{ph.}^2} - \frac{\beta_1}{4\beta_0^3 t^2} \ln g_{ph}^2 + \frac{\beta_1^2}{4\beta_0^3 t^2} \ln(\beta_0 + \beta_1 g_{ph.}^2) \right\} \gamma_0 \qquad (5.71)$$

while the corresponding term in eq(5.65) is :

$$\frac{C(g_{ph})}{t} = \left\{ -\frac{1}{4\beta_0^2 g_{ph}^2 t} - \frac{\beta_1}{4\beta_0^3 t} \ln g_{ph.}^2 + \frac{\beta_1^2}{4\beta_0^3 t} \ln(\beta_0 + \beta_1 g_{ph.}^2) \right\} \gamma_0 \qquad (5.72)$$

(5.73)

We give the value of the ratio R for some values of $g^2/4\pi$.

$g^2/4\pi$	1.0	0.7	0.5
R	26	27	29

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Appendix A

The generators t^a of SU(3) obey the follwing relations:

$$[t^a, t^b] = i f^{abc} t^c \qquad \{t^a, t^b\} = \tfrac{1}{3}\delta^{ab} I_3 + d^{abc} t^c \qquad (A.1)$$

where f^{abc} is totally antisymmetric while d^{abc} is totally symmetric. The non vanishing elements of f^{abc} and d^{abc} are:

a	b	c	f^{abc}
1	2	3	1
1	4	7	1/2
1	5	6	-1/2
2	4	6	1/2
2	5	7	1/2
3	4	5	1/2
3	6	7	-1/2
4	5	8	$\sqrt{3}/2$
6	7	8	$\sqrt{3}/2$

a	b	c	d^{abc}
1	1	8	$1/\sqrt{3}$
1	4	6	1/2
1	5	7	1/2
2	2	8	$1/\sqrt{3}$
2	4	7	-1/2
2	5	6	1/2
3	3	8	$1/\sqrt{3}$
3	4	4	1/2
3	5	5	1/2
3	6	6	-1/2
3	7	7	-1/2
4	4	8	$-1/(2\sqrt{3})$
5	5	8	$-1/(2\sqrt{3})$
6	6	8	$-1/(2\sqrt{3})$
7	7	8	$-1/(2\sqrt{3})$
8	8	8	$-1/\sqrt{3}$

We quote some useful relations between f^{abc} and d^{abc}.

$$d^{iik} = 0 \quad , \quad d^{ijk} f^{\ell jk} = 0 \quad , \quad f^{ijk} f^{\ell jk} = 3\delta^{i\ell}$$

$$d^{ijk} d^{\ell jk} = 5/3\, \delta^{i\ell} \quad , \quad f^{piq} f^{qjr} f^{rkp} = -3/2\, f^{ijk}$$

$$d^{piq} f^{qjr} f^{rkp} = -3/2\, d^{ijk} \quad , \quad d^{piq} d^{qjr} f^{rkp} = 5/6\, f^{ijk}$$

$$d^{piq} d^{qjr} d^{rkp} = -1/2\, d^{ijk} \ .$$

Combining eqs(A.1) we get the following useful formulae:

$$t^i t^j = 1/6\,\delta^{ij}\, I_3 + 1/2\,(d^{ijk} + i f^{ijk})\, t^k$$

$$t^i t^i = 4/3\, I_3 \qquad\qquad t^i t^j t^i = -1/6\, t^j$$

Using the above we can easily prove:

$$(t^a t^b)_{ij}\;(t^a t^b)_{k\ell} = 2/9\;(I)_{ij}\;(I)_{k\ell} - 1/3\;(t^a)_{ij}\;(t^a)_{k\ell}$$

$$(t^a t^b)_{ij}\;(t^b t^a)_{k\ell} = 2/9\;(I)_{ij}\;(I)_{k\ell} + 7/6\;(t^a)_{ij}\;(t^a)_{k\ell}$$

<u>Fierz identities for t-matrices:</u>

$$(t^a)_{ij}\;(t^a)_{k\ell} = 1/2\;(I)_{i\ell}\,(I)_{kj} - 1/6\,(I)_{ij}\;(I)_{k\ell}$$

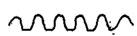
$$(t^a)_{ij}\;(t^a)_{k\ell} = 4/9\;(I)_{i\ell}(I)_{kj} - 1/3\,(t^a)_{i\ell}\;(t^a)_{kj}$$

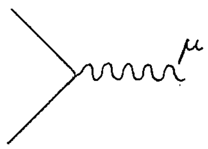
$$(I)_{ij}\;(I)_{k\ell} = 1/3\,(I)_{i\ell}\,(I)_{kj} + 2\,(t^a)_{i\ell}\;(t^a)_{kj}$$

(summation convention is understood over repeated indices)

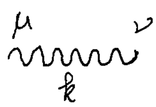
Appendix B

We quote the Feynman rules for Q.E.D. (in the Feynman gauge):

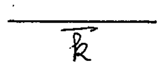
 fermion lime ,  photon lime



$$-ie\gamma^{\mu}$$



$$\frac{-ig^{\mu\nu}}{k^2}$$



$$\frac{i(\not{k}+m)}{k^2-m^2}$$

fermion Loop (-1)

Loop-momentum K $\int \frac{d^n k}{(2\pi)^n}$

Appendix C

Useful formulae on n-dimensional integrals are the following:

$$\int \frac{d^n k}{(k^2+2k\cdot p+M^2)^\alpha} = \frac{i\pi^{\frac{n}{2}}}{\Gamma(\alpha)} \Gamma(\alpha-\tfrac{n}{2}) \frac{1}{(M^2-p^2)^{\alpha-\frac{n}{2}}}$$

$$\int \frac{d^n k \; k^\mu}{(k^2+2k\cdot p+M^2)^\alpha} = \frac{i\pi^{\frac{n}{2}}}{\Gamma(\alpha)} \Gamma(\alpha-\tfrac{n}{2}) \frac{-p^\mu}{(M^2-p^2)^{\alpha-\frac{n}{2}}}$$

$$\int \frac{d^n k \; k^\mu k^\nu}{(k^2+2k\cdot p+M^2)^\alpha} = \frac{i\pi^{\frac{n}{2}}}{\Gamma(\alpha)} \left\{ \frac{\Gamma(\alpha-\frac{n}{2}) p^\mu p^\nu}{(M^2-p^2)^{\alpha-\frac{n}{2}}} + \frac{\Gamma(\alpha-1-\frac{n}{2}) \frac{1}{2} g^{\mu\nu}}{(M^2-p^2)^{\alpha-1-n/2}} \right\}$$

$$\int \frac{d^n k \; k^\mu k^\nu k^\lambda}{(k^2+2k\cdot p+M^2)^\alpha} = \frac{i\pi^{\frac{n}{2}}}{\Gamma(\alpha)} \left\{ -\frac{\Gamma(\alpha-\frac{n}{2})}{(M^2-p^2)^{\alpha-\frac{n}{2}}} p^\mu p^\nu p^\lambda \right.$$
$$\left. - \frac{\Gamma(\alpha-1-\frac{n}{2})}{(M^2-p^2)^{\alpha-1-\frac{n}{2}}} \frac{1}{2} \left(g^{\mu\nu} p^\lambda + g^{\mu\lambda} p^\nu + g^{\nu\lambda} p^\mu \right) \right\}$$

$$\int \frac{d^n k \; k^\mu k^\nu k^\lambda k^\sigma}{(k^2+2kp+M^2)^\alpha} = \frac{i\pi^{\frac{n}{2}}}{\Gamma(\alpha)} \left\{ \frac{\Gamma(\alpha-\frac{n}{2})}{(M^2-p^2)^{\alpha-\frac{n}{2}}} p^\mu p^\nu p^\lambda p^\sigma \right.$$
$$+ \frac{\Gamma(\alpha-1-\frac{n}{2})}{(M^2-p^2)^{\alpha-1-n/2}} \frac{1}{2} \left(g^{\mu\sigma} p^\nu p^\lambda + g^{\nu\sigma} p^\mu p^\lambda + g^{\mu\nu} p^\lambda p^\sigma + \right.$$
$$\left. g^{\lambda\sigma} p^\mu p^\nu + g^{\mu\lambda} p^\nu p^\sigma + g^{\nu\lambda} p^\mu p^\sigma \right)$$
$$\left. + \frac{\Gamma(\alpha-2-\frac{n}{2})}{(M^2-p^2)^{\alpha-2-\frac{n}{2}}} \frac{1}{4} \left(g^{\mu\nu} g^{\lambda\sigma} + g^{\mu\lambda} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\lambda} \right) \right\}$$

$$\int \frac{d^n k \; k^\mu k^\nu k^\lambda k^\phi k^\sigma}{(k^2+2k\cdot p+M^2)^\alpha} = \frac{i\pi^{\frac{n}{2}}}{\Gamma(\alpha)} \Bigg\{ -\frac{\Gamma(\alpha-\frac{n}{2})}{(M^2-p^2)^{\alpha-\frac{n}{2}}} \, p^\mu p^\nu p^\lambda p^\phi p^\sigma$$

$$-\frac{\Gamma(\alpha-1-\frac{n}{2})}{(M^2-p^2)^{\alpha-1-\frac{n}{2}}} \, \frac{1}{2} \left\{ \begin{array}{l} g^{\mu\phi} p^\nu p^\lambda p^\sigma + g^{\nu\phi} p^\mu p^\lambda p^\sigma + g^{\phi\lambda} p^\mu p^\nu p^\sigma + g^{\phi\sigma} p^\mu p^\nu p^\lambda \\ + g^{\mu\sigma} p^\nu p^\lambda p^\phi + g^{\nu\sigma} p^\mu p^\lambda p^\phi + g^{\lambda\sigma} p^\mu p^\nu p^\phi + g^{\nu\mu} p^\lambda p^\sigma p^\phi \\ + g^{\mu\lambda} p^\nu p^\sigma p^\phi + g^{\nu\lambda} p^\mu p^\sigma p^\phi \end{array} \right\}$$

