**Introduction to the AdS/CFT correspondence**

## Outline

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II AdS/CFT correspondence

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III Large N review

IV D3 branes and $AdS_5 \times S^5$

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V Finite T, probe branes, quarks, other extensions

## 0 References

Descriptive:

• B. Zwiebach, "A first course in string theory," 2nd ed., (Cambridge, 2004).

• S. S. Gubser and A. Karch, "From gauge-string duality to strong interactions: a pedestrian's guide," [arXiv:0901.0935].

Detailed:

• O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," [arXiv:hep-th/9905111].

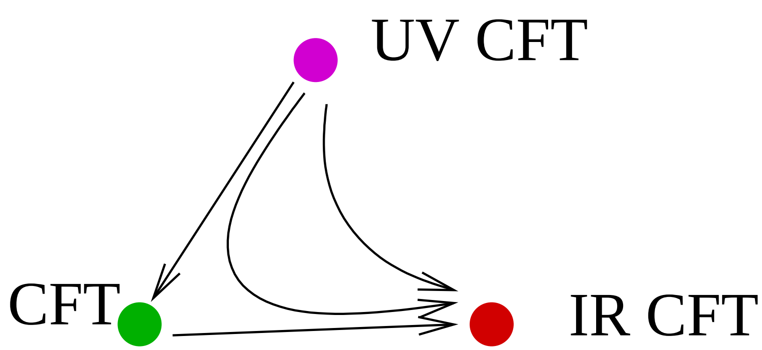
• I. R. Klebanov and E. Witten, "AdS/CFT correspondence and symmetry breaking," [arXiv:hep-th/9905104].

• E. D'Hoker and D. Z. Freedman, "Supersymmetric gauge theories and the AdS/CFT correspondence," [arXiv:hep-th/0201253].

• K. Skenderis, "Lecture notes on holographic renormalization," [arXiv:hep-th/0209067].

## I CFT review

Scale-invariant quantum field theories are important as possible end-points of RG flows. Knowledge of them and their relevant deformations (partially) organizes the space of QFTs.




It is believed that unitary scale-invariant theories are also conformally invariant: the space-time symmetry group Poincaré$_d \times$Dilatations enlarges to the Conformal$_d$ group.

($d =$ no. of space-time dimensions.)

Equivalently, there exists a local, conserved, traceless energy-momentum tensor $T^{\mu\nu}(x)$.

## CFT review: conformal algebra

Conformal$_d$ = group of reparameterizations which preserve the (flat) space-time metric up to a local scale factor. Generators are (Lorentz) rotations, $M^{\mu\nu}$, plus translations, dilatations, and special conformal transformations ($d > 2$):

$$\begin{aligned} P^\mu &: x^\mu \rightarrow x^\mu + \alpha^\mu \\ D &: x^\mu \rightarrow (1+\epsilon)x^\mu \qquad\qquad (\mu, \nu = 1, \ldots, d) \\ K_\nu &: x^\mu \rightarrow x^\mu + \epsilon_\nu (g^{\mu\nu} x^2 - 2x^\mu x^\nu) \end{aligned}$$

and have algebra (other commutators vanish or follow from rotational invariance)

$$[D, K_\mu] = iK_\mu \qquad [D, P_\mu] = -iP_\mu \qquad [P_\mu, K_\nu] = 2iM_{\mu\nu} - 2ig_{\mu\nu} D$$

Conformal$_d \simeq SO(d,2)$ (Minkowski) or $\simeq SO(d+1,1)$ (Euclidean) by defining $2d+1$ other rotations

$$M_{\mu\, d+1} \equiv (K_\mu - P_\mu)/2, \qquad M_{\mu\, d+2} \equiv (K_\mu + P_\mu)/2, \qquad M_{d+1\, d+2} \equiv D$$

## CFT review: primary operators

In radial quantization each local operator $\mathcal{O}(x^\mu)$ defines a state by $\mathcal{O}(0)|0\rangle$. In a CFT, by scale invariance this is a state on any size $S^{d-1}$ around the origin. Conversely *any* state on an $S^{d-1}$ can be written this way by shrinking the sphere.

Classify local operators by their Lorentz rep, any internal quantum no.s, and their scaling dimension, $\Delta$,

$$\mathcal{O}_\Delta(\lambda x^\mu) = \lambda^{-\Delta}\mathcal{O}_\Delta(x^\mu) \quad \Leftrightarrow \quad [D, \mathcal{O}_\Delta(0)] = -i\Delta\mathcal{O}_\Delta(0)$$

Unitarity puts lower bounds on $\Delta$. Acting with $P^\mu$ ($K^\mu$) increases (decreases) $\Delta$ by 1.

*Primary* operators are those annihilated by $K^\mu$. Their *descendants* are those made by acting on the primary with $P^\mu$'s. All local operators are found in this way.

CFT review: conformal bootstrap

OPE of two (scalar) primaries:

$$\mathcal{O}_{\Delta_i}(x)\mathcal{O}_{\Delta_j}(0) \sim \sum_k c_{ijk}|x|^{-\Delta_i-\Delta_j+\Delta_k}(\mathcal{O}_{\Delta_k}(0) + \text{descendants})$$

OPE plus conformal invariance determines all correlators:

$$\begin{aligned}
\langle\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\rangle &= \delta_{\Delta_1,\Delta_2}\ |x_{12}|^{-\Delta_1-\Delta_2} \\
\langle\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle &= c_{123}\ |x_{12}|^{-\Delta_1-\Delta_2+\Delta_3}|x_{23}|^{-\Delta_2-\Delta_3+\Delta_1}|x_{31}|^{-\Delta_3-\Delta_1+\Delta_2} \\
\langle\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\mathcal{O}_{\Delta_4}(x_4)\rangle &= |x_{12}|^{-\Delta_1-\Delta_2}|x_{34}|^{-\Delta_3-\Delta_4}\ \mathcal{F}_{1234}\left(\frac{x_{12}x_{34}}{x_{13}x_{24}}, \frac{x_{14}x_{23}}{x_{13}x_{24}}\right) \\
&\vdots
\end{aligned}$$

So the list of the quantum numbers ($\Delta_i$, spins, ...) of the primaries together with the $c_{ijk}$ is sufficient to determine a CFT. However, it is not necessary: unitarity and associativity (“crossing symmetry”) of the OPE puts many non-trivial relations among these numbers, so arbitrary lists will not in general define consistent CFTs.

