Generalized Holography

Ioannis Papadimitriou

Swansea University

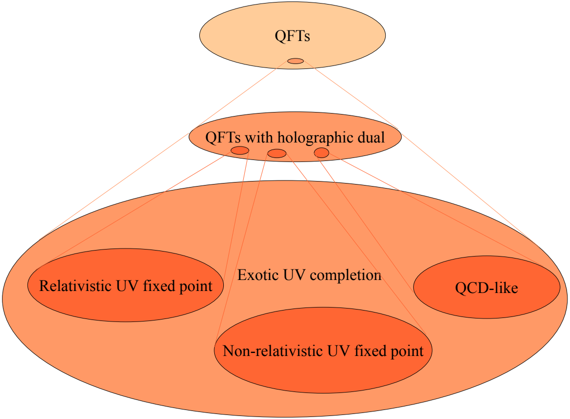
United Kingdom

Fifth Aegean Summer School
From Gravity to Thermal Gauge Theories: The AdS/CFT correspondence

Milos 2009

22 September 2009

09xx.xxxx [hep-th] [I. P.] to appear


QFTs
QFTs with holographic dual
Relativistic UV fixed point
Exotic UV completion
QCD-like
Non-relativistic UV fixed point

QFT UV completion vs string background asymptotics

- The holographic (radial) coordinate of the string dual of a QFT measures the energy scale of the QFT (i.e. holography geometrizes the RG flow)
- Classification of asymptotics of string backgrounds corresponds to a classification of UV completions of QFTs that admit a holographic dual
- One can identify the following possibilities:

Relativistic UV fixed point	$\leftrightarrow$	AdS
Non-relativistic UV fixed point	$\leftrightarrow$	Lifshitz Schrödinger
QCD-like	$\leftrightarrow$	logarithmically running dilaton
Exotic UV completion (e.g. higher dimensional QFT, LST,...)	$\leftrightarrow$	Klebanov-Strassler Maldacena-Núñez ...

Why generalized holography?

- From the above possibilities only the case of a relativistic UV fixed point, corresponding to an asymptotically locally AdS string background is well understood (see Kostas' lectures)
- The possibility of non-relativistic UV fixed points arises in various interesting condensed matter systems
- Attempts to find a holographic dual of QCD typically lead to some QFT with exotic UV completion
- To use holography to study QFTs with any UV completion other than a relativistic fixed point we need to be able to establish a holographic dictionary and do holographic renormalization for the corresponding non-asymptotically AdS backgrounds

Supergravity approximation

- Ultimately an holographic duality involves a QFT on the one side and a quantum theory of gravity, such as string theory, on the other
- There is usually some limit of the QFT in question (e.g. large N, large 't Hooft coupling) that reduces the dual string theory to supergravity
- I will assume for a start that such a limit exists and I will consider only holographic models within supergravity

Classification of supergravities with an holographic dual

- Since the string background asymptotics determines the UV behavior of the QFT, we can classify QFTs by classifying supergravity theories in non-compact spaces and their asymptotic boundary conditions
- One is then faced with the question: *do all supergravities and/or boundary conditions correspond to the supergravity approximation of the string holographic dual of a quantum theory that does not contain gravity?*
- Lacking the general answer to this question, we will make some extra assumptions about the nature of the holographic dictionary, which will restrict the admissible supergravity/boundary conditions possibilities

Assumptions on the holographic dictionary

1. The holographic model we are seeking is supposed to describe a local QFT, at least over a range of energy scales.
2. The observable quantities of the local QFT, as captured by correlation functions of gauge invariant operators are obtained from the supergravity theory via the standard relation

$$W[J^\alpha] = -S_{\text{on-shell}}[\phi^\alpha; \phi \sim J^\alpha].$$

   where $W[J^\alpha]$ indicates the generating functional of connected correlation functions of a local gauge-invariant operator $\mathcal{O}_\alpha$, J^α is the source, and $\phi \sim J^\alpha$ indicates that the source enters in the supergravity description as some form of a boundary condition.
3. Any long-distance divergences of the on-shell supergravity action should be removable with a *finite* number of *local* in the sources counterterms, i.e.