$$-\frac{\Gamma(\alpha-2-\frac{n}{2})}{(M^2-p^2)^{\alpha-2-\frac{n}{2}}} \, \frac{1}{4} \left\{ \begin{array}{l} (g^{\mu\sigma} g^{\phi\nu} + g^{\mu\phi} g^{\sigma\nu} + g^{\mu\nu} g^{\sigma\phi}) \, p^\lambda \\ + (g^{\nu\sigma} g^{\phi\lambda} + g^{\nu\lambda} g^{\sigma\phi} + g^{\nu\phi} g^{\sigma\lambda}) \, p^\mu \\ + (g^{\mu\sigma} g^{\phi\lambda} + g^{\mu\phi} g^{\sigma\lambda} + g^{\mu\lambda} g^{\sigma\phi}) \, p^\nu \\ + (g^{\nu\lambda} g^{\phi\mu} + g^{\nu\phi} g^{\lambda\mu} + g^{\nu\mu} g^{\phi\lambda}) \, p_\sigma \\ + (g^{\mu\nu} g^{\lambda\sigma} + g^{\mu\sigma} g^{\lambda\nu} + g^{\mu\lambda} g^{\nu\sigma}) \, p_\phi \end{array} \right\} \Bigg\}$$

$$\int \frac{d^n k \; k^2}{(k^2+2kp+M^2)^\alpha} = \frac{i\pi^{\frac{n}{2}}}{\Gamma(\alpha)} \left\{ \frac{\Gamma(\alpha-\frac{n}{2})}{(M^2-p^2)^{\alpha-\frac{n}{2}}} \, p^2 - \frac{\Gamma(\alpha-1-\frac{n}{2})}{(M^2-p^2)^{\alpha-1-\frac{n}{2}}} \, \frac{1}{2} n \right\}$$

$$\int \frac{d^n k \; k^2 k^\mu k^\nu k^\lambda k^\phi}{(k^2+M^2)^\alpha} = \frac{i\pi^{\frac{n}{2}}}{\Gamma(\alpha)} \; \frac{\Gamma(\alpha-3-\frac{n}{2})}{(M^2)^{\alpha-3-\frac{n}{2}}} \; \frac{1}{8} \, (g^{\mu\nu} g^{\lambda\phi} + g^{\mu\lambda} g^{\nu\phi} + g^{\mu\phi} g^{\nu\lambda})$$

$$\int \frac{d^n k \; k^2 k^\mu k^\nu}{(k^2+2kp+M^2)^\alpha} = \frac{i\pi^{\frac{n}{2}}}{\Gamma(\alpha)} \left\{ \begin{array}{l} \dfrac{\frac{1}{2} g^{\mu\nu}}{(M^2-p^2)^{\alpha-\frac{n}{2}-1}} \left\{ \begin{array}{l} \Gamma(\alpha-\frac{n}{2}-1) \, p^2 \\ + \Gamma(\alpha-2-\frac{n}{2}) \, \frac{1}{2}(n+2)(M^2-p^2) \end{array} \right\} \\ + \dfrac{p_\mu p_\nu}{(M^2-p^2)^{\alpha-\frac{n}{2}}} \left\{ \begin{array}{l} \Gamma(\alpha-\frac{n}{2}) \, p^2 \\ + (\alpha-1-\frac{n}{2}) \, \Gamma(\alpha-2-\frac{n}{2}) \, \frac{1}{2}(n+2)(M^2-p^2) \end{array} \right\} \\ + \dfrac{p_\mu p_\nu}{[M^2-p^2]^{\alpha-1-\frac{n}{2}}} \left\{ \begin{array}{l} \Gamma(\alpha-1-\frac{n}{2}) \\ - \Gamma(\alpha-2-\frac{n}{2}) \, \frac{1}{2}(n+2) \end{array} \right\} \end{array} \right\}$$

In dimensional regularization, ultra-violet divergences appear as poles of the Γ-functions. We give the expansions of n-dependent Γ-functions about n=4. We define $\epsilon \equiv n-4$, and γ is the Euler constant.

$$\Gamma(4-n) = -\frac{1}{\epsilon} - \gamma + O(\epsilon)$$

$$\Gamma(2-n/2) = -\frac{2}{\epsilon} - \gamma + O(\epsilon)$$

$$\Gamma(1-n/2) = \frac{2}{\epsilon} - 1 + \gamma + O(\epsilon)$$

$$\Gamma(1-n/2)\,\Gamma(4-n) = -\frac{2}{\epsilon^2} + \frac{1}{\epsilon} - \frac{3\gamma}{\epsilon} + O(\epsilon^0)$$

$$\Gamma(2-n/2)\,\Gamma(4-n) = \frac{2}{\epsilon^2} - \frac{3\gamma}{\epsilon} + O(\epsilon^0)$$

$$\Gamma(3-n/2) = 1 - \frac{\gamma}{2}\epsilon + O(\epsilon)$$

$$\Gamma(5-n) = 1 + \gamma\epsilon + O(\epsilon)$$

$$\Gamma(4-n/2) = 1 - \frac{\epsilon}{2} + (\gamma\cdot\epsilon\text{-terms}) + O(\epsilon^2)$$

$$\Gamma(5-n/2) = 2 - \frac{3}{2}\epsilon + (\gamma\cdot\epsilon\text{-terms}) + O(\epsilon^2)$$

Useful formulae for the γ-algebra in n-dimensions are:

$$\gamma^\mu \not{p}\, \gamma_\mu = (2-n)\not{p} \qquad (\not{p} \equiv p\cdot\gamma)$$

$$\gamma^\mu \not{p}\not{q}\,\gamma_\mu = 4\, p\cdot q + (n-4)\not{p}\not{q}$$

$$\mathrm{Tr}(S) = 0 \qquad S \text{ is an odd string of } \gamma\text{'s}$$

$$\mathrm{Tr}(I) = 4$$

(the last relation is true only when n=4. But as G. t' Hooft proves (Nucl. Phys. B44, 189, 1972) this is of no importance.

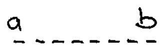
Appendix D

The Feynman rules for Q.C.D. in the Feynman gauge.

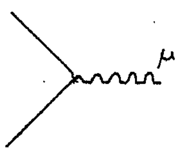
fermion line , gluon line , ghost line

$$\frac{i(\not{k}+m)}{k^2-m^2}$$

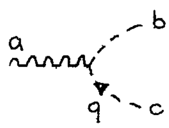
$$\frac{-ig^{\mu\nu}}{k^2}$$



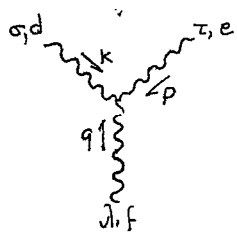
$$\frac{i}{k^2}\delta_{ab}$$



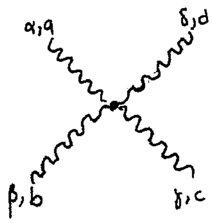
$$-ig\gamma^\mu t^a$$



$$-gf^{abc}q_\mu$$



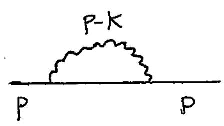
$$-gf^{def}\left[(q-k)_\tau g_{\sigma\lambda}+(p-q)_\sigma g_{\tau\lambda}+(k-p)_\lambda g_{\tau\sigma}\right]$$



$$ig^2\quad\begin{array}{l} f^{lac}f^{lbd}(g_{\alpha\beta}g_{\gamma\delta}-g_{\alpha\delta}g_{\beta\gamma}) \\ f^{lad}f^{lbc}(g_{\alpha\beta}g_{\gamma\delta}-g_{\alpha\gamma}g_{\beta\delta}) \\ f^{lab}f^{lcd}(g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma}) \end{array}$$

<u>Appendix E</u>

Calculation of γ_F to order g^2.



$$-g^2 \int \frac{d^n k}{(2\pi)^n} \frac{\gamma^\mu(\not k+m)\gamma_\mu}{(k^2-m^2)\,(p-k)^2}\,(t^a t^a)$$

We use Feynman parameters to combine the propagators:

$$\frac{1}{(k^2-m^2)(p-k)^2} = \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)}\int_0^1 dx \;\frac{1}{\left[k^2-2pkx+p^2x-m^2(1-x)\right]^2}$$

Since we are interested in the wave function renormalization constant, we only need the integral:

$$-g^2(t^a t^a)\int \frac{d^n k}{(2\pi)^n}\cdot\int_0^1 dx \;\frac{\gamma^\mu \not k \gamma_\mu}{\left[k^2-2pkx+p^2x-m^2(1-x)\right]^2} =$$

$$= -g^2\left(\frac{4}{3}\right)\int \frac{d^n k}{(2\pi)^n}\int_0^1 dx\;\frac{(2-n)\,\not k}{\left[k^2-2pkx+p^2x-m^2(1-x)\right]^2} =$$

$$= -g^2\left(\frac{4}{3}\right)\frac{i\pi^{n/2}}{(2\pi)^n}\int dx\,\frac{1}{\Gamma(2)}\;\frac{\Gamma(2-n/2)\;\not p x\,(2-n)}{\left[p^2x-m^2(1-x)-p^2x^2\right]^{2-n/2}}\,.$$