## CFT review: partition function

A useful way of encoding a CFT is with its partition function

$$Z[\phi_{\Delta_i}] \equiv \left\langle \exp\left(\int d^dx\, \phi_{\Delta_i}(x)\mathcal{O}_{\Delta_i}(x)\right)\right\rangle_{\text{CFT}}$$

which generates correlation functions by taking derivatives wrt sources $\phi_\Delta$ for each primary

$$\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\ldots\rangle = \left.\frac{\partial^n Z[\phi_{\Delta_i}]}{\partial\phi_{\Delta_1}(x_1)\partial\phi_{\Delta_2}(x_2)\cdots}\right|_{\phi_i=0}$$

Conformal (and other) invariance of the correlators is reflected in the the conformal invariance of $Z[\phi_\Delta]$. E.g., under scaling,

$$\int d^dx\, \phi_\Delta(x)\mathcal{O}_\Delta(x) = \int d^d(\lambda x)\, \phi_\Delta(\lambda x)\mathcal{O}_\Delta(\lambda x) = \lambda^{d-\Delta}\int d^dx\, \phi_\Delta(\lambda x)\mathcal{O}_\Delta(x)$$

so $Z$ invariant under

$$\phi_\Delta(x) \to \lambda^{d-\Delta}\phi_\Delta(\lambda x).$$

In general, the sources transform in field representations of the conformal group (plus any internal symmetry group) and $Z[\phi_\Delta]$ is an invariant combination of these fields.

## **II** AdS/CFT correspondence

We now introduce a trick to generate conformally invariant $Z[\phi_\Delta]$'s. It will not generate *all* CFTs, just a special subset.

The idea is to copy the way we write actions as invariants of field representations of the Poincaré group, and apply it instead to the conformal group.

Local field representations of the Poincaré group are functions on $\mathbb{R}^d$ valued in finite dimensional representations of the Lorentz group. Poincaré invariant actions are formed by taking translationally invariant integrals over $\mathbb{R}^d$ of scalar combinations of fields and their derivatives formed using the Minkowski (Euclidean) metric on $\mathbb{R}^d$.

The key point here is that $d$-dimensional Minkowski (Euclidean) space is the space whose isometry group is Poincaré$_d$.

## AdS/CFT correspondence: AdS geometry

So, find the space whose isometry group is Conformal$_d \simeq SO(d+1,1)$. (Work in Euclidean signature for now.)

This looks just like the $d+2$-dimensional rotation group, so flat $\mathbb{R}^{d+2}$ might work. But it has additional translational symmetries. Remove them by restricting to a $d+1$-dimensional "sphere" of radius $R$: this preserves the rotational symmetry, but leaves no translations.

For $SO(d+1,1)$, really want a Lorentzian "sphere" = hyperboloid

$$\vec{X}^2 + V_+^2 - V_-^2 = R^2$$

where $\vec{X} = \{X_1, \ldots, X_d\}$ and $V_\pm$ are Cartesian coordinates on $\mathbb{R}^{d+1,1}$ with metric

$$ds^2 = d\vec{X}^2 + dV_+^2 - dV_-^2.$$

This is called anti de Sitter space, or AdS$_{d+1}$, of radius $R$.

## AdS/CFT correspondence: AdS geometry (cont.)

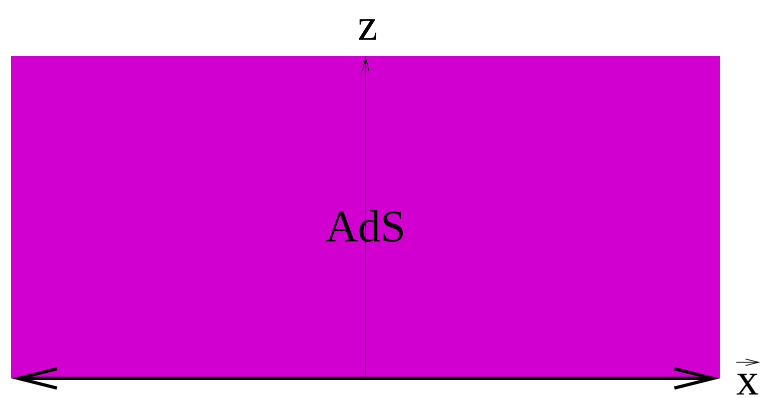
Useful coordinates are $\{\vec{x}, z\}$ defined by

$$\vec{X} = \frac{R}{z}\vec{x}, \qquad V_{\pm} = \frac{1}{2}\left(z + \frac{\vec{x}^2 \pm R^2}{z}\right)$$

which parametrizes solutions of the hyperboloid constraint for $\vec{x} \in \mathbb{R}^d$ and $z > 0$. In these coordinates (called Poincaré patch coordinates), the AdS metric reads

$$ds^2 = \frac{R^2}{z^2}(dz^2 + d\vec{x}^2).$$

Thus AdS$_{d+1}$ is conformal to the upper-half space $z > 0$ of $\mathbb{R}^{d+1}$



## AdS/CFT correspondence: AdS geometry (cont.)

[ A closely related set of coordinates are $\vec{x}$ and $r = R^2/z$, in which

$$ds^2 = \frac{R^2}{r^2} dr^2 + \frac{r^2}{R^2} d\vec{x}^2.$$

$r$ is often called the radial coordinate of the AdS. Here $r = \infty$ is the boundary of AdS, and $r = 0$ can be thought of as a horizon. Note that both are infinitely distant from any finite $r$.

In Minkowski signature, simply change in the AdS metric

$$d\vec{x}^2 \to -dt^2 + d\vec{x}^2$$

the $d$-dimensional Minkowski metric, where now on the right side $\vec{x} = (x_1, \ldots x_{d-1})$. Minkowski-AdS has the interesting feature that though the boundary is radially infinitely far from any interior point, it can be reached in finite time by radial light-like signals.

I will stick mostly to Euclidean-signature AdS from now on. ]

## AdS/CFT correspondence: partition function

Any generally covariant function of (tensor) fields $\phi(z, \vec{x})$ on AdS will be conformally invariant. But these fields live in one more dimension ($z$) than we want. So we want to restrict to fields on a $d$-dimensional subspace of AdS$_{d+1}$ while keeping general covariance in the full AdS$_{d+1}$.

The boundary, $\partial$AdS $\simeq \{z{=}0, \vec{x}\}$, is special because the group of conformal isometries acts on it in the same way as the conformal group acts in $d$-dimensional space-time: (Lorentz) rotations acting as $\{z, \vec{x}\} \to \{z, \Lambda\vec{x}\}$ and the scaling transformation $\{z, \vec{x}\} \to \{\lambda z, \lambda\vec{x}\}$ are clearly isometries. (The special conformal transformations are more complicated.)