$$W_{\text{ren}}[J^\alpha] = -S_{\text{on-shell}}[\phi^\alpha; \phi \sim J^\alpha] - S_{\text{local}}[J^\alpha].$$

Generalized holographic renormalization

- To address the long-distance divergences of the supergravity on-shell action we need a generalization of holographic renormalization, that is a systematic analysis of the long-distance properties of supergravities in non-compact spacetimes
- There is a natural way to formulate this problem:
  - In non-compact spacetimes there is a notion of asymptotic infinity where spacetime locally looks like
    $$\mathcal{M} \sim \mathbb{R} \times \Sigma_r,$$
    where r is a radial coordinate along which asymptotic infinity is approached, and Σ_r is a constant r slice.
  - This facilitates a Hamiltonian analysis of the supergravity theory in a $d+1$ decomposition, where the bulk supergravity fields are traded for the induced fields on the d-dimensional slices Σ_r and r serves as the Hamiltonian "time".
  - Hamilton-Jacobi theory gives directly the on-shell action
  - This provides us with a clear-cut first criterion for which supergravity theories/backgrounds can serve as legitimate holographic duals of local QFTs:

    Such theories must admit a radial Hamiltonian formulation and the corresponding Hamilton-Jacobi equation must admit solutions of the form
    $$\mathcal{S}_r = \mathcal{S}_{\rm local} + \mathcal{S}_{\rm finite},$$
    where $\mathcal{S}_{\rm local}$ is local in the induced fields on Σ_r and $\mathcal{S}_{\rm finite}$ admits a well-defined non-zero limit as $r \to \infty$.

Toy example...

- To illustrate how this criterion imposes constraints on the possible supergravity theories and backgrounds, let us consider a free massive scalar field in a pure gravitational background

$$ds^2_{d+1} = dr^2 + \gamma_{ij} dx^i dx^j = dr^2 + e^{2A(r)} d\vec{x}^2.$$

- The equation of motion following from the Lagrangian

$$L = \frac{1}{2} \int d^d x \sqrt{\gamma} \left(\dot{\varphi}^2 + \gamma^{ij} \partial_i \varphi \partial_j \varphi + m^2 \varphi^2 \right),$$

  is

$$\left(\partial_r^2 + d\dot{A}\partial_r - p^2 e^{-2A} - m^2 \right) \tilde{\varphi} = 0.$$

- The corresponding on-shell action is given by

$$\mathcal{S}_r = \frac{1}{2} \int_{\Sigma_r} d^d x \sqrt{\gamma} \varphi \dot{\varphi}.$$

- From Hamilton-Jacobi theory and the fact that the e.o.m. is linear we know we can write

$$\dot{\varphi} = f(A; p^2) \varphi,$$

  where the function $f(A; p^2)$ satisfies the equation

$$(f' + df)\dot{A} + f^2 - p^2 e^{-2A} - m^2 = 0.$$

- The function $A(r)$, or equivalently the function $\omega(A) \equiv \dot{A}$, specifies the background, while $f(A;p^2)$ determines the on-shell action:

$$\mathcal{S}_r = \frac{1}{2}\int_{\Sigma_r} d^d x\sqrt{\gamma}\varphi f(A;p^2)\varphi.$$

- For backgrounds that satisfy the above assumptions about the holographic dictionary it should be possible to find solutions to the above equation for f that take the form

$$f = \sum_{k=0}^{n} f_k(A)p^{2k} + \tilde{f}, \quad n < \infty.$$

- In order for $\tilde{f}$ to give a finite contribution to the on-shell action we must also have

$$\tilde{f}(A;p^2) \sim e^{-\int dA(d+2f_0/\omega)}, \text{ as } A \to \infty.$$

- f_0 satilfies

$$(f_0' + df_0)\omega(A) + f_0^2 - m^2 = 0,$$

while f_k, $k > 0$, satisfy *linear* equations:

$$\omega(A)f_k' + (d\omega(A) + 2f_0)f_k = \mathcal{R}_k, \quad k > 0,$$

where

$$\mathcal{R}_1 = e^{-2A}, \quad \mathcal{R}_k = -\sum_{l=1}^{k-1} f_l f_{k-l}, \quad k > 1.$$

- The homogeneous solution of the linear equations can be discarded as they are finite:

$$f_k^{\text{hom}} = c_k e^{-\int dA(d+2f_0/\omega)}, \text{ as } A \to \infty.$$

- The inhomogeneous solutions are given by

$$f_k = U(A)\int^A \frac{d\bar{A}}{U(\bar{A})}\mathcal{R}_k(\bar{A}), \quad U(A) \equiv e^{-\int dA(d+2f_0/\omega)}.$$

- The condition that $n < \infty$ requires that there exists n s.t.

$$f_{n+1} = o\left(e^{-\int dA(d+2f_0/\omega)}\right).$$

- This essentially restricts the background geometry to be asymptotically AdS: $A \sim r$

Improved Holographic QCD

- Improved Holographic QCD (IHQCD) [Gursoy, Kiritsis, Nitti] was put forward as an effective 5-dimensional supergravity holographic model for low energy QCD, allowing for the logarithmic running of the gauge coupling in the UV.
- The above approach lets us decide if it satisfies the above assumptions about the holographic dictionary and, if it does, to construct the local counterterms.
- See Elias' talk later this week...

IHQCD supergravity action

- IHQCD is described by the action

$$S = -\frac{1}{2\kappa^2}\left(\int_{\mathcal{M}} d^{d+1}x\sqrt{g}\,(R - \partial_\mu\varphi\partial^\mu\varphi - Z(\varphi)\partial_\mu\chi\partial^\mu\chi + V(\varphi)) + G.H.\right),$$

  where $(2\kappa^2)^{-1} = (16\pi G_5)^{-1} = M_{pl}^3 N_c^2$, $\varphi = \xi\log\lambda$ is the dilaton, dual to the operator $\mathrm{Tr}\,F^2$, and χ is the axion, dual to $\mathrm{Tr}\,F\wedge F$.
- $V(\lambda)$ and $Z(\lambda)$ completely determine the bulk action. They are respectively related to the exact beta functions of the 't Hooft coupling and the axion in the gauge theory.
- Asymptotic freedom requires that they admit expansions around $\lambda = 0$ of the form

$$Z(\lambda) = (M_{pl}^3 N_c^2)^{-1}\sum_{n=0}^{\infty} Z_n\lambda^n, \qquad V(\lambda) = \frac{12}{\ell^2}\left(1 + \sum_{n=1}^{\infty} V_n\lambda^n\right),$$

  where ℓ is the radius of the AdS corresponding to the UV (free) fixed point of the gauge theory. These expansions correspond to the purturbative expansions of these beta functions.
- Z_n are $\mathcal{O}(N_c^0)$, since the axion is in the RR sector of the string theory and so its kinetic term is $\mathcal{O}(1/N_c^2)$ relative to the rest of the terms in the action.

Radial Hamiltonian analysis

- To proceed with the Hamiltonian analysis of this action, we start with the standard ADM-like decomposition of the metric

$$ds^2 = (N^2 + N_i N^i)dr^2 + 2N_i dr dx^i + \gamma_{ij} dx^i dx^j,$$

where N and N_i are respectively the lapse and shift functions, and γ_{ij} is the induced metric on the hypersurfaces Σ_r of constant radial coordinate r.

- The Lagrangian then takes the form

$$\begin{aligned} 2\kappa^2 L \quad = \quad & -\int_{\Sigma_r} d^d x \sqrt{\gamma} N \left(R[\gamma] + K^2 - K^i_j K^j_i \right) \\ & + \int_{\Sigma_r} d^d x \sqrt{\gamma} N \left\{ \frac{1}{N^2}(\dot{\varphi}^2 + Z(\varphi)\dot{\chi}^2) - \frac{2N^i}{N^2}(\dot{\varphi}\partial_i\varphi + Z(\varphi)\dot{\chi}\partial_i\chi) \right. \\ & \left. + \left(\gamma^{ij} + \frac{N^i N^j}{N^2} \right) (\partial_i\varphi\partial_j\varphi + Z(\varphi)\partial_i\chi\partial_j\chi) - V(\varphi) \right\}. \end{aligned}$$

where K_{ij} is the extrinsic curvature of the hypersurface Σ_r and $K \equiv K^i_i$.

- The canonical momenta conjugate to γ_{ij}, φ and χ are now obtained respectively as

$$\pi^{ij} = -\frac{1}{2\kappa^2}\sqrt{\gamma}(K\gamma^{ij} - K^{ij}),$$
$$\pi_\varphi = \frac{1}{\kappa^2 N}\sqrt{\gamma}\left(\dot{\varphi} - N^i\partial_i\varphi\right),$$
$$\pi_\chi = \frac{1}{\kappa^2 N}\sqrt{\gamma}Z(\varphi)\left(\dot{\chi} - N^i\partial_i\chi\right).$$

- The Hamiltonian is given by

$$H = \int_{\Sigma_r} d^dx\left(\pi^{ij}\dot{\gamma}_{ij} + \pi_\varphi\dot{\varphi} + \pi_\chi\dot{\chi}\right) - L = \int_{\Sigma_r} d^dx\left(N\mathcal{H} + N_i\mathcal{H}^i\right),$$

where

$$\begin{aligned}\mathcal{H} &= 2\kappa^2\gamma^{-\frac{1}{2}}\left(\pi^i_j\pi^j_i - \frac{1}{d-1}\pi^2 + \frac{1}{4}\pi_\varphi^2 + \frac{1}{4}Z^{-1}(\varphi)\pi_\chi^2\right) \\ &\quad + \frac{1}{2\kappa^2}\sqrt{\gamma}\left(R[\gamma] - \partial_i\varphi\partial^i\varphi - Z(\varphi)\partial_i\chi\partial^i\chi + V(\varphi)\right), \\ \mathcal{H}^i &= -2D_j\pi^{ij} + \pi_\varphi\partial^i\varphi + \pi_\chi\partial^i\chi.\end{aligned}$$

The Hamilton-Jacobi equation

- Hamilton-Jacobi theory now gives the canonical momenta as derivatives of the on-shell action:

$$\pi^{ij} = \frac{\delta \mathcal{S}_r}{\delta \gamma_{ij}}, \quad \pi_\varphi = \frac{\delta \mathcal{S}_r}{\delta \varphi}, \quad \pi_\chi = \frac{\delta \mathcal{S}_r}{\delta \chi},$$

  where γ_{ij}, ϕ^α should be understood as “boundary conditions” on Σ_r.
- This follows from the identity

$$\pi^{ij}\delta\gamma_{ij} + \pi_\varphi \delta\varphi + \pi_\chi \delta\chi = \delta\mathcal{L} + \partial_i v^i(\delta\gamma, \delta\varphi, \delta\chi),$$

  for some local $v^i(\delta\gamma, \delta\phi)$, where $\mathcal{S}_r = \int_{\Sigma_r} d^d x \mathcal{L}(\gamma, \phi)$.
- Inserting these momenta into the Hamiltonian constraint $\mathcal{H} = 0$ leads to the Hamilton-Jacobi equation

$$2\kappa^2\gamma^{-\frac{1}{2}}\left(\left(\gamma_{ik}\gamma_{jl} - \tfrac{1}{d-1}\gamma_{ij}\gamma_{kl}\right)\tfrac{\delta\mathcal{S}_r}{\delta\gamma_{ij}}\tfrac{\delta\mathcal{S}_r}{\delta\gamma_{kl}} + \tfrac{1}{4}\left(\tfrac{\delta\mathcal{S}_r}{\delta\varphi}\right)^2 + \tfrac{1}{4}Z^{-1}(\varphi)\left(\tfrac{\delta\mathcal{S}_r}{\delta\chi}\right)^2\right)$$
$$+\tfrac{1}{2\kappa^2}\sqrt{\gamma}\left(R[\gamma] - \partial_i\varphi\partial^i\varphi - Z(\varphi)\partial_i\chi\partial^i\chi + V(\varphi)\right) = 0$$

Long-distance solution of the H-J equation

- The aim now is to solve this equation for long distances, or for $r \to \infty$. In this limit the on-shell action takes the form

$$\mathcal{S}_{(0)} = \frac{1}{\kappa^2} \int_{\Sigma_r} d^d x \sqrt{\gamma} U(\varphi),$$

  for some function $U(\varphi)$ – without loss of generality U can be taken to be a function of φ only.
- Note that this is a non-trivial statement about the asymptotic form of the induced fields. It implies, for example, that

$$\gamma_{ij} \sim e^{2A} g_{(0)ij}(x), \quad A = -\frac{1}{d-1} \int^{\varphi} \frac{d\bar{\varphi}}{U'(\bar{\varphi})} U(\bar{\varphi}),$$

  where $g_{(0)ij}(x)$ is an arbitrary metric independent of the radial coordinate.

- We can systematically compute corrections to this action as egenfunctions of the operator

$$\delta_\gamma = \int d^d x 2\gamma_{ij} \frac{\delta}{\delta\gamma_{ij}},$$

namely

$$\mathcal{S} = \mathcal{S}_{(0)} + \mathcal{S}_{(2)} + \mathcal{S}_{(4)} + \cdots , \quad \delta_\gamma \mathcal{S}_{(2n)} = (d-2n)\mathcal{S}_{(2n)}.$$

- Applying the above identity to the variation δ_γ we obtain

$$2\pi_{(2n)} = (d-2n)\mathcal{L}_{(2n)} + \partial_i {v^i}_{(n)}.$$

Since $\mathcal{L}$ is defined up to a total derivative, we can absorb the last term in $\mathcal{S}_{(2n)}$ such that

$$2\pi_{(2n)} = (d-2n)\mathcal{L}_{(2n)}.$$

- Inserting the above expansion of the on-shell action in the Hamiltonian constraint and matching terms of equal δ_γ eigenvalue we obtain

$$(\partial_\varphi U)^2 + Z^{-1}(\varphi)(\partial_\chi U)^2 - \frac{1}{d-1}U^2 + V(\varphi) = 0,$$
$$U'(\varphi)\frac{\delta}{\delta\varphi}\int d^dx\mathcal{L}_{(2n)} - \left(\frac{d-2n}{d-1}\right)U(\varphi)\mathcal{L}_{(2n)} = \mathcal{R}_{(2n)}, \quad n>0,$$

where

$$\begin{aligned}\mathcal{R}_{(2)} &= -\frac{1}{2\kappa^2}\sqrt{\gamma}\left(R[\gamma] - \partial_i\varphi\partial^i\varphi - Z(\varphi)\partial_i\chi\partial^i\chi\right),\\ \mathcal{R}_{(2n)} &= -2\kappa^2\gamma^{-\frac{1}{2}}\sum_{m=1}^{n-1}\Bigg(\pi_{(2m)}{}^i_j\pi_{(2(n-m))}{}^j_i - \frac{1}{d-1}\pi_{(2m)}\pi_{(2(n-m))}\\ &\quad +\frac{1}{4}\pi_{\varphi(2m)}\pi_{\varphi(2(n-m))} + \frac{1}{4}Z^{-1}(\varphi)\pi_{\chi(2m)}\pi_{\chi(2(n-m))}\Bigg), \quad n>1.\end{aligned}$$

- The linear equation for $\mathcal{L}_{(2n)}$, $n > 0$, admits the homogeneous solution

$$\mathcal{L}_{(2n)}^{hom} = F_{(2n)}[\gamma] \exp\left(\left(\frac{d-2n}{d-1}\right)\int^{\varphi} \frac{d\bar{\varphi}}{U'(\bar{\varphi})} U(\bar{\varphi})\right) = F_{(2n)}[\gamma] e^{-(d-2n)A},$$

where $F_{(2n)}[\gamma]$ is a covariant function of the induced metric of weight $d-2n$. As in the toy model, the homogeneous solution contributes only to *finite* local terms and can therefore be discarded.

- We are therefore only interested in the inhomogeneous solution:

$$\mathcal{L}_{(2n)} = e^{-(d-2n)A(\varphi)} \int^{\varphi} \frac{d\bar{\varphi}}{U'(\bar{\varphi})} e^{(d-2n)A(\bar{\varphi})} \mathcal{R}_{(2n)}(\bar{\varphi}).$$

- Evaluating this integral is straightforward if $\mathcal{R}_{(2n)}$ does not involve derivatives of the dilaton, φ, but it requires some caution when it does. In that case, one needs to use the freedom to add total derivatives to $\mathcal{L}_{(2n)}$ to write the integrand as

$$\frac{\delta\varphi}{U'(\varphi)} e^{(d-2n)A(\varphi)} \mathcal{R}_{(2n)}(\varphi) = \delta_{\varphi} F_{(2n)} + e^{(d-2n)A(\varphi)} \partial_i {v_{(2n)}}^i(\delta\varphi), \tag{1}$$

where $F_{(2n)}$ and v^i are to be determined. Then,

$$\mathcal{L}_{(2n)} = e^{-(d-2n)A(\varphi)} F_{(2n)}. \tag{2}$$

- These integrals can be evaluated systematically to obtain the local divergent part of the on-shell action.