Therefore the infinite part is:

$$\text{I.P.} = -\frac{i\pi^2}{(2\pi)^4}\,g^2\left(\frac{4}{3}\right)\left(-\frac{2}{\epsilon}\right)\frac{1}{2}\,(-2) = -\frac{i\pi^2}{(2\pi)^4}\,g^2\,\frac{8}{3}\,\frac{1}{\epsilon}$$

and

$$Z_F = \frac{\pi^2}{(2\pi)^4}\,g^2\,\frac{8}{3} \qquad \text{and} \qquad \gamma_F = \frac{1}{2}\,g\,\frac{\partial Z_F}{\partial g} = \frac{\pi^2}{(2\pi)^4}\,\frac{8}{3}\,g^2$$

We give the results for the 6 one-loop diagrams :

Diagram 1a
Diagram 1b
$$(-ig^2)\left(-\frac{i\pi^2}{(2\pi)^4}\right)\left(-\frac{1}{\epsilon}\right)(+8)\quad (t)(t)$$

Diagram 2a
Diagram 2b
$$(-ig^2)\left(\frac{+i\pi^2}{(2\pi)^4}\right)\left(-\frac{1}{\epsilon}\right)(2)\left(\frac{4}{3}\right)\quad (I)(I)$$

Diagram 3a
Diagram 3b
$$(-ig^2)\left(\frac{+i\pi^2}{(2\pi)^4}\right)\left(-\frac{1}{\epsilon}\right)(2)\quad (t)(t)$$

Diagram 4a
Diagram 4b
$$(-ig^2)\left(\frac{-i\pi^2}{(2\pi)^4}\right)\left(-\frac{1}{\epsilon}\right)(+8)\left\{\begin{array}{l} 2/9\,(I)(I) \\ -1/3\,(t)(t)\end{array}\right\}$$

Diagram 5a
Diagram 5b
$$(-ig^2)\left(\frac{i\pi^2}{(2\pi)^4}\right)^2\left(-\frac{1}{\epsilon}\right)(2)\left(-1/6\right)\quad (t)(t)$$

Diagram 6a
Diagram 6b
$$(-ig^2)\left(\frac{i\pi^2}{(2\pi)^4}\right)\left(-\frac{1}{\epsilon}\right)(2)\left\{\begin{array}{l} 2/9\,(I)(I) \\ 7/6\,(t)(t)\end{array}\right\}$$

<u>Appendix F</u>

We shall try to explain here why operators of dimension 4 (or less) in the Wilson expansion can be ignored (Nanopoulos D. V. Nuovo Cimento Letters <u>8</u> , 873, 1973, and S. Weinberg Phys. Rev. Letters <u>31</u>, 494, 1973 and Phys. Rev. <u>D8</u>, 605, 4482, 1973)

Consider a renormalizable gauge theory of strong, electromagnetic and weak interactions. Let the gauge group G be the direct product of G_S and G_W, where G_S is the gauge group of strong interactions and G_W the gauge group of weak interactions:

$$G = G_S \otimes G_W \qquad (F.\,1)$$

and the generators of G_S commute with those of G_W. Let the coupling of G_S be of the order O(1) and that of G_W of order O(e). The spin-$\frac{1}{2}$ hadrons "see" both G_S and G_W, but the leptons "see" only the G_W. The left- and right-handed parts of spin-$\frac{1}{2}$ hadrons transforms in the same way under G_S. We also have the Higgs scalars which see only G_W, and in order to have mass M for the weak intermediate bosons, they must have a vacuum expectation value of the order M/e. Also the Higgs scalars produce or contribute to the masses of the fermions. Then the most general form of a strong-interaction Lagrangian which is allowed by G_S invariance and renormalizability is:

$$\mathcal{L}_{STRON} = \bar{\psi}\, C_1 \gamma^\mu D_\mu \psi + \bar{\psi}\, C_2 \psi + \frac{1}{2} C_3 (D_\mu \phi_a)(D^\mu \phi_a) + P(\phi) + \frac{1}{4} C_4 F_{a\mu\nu} F_a^{\mu\nu} \qquad (F.2)$$

where ψ and ϕ are the fermion and Higgs scalar fields, $P(\phi)$ is G_S-invariant quatric polynomial in ϕ, $D_\mu\psi$ and $D_\mu\phi$ are gauge-covariant derivatives of G_S and C_1 and C_2 are gauge G_S-invariant matrices (which may contain γ_5 and I)

Now because C_1 and C_2 are G_5-invariant matrices, we can redefine the fields ψ in such a way that $C_1=1$ and C_2 is a real matrix m, free of γ_5, without introducing γ_5 into $D_\mu\psi$. So, with a redefinition of the fields ψ ,parity is conserved, to the zeroth order, provided that the parities of the fields are chosen consistenly (spin-$\frac{1}{2}$ and -0 fields with parity +, G_5-gauge fields are polar)

Now we are going to study the second-order corrections coming from the weak and electromagnetic interactions (and to all orders in the strong interaction). The change in S-matrix from corrections coming from gauge vector boson exchange is (in Feynman gauge):

$$\delta S_{fi} = (2\pi)^4 \delta(P_f - P_i) \int d^4k \;\; G^{\alpha\beta}_{if} \, (k^2 + M^2)^{-1}_{\alpha\beta} \qquad (F.3)$$

where

$$G^{\alpha\beta}_{if} = \frac{i}{2(2\pi)^4} \int d^4x \, \langle f \,|\, T(J^\alpha_\mu(x)\, J^{\mu\beta}(y) \,|\, i\rangle \qquad (F.4)$$

J^α_μ are weak hadronic currents (formed by spin-$\frac{1}{2}$ hadronic fields) and M^2 is the weak vector boson mass matrix

From eq(F.3) is obvious that we are going to have corrections of order α , only when $G(k)$ goes not faster than k^{-2} :

$$G(k) \xrightarrow[K\to\infty]{} 1/K^\nu \qquad \nu \leq 2 \qquad (F.5)$$

We use the O.P.E. of the matrix element (averaged over all directions)

$$\int d\omega_k \; G_{if}(k) \xrightarrow[K\to\infty]{} \sum_\tau \langle f \,|\, O_\tau \,|\, i\rangle \, g_\tau^{\alpha\beta}(k) \qquad (F.6)$$

Now in an asymptotically free theory, $g_\tau^{\alpha\beta}$ has dimension (apart from logarithms) :

$$g_\tau^{\alpha\beta}(k) = O(k^{2-d_\tau}) \qquad (F.7)$$

where d_τ is the canonical dimension of O_τ. Thus, the terms which contribute to order α violation of parity are:

$$2-d_\tau \geqslant -2 \quad \text{or} \quad d_\tau \leq 4 \tag{F.8}$$

Further we can express the change in the S-matrix with an equivalent change in the $\mathcal{L}_{\text{STRONG}}$:

$$\delta\mathcal{L}_{\text{STRONG}} = \sum_\tau O_\tau \int_0^\infty g_\tau^{\alpha\beta}(k)\,(K^2+M^2)^{-1}\,K^3 dK \tag{F.9}$$

where the operators O_τ has dimension $\leqslant 4$ (and must be renormalizable). Now since i) O_τ are G_S-invariant operators (because J^α_μ does not "see" the G_S) and ii) O_τ are Lorentz invariant (because we averaged over momentum space directions), the order α correction term in eq(F.9) must be precisely of the same form as the original Lagrangian (F.2) and therefore can be eliminated by redefining the spin-$\frac{1}{2}$ fields as before.

Appendix G

Consider the inserted (amputated) Green's function Γ_0 with the insertion of the operator O. If Z is the wave function renormalization constant of the field ψ:

$$\psi_B = Z^{1/2} \psi_R$$

and Z_o is the renormalization constant of the operator O:

$$O_B = Z_o O_R$$

then:

$$\Gamma_{oR}^{(N)} = Z^{N/2} (Z_o)^{-1} \Gamma_{o,B}^{(N)}$$

where $\Gamma^{(N)}$ is the Green's function with N external legs. Then, by using the definition of γ-function (see eq(3.7)), we get:

$$\begin{aligned} \gamma &= \mu \frac{\partial}{\partial \mu} \ln (Z^{N/2} Z_o^{-1}) = \\ &= \frac{N}{2} \mu \frac{\partial}{\partial \mu} \ln Z - \mu \frac{\partial}{\partial \mu} \ln Z_o \\ &\equiv \frac{N}{2} \gamma_F - \gamma_o \end{aligned}$$

Appendix H

Renormalization Group Equation for the Wilson Coefficient.

Consider the inserted Green's function $\Gamma^{(N)}_{A,B}$ where A and B are currents and therefore having zero anomalous dimensions. Then $\Gamma^{(N)}_{A,B}$ satisfies the renormalization group equation:

$$\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g_R}+\gamma_m\, m\frac{\partial}{\partial m}-\frac{N}{2}\gamma\right]\Gamma^{(N)}_{A,B}=0 \qquad (H.1)$$

Now Wilson expansion for $\Gamma^{(N)}_{A,B}$ gives:

$$\Gamma^{(N)}_{A,B}=\sum C_i\,\Gamma^{(N)}_{O_i} \qquad (H.2)$$

$\Gamma^{(N)}_{O_i}$'s satisfy also the R.G.E. :

$$\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g_R}+\gamma_m\, m\frac{\partial}{\partial m}-\frac{N}{2}\gamma+\gamma_{Oi}\right]\Gamma^{(N)}_{O_i}=0 \qquad (H.3)$$

(where, for simplicity, we have assumed that O_i's do not mix).