So boundary values of (tensor) functions on AdS transform as representations of the conformal group on space-time, and the partition function can be any generally covariant (on AdS$_{d+1}$) function of the boundary values.

## AdS/CFT correspondence: partition function (cont.)

A way of writing a general class of such functions is as a functional integral over fields $\phi(z,\vec{x})$ on AdS$_{d+1}$ with a generally covariant measure keeping the boundary values $\phi(0,\vec{x}) \sim \overline{\phi}(\vec{x})$ fixed:

$$Z[\overline{\phi}(\vec{x})] = \int_{\phi|_\partial = \overline{\phi}} \mathcal{D}\phi(z,\vec{x})\, e^{-S[\phi(z,\vec{x})]}$$

Then, *e.g.*, $Z$ is invariant under scale transformations taking scalar fields $\phi(z,\vec{x}) \to \phi(\lambda z, \lambda\vec{x})$. In particular, if $\phi$ behaves near the boundary like

$$\phi(z,\vec{x}) \sim z^{d-\Delta}\overline{\phi}(\vec{x}) + \mathrm{O}(z^{d-\Delta+1})$$

it follows that the boundary value $\overline{\phi}(\vec{x})$ transforms under scaling as

$$\overline{\phi}(\vec{x}) \to \lambda^{d-\Delta}\overline{\phi}(\lambda\vec{x}).$$

Comparing to CFT scaling(p. 7) it follows that $\overline{\phi} \equiv \overline{\phi}_\Delta$ is the source of a (scalar) primary operator $\mathcal{O}_\Delta$ of dimension $\Delta$ in the CFT.

(Similarly for non-scalar fields ...)

## AdS/CFT correspondence: partition function (cont.)

Since $Z$ given as a path integral, it defines a quantum theory if the $\phi$ have kinetic terms (fluctuate).

E.g., if $\phi$ is free scalar on AdS,

$$S[\phi] = \frac{1}{2}\int_{\mathrm{AdS}_{d+1}} \sqrt{g} \quad \phi(-D_\mu D^\mu + m^2)\phi$$

then eom has 2 independent solutions scaling near the boundary as $z^{d-\Delta}$ and $z^{\Delta}$ so asymptotically

$$\phi(z, \vec{x}) \sim z^{d-\Delta}\overline{\phi}(\vec{x}) + \cdots + z^{\Delta}\varphi(\vec{x}) + \cdots$$

with

$$\Delta = \frac{d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 + m^2R^2}.$$

and $\overline{\phi}$ is the source for $\mathcal{O}_\Delta$ in the CFT. In the limit $m \to \infty$ where $\phi$-fluctuations turn off, $\Delta \to \infty$.

## AdS/CFT correspondence: partition function (cont.)

Every CFT has a local energy-momentum tensor, $T^{\mu\nu}(\vec{x})$, as a primary operator of dimension $\Delta = d$. It is sourced by $\int h_{\mu\nu} T^{\mu\nu}$, so $h_{\mu\nu}$ is the boundary value of a spin-2 field on AdS. An argument similar to the one for the scalar field shows that $\Delta = d$ for $T^{\mu\nu}$ implies that $m^2 = 0$ for $h_{\mu\nu}$.

So the AdS theory must have a dynamical, massless spin-2 field: a graviton.

The partition function of a quantum gravity theory on an asymptotically AdS$_{d+1}$ space-time as a function of the boundary values of its fields is the partition function of a CFT$_d$ with the boundary values acting as sources for the primary operators:

$$Z_{\mathrm{qu-grav}}[\overline{\phi}] = Z_{\mathrm{CFT}}[\overline{\phi}]$$

## AdS/CFT correspondence: partition function (cont.)

This result begs the questions:

(1) Are there consistent quantum gravity theories in which we can compute the lhs?

(2) What class of CFTs do they correspond to?

(1a): The only examples of quantum gravity theories whose consistency we have confidence in are string theories.

(1b): It is difficult to compute string theory partition functions on AdS backgrounds, except in the weak-coupling, low-energy limit, in which case is reduces to classical Einstein gravity coupled to other massless string fields. (more below...)

(2): We need specific computable examples to answer this question (tomorrow's lecture).

## AdS/CFT correspondence: partition function (cont.)

[Some comments:

• If replace AdS$_{d+1}$ by AdS$_{d+1} \times X$ with $X$ any space without boundary, same procedure still works. Isometry group of $X$ becomes global internal symmetry of CFT.

• If define a partition function by the same procedure but cut off the AdS at $z = \epsilon > 0$ (*i.e.*, make $z = \epsilon$ the boundary), then keep Poincaré invariance, but break conformal invariance, locality, on length scales smaller than $\epsilon$.

• For $0 > m^2 > -d^2/4R^2$, scaling dimns $d > \Delta > d/2$, real. AdS scalars stable for negative mass-squared in this range (Breitenlohner-Friedmann).

• For $(d+2)/2 > \Delta > d/2$, $z^{d-\Delta}$ as well as $z^{\Delta}$ solution is normalizable, so can use either as source of operator $\mathcal{O}_\Delta$ or $\mathcal{O}_{d-\Delta}$, resp., allowing operators down to unitarity bound $(d-2)/2$ (Klebanov-Witten).]

## AdS/CFT correspondence: semi-classical gravity limit

Gravitational theory has low-energy effective action

$$\frac{1}{\kappa^2}\int d^{d+1}x\,\sqrt{g}\left(\mathcal{R}+\alpha'\mathcal{R}^2+\cdots\right)$$

Define length scales governing strength of gravity and size of higher-derivative terms by

$$\text{Planck length:}\quad \ell_p \sim \kappa^{\frac{2}{d-1}}, \qquad \text{String length:}\quad \ell_s \sim \sqrt{\alpha'}.$$

In AdS background $\mathcal{R} \sim R^{-2}$, so weak gravity and small higher-derivative terms when

$$\ell_p \ll R, \quad \text{and} \quad \ell_s \ll R.$$

Then semi-classical (saddle-point) approximation to gravitational partition function is

$$Z_{\mathsf{qu-grav}}[\overline{\phi}] \sim \sum_{\{\phi_{cl}\}} e^{-S_{\mathsf{Einstein}}[\phi_{cl}]}$$

where $\{\phi_{cl}\}$ are the classical field values from extremizing the action subject to the $\phi|_\partial = \overline{\phi}$ boundary conditions, and the sum is over al the such extrema.

