Low-Energy Theorems and Spectral Density of the Dirac Operator in AdS/QCD

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AdS/QCD Models

- AdS/QCD – a new approach to studying strong interactions [J. Erlich, E. Katz, D. T. Son, M. A. Stephanov, 2005; A. Karch, E. Katz, D. T. Son, M. A. Stephanov, 2006].
- In order to understand the underlying structure of the QCD dual two basic types of models have been developed:
  - ten-dimensional "top-down" models [hep-th/0412141, hep-th/0205236],
  - simpler "bottom-up" AdS/QCD models [hep-ph/0501128, hep-ph/0602229, hep-ph/0501218, 0902.1998 [hep-ph]].
- Low-energy theorems (precise equations in QCD) are a good test of holographic models.

Five-Dimensional Effective Action

$$S_{5D} = \int d^5x\sqrt{g}e^{-\Phi}\mathrm{tr}\left\{\Lambda^2\left(|DX|^2 + \frac{3}{R^2}|X|^2 + \frac{\kappa}{R^2}|X|^4\right) - \frac{1}{4g_5^2}(F_L^2 + F_R^2)\right\}$$

with a metric $ds^2 = \dfrac{R^2}{z^2}(-dz^2 + dx_\mu dx^\mu)$.

- "Hard-wall": $\Phi(z) \equiv 0,\ \kappa = 0,\ 0 \leqslant z \leqslant z_m$,
  background $X(z) = \frac{1}{2}(mz + \sigma z^3)$,
- "Soft-wall": $\Phi(z) \sim \lambda z^2 (z \to \infty),\ \kappa \neq 0,\ 0 \leqslant z < \infty$,
  background $X(z) \sim \frac{1}{2}(mz + \sigma z^3)$ for $z \to 0$.

$$\begin{aligned}
L_\mu^a(x, z=0) &= \text{source of } \bar{q}_L(x)\gamma_\mu t^a q_L(x), \\
R_\mu^a(x, z=0) &= \text{source of } \bar{q}_R(x)\gamma_\mu t^a q_R(x), \\
\lim_{z\to 0} \frac{2}{z} X^{\alpha\beta}(x, z) &= \text{source of } \bar{q}_L^\alpha(x) q_R^\beta(x) \\
&= m\delta^{\alpha\beta} \text{ in the absence of (pseudo)scalar currents.}
\end{aligned}$$

Parameters of the model

There are several free parameters:

- g_5: $\dfrac{g_5^2}{R} = \dfrac{12\pi^2}{N_c}$ from QCD Sum rules,
- Λ: $\Lambda^2 R^3 = \dfrac{N_c}{4\pi^2}$ from QCD Sum rules,
- z_m is fixed by the ρ-meson mass,
- λ is fixed by the slope of the Regge trajectory,
- κ by the mass splitting of the axial and vector mesons.
- Quark condensate: $\Sigma = N_f C = N_f \sigma \Lambda^2 R^3$.

Low-Energy Theorems

Corollaries of the axial Ward identities in terms of:

- Quark currents $S_i(x)$, $P_j(y)$,
- Chiral Lagrangian (L_i – parameters of the NLO Lagrangian of order of $\mathcal{O}(p^4)$, $B = G_\pi/F_\pi$),
- Spectral density $\rho(\lambda, m)$,
- Topological charge density $Q(x) \sim \mathrm{tr} F_{\mu\nu}(x)\tilde{F}^{\mu\nu}(x)$.

$$i\int d^4x\,\langle \delta_{ij}S_0(x)S_0(0) - P_i(x)P_j(0)\rangle = -\frac{G_\pi^2\delta_{ij}}{m_\pi^2} + \delta_{ij}\frac{B^2}{8\pi^2}(L_3 - 4L_4 + 3)$$

$$= 2\delta_{ij}\int d\lambda \left(\frac{m\frac{\partial}{\partial m}\rho(\lambda, m)}{(\lambda^2 + m^2)} - \frac{2m^2\rho(\lambda, m)}{(\lambda^2 + m^2)^2} \right), \quad (1)$$

$$i\int d^4x\,\langle S_i(x)S_j(0) - \delta_{ij}P_0(x)P_0(0)\rangle$$

$$= \delta_{ij}\int d\lambda \frac{4m^2\rho(\lambda, m)}{(\lambda^2 + m^2)^2} - 2\delta_{ij}\frac{\int d^4x\,\langle Q(x)Q(0)\rangle}{m^2 V}, \quad (2)$$

$$i\int d^4x\,\langle P_3(x)P_0(0)\rangle$$

$$= 2(m_u - m_d)m\int d\lambda \frac{\rho(\lambda, m)}{(\lambda^2 + m^2)^2} - (m_u - m_d)\frac{\int d^4x\,\langle Q(x)Q(0)\rangle}{m^3 V}. \quad (3)$$

[J. Gasser and H. Leutwyler, 1984]