Thus eqs(H.1)-(H.3) give:

$$\begin{aligned}
0&=\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g_R}+\gamma_m\, m\frac{\partial}{\partial m}-\frac{N}{2}\gamma\right]\Gamma^{(N)}_{A,B}=\\
&=\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g_R}+\gamma_m\, m\frac{\partial}{\partial m}-\frac{N}{2}\gamma\right]\sum_i C_i\,\Gamma^{(N)}_{O_i}=\\
&=\sum_i\Big\{\Gamma^{(N)}_{O_i}\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g_R}+\gamma_m\, m\frac{\partial}{\partial m}\right]C_i+\\
&\qquad+C_i\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g_R}+\gamma_m\, m\frac{\partial}{\partial m}\right]\Gamma^{(N)}_{O_i}+\left(-\frac{N}{2}\right)\gamma\, C_i\,\Gamma^{(N)}_{O_i}\Big\}\\
&=\sum_i\Big\{\Gamma^{(N)}_{O_i}\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g_R}+\gamma_m\, m\frac{\partial}{\partial m}\right]C_i+C_i\left[\frac{N}{2}\gamma-\gamma_{O_i}\right]\Gamma^{(N)}_{O_i}-\frac{N}{2}\gamma\, C_i\,\Gamma^{(N)}_{O_i}\Big\}=\\
&=\sum_i\Gamma^{(N)}_{O_i}\left[\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g_R}+\gamma_m\, m\frac{\partial}{\partial m}-\gamma_{O_i}\right]C_i
\end{aligned} \qquad (H.4)$$

Since C_i are coefficients of the expansion of the product $A\cdot B$ with respect to O_i 's, which form a complet basis, then eq(H.9) tells us that:

$$\left[\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g_R} + \gamma_m m \frac{\partial}{\partial m} - \gamma_{0i} \right] C_i = 0 \qquad (H.5)$$

Appendix I

We have used throughout the calculations two Feynman parameters, x, and y, to combine the propagators. Although we have 34 different types of diagrams, eachone decomposed in several terms, having a different Feynman parameter integral, it appears, at the end, that we have only 25 integrals to evaluate. We give the values of these integrals in the table below.

general form: $$\int_0^1 dx \int_0^1 dy \, \frac{x^a (1-x)^b y^c (1-y)^d}{[x(1-x)y+1-y]^e}$$

(note the symmetry under the transformation $x \to (1-x)$ when $a=b$)

a	b	c	d	e	value
0	0	0	1	1	0.781
0	0	0	1	2	1.480
0	0	0	3	2	0.410
0	0	0	3	3	0.590
0	0	0	3	4	1.033
0	1	0	2	2	1/3
0	1	0	2	3	0.615
0	1	0	3	4	0.516
1	0	0	3	3	0.295
1	1	0	1	2	0.219
1	1	0	2	3	0.166
1	1	0	3	3	0.090

a	b	c	d	e	value
1	1	0	3	4	0.137
1	2	0	3	5	1/8
1	2	1	1	3	0.166
2	0	0	1	2	0.551
2	0	0	3	3	0.205
2	0	0	3	4	0.380
2	1	0	2	3	0.083
2	1	0	2	4	1/6
2	1	0	3	4	0.068
2	2	0	3	4	0.025
2	2	1	2	4	0.030
2	3	2	1	4	0.068
3	2	1	1	4	1/12

In general the exponents in the Feynman parameter integrals depends on ϵ. Diagrams, in which momentum integration gives $1/\epsilon^2$, can give $1/\epsilon$-term when the Feynman parameter integral is expanded, with respect to ϵ, and take the term proportional to ϵ. Thus we have to evaluate Feynman parameter integrals of the form:

$$\int_0^1 dx \int_0^1 dy \; \frac{x^a(1-x)\, y^c (1-y)^d}{[x(1-x)y+1-y]^e} \left[\ln\left(x(1-x)y+1-y\right) + \ln y\right]$$

We give the values of the integrals which appeared in our calculations.

$$\int_0^1 dx \int_0^1 dy \; \frac{x^a(1-x)\, y^c (1-y)^d}{[x(1-x)y+1-y]^e} \ln\left(x(1-x)y+1-y\right)$$

a	b	c	d	e	value
1	1	0	1	3	-0.531
1	2	0	2	4	-0.454
2	1	0	1	3	-0.266
2	1	0	3	5	-0.098

a	b	c	d	e	value
2	2	0	1	3	-0.068
2	2	0	3	5	-0.024
3	1	0	1	3	-0.198

$$\int_0^1 dx \int_0^1 dy \; \frac{x^a(1-x)\, y^c (1-y)^d}{[x(1-x)y+1-y]^d} \ln y$$

a	b	c	d	e	value
1	1	0	1	3	-0.281
1	2	0	2	4	-0.122
2	1	0	1	3	-0.141
2	1	0	3	5	-0.108

a	b	c	d	e	value
2	2	0	1	3	-0.054
2	2	0	3	5	-0.041
3	1	0	1	3	-0.087

In the integrals of the form

$$\int_0^1 dx\,dy \; \frac{x^a(1-x)^b\, y^c(1-y)^d}{[x(1-x)y+1-y]^e} \quad \text{and} \quad \int_0^1 dx\,dy \; \frac{x^a(1-x)^b\, y^c(1-y)^d}{[x(1-x)y+1-y]^e} \ln[(1-x)xy+1-y]$$

We can perform the y-integration. In that case we are left with the following types of integrals with respect to x :

$$(\mathrm{I}) \int_0^1 dx \; \frac{x^a(1-x)^b}{[x(1-x)-1]^c} \qquad \text{or} (\mathrm{II}) \int_0^1 dx \; \frac{x^a(1-x)^b}{[x(1-x)-1]^c} \ln x$$

In the following tables we give the valus of those integrals for some values of a, b, c, d and e.

a	b	c	value
0	0	1	-.1209E+01
1	0	1	-.6046E+00
1	1	1	-.2092E+00
1	2	1	-.1046E+00
1	3	1	-.6207E-01
2	0	1	-.3954E+00
2	1	1	-.1046E+00
2	2	1	-.4253E-01
2	3	1	-.2127E-01
3	0	1	-.2908E+00
3	1	1	-.6207E-01
3	2	1	-.2127E-01
3	3	1	-.9200E-02
0	0	2	.1473E+01
1	0	2	.7364E+00
1	1	2	.2636E+00
1	2	2	.1318E+00
1	3	2	.7740E-01
2	0	2	.4728E+00
2	1	2	.1318E+00
2	2	2	.3440E-01
2	3	2	.2720E-01
3	0	2	.3410E+00
3	1	2	.7740E-01
3	2	2	.2720E-01
3	3	2	.1187E-01
0	0	3	-.1806E+01
1	0	3	-.9031E+00
1	1	3	-.3333E+00
1	2	3	-.1667E+00
1	3	3	-.9693E-01
2	0	3	-.5697E+00
2	1	3	-.1667E+00
2	2	3	-.6973E-01
2	3	3	-.3487E-01
3	0	3	-.4031E+00
3	1	3	-.9693E-01
3	2	3	-.3487E-01
3	3	3	-.1533E-01
0	0	4	.2229E+01
1	0	4	.1115E+01
1	1	4	.4229E+00
1	2	4	.2115E+00
1	3	4	.1219E+00
2	0	4	.6916E+00
2	1	4	.2115E+00
2	2	4	.8957E-01
2	3	4	.4479E-01
3	0	4	.4802E+00
3	1	4	.1219E+00
3	2	4	.4479E-01
3	3	4	.1984E-01
0	0	5	-.2767E+01
1	0	5	-.1384E+01
1	1	5	-.5382E+00
1	2	5	-.2691E+00
1	3	5	-.1538E+00
2	0	5	-.8454E+00
2	1	5	-.2691E+00
2	2	5	-.1153E+00
2	3	5	-.5763E-01
3	0	5	-.5763E+00
3	1	5	-.1538E+00
3	2	5	-.5763E-01
3	3	5	-.2570E-01

(I)