# AdS/CFT correspondence: correlation functions

In this semi-classical gravity limit, there is now a concrete calculational procedure for extracting correlation functions of the associated CFT via the AdS/CFT correspondence.

(i) Find the dominant saddle point.

(ii) Each $d+1$-dimensional (or "bulk") field, $\phi$, obeys a 2nd order pde eom on the (asymptotically) AdS space, with near-boundary asymptotic expansion

$$\phi(z,\vec{x}) \sim z^{d-\Delta}\overline{\phi}(\vec{x}) + \cdots + z^{\Delta}\varphi(\vec{x}) + \cdots$$

(iii) So $\phi \equiv \phi_\Delta$ is associated to the source for an $\mathcal{O}_\Delta$ CFT primary.

(iv) With boundary conditions fixing $\overline{\phi}(\vec{x})$, then $\varphi(\vec{x})$ is determined by the eom, so $\phi = \phi[\overline{\phi}]$.

(v) Evaluate $S_{\text{Einstein}}$ on these solutions to get, by AdS/CFT,

$$S_{\text{Einstein}}[\overline{\phi}] = -\ln\left(Z_{\text{CFT}}[\overline{\phi}]\right) \equiv -W_{\text{CFT}}[\overline{\phi}].$$

(vi) So "connected" correlators in CFT are

$$\left\langle \mathcal{O}_{\Delta_1}(\vec{x}_1)\cdots\mathcal{O}_{\Delta_n}(\vec{x}_n)\right\rangle_{\text{CFT}-\text{conn}} = -\left.\frac{\partial^n S_{\text{Einstein}}[\{\overline{\phi}\}]}{\partial\overline{\phi}_{\Delta_1}(\vec{x}_1)\cdots\partial\overline{\phi}_{\Delta_n}(\vec{x}_n)}\right|_{\overline{\phi}=0}$$

## AdS/CFT correspondence: correlation functions (cont.)

To actually compute $S_{\text{Einstein}}[\overline{\phi}]$ and its derivatives, one must regulate (by cutting off consistently at $z = \epsilon > 0$), renormalize (by adding local counterterms on the $z = \epsilon$ boundary) to preserve conformal invariance, then take the $\epsilon \to 0$ limit to extract finite answers.

See K. Skenderis' lectures for details. E.g., for a free massive scalar plus $-(g/4)\phi^4$ interaction term:

$$\begin{aligned}
\langle \mathcal{O}_\Delta(\vec{x}) e^{\int \overline{\phi}_\Delta \mathcal{O}_\Delta} \rangle_{\text{CFT}} &= (2\Delta - d)\varphi(\vec{x}) \\
\langle \mathcal{O}_\Delta(\vec{x}_1)\mathcal{O}_\Delta(\vec{x}_2) \rangle_{\text{CFT}} &= (2\Delta - d)\frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta - \frac{d}{2})}\frac{1}{|x_{12}|^{2\Delta}} \\
\langle \mathcal{O}_\Delta(\vec{x}_1)\mathcal{O}_\Delta(\vec{x}_2)\mathcal{O}_\Delta(\vec{x}_3) \rangle_{\text{CFT}} &= 0 \\
\langle \mathcal{O}_\Delta(\vec{x}_1)\mathcal{O}_\Delta(\vec{x}_2)\mathcal{O}_\Delta(\vec{x}_3)\mathcal{O}_\Delta(\vec{x}_4) \rangle_{\text{CFT}} &= \ldots \text{explicit but complicated} \ldots
\end{aligned}$$

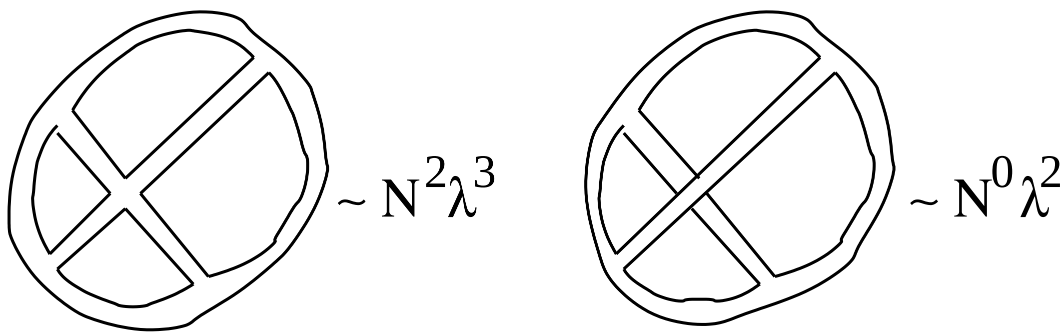
# **III** Large N review

$U(N)$ YM in 't Hooft limit: $N \to \infty$ keeping $\lambda \equiv g_{ym}^2 N$ fixed.
Fields $\phi$ in adjoint representation of $U(N)$ with action of the form

$$\mathcal{L} \sim \frac{N}{\lambda} \mathrm{tr} \left( d\phi d\phi + \phi^2 d\phi + \phi^4 \right),$$

so the 't Hooft limit looks classical.

Write adjoint indices as $\phi^i_j$ , $i, j = 1, \ldots, N$, corresponding to the $N \otimes \overline{N}$ decomposition of the adjoint representation. Then $\langle \phi^i_j \phi^k_\ell \rangle \propto \delta^i_\ell \delta^j_k$, so notate propagators in Feynman diagrams as double lines with each line tracking the fundamental or antifundamental indices.




Associate to each interaction a vertex (no.$=V$), to propagators edges (no.$=E$), and to loops faces (no.$=F$), to form a triangulation of some oriented surface.

## Large N review: planar limit

Count powers of $N$ and $\lambda$: each vertex has $N/\lambda$, each propagator $\lambda/N$, each face has $N$ (from trace over fundamental indices). So diagram has weight $N^{V-E+F}\lambda^{E-V}$.

Since $V - E + F = 2 - 2g$ where $g$ is the genus of the surface, and extending to diagrams with $n$ external propagators (with appropriate normalization), you find that connected correlators have the general form

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \sim \sum_{g=0}^{\infty} N^{2-2g-n} F_{g,n}(\lambda)$$

So are dominated by $g = 0$, "planar", diagrams in 't Hooft limit.

In this limit, 2-point functions $\sim N^0$, 3-point functions $\sim N^{-1}$, *etc.*, so $1/N$ acts like an effective coupling constant in large N YM (in addition to $\lambda$).