Zero momentum correlation functions of QCD currents

Correlation functions are calculated via the AdS/CFT prescription:

$$\mathcal{Z}_{QCD}[\mathcal{J}_I(x_\mu)] = \exp\left(iS_{5D\ \ classical}\right)\Big|_{\Phi_I(0,x_\mu)=\mathcal{J}_I(x_\mu)} \Rightarrow$$

$$\langle \mathcal{O}_I(x)\mathcal{O}_J(y)\rangle = -\ e^{-iS_{5D\ \ classical}} \frac{\delta}{\delta\Phi_I(0,x)} \frac{\delta}{\delta\Phi_J(0,y)} e^{iS_{5D\ \ classical}}\Bigg|_{\Phi_I(0,x_\mu)=0}$$

At zero momentum for N_f quark flavors with equal masses

$$i\,\langle P_i(0)P_j(0)\rangle = \delta_{ij}\frac{C}{m} = \delta_{ij}\frac{G_\pi^2}{m_\pi^2},$$

where $\left\langle \bar{q}^\alpha q^\beta \right\rangle = C\delta^{\alpha\beta}$ in the chiral limit, $\Sigma = \langle \bar{q}^\alpha q_\alpha \rangle$. One can see that we obtain a singularity corresponding to the pion exchange. The pole residue is the same for the middle and left-hand sides of Eqn. (1).

In the particular $N_f = 2,\ m_u \neq m_d$ case

$$i\left\langle P_i(0)P_j(0)\right\rangle = \delta_{ij}\frac{C}{m} - \frac{C\Delta m}{2m^2}(\delta_{i0}\delta_{3j} + \delta_{j0}\delta_{3i} - i\delta_{i1}\delta_{2j} - i\delta_{j1}\delta_{2i}),$$

where $m = \dfrac{m_u + m_d}{2},\ \Delta m = m_u - m_d.$
Scalar current correlators are calculated analogously and are regular in the chiral limit. Thus for a zero momentum

$$i\left\langle \delta_{ij}S_0(0)S_0(0) - P_i(0)P_j(0)\right\rangle \sim i\left\langle S_i(x)S_j(0) - \delta_{ij}P_0(x)P_0(0)\right\rangle$$
$$\sim -\delta_{ij}\frac{C}{m},$$
$$i\left\langle P_3(0)P_0(0)\right\rangle \sim -\frac{C\Delta m}{2m^2}.$$

Parameters of the Chiral Lagrangian

Another point of view on the holographic models:

- Five-dimensional fields on the AdS boundary are sources of the QCD currents.
- QCD currents are sources of the mesons.
- Kaluza–Klein modes $\propto$ meson wavefunctions with corresponding quantum numbers.
- If we integrate out the dynamics along the z axis, the 5D action will generate an effective chiral Lagrangian.

Similar to the "top-down" models [T. Sakai, S. Sugimoto, 2005].

Gauge fields can be combined into

$$V_\mu^a = L_\mu^a + R_\mu^a,\ A_\mu^a = L_\mu^a - R_\mu^a$$

One can fix the gauge $V_z = A_z = \partial^\mu V_\mu = 0$, so that A_μ retains a longitudinal component: $A_\mu = A_{\perp\mu} + \partial_\mu \phi$. $\phi \propto$ the source of the pseudoscalar current.

KK decomposition:

$$\phi^a(z,x) = \sum_n f_\phi^{(n)}(z)\phi^{a(n)}(x),$$

$$f_\phi^{(n)}(z) - \text{E.o.M. solution in AdS},\ \phi^{a(0)}(x) \propto \pi^a(x).$$

Having integrated out all z-dependence in the 5D action of the lowest KK-mode we obtain

$$S_{5D} \to A_1 \cdot \frac{1}{2}\partial_\mu\phi^{(0)}\partial^\mu\phi^{(0)} + A_2 \cdot [\partial_\mu\phi^{(0)}, \partial_\nu\phi^{(0)}]^2 + A_3 \cdot m\partial_\mu\phi^{(0)}\partial^\mu\phi^{(0)}.$$

This allows to find the parameters explicitly $L_{1,2,3} \propto \frac{A_2}{A_1^2},\ L_4 \propto \frac{A_3}{A_1}$. The following (universal for holographic models) equation holds

$$L_3 = -3L_2 = -6L_1.$$

Parameters L_i are regular in the chiral limit.

Spectral Density of the Dirac Operator

The Dirac operator $\hat{D} \equiv \gamma^\mu(\partial_\mu + ig_s A_\mu)$ has no dual AdS description, and its spectral density $\rho(\lambda) = \frac{1}{V}\left\langle \sum_n \delta(\lambda - \lambda_n) \right\rangle_A$, where $\{\lambda_n\}$ are the eigenvalues of $i\hat{D}$, has no direct AdS/QCD interpretation similar to the Chiral Lagrangian and QCD partition function.