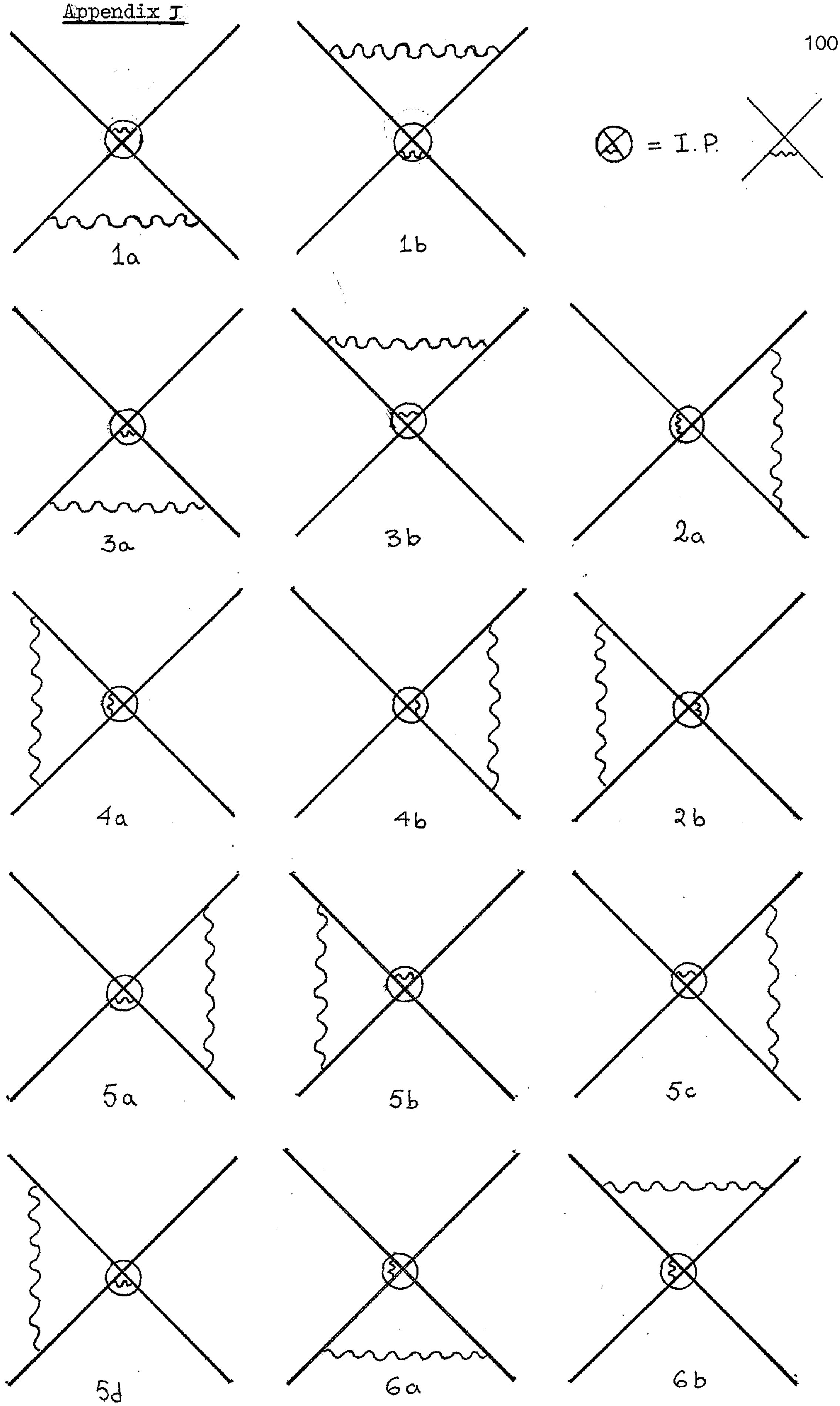
$$\int_0^1 dx \, \frac{x^a (1-x)^b}{[x(1-x)-1]^c}$$

a	b	c	value
0	0	1	.1172E+01
1	0	1	.3118E+00
1	1	1	.1720E+00
1	2	1	.1101E+00
1	3	1	.7707E-01
2	0	1	.1399E+00
2	1	1	.6182E-01
2	2	1	.3306E-01
2	3	1	.1985E-01
3	0	1	.7805E-01
3	1	1	.2876E-01
3	2	1	.1321E-01
3	3	1	.6954E-02
0	0	2	-.1386E+01
1	0	2	-.3907E+00
1	1	2	-.2139E+00
1	2	2	-.1351E+00
1	3	2	-.9312E-01
2	0	2	-.1767E+00
2	1	2	-.7883E-01
2	2	2	-.4199E-01
2	3	2	-.2499E-01
3	0	2	-.9787E-01
3	1	2	-.3684E-01
3	2	2	-.1701E-01
3	3	2	-.8930E-02
0	0	3	.1653E+01
1	0	3	.4914E+00
1	1	3	.2674E+00
1	2	3	.1667E+00
1	3	3	.1132E+00
2	0	3	.2240E+00
2	1	3	.1008E+00
2	2	3	.5348E-01
2	3	3	.3155E-01
3	0	3	.1232E+00
3	1	3	.4728E-01
3	2	3	.2194E-01
3	3	3	.1149E-01
0	0	4	-.1989E+01
1	0	4	-.6205E+00
1	1	4	-.3357E+00
1	2	4	-.2066E+00
1	3	4	-.1383E+00
2	0	4	-.2848E+00
2	1	4	-.1291E+00
2	2	4	-.6829E-01
2	3	4	-.3996E-01
3	0	4	-.1557E+00
3	1	4	-.6081E-01
3	2	4	-.2834E-01
3	3	4	-.1481E-01
0	0	5	.2412E+01
1	0	5	.7863E+00
1	1	5	.4231E+00
1	2	5	.2574E+00
1	3	5	.1700E+00
2	0	5	.3631E+00
2	1	5	.1658E+00
2	2	5	.8741E-01
2	3	5	.5076E-01
3	0	5	.1974E+00
3	1	5	.7835E-01
3	2	5	.3665E-01
3	3	5	.1912E-01

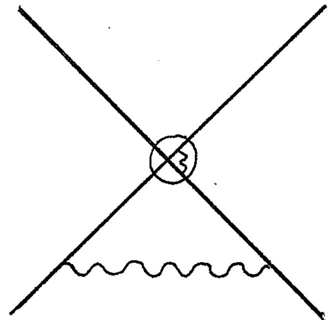
(II)