## IV D3 branes and $\mathrm{AdS}_5 \times \mathrm{S}^5$

Now we will use a specific quantum gravity theory—string theory—to derive a specific example of the AdS/CFT correspondence. In particular:

$$\text{type IIB string on } \mathrm{AdS}_5 \times S^5$$
$$\Leftrightarrow$$
$$\text{4-dimensional } \mathcal{N} = 4 \text{ supersymmetric SU(N) YM}$$

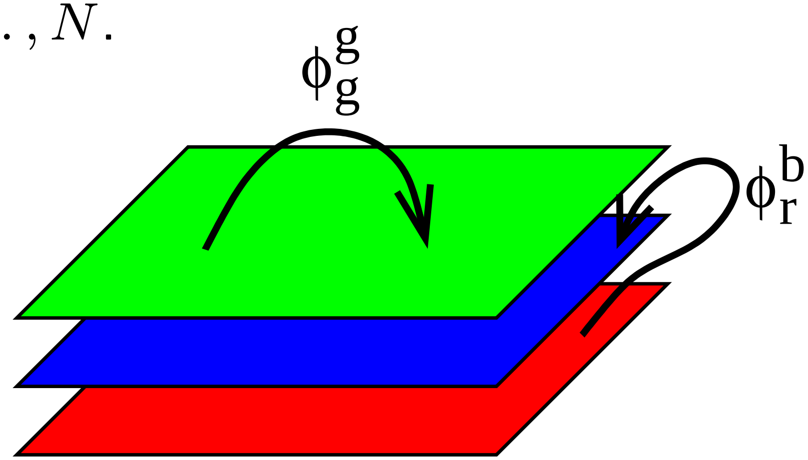
The same type of argument can be used to find many other examples. The basic steps:

- SU(N) SYM from low energy limit of open strings on N D3-branes
- Near horizon AdS geometry and decoupling of the distant (asymtotically flat) gravitational modes
- Mapping of parameters, and strong/weak coupling duality.

## D3 branes and $AdS_5 \times S^5$: SYM from D-branes

In IIB string theory place a stack of $N$ parallel D3-branes extended along the $\vec{x} = x_{0,1,2,3}$ directions, and take the limit in which they become coincident.

At weak enough string coupling, $g_s \ll 1$, this stack is well-described as 4-manifolds where open strings may end. Label the D3-branes by indices $i, j, \ldots = 1, \ldots, N$.




Lightest modes of open strings connecting the branes then carry adjoint $U(N)$ labels and fill out massless $\mathcal{N} = 4$ supermultiplet of $U(N)$ SYM.

The SYM coupling is $g_{ym} \sim \sqrt{g_s}$ (the open string coupling).

## D3 branes and AdS$_5\times$S$^5$: SYM from D-branes (cont.)

For small enough $g_s$ at fixed $N$, the gravitational backreaction of the Dbranes is small, since even though the Dbrane tension $\propto N g_s^{-1}$, the gravitational coupling $\kappa^2 \propto g_s^2$.

Take the low energy limit, $E \ll \ell_s^{-1}$ or $E^2\alpha' \to 0$, so only excite massless string modes. Then there are only the bulk (10-d) supergravity modes of the closed strings and the $U(N)$ SYM modes of the open strings.

The couplings between the two sectors are higher-derivative, and so vanish in the low-energy limit.

The supergravity sector is IR-free. ($\kappa \sim g_s\alpha'^2$, so the dimensionless coupling is $E^4\kappa \to 0$ as $E \to 0$.)

The SYM sector stays at fixed coupling, $g_{ym}^2$, since $\mathcal{N} = 4$ SYM is a CFT for all $g_{ym}$. (I.e., the gauge coupling is an exactly marginal coupling.)

## D3 branes and AdS$_5\times$S$^5$: SYM from D-branes (cont.)

The diagonal U(1) $\subset$ U(N) also decouples and is free. This corresponds to the overall translational degrees of freedom of the N D3-branes.

Net result:

In the small $g_s$, fixed N limit, the low energy effective action of IIB strings in the presence of N coincident D3-branes is three decoupled sectors:

(free 10-d supergravity) $\times$

(free 4-d U(1) SYM) $\times$

(4-d SU(N) SYM CFT with $g_{ym}^2 \sim g_s$)

## D3 branes and $\mathrm{AdS}_5 \times \mathrm{S}^5$: D-brane near horizon geometry

Now look at the same system in the $g_s \to 0$, $N \to \infty$ limit, keeping $g_s N \equiv \lambda$ fixed.

In this limit the gravitational backreaction of the Dbranes cannot be neglected, since Dbrane tension $\times$ gravitational coupling $\propto N g_s^{-1} \times g_s^2 = \lambda$.

In the low energy, $g_s \to 0$ limit, the D3-brane classical supergravity solution is

$$
\begin{aligned}
ds^2 &= f^{-1/2}(-dt^2 + d\vec{x}^2) + f^{1/2}(dr^2 + r^2 d\Omega_5^2) \\
F_5 &= (1 + *) dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge df^{-1} \\
f &\equiv 1 + \frac{R^4}{r^4}, \qquad R^4 \equiv 4\pi g_s N \alpha'^2 .
\end{aligned}
$$

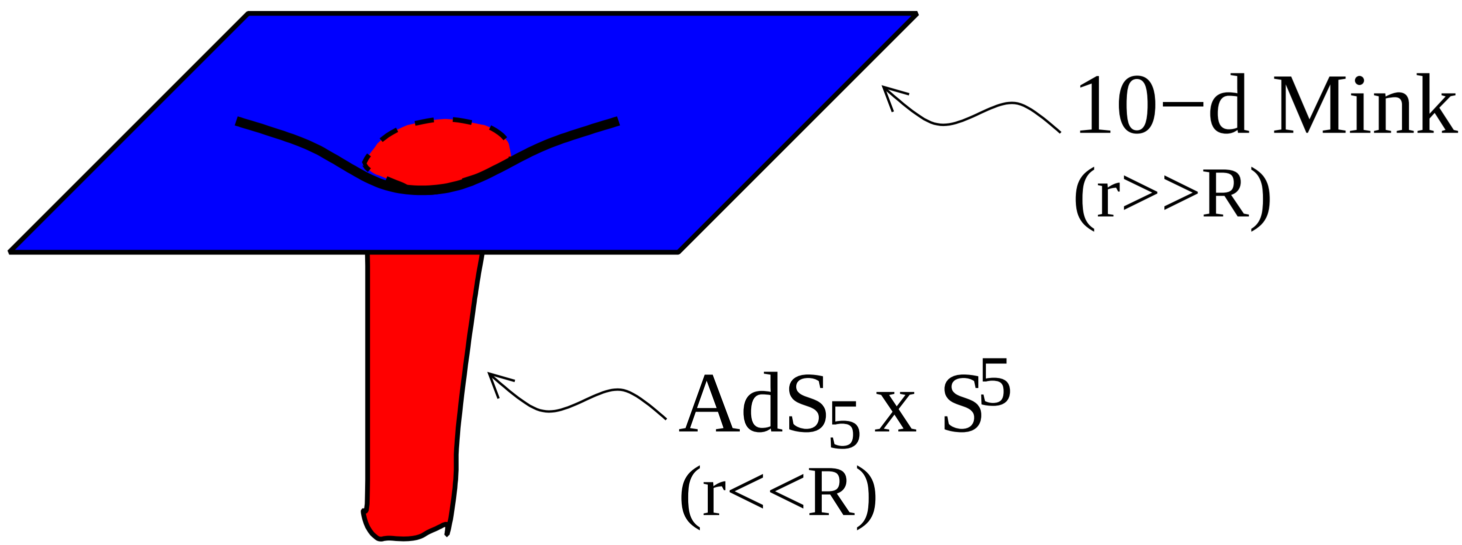
$F_5$ is the RR 5-form flux sourced by the N D3-branes.

## D3 branes and AdS$_5\times$S$^5$: D-brane near horizon geometry (cont.)

For $r \gg R$, $f \to 1$ solution asymptotes to flat $\mathbb{R}^{9,1}$.
For $r \ll R$, $f^{1/2} \to R^2/r^2$, so metric becomes

$$ds^2 \sim \frac{R^2}{r^2} dr^2 + \frac{r^2}{R^2}(-dt^2 + d\vec{x}^2) + d\Omega_5^2$$

which we recognize as AdS$_5\times$S$^5$ in Poincaré patch coordinates.




The AdS boundary ($r \to \infty$) is replaced by the transition to flat $\mathbb{R}^{9,1}$.

## D3 branes and $\mathrm{AdS}_5 \times \mathrm{S}^5$: D-brane near horizon geometry (cont.)

We now want to consider the low energy effective description of physics in this geometry.

There are massless supergravity scattering states in the $\mathbb{R}^{9,1}$ asymptotic region.

There are also arbitrarily low-energy modes far down the AdS "throat".

To see this:

Consider an object of fixed energy $E_*$ as measured in the frame of a co-moving observer. From the point of view of an asymptotic observer at $r = \infty$ (where we are measuring all our observables) it has redshifted energy $E = \sqrt{g_{tt}} E_* = f^{-1/4} E_*$. So $E \sim r E_* / R \to 0$ as $r \to 0$. Therefore states of *arbitrary* finite energy $E_*$ in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ throat are low-energy excitations.

Finally there are also a few massless modes corresponding (from the asymptotic $\mathbb{R}^{9,1}$ perspective) to the translational zero-modes of the whole throat. (They are the "singleton" boundary modes on the AdS.) Since they are the translational modes of the whole stack of D3-branes, they are equivalent to a free 4-d U(1) SYM theory.

## D3 branes and AdS$_5\times$S$^5$: D-brane near horizon geometry (cont.)

In summary,

In the small $g_s$, fixed $\lambda = g_s N$ limit, the low energy effective action of IIB strings in the presence of N coincident D3-branes has three sectors:

(free 10-d supergravity) $\times$
(free 4-d U(1) SYM) $\times$
(IIB string theory on AdS$_5\times$S$^5$
with $R^4 = 4\pi\lambda\alpha'^2$ and $\kappa = g_s\alpha'^2$)

## D3 branes and $AdS_5 \times S^5$: D-brane near horizon geometry (cont.)

Furthermore, these three sectors decouple in the low-energy limit:

The cross-section for an $\mathbb{R}^{9,1}$ supergravity wave of frequency $\omega$ to scatter off the throat $(r < R)$ region is $\sigma \sim \omega^3 R^8$, so vanishes in the low-energy $\omega \to 0$ limit.

As the low-energy throat modes are localized closer to $r = 0$, escape to the asymptotically flat region is energetically suppressed.

Low energy $\mathbb{R}^{9,1}$ gravitational waves can't excite the massive throat translational modes, while the associated singleton modes on the boundary of AdS decouple from bulk AdS modes.

Thus all three sectors decouple.

## D3 branes and $AdS_5 \times S^5$: strong/weak duality

Comparing these two low energy descriptions leads to the "Maldacena conjecture":

$$\text{type IIB string on } \mathrm{AdS}_5 \times S^5$$
$$\Leftrightarrow$$
$$\text{4-dimensional } \mathcal{N} = 4 \text{ supersymmetric SU(N) YM}$$

with parameters identified as

$$4\pi g_s = g_{ym}^2, \qquad R^4/\alpha'^2 = \lambda \equiv g_{ym}^2 N$$

It is just a conjecture because the AdS geometry came from a classical supergravity solution, *i.e.*, did not contain $g_s$ or $\alpha'$ corrections.

Note that o($\alpha'$) string worldsheet corrections are o($1/\sqrt{\lambda}$), and a fixed $\lambda$, o($g_s$) string loop corrections are o($1/N$).

# D3 branes and AdS$_5\times$S$^5$: strong/weak duality

The string theory on AdS$_5\times$S$^5$ is calculable in the classical supergravity limit where $g_s \ll 1$ (so no string loops) and $\ell_s \ll R$ (so no $\alpha'$ corrections).

In terms of YM parameters this means that $N \gg \lambda \gg 1$, which is the planar limit, but at *strong* 't Hooft coupling.

There are different possible versions of the conjecture:
**Weak**—valid only for $g_s N \to \infty$: neither $\alpha'$ nor $g_s$ corrections agree.
**Medium**—valid $\forall\ g_s N$, but only for $N \to \infty$: only $\alpha'$ corrections agree.
**Strong**—valid $\forall\ g_s N$ and $\forall\ N$: exact.

The strong version of the conjecture allows *all* kinds of finite energy interior processes and objects in the AdS space-time, including space-times with different topologies (*e.g.*, balck holes). So in this version,

$$Z_{\text{CFT}} = \sum \forall \text{ asymptotic AdS geometries}$$

Most tests of the conjecture are by computing at large $N$ quantities whose $\lambda$-dependence is determined by supersymmetry, though a few are also checks that $1/N$ corrections match as well.

## V Extensions

There are many examples, refinements, extensions and deformations of the AdS/CFT correspondence ... see the other talks in this school. Two extensions which are basic and play an important role in many other applications are adding:

- finite temperature, and
- brane probes.

