Hints of Integrability Beyond the Planar Limit

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Based on : On going project

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Motivation

- To study the string theory on non-trivial background is very difficult due to lack of systemetic formulation.
- The main motivation of this talk is to understand of such string theories.
- This will be done by exploiting AdS/CFT correspondence

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Example:

Type IIB String theory on Asymptotically $AdS_5 \times S^5$ is dual to $U(N)$, $\mathcal{N} = 4$ SYM (CFT) on the boundary $3 + 1$ dimensional Minkowski space.

Plan

1. Review: AdS/CFT; Conformal Dimension
   - AdS/CFT Correspondence
   - Conformal Dimension
2. Integrability beyond Planar limit
   - Conformal Dimension in nontrivial background
   - Integrability in the large M limit:

AdS/CFT

- In Euclidean Space:

String theory on curved AdS_5 background with boundary $R \times S^3$

N=4 SYM theory is a CFT on $R \times R^3$

String coupling g_s and string length l_s

’t Hooft coupling λ and rank of the gauge group N

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Relation

$$g_s = g_{YM}^2 \sim \frac{1}{N} \quad \text{and} \quad \left(\frac{R}{l_s}\right)^4 = 4\pi\lambda$$

- In the $g_s \to 0$ or $N \to \infty$ limit, string theory side correspond to free string moving on $AdS_5 \times S^5$ geometry, while the gauge theory side reduce to the planar limit.
- Correspondence is a strong-weak coupling duality in which a strongly coupled field theory is dual to a gravitational theory with small curvature corrections and vice versa.

AdS/CFT

- Part of the dictionary says:-

1. Every individual state of the string theory ⇔ A gauge invarient operator

Map

2. Energy of a string state ⇔ conformal dimension of the corresponding operator

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2. Energy of a string state ⇔ conformal dimension of the corresponding operator

- This is a consequence of the fact that the generator of time translations in the dual CFT, $\mathcal{N} = 4$ SYM defined on $R \times S^3$ maps in to the generator of dilatations in $\mathcal{N} = 4$ SYM defined on R^4

- Thus, by studying the string theory energy spectrum, we can have the conformal dimension of the corresponding gauge theory operator and vice versa.

$\mathcal{N} = 4$ SYM

- We study the field theory side of the correspondence, i.e. $\mathcal{N} = 4$ Super Yang Mills theory (SYM)
- This is a $4-$dimensional, supersymmetric, conformal field theory with the action

Action

$$S = \frac{1}{2}\int \frac{d^4x}{(2\pi)^2}\mathrm{Tr}\Big(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\Phi_m D_\mu\Phi_m - \frac{1}{4}g^2\left[\Phi_m, \Phi_n\right]\left[\Phi_m, \Phi_n\right] + \quad Fermions\Big)$$

- We construct and study operators built out by complex linear combinations of these:

$$Z = \Phi_1 + \Phi_2,\ Y = \Phi_3 + \Phi_4 \quad \text{and} \quad X = \Phi_5 + \Phi_6$$

Conformal Dimensions

- In a CFT, the conformal dimension of an operator is defined by the 2-point correlation function:

$$\left\langle \mathcal{O}_\alpha(x)\mathcal{O}_\beta(0) \right\rangle = \frac{\delta_{\alpha\beta}}{|x|^{2\Delta}}$$

- The full conformal dimension is combination of the classical scaling dimension and anomalous dimension:

$$\Delta = \Delta_0 + \gamma$$

- The classical scaling dimension simply corresponds to the number of fields need to built the operator.
- Determining γ is more subtle and requires an understanding of the dynamics of the theory.

Anomalous Dimensions

- Its an eigenvalue of an operator $\mathcal{O}_\alpha(x)$ that corresponds to an eigenvector of matrix of anomalous dimension

$$\Gamma = \frac{d \log Z}{d \log \Lambda}.$$

Where Λ is the UV cut off imposed in regularization and Z is the wave function renormalization factor needs to cancel the divergent part of the Feynman diagrams of certain correlators to some order in the perturbation theory rendering them finite.

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- Once we know the matrix of anomalous dimensions, then we have to diagonalize it to get the eigen values and this a highly non-trivial task.

- One case where the matrix was successfully diagonalized is in the scalar $SO(6)$ sector to one-loop.

Diagonalization

- Minahan & Zarembo brilliantly shown:

Map

The matrix of anomalous dimensions can be mapped into the Hamiltonian of a closed spin chain. *Which turned out to be integrable system*.

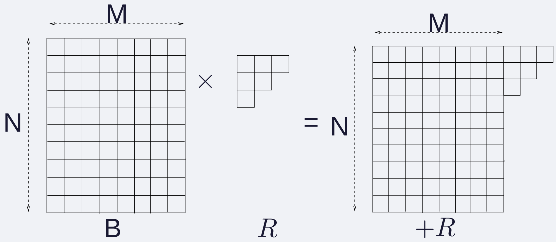
- Integrable system means, there is a tower of commuting charges.
- This allowed the use of powerful Bethe Ansatz techniques in diagonalizing the matrix.

Diagonalization

- So, the problem of finding the anomalous dimension and hence the spectrum of string state is modified to
  1. find out integrability
  2. Then find out the eigen state and the eigen values of the spin chain.
- In the rest of the talk we only concentrate on the integrability part.

Conformal Dimensions of Operators in presence of Background

- We wish to study the conformal dimensions of scalar operators multiplied by background Schur polynomial operators.



- These Shur polynomials (built out of a single matrix field, e.g.Z) with R charge of order N^2 are dual to a class of solutions of type IIB supergravity, these are the LLM geometries i.e.

$$\chi_B(Z) \leftrightarrow \boxed{\text{New Geometry}}$$

R. de Mello Koch

Conformal Dimensions

- Investigating the string theory dynamics in these new classical background geometries using the dual conformal field theory corresponds to evaluating correlators of the following form:

$$\Big\langle \mathcal{O}(Z,Y)\mathcal{O}(Z^\dagger Y^\dagger)\Big\rangle_B = \frac{\Big\langle \chi_B(Z)\mathcal{O}(Z,Y)\mathcal{O}(Z^\dagger Y^\dagger)\chi_B^\dagger(Z)\Big\rangle}{\Big\langle \chi_B(Z)\chi_B^\dagger(Z)\Big\rangle}$$

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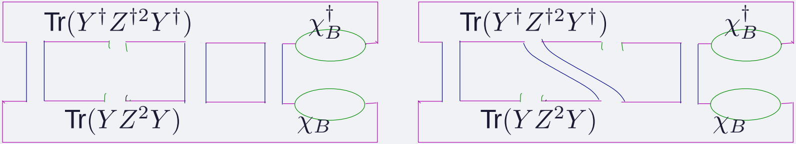

$$\left\langle \mathcal{O}(Z,Y)\mathcal{O}(Z^\dagger Y^\dagger)\right\rangle_B = NM(N+M) \qquad = M^2(N+M)$$

Delatation Operator

- Beisert, Kristjansen and Staudacher (BKS) developed a far more efficient means of calculating the matrix of anomalous dimensions for the vacuum $\mathcal{N} = 4$ SYM case corresponding to studying the dual string theory in the trivial $AdS_5 \times S^5$ background, by introducing a dilatation operator:

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- This operators admits a perturbative expansion:

$$D = \sum_{k=0}^{\infty} \left(\frac{g_{YM}^2}{16\pi^2} \right)^k D_{2k}$$

- In the derivation of the D_{2k}, the k-loop renormalized two point correlation function of the operators $\mathcal{O}_\alpha$ is expressed in a general and abstract way.

Delatation Operator

- For an example, the one loop renormalized two point correlator can be written in the form:

$$\left\langle \mathcal{O}_\alpha(x)\mathcal{O}_\beta(0)\right\rangle_{\text{one}-\text{loop}} = \exp(W_0)\exp\left[\log\left(\frac{x_0^2}{x^2}\right)g^2 D_2(x)\right]\mathcal{O}_\alpha(x)\mathcal{O}_\beta(0)\Big|_{\Phi=0}$$

- Where, in the dimensional regularization scheme,

$$D_2 = \lim_{\epsilon\to 0}\epsilon\big(\text{One Loop Regularized Connected Feynman diagrams}\big)$$

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- Finally, the dilatation operator defined by BKS for the one loop connected Feynman diagrams is:

$$D_2 = - : \text{Tr}\left[\Phi_m, \Phi_n\right]\left[\frac{d}{d\Phi_m}, \frac{d}{d\Phi_n}\right] : -\frac{1}{2} : \text{Tr}\left[\Phi_m, \frac{d}{d\Phi_n}\right]\left[\Phi_m, \frac{d}{d\Phi_n}\right] :$$

- In the $SU(2)$ i.e. two matrix sector that we are interested in:

$$D_2 = -2 : \text{Tr}\left[Z, Y\right]\left[\frac{d}{dZ}, \frac{d}{dY}\right] :$$

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- Using this operator, we will try to find out the integrability in our non-trivial backgrounds.

Effective Dilatation Operator

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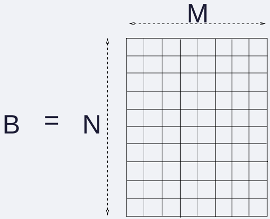
- We define an effective dilatation operator as:

$$D\left(\chi_B(Z)\mathcal{O}(Z,Y)\right) = \chi_B(Z)D_{\text{eff}}\mathcal{O}(Z,Y) \Rightarrow D_{\text{eff}} = \frac{1}{\chi_B(Z)}D\chi_B(Z)$$

- Using:

$$\chi_B = \left[det(Z)\right]^M$$



Effective Dilatation Operator

- Using:

$$\frac{\partial}{\partial Z_{ij}}\chi_B(Z) = M Z^{-1}_{ji}\chi_B(Z)$$

- We see that to obtain the effective dilatation operator from the original dilatation operator is to replace $\frac{\partial}{\partial Z_{ij}}$ in the original operator with:

$$\frac{\partial}{\partial Z_{ij}} \to \frac{\partial}{\partial Z_{ij}} + M Z^{-1}_{ji}$$

- This yields the following one-loop effective dilatation operator:

$$D_{2,\text{eff}} = -2 : \text{Tr}[Z,Y]\left[\frac{d}{dZ},\frac{d}{dY}\right] : -2M : \text{Tr}\left(\left(ZYZ^{-1} + Z^{-1}YZ - 2Y\right)\frac{d}{dY}\right)$$

Large M limit

- The delatation operator, after subtracting the classical dimension out, can be written as

$$D = D\left(Z, Y, \frac{d}{dZ}, \frac{d}{dY}\right)$$

and

$$D_{\text{eff}} = D_{\text{eff}}\left(Z, Y, (\frac{d}{dZ} + MZ^{-1}), \frac{d}{dY}\right)$$

- At Large M limit, i.e. $M, N \to \infty$ and in addition, the ratio $\frac{N}{M} \to 0$. The effective delatation operator becomes as

$$\tilde{D}_{\text{eff}} = \tilde{D}_{\text{eff}}\left(Z, Y, MZ^{-1}, \frac{d}{dY}\right)$$

- Expand as

$$\tilde{D}_{\text{eff}} = \sum_n \tilde{D}_{\text{eff n}}$$

- Wher $\tilde{D}_{\text{eff n}}$ has a total of n derivatives with respect to Y.

Large M limit

- From the general stucture of a connected planar l-loop vertex we know that D act on $l+1$ adjacent sites. Thus D should contains $l+1$ derivative of Z and Y.
- The leading contribution in the large M limit comes from $\tilde{D}_{\text{eff } 0}$ term. However, the dimension of the operator $\text{Tr}(Z^J)$ is not corrected, it must be annihilated by D and hence, at large M limit by $\tilde{D}_{\text{eff } 0}$.
- This implies $\tilde{D}_{\text{eff } 0}$ must be vanishes. Thus the leading contribution will infact come from the the next term $\tilde{D}_{\text{eff } 1}$.