$$\int_0^1 dx \; \frac{x^a (1-x)^b}{[x(1-x)-1]^c} \ln x$$

Appendix J



Counter-terms considered in the two-loop calculation.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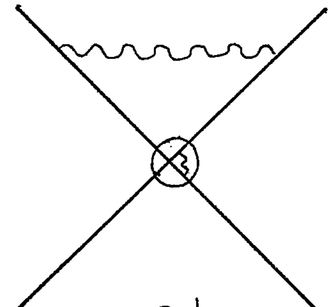
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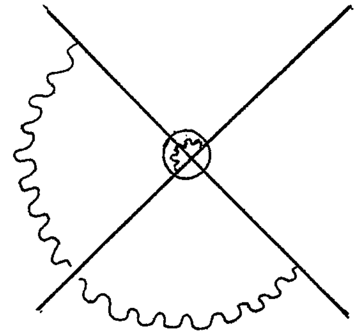
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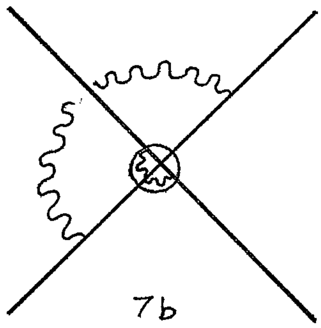
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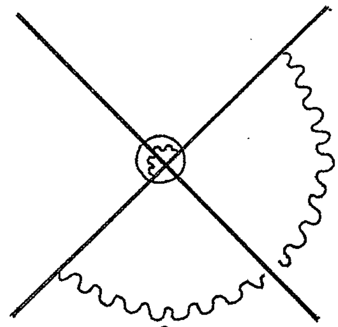
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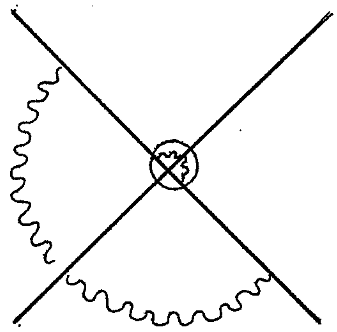
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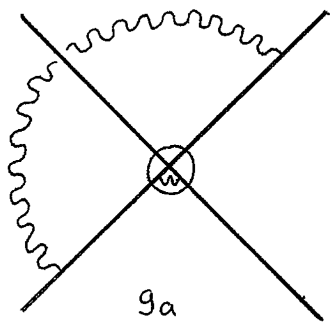
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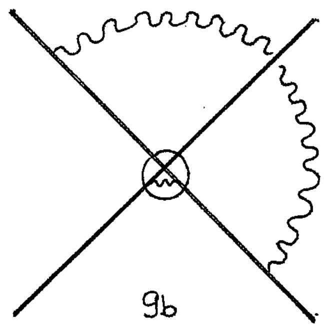
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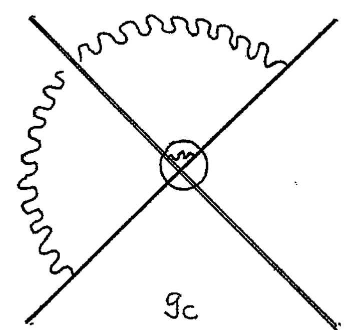
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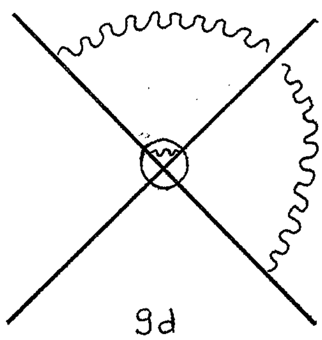
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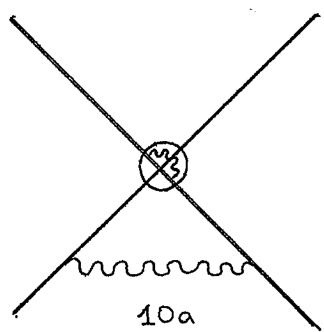
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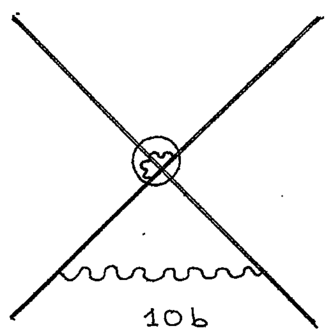
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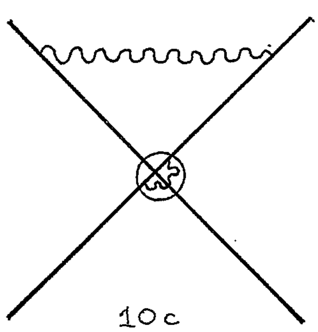
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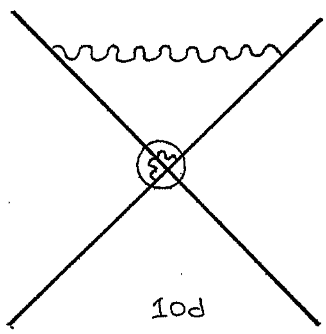
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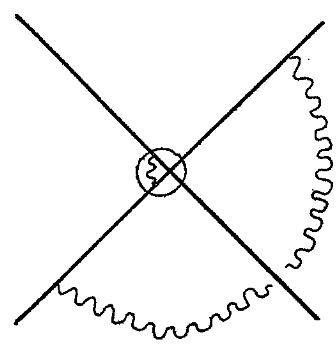
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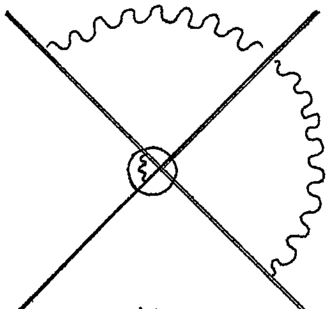
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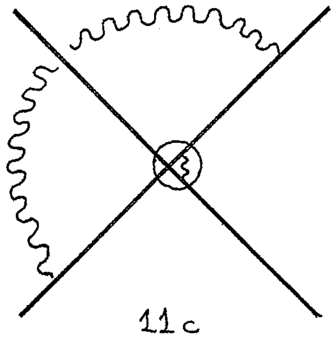
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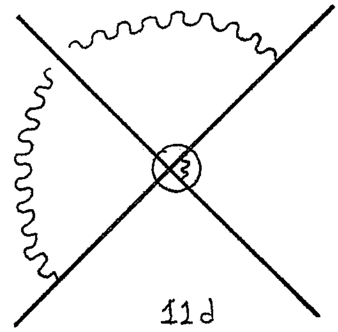
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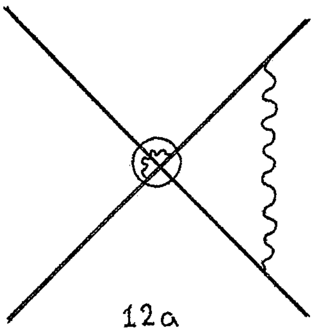
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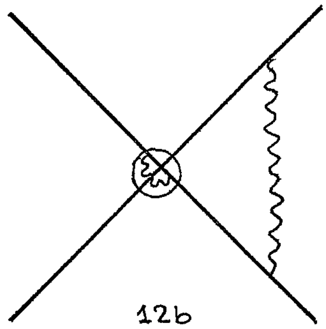
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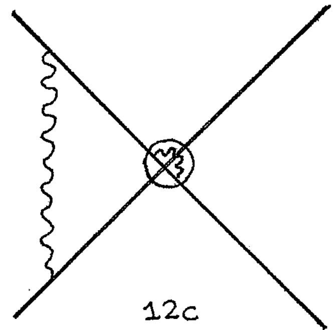
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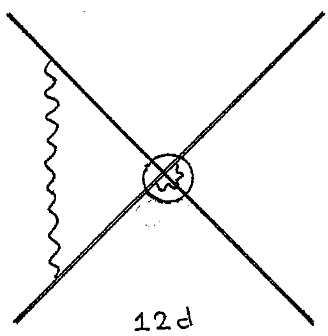
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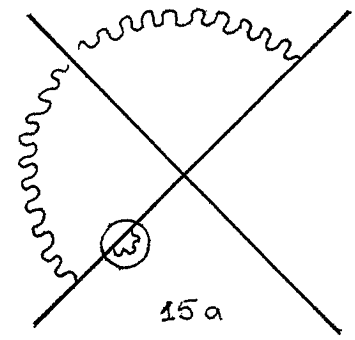
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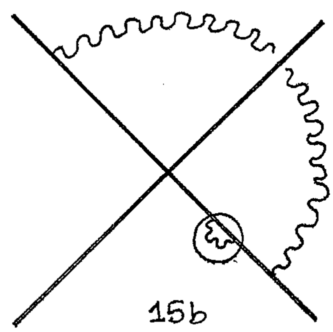
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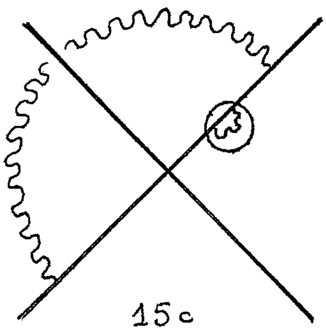
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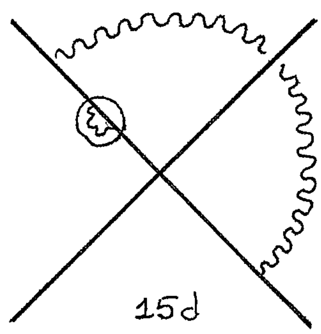
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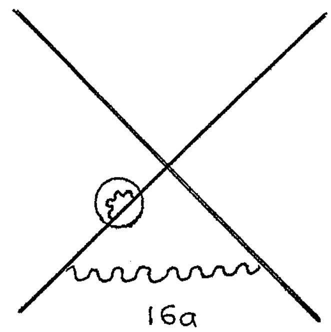
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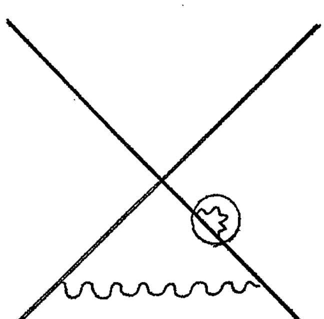
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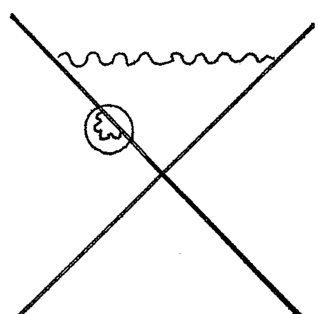
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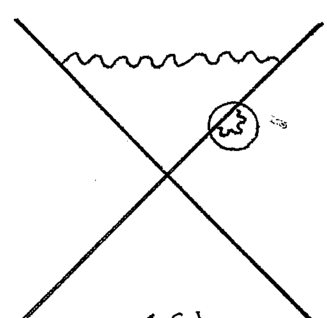
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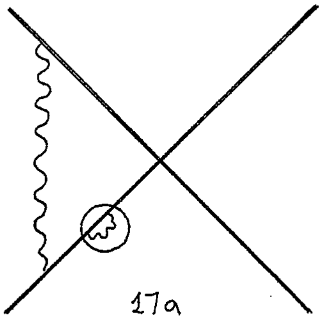

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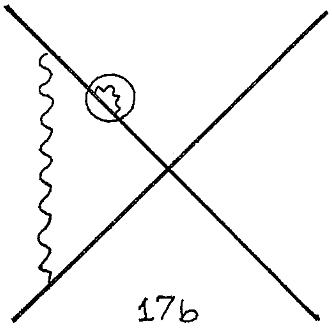

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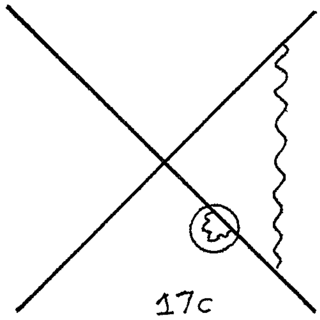

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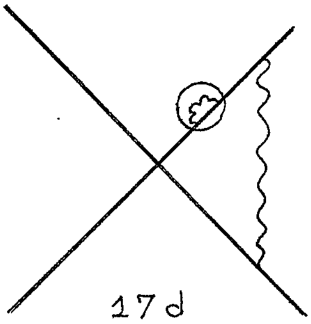

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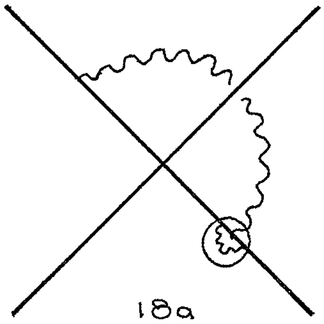

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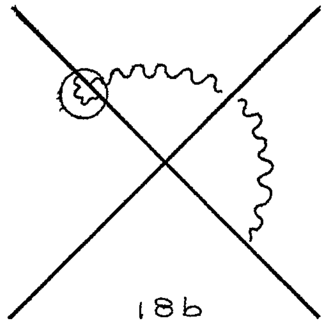

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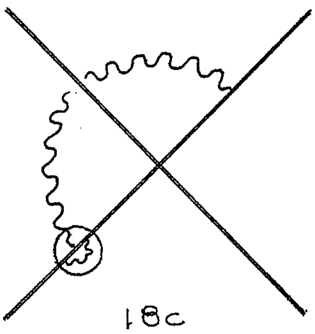

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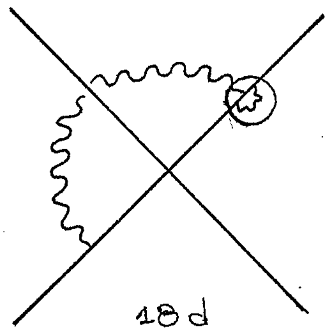

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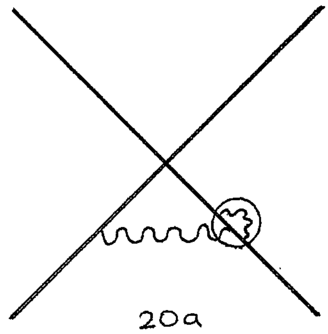