However one can express $\rho(\lambda)$ via a partition function of a QCD-like theory whith a dual description
[S. F. Edwards and P. W. Anderson, 1975;
J. J. M. Verbaarschot and M. R. Zirnbauer, 1984;
K. B. Efetov, 1983].

$$\rho(\lambda) = \frac{1}{V}\left\langle \sum_n \delta(\lambda - \lambda_n) \right\rangle_A = \frac{1}{\pi V}\left\langle \lim_{\mu\to 0} \sum_n \frac{\mu}{\mu^2 + (\lambda - \lambda_n)^2} \right\rangle_A$$

$$= \frac{1}{2\pi V} \lim_{\mu\to 0} \frac{\partial}{\partial\mu} \left\langle \log Det[i\hat{D} - \lambda - i\mu] + \log Det[i\hat{D} - \lambda + i\mu] \right\rangle_A$$

We use a so-called "replica trick" : $\log z = \frac{\partial}{\partial n} z^n \Big|_{n=0}$

$$\rho(\lambda) = \frac{1}{\pi V} \lim_{\mu \to 0} \frac{\partial}{\partial \mu} \lim_{n \to 0} \frac{\partial}{\partial n} \Re \left\langle Det^n [i\hat{D} - \lambda - i\mu] \right\rangle_A$$
$$= \frac{1}{\pi V} \lim_{\mu \to 0} \frac{\partial}{\partial \mu} \lim_{n \to 0} \frac{\partial}{\partial n} \Re \int DA\ e^{iS_{YM}[A]} \times \mathcal{Z}_{quarks}(m_q = m,\ N_f)[A]$$
$$\times \mathcal{Z}_{ghosts}(m_q = \lambda + i\mu,\ n \cdot N_f)[A].$$

This partition function can be calculated in "hard-wall" AdS/QCD, with a boundary condition on the scalar field:

$$\lim_{z \to 0} \frac{2}{z} X = \begin{pmatrix} m & & & & & \\ & \ddots & & & 0 & \\ & & m & & & \\ & & & \lambda + i\mu & & \\ & 0 & & & \ddots & \\ & & & & & \lambda + i\mu \end{pmatrix} \begin{matrix} 1 \\ \vdots \\ N_f \\ N_f + 1 \\ \vdots \\ N_f(n+1) \end{matrix}$$

Result is the following:

$$\rho(\lambda) = -\frac{1}{\pi}\left(\Sigma(m) + m\frac{d}{dm}\Sigma(m)\right)\Bigg|_{m=\lambda}.$$

A QCD result [V. A. Novikov, M. A. Shifman, A. I. Vainshtein, V. I. Zakharov, 1981] $\Sigma(m) = \Sigma(0)\left(1 - \frac{3m_\pi^2 \log m_\pi^2/\mu_{hadr}^2}{32\pi^2 F_\pi^2}\right)$ leads to a formula

$$\rho(\lambda) = -\frac{1}{\pi}\Sigma(0)\left(1 - \frac{3\Sigma(0)}{8\pi^2 N_f F_\pi^4}\lambda - \frac{3\Sigma(0)}{4\pi^2 N_f F_\pi^4}\lambda \log\lambda\right), \lambda > 0.$$

So:

- the result agrees with the Casher–Banks identity $\rho(0) = -\Sigma(0)/\pi$ [T. Banks and A. Casher, 1980],
- up to a $\propto N_f^2 - 4$ factor reproduces the result of Smilga and Stern [A. Smilga and J. Stern, 1993] for the term linear in λ,
- has terms $\propto \lambda \log \lambda$.

Drawbacks of this result are the following:

- No dependence on mass;

- No terms $\propto \lambda^2$ and higher.

It seems that we might obtain a more precise result for $\rho'(0)$ if we either manage to formulate a consistent "soft-wall" model with different flavors or take into account the N_f–dependence of the 5D metric ($\sim$ to the flavor brane back-reaction).

Compatibility of AdS/QCD models with low-energy theorems in the chiral limit

Integrals with $\rho(\lambda)$ in the right-hand sides of the theorems $(1-3)$ yield

$$\int d\lambda \, \frac{m^2\rho(\lambda)}{(\lambda^2+m^2)^2} \sim \frac{C}{m}, \; \int d\lambda \, \frac{m\Delta m\rho(\lambda)}{(\lambda^2+m^2)^2} \sim \frac{C\Delta m}{m^2},$$

and we use a result by [Katz, Schwarz] for the topological charge density 2-point correlator.

In the $m \to 0$ limit for each theorem (1, 2, 3) we get equal pole residues on all sides. Thus, AdS/QCD models are compatible with low-energy theorems in the chiral limit.

Conclusion

- We have demonstrated the agreement of the AdS/QCD model in the chiral limit with the low-energy theorems.
- We have proposed a way to calculate the spectral density of the Dirac operator in AdS/QCD.
- Result in the "hard-wall" model agrees with the Casher–Banks identity and up to a numerical N_f–depending factor reproduces the Smilga and Stern result.
- Testing the theorems beyond the chiral limit might probably bring us closer to understanding the structure of the AdS/QCD models.