Large M limit

- At the leading order

$$\tilde{D}_{\text{eff}} = D_{\text{eff }1} = \sum_n c_n \text{Tr}\left(Z^n Y Z^{-n} \frac{d}{dY} \right)$$

- Where c_n is function of $g^2 M$
- Trivially it can be shown that

$$\left[\text{Tr}\left(Z^n Y Z^{-n} \frac{d}{dY} \right), \text{Tr}\left(Z^{-m} Y Z^m \frac{d}{dY} \right) \right] = 0$$

- So,

$$\left[\tilde{D}_{\text{eff}}, \text{Tr}\left(Z^{-m} Y Z^m \frac{d}{dY} \right) \right] = 0$$

- Which clearly shows an infinite number of conserved quantities to all loops.

Integrability

- To find the integrability of a system, we have to construct an $R_{12}(u)$-matrix which acts a tensor product of two n-dimensional vector spaces. Where u is the spectral parameter.
- R-matrix should satisfy the *Yang-Baxter equation*

$$R_{12}(u)R_{123}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u)$$

- Construct transfer matrix $T(u)$ from R-matrix as

$$T(u) = R_{01}(u)R_{02}(u).....R_{0L}(u)$$

  which acts on a tensor product of $L+1$, n-dimensional vector spaces.
- Using *Yang-Baxter equation* one can show

$$[\text{Tr}(T(u)), \text{Tr}(T(v))] = 0$$

Integrability

- Transfer matrix $T(u)$ is a polynomial of u. So,

$$\text{Tr}(T(u)) = \sum_m u^m t_m$$

- So $t_1, t_2....$ are among a set of commuting operators and t_1 is identified with the D_2 plus a constant. Minahan & Zarembo
- t_2 is identified with the U_2 plus a constant. BKS
- So the integrability condition reduces at one loop as

$$[D_2, U_2] = 0$$

- Where $U_2 = U(Z, Y, \frac{d}{dZ}, \frac{d}{dY})$
- $U_2 = M^0 U_{2,0} + M^1 U_{2,1} + M^2 U_{2,2}$ and

$$U_{2,2} = \text{Tr}(Z^{-2} Y Z^2 \frac{d}{dY}) - 2\text{Tr}(Z^{-1} Y Z^1 \frac{d}{dY}) + 2\text{Tr}(Z^1 Y Z^{-1} \frac{d}{dY}) - \text{Tr}(Z^2 Y Z^{-2} \frac{d}{dY})$$

Physical Meaning

- Delatation operators are made from the two point functions of the theory by two complex fields Y and Z.
- Thus, delatation operators can be understood as Wick contractions between the two fields of the two operators whose two point function we are computing.
- Example,

$$\langle Z^{\dagger}_{ij} Z_{kl} \rangle = \delta_{jk}\delta_{il} = \frac{d}{dZ_{ji}} Z_{kl}$$

- Thus, $\frac{d}{dZ}$ represnts the $Z^{\dagger}$.
- Recall that we replace

$$\frac{d}{dZ} = \frac{d}{dZ} + MZ^{-1}$$

- In the same analogy MZ^{-1} corresponds to the $Z^{\dagger}$ of the background. So, the corresponding contractions will be between a $Z^{\dagger}$ of the background and a Z field of the operator.

Physical Meaning

- In the large M limit, the contractions with the background dominate as compared to contractions with fields belonging to the operators of the two point functions we are computing.
- One can think that the matrices entering into the operators are “bits of a string". In this limit, the different bits in the string do not interact with each other - they interact only with the background.

Conclusion

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- At $M = 0$ also have integrability as proved by BKS

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- Can we find integrability in the intervening region?
- We are currently exploring this for our upcoming publication....