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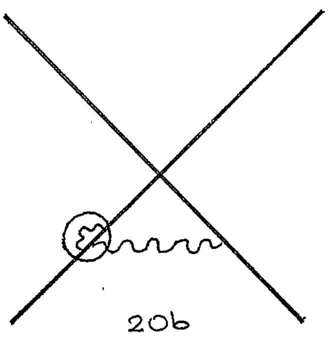

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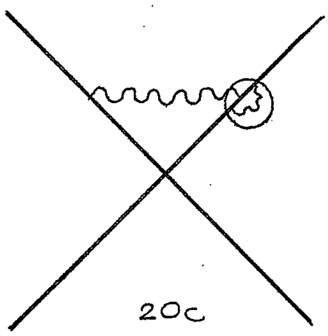

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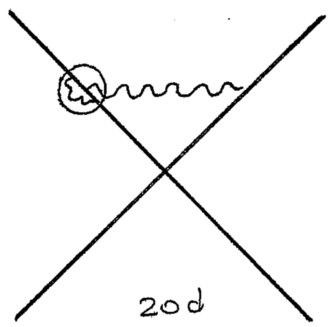

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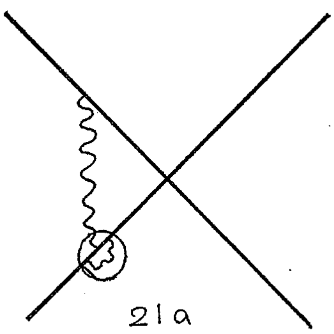

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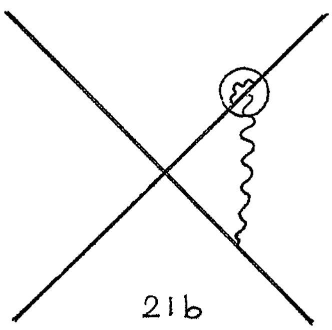

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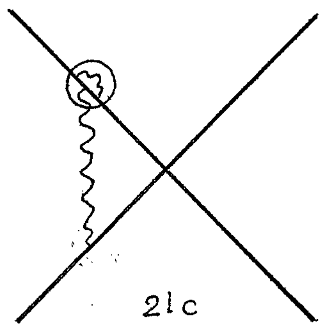

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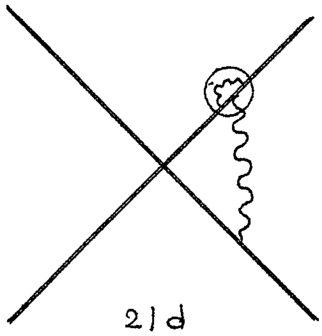

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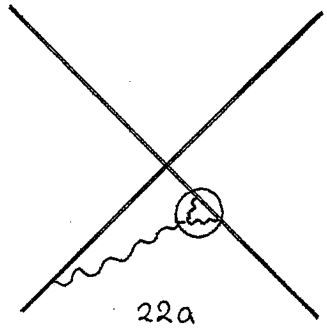

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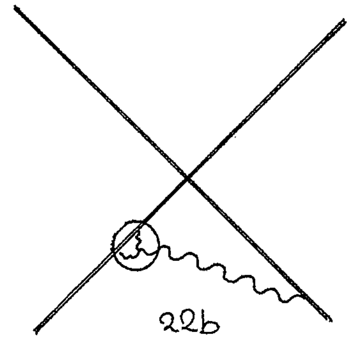

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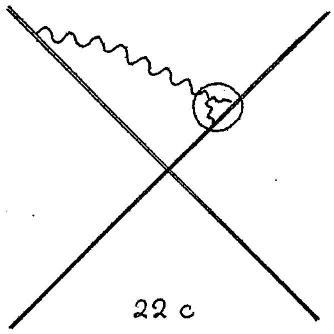

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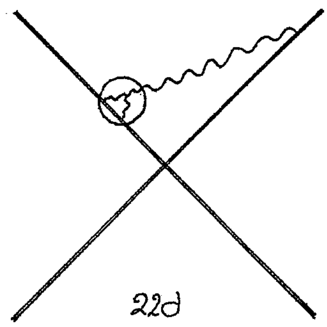

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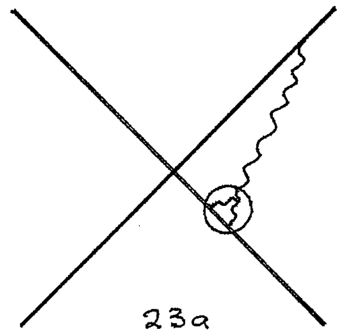

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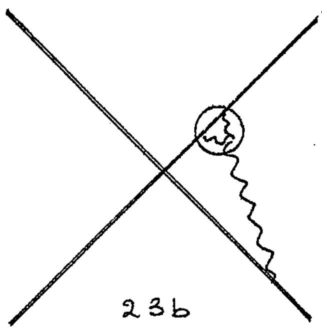

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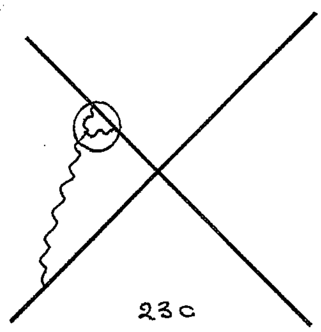

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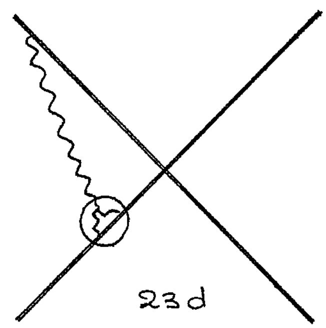

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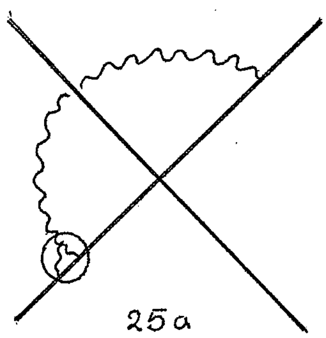

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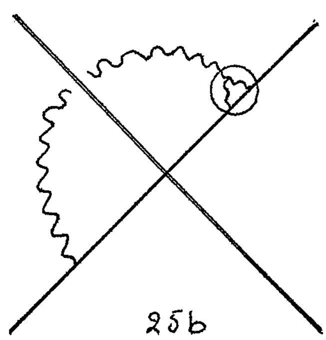

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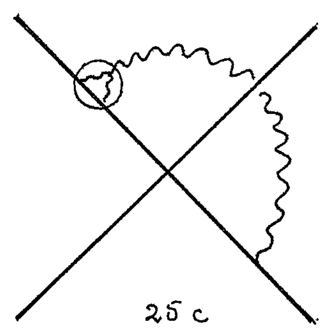

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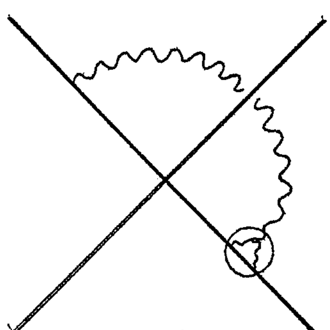

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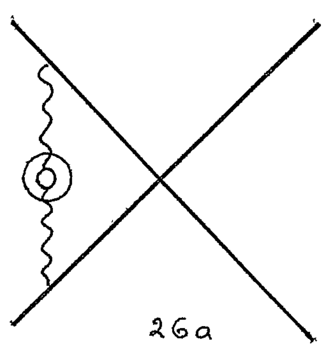

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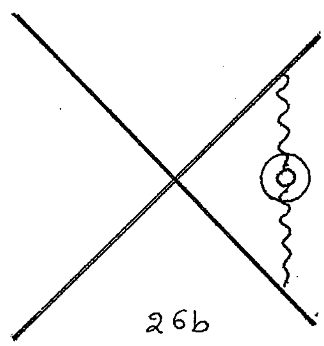

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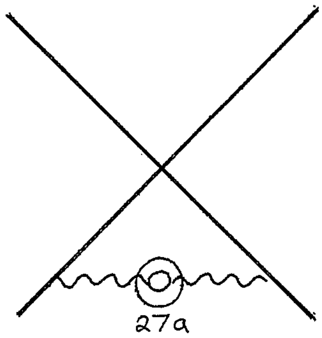

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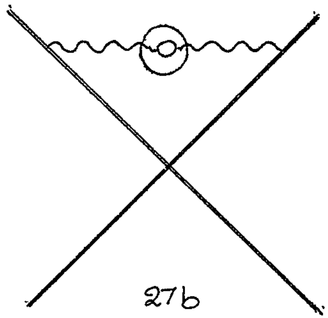

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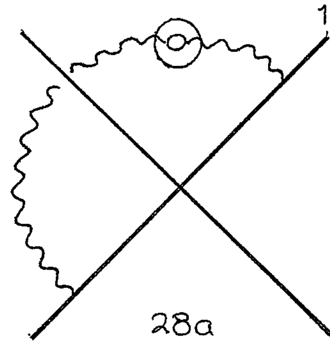

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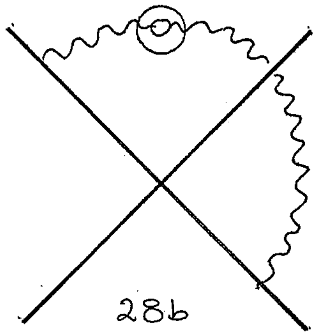