The finite temperature extension can be derived by the same argument, but keeping a finite energy density on the D3-branes. In this case the supergravity solution become a black 3-brane, and the near-horizon limit is the $AdS_5$-black hole $\times$ $S^5$ geometry. The AdS black hole Hawking temperature is the same as the temperature of the SYM theory.

In what follows I will describe the brane probe approximation.

Extensions: brane probes

How can one add fundamental (as opposed to adjoint) matter to the SYM theory in the AdS/CFT correspondence?

In the weak coupling limit, we need string states with one end on the D3-brane stack (carrying a fundamental color index) and the other end elsewhere.

Since fundamental strings end on Dbranes, we should add other kinds of Dbranes to the initial setup. We call these other branes "flavor branes" since they will label different flavors of fundamental matter.

In general it is hard to find supergravity solutions for the gravitational backreaction of adding flavor branes. However, if the number, $N_f$, of flavor branes is much smaller than the number, $N$, of (color) D3-branes, this backreaction can be ignored in the large $N$ limit.

Extensions: brane probes (cont.)

To see this, recall that:

• Newton's constant $\kappa^2 \sim g_s^2 \sim N^{-2}$ in the 't Hooft limit (where $\lambda = g_s N$ is kept fixed),

• the tension of $N_f$ Dp-branes is $N_f T_{\mu\nu}^{\text{Dp}} \sim N_f g_s^{-1} \sim N_f N$,

• so the gravitational backreaction of the flavor branes is $\kappa^2 N_f T_{\mu\nu}^{\text{Dp}} \sim N_f/N$ which vanishes as $N \to \infty$ with $N_f$ fixed.

Thus such probe branes need only satisfy their classical equations of motion in the unperturbed background space-time generated by the $N$ color branes. These equations come from extremizing the probe branes' worldvolume (or the DBI action if the brane's U(1) field is turned on).

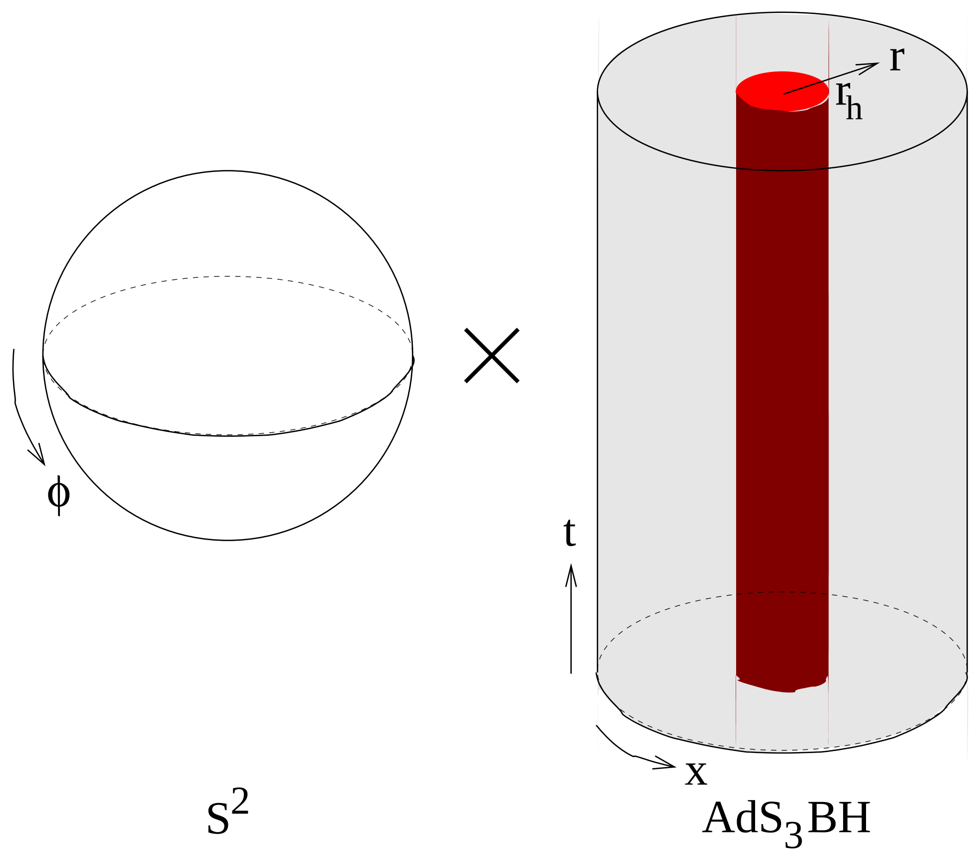
Extensions: D7 example

$\mathcal{N}$=2 SU($N_c$) SYM with $N_f$ massive quarks at temperature $T$

equivalent to

IIB strings on (AdS$_5$ black hole)$\times$S$^5$ background with $N_f$ D7-branes.

$T \propto r_h$, black hole horizon:



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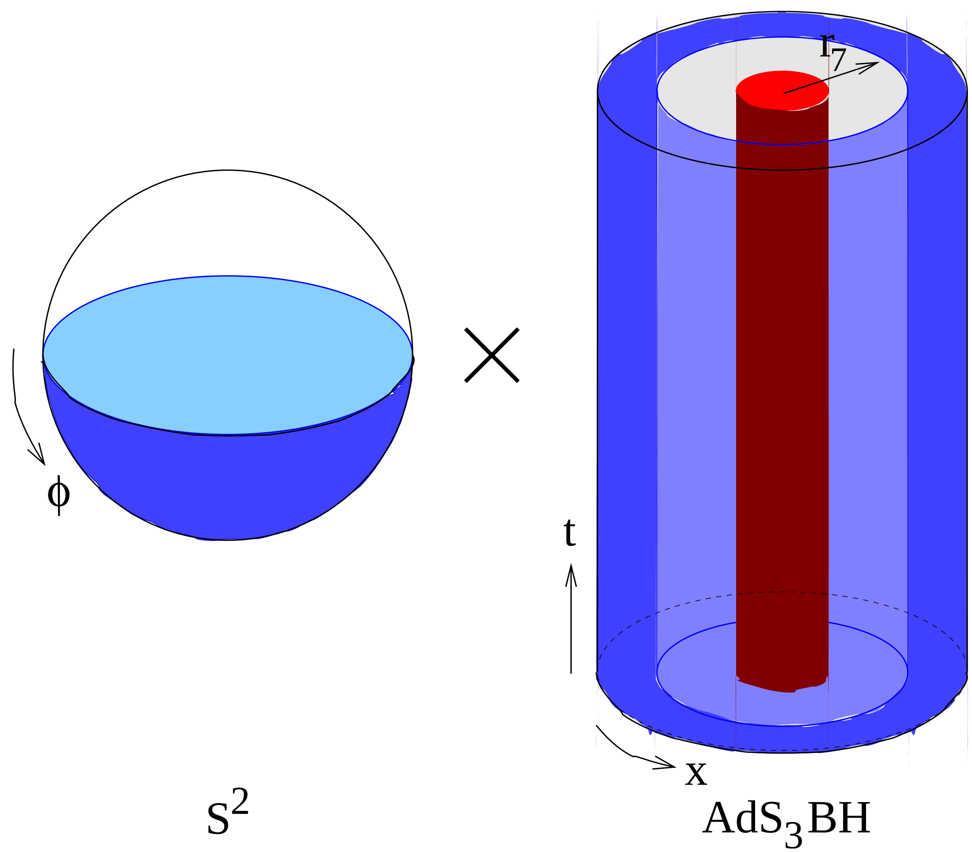
D7 $\subset$ AdS$_5\times$ S$^5$. Quark mass $\propto r_7$. [Karch...0205236, Babington...0306018]



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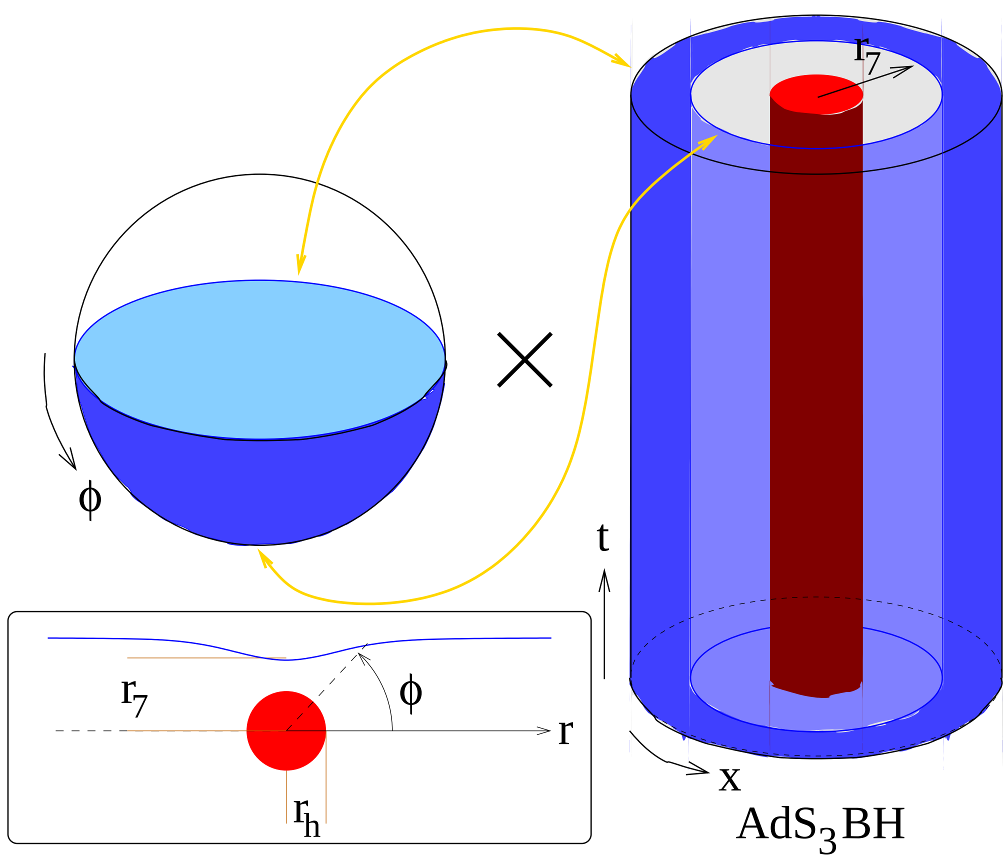
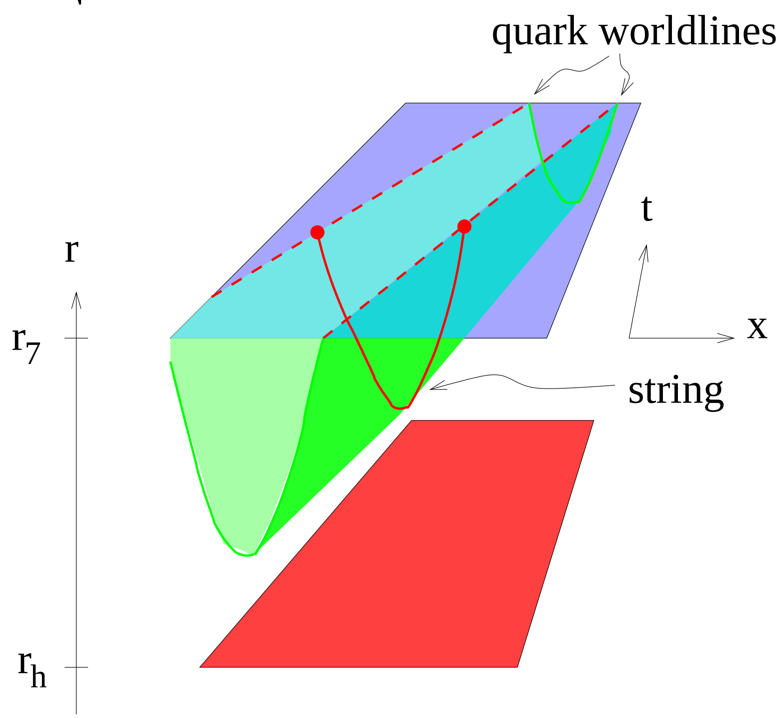


Extensions: probes on probes

(Anti)quarks = endpoints of strings on 7 branes

Quark dynamics = extremize string worldsheet area

$$S_{NG} = \frac{-1}{2\pi\alpha'} \int d^2\sigma \sqrt{-\det\left[g_{\mu\nu}\frac{\partial x^\mu}{\partial \sigma^\alpha}\frac{\partial x^\nu}{\partial \sigma^\beta}\right]}, \qquad x^\mu \in \{t, \vec{x}, r, \phi, \ldots\}.$$




Gives rise to much interesting work: *e.g.*, Herzog et.al., Gubser et.al., H.Liu et.al., Chesler et.al. ...