- The resulting full expression for the covariant counterterms for $d = 4$ is:

$$
\begin{aligned}
S_{ct} \quad = \quad & -\frac{1}{8\pi G_5}\int_{\Sigma_r} d^4x\sqrt{\gamma}\lambda\left\{U(\lambda) - \frac{1}{2}\xi^2 e^{-2A}\int^{\lambda}\frac{d\bar{\lambda}}{\bar{\lambda}^2 U'(\bar{\lambda})}e^{2A(\bar{\lambda})}R\right. \\
& +\xi^2\frac{U}{U'}\Xi'\lambda^{-2}\partial^i\lambda\partial_i\lambda + \frac{1}{2}\xi^2 e^{-2A}\int^{\lambda}\frac{d\bar{\lambda}}{\bar{\lambda}^2 U'(\bar{\lambda})}e^{2A(\bar{\lambda})}Z(\bar{\lambda})\partial^i\chi\partial_i\chi \\
& +\log e^{-2r}\frac{\ell^3}{16}\left[R_{ij}R^{ij} - \frac{1}{3}R^2 - \frac{Z_0}{M_{pl}^3 N_c^2}\left(R^{ij}\partial_i\chi\partial_j\chi - \frac{1}{6}R\partial^i\chi\partial_i\chi\right.\right. \\
& \left. -6b_0^{-2}\lambda^{-4}\partial_i\lambda\partial^i\lambda\partial_j\chi\partial^j\chi + 6b_0^{-1}\lambda^{-2}\partial_i\lambda\partial_j\chi D^i D^j\chi + D^i D^j\chi D_i D_j\chi\right) \\
& \left.\left. -\frac{2}{3}\frac{Z_0^2}{(M_{pl}^3 N_c^2)^2}\partial_i\chi\partial^i\chi\partial_j\chi\partial^j\chi\right]\right\},
\end{aligned}
$$

- In this expression

$$\Xi(\lambda) = -\frac{1}{2}\xi^2 e^{-(d-2)A} \int^{\lambda} \frac{d\bar{\lambda}}{\bar{\lambda}^2 U'(\bar{\lambda})} e^{(d-2)A(\bar{\lambda})},$$

  and b_0 is related to the logarithmic asymptotics of the dilaton via the relation

$$U(\lambda) = -\frac{d-1}{\ell} - \frac{\xi^2 b_0}{\ell}\lambda + \mathcal{O}(\lambda^2).$$

- Having constructed the local counterterms one can systematically deduce the asymptotic expansions of the bulk fields (Fefferman-Graham like expansions) from the canonical momenta using the relations

$$\dot{\varphi} \sim \pi_\varphi \sim -\pi_\varphi^{ct} = -\frac{\delta \mathcal{S}_{ct}}{\delta \varphi}$$

- This leads to the identification of the sources and a derivation of the renormalized Ward identities.

Summary & Conclusions

- QFTs with relativistic UV fixed points, corresponding to asymptotically AdS string backgrounds, are not the only interesting QFTs admitting a holographic dual
- Various examples of QFTs with non-relativistic UV fixed points or more exotic UV completions arise often in the context of condensed matter systems and QCD-like theories.
- To apply holographic techniques to study these QFTs, it is necessary to develop a systematic way of constructing the holographic dictionary
- This requires a systematic study of the long-distance properties of supergravity, which can be carried out efficiently using a radial Hamiltonian formulation of supergravity
- Such an approach makes manifest the crucial interplay between the supergravity Lagrangian, the boundary conditions, and the locality of the divergences of the on-shell action, allowing for an easy classification of supergravity theories that can potentially serve as the holographic dual of a local QFT.