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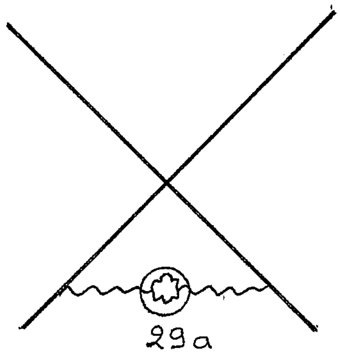

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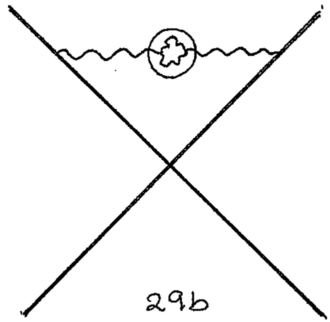

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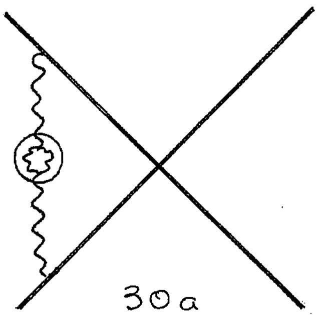

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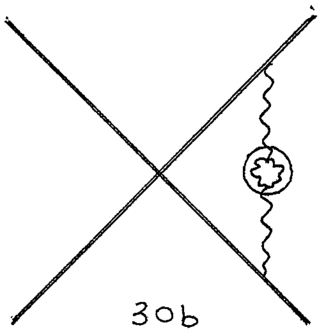

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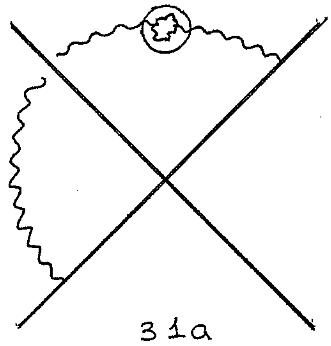

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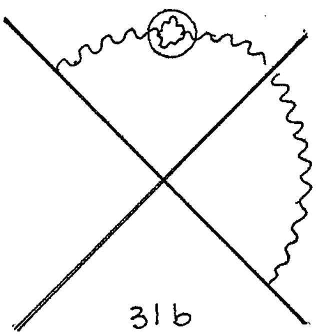

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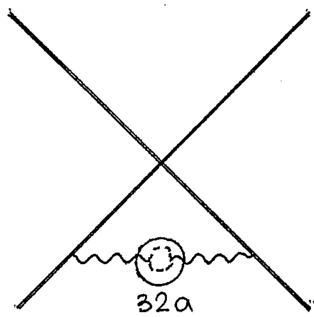

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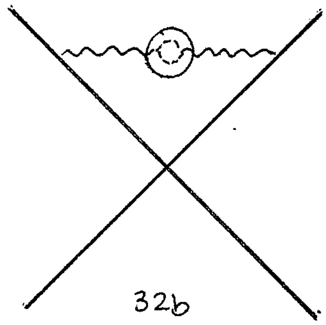

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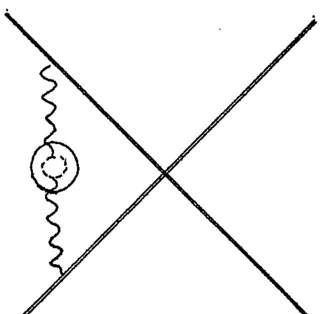

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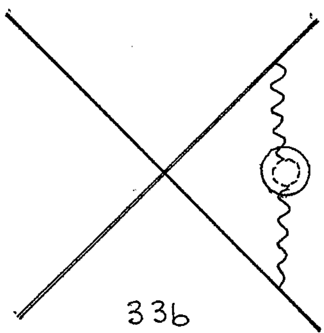

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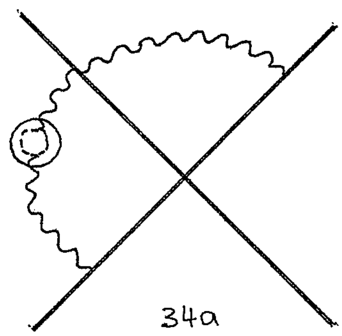

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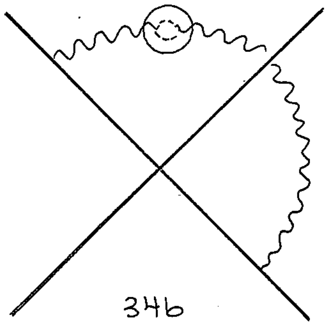

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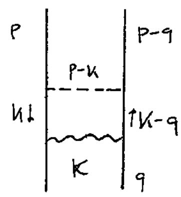
Appendix K

We evaluate the diagrams in Fig(5.3) in $\bar{g}^2$ order where $q^2 = p^2 = M_w^2 = (2pq)$



Fig(I.1)

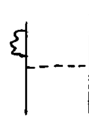
Throwing away terms proportional to $\not q$ and $\not p$ (from the equation of motion) we get for the diagram in Fig(I.1) :

$$(-g^2)\frac{1}{M_w^2}\, g_w^2 \left\{ \frac{i\pi^2}{(2\pi)^4}\,\frac{1}{2}\, 16 \int_0^1 dx\,dy \frac{(1-x)\,y^2}{[x(1-x)y^2-(1-y^2)]} \right\} (t^a)(t^a)$$

$$= (-g^2)\frac{1}{M_w^2}\, g_w^2 \left\{ \frac{i}{16\pi^2}\; 8\cdot(1.4464) \right\} (t^a)(t^a) \qquad (K.1)$$

Diagram  will give :

$$(-g^2)\frac{1}{M_w^2}\, g_w^2 \left\{ \frac{i}{16\pi^2}\, 2\,(1.4464) \right\} (t^a)(t^a) \qquad (K.2)$$

The remaining diagram  has an infinite part proportional to $\Gamma(2-n/2)$, which cansels when the diagram  is taken into account.

Thus the sum of diagrams in Fig(5.3) is :

$$(-g^2)\frac{1}{M_w^2}\, g_w^2\, \frac{i}{16\pi^2}\; 20\cdot(1.4464)\,(t^a)(t^a) =$$

$$= (-g^2)\frac{1}{M_w^2}\, g_w^2\, \frac{i}{16\pi^2}\; 20\,(1.4464)\, \left\{ \tfrac{1}{6}\,\hat{O}^+ - \tfrac{1}{3}\,\hat{O}^- \right\} \qquad (K.3)$$

So we get for $\dfrac{\partial \tilde{C}}{\partial g^2}$:

$$\frac{\partial \tilde{C}}{\partial g^2} = 20 \cdot (1.4464) \frac{1}{3} \quad \text{for} \quad C_{\hat{6}-} \tag{K.4}$$

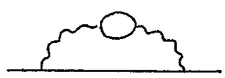
$$\frac{\partial \tilde{C}}{\partial g^2} = -20 \cdot (1.4464) \frac{1}{6} \quad \text{for} \quad C_{\hat{6}+} \tag{K.5}$$

(the other factors in eq(I.3) will appear when the final integration (eq(5.1)) is performed).

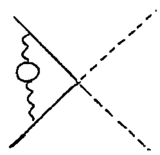
Appendix L

The Ward identity provides a check for a number of diagrams, namely those in which the gluon exchange is within the same current (I would like to thank Dr C. T. Sachrajda for suggesting this check). For example the sum of the contribution of the diagrams 4a, 17a and 17b must be minus the contribution of diagram  to γ_F. From reference (5), chapter 5, the contribution of the above diagram to γ_F is : $\frac{1}{2}(\frac{4}{3})^2 \frac{g^4}{(16\pi^2)^2} = \frac{g^4}{(16\pi^2)^2}\,0.8888$, while the sum of the contribution of 4a+17a+17b is: $-\frac{g^4}{(16\pi^2)^2}\,0.9821$. The difference is due to the numerical evaluation of the Feynman parameter integrals.

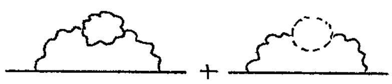
Checking the other diagrams we get:



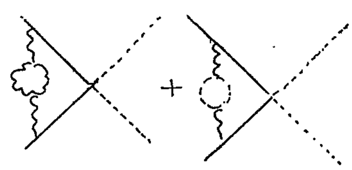
$\frac{g^4}{(16\pi^2)^2}\ 5.333$



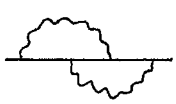
$-5.333\,\frac{g^4}{(16\pi^2)^2}$



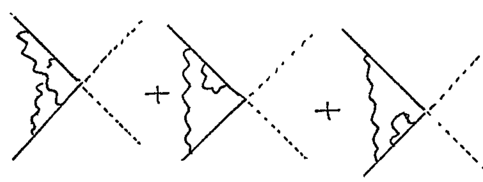
$-\frac{g^4}{(16\pi^2)^2}\,10.00$



$+\,10.58\,\frac{g^4}{(16\pi^2)^2}$

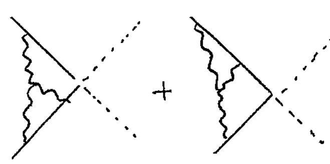


$-\frac{g^4}{(16\pi^2)^2}\ 0.22$



$+0.23\,\frac{g^4}{(16\pi^2)^2}$

$-\frac{g^4}{(16\pi^2)^2}\ 22.00$



$+\,21.63\,\frac{g^4}{(16\pi^2)^2}$

Therefore, within the computer accuracy, the Ward identity is satisfied.