Correlators in Non-Trivial Backgrounds

R. de Mello Koch, T. Dey, N. Ives, M. Stephanou

University of Witwatersrand

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4. Summary

AdS/CFT Correspondence

- The presently described research is set in the context of the AdS/CFT correspondence.
- This correspondence is a duality between a string theory defined on a certain space (the product of Anti de-Sitter space and a closed manifold) and a conformal field theory on the boundary of this space.
- The AdS/CFT correspondence, in its original form, establishes a duality between Type IIB string theory defined on $AdS_5 \times S^5$ space and $\mathcal{N} = 4$ Super Yang-Mills defined on the boundary of this space.

$\mathcal{N} = 4$ Super Yang-Mills

- Our research takes place on the field theory side of the correspondence, i.e. in $\mathcal{N} = 4$ Super Yang Mills theory (SYM), a 4 dimensional, supersymmetric QFT.

- The theory has matching bosonic and fermionic degrees of freedom (hence SUSY) and includes 6 real scalar fields transforming in the adjoint of U(N). We construct and study operators built out of complex linear combinations of these:

$$Z = \Phi_1 + i\Phi_2, \; Y = \Phi_3 + i\Phi_4, \; X = \Phi_5 + i\Phi_6$$

- These fields have the following two point functions:

$$\left\langle Z_{ij} Z^\dagger_{kl} \right\rangle = \delta_{il}\delta_{jk}, \; \left\langle Y_{ij} Y^\dagger_{kl} \right\rangle = \delta_{il}\delta_{jk}, \; \left\langle X_{ij} X^\dagger_{kl} \right\rangle = \delta_{il}\delta_{jk}$$

BPS operators of $\mathcal{N} = 4$ SYM

- In particular we are concerned with half-BPS operators (operators comprised of a single matrix field) and near BPS operators (operators predominantly composed of a single matrix field with a sprinkling of the other complex matrix fields).
- Schur polynomials provide a very useful basis in which to express half-BPS operators due to certain special properties.

Definition and Properties of Schur Polynomials

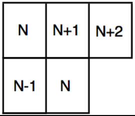
- The Schur Polynomial is defined as:

$$\chi_R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{Tr}\left(\Gamma_R(\sigma)\right) Z^{i_1}_{i\sigma(1)} Z^{i_2}_{i\sigma(2)} \cdots Z^{i_n}_{i\sigma(n)}$$

- Useful features include the fact that the two point function of Schur Polynomials is known to all orders in N:

$$\left\langle \chi_R(Z)\, \chi_S\left(Z^\dagger\right) \right\rangle = \delta_{RS} f_R$$

- The product of weights f_R is defined by considering the Young diagram R:



f_R = N^2 (N+1) (N+2) (N-1)

Definition and Properties of Schur Polynomials

- The Schur polynomials also satisfy a product rule:

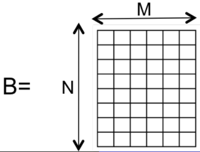
$$\chi_R(Z)\,\chi_S(Z) = \sum_T g_{RST}\,\chi_T(Z)$$

- This product rule can be understood as resulting from the fact that Schur polynomials are themselves characters of SU(N).

- The Littlewood-Richardson numbers g_{RST} count the number of times the irreducible representation T appears in the direct product of the irreducible representations R and S.

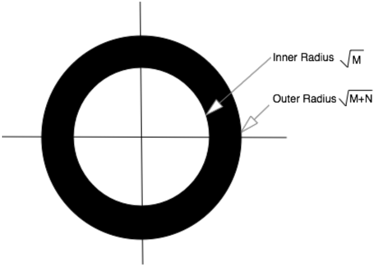
LLM Backgrounds

- Schur Polynomials with $O(N^2)$ fields comprising them are dual to the LLM geometries (asymptotically AdS geometries).
- The particular Young diagram labeling the Schur Polynomial corresponds to a LLM boundary condition which in turn determines the metric of the LLM geometry.
- We consider the following Young diagram label for the background Schur Polynomial.



LLM Backgrounds

- This corresponds to the LLM geometry which is obtained by taking the LLM boundary condition to be a black annulus on a white plane.



The Objective

- Obtaining the result of probing the new background geometry with gravitons, strings or branes on the string theory side of the AdS/CFT correspondence amounts to calculating correlators of the following form on the field theory side:

$$\langle \mathcal{O} \rangle_B = \frac{\left\langle \chi_B \left(Z \right) \mathcal{O} \chi_B \left(Z^\dagger \right) \right\rangle}{\left\langle \chi_B \left(Z \right) \chi_B \left(Z^\dagger \right) \right\rangle}$$

- The inherent difficulties in evaluating such a correlator are clear. Since there are $O(N^2)$ fields comprising $\chi_B(Z)$ non-planar diagrams can certainly not be neglected.
- To evaluate correlators of this form using perturbation theory requires a reorganization corresponding to re-summing an infinite number of diagrams.

Solution

- Fortunately the Schur Polynomial basis for multi-trace half-BPS operators makes the calculation easily tractable.
- Express two arbitrary half-BPS operators in the Schur basis:

$$\prod_i \mathrm{Tr}\left(Z^{n_i}\right) = \sum_R \alpha_R \chi_R(Z), \quad \prod_j \mathrm{Tr}\left(Z^{\dagger m_j}\right) = \sum_R \beta_R \chi_R\left(Z^\dagger\right)$$

- In the trivial background:

$$\mathcal{A}(N) = \left\langle \prod_i \mathrm{Tr}\left(Z^{n_i}\right) \prod_j \mathrm{Tr}\left(Z^{\dagger m_j}\right) \right\rangle = \sum_{R,S} \alpha_R \beta_S \left\langle \chi_R \chi_S^\dagger \right\rangle = \sum_R \alpha_R \beta_R f_R$$

Solution

- In the annulus background:

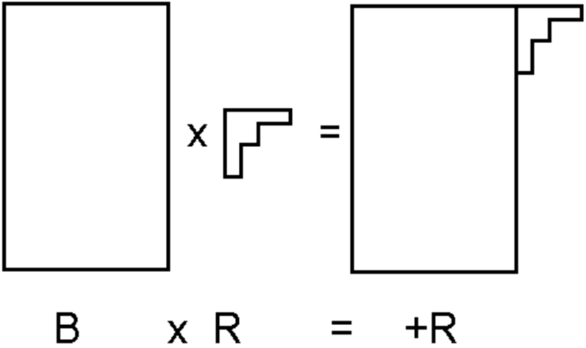
$$\mathcal{A}_B(N) = \left\langle \prod_i \mathrm{Tr}\left(Z^{n_i}\right) \prod_j \mathrm{Tr}\left(Z^{\dagger m_j}\right) \right\rangle_B$$
$$= \sum_{R,S} \alpha_R \beta_S \frac{\langle \chi_B(Z)\chi_R(Z)\chi_S(Z^\dagger)\chi_B(Z^\dagger)\rangle}{f_B}$$

- Now, due to the fact that every column of the Young diagram labeling the background Schur polynomial has N boxes, it is a singlet of SU(N) and thus the product of the background Schur polynomial and any other Schur has the simple result:

$$\chi_B(Z)\chi_R(Z) = \chi_{+R}(Z)$$

Solution

- Graphically:

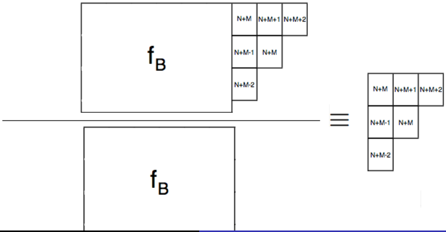


Solution

$$\mathcal{A}_B(N) = \sum_{R,S} \alpha_R \beta_S \frac{\langle \chi_{+R}(Z)\, \chi_{+S}(Z^\dagger) \rangle}{f_B}$$

$$= \sum_R \alpha_R \beta_R \frac{f_{+R}}{f_B} = \sum_R \alpha_R \beta_R f'_R$$

- Where f'_R is precisely equal to f_R after sending $N \to N + M$



Solution

- Thus we have:

$$\mathcal{A}_B\left(N\right) = \mathcal{A}\left(N + M\right)$$

- $\mathcal{A}(N)$ admits an expansion in $\frac{1}{N}$ which implies that $\mathcal{A}(N+M)$ admits an expansion in $\frac{1}{N+M}$.

- This is not a property of the full theory, and pertains to amplitudes of half-BPS operators that can be rewritten in the Schur polynomial basis.

Field Theory Side

- We wish to calculate three point functions of the form:

$$\left\langle \mathrm{Tr}\left(Z^{n}\right)\mathrm{Tr}\left(Z^{m}\right)\mathrm{Tr}\left(Z^{\dagger m+n}\right)\right\rangle$$

- If the holomorphic fields are inserted at one spacetime event and the anti-holomorphic fields are inserted at a second spacetime event, this correlator is extremal and protected by a non-renormalization theorem.
- This means that free field theory results can be faithfully compared to supergravity calculations at large 't Hooft coupling.
- The particular correlator that will be used for comparison is:

$$\left\langle \mathrm{Tr}\left(Z^{2}\right)\mathrm{Tr}\left(Z^{2}\right)\mathrm{Tr}\left(Z^{\dagger 4}\right)\right\rangle_{B}$$

Field Theory Side

- Utilizing our previous results:

$$\left\langle \chi_B(Z)\,\mathrm{Tr}\left(Z^2\right)\mathrm{Tr}\left(Z^2\right)\mathrm{Tr}\left(Z^{\dagger 4}\right)\chi_B\left(Z^\dagger\right)\right\rangle = 16(N+M)^3 f_B$$

- Keeping the comparison with the supergravity in mind, we relate the three point function to a one point function of $\mathrm{Tr}\left(Z^2\right)$ in the following normalized state:

$$|\Phi\rangle = \mathcal{N}\left(\mathrm{Tr}\left(Z^2\right)+\mathrm{Tr}\left(Z^4\right)\right)\chi_B(Z)\,|0\rangle$$

- Now:

$$\langle\Phi|\,\mathrm{Tr}\left(Z^2\right)|\Phi\rangle = \mathcal{N}^2\left\langle \chi_B(Z)\,\mathrm{Tr}\left(Z^2\right)\mathrm{Tr}\left(Z^2\right)\mathrm{Tr}\left(Z^{\dagger 4}\right)\chi_B\left(Z^\dagger\right)\right\rangle$$

Field Theory Side

- Where, using our previous techniques we obtain:

$$\frac{1}{\mathcal{N}^2} = 4(N+M)^4 f_B(1 + O((N+M)^{-2}))$$

- Thus:

$$\langle\Phi|\,\mathrm{Tr}\left(Z^2\right)|\Phi\rangle = \frac{4}{M+N}$$

- This is the result we wish to compare to the supergravity result.

Supergravity Side

- The supergravity field that couples to the boundary operator we are considering, $\mathrm{Tr}\left(Z^2\right)$, is a mixture of components of the fluctuations of the metric on the S^5 and the five form field strength, denoted as S^{22} by Skenderis and Taylor, whose results we now apply.

- The one point function we desire is obtained by variation of the renormalized on-shell supergravity action with respect to the boundary value of S^{22}. The result obtained by Skenderis and Taylor was:

$$\langle \mathcal{O}_{S^{22}} \rangle = N^2 \int r^3 \rho \, e^{2i\phi} dr d\phi$$

Supergravity Side

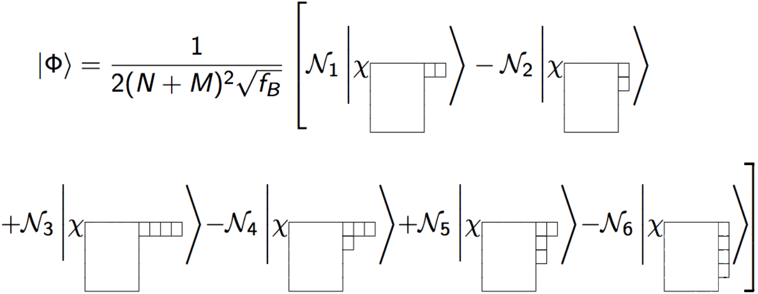
- To determine ρ we utilize the free fermion description of the half-BPS sector of $\mathcal{N} = 4$ SYM.
- The Schur polynomial labelled by Young diagram λ (once normalized) corresponds to the state:

$$\hat{C}^{\dagger}_{N-1+\lambda_1} ... \hat{C}^{\dagger}_{\lambda_N} |0\rangle$$

- Where the fermion creation and annihilation operators, $\hat{C}^{\dagger}_{l}$ and $\hat{C}_{m}$ satisfy: $\{\hat{C}^{\dagger}_{l}, \hat{C}_{m}\} = \delta_{lm}, \quad \hat{C}_{m} |0\rangle = 0$
- The energies of the free fermions are given by: $E_i = \lambda_i + N - i + 1, \; i = 1, ..., N$

Supergravity Side

- To utilize the free fermion description we rewrite the the state $|\Phi\rangle$ in terms of normalized Schur polynomials:



$$\equiv \sum_{i=1}^{6} \tilde{\mathcal{N}}_i \, |\chi_i\rangle$$

Supergravity Side

- We have normalized the Schur polynomials utilizing the correlator result e.g.:

$$\chi_{\boxed{}\,\square\square} = \sqrt{f_B(N+M)(N+M+1)}\,\left|\chi_{\boxed{}\,\square\square}\right\rangle$$

- since,

$$\left\langle \chi_{\boxed{}\,\square\square}(Z)\,\chi_{\boxed{}\,\square\square}\left(Z^{\dagger}\right)\right\rangle = f_B(N+M)(N+M+1)$$

Supergravity Side

- The supergravity density function is given by:

$$\rho = \frac{e^{-Nr^2}}{\pi} \sum_{l,m} (z^*)^l (z)^m \sqrt{\frac{2^{l+m}}{l!m!}} \langle \Phi | \hat{C}_l^\dagger \hat{C}_m | \Phi \rangle , \quad z = \sqrt{\frac{N}{2}} r e^{i\phi}$$

- In evaluating $\langle \Phi | \hat{C}_l^\dagger \hat{C}_m | \Phi \rangle$ we note that there are two types of terms that contribute, those where $l = m$ or where l and m are such that the fermion occupation of $\hat{C}_m |\chi_i\rangle$ matches that of $\langle \chi_i | \hat{C}_l^\dagger$.
- We denote the sum of terms of the first type ρ_1 and the sum of terms of the second type $\tilde{\rho}_2 = \cos 2\phi \rho_2$

Supergravity Side

- In this way we obtain a density that has the form:

$$\rho = \rho_1 + \cos 2\phi \rho_2$$

- The integral over ρ_1 vanishes and finally evaluating the integral of ρ_2, we obtain:

$$\langle \Phi | \operatorname{Tr}\left(Z^2\right) | \Phi \rangle = \frac{4}{M+N}$$

- This matches the field theory result precisely!

Summary

- We have introduced a means to easily calculate half-BPS correlators in the presence of certain Schur polynomial operators dual to new background geometries.
- In particular the annulus background was presented here but we have also extended the technology to multi-ring backgrounds as well as multi-charge backgrounds.
- Confirming the correctness of the results obtained involved calculating holographically renormalized correlators for comparison. We have agreement.

References

1. K. Skenderis and M. Taylor, "Anatomy of Bubbling Solutions," JHEP 09 (2007) 019 [arXiv:0706.0216]

2. R. de Mello Koch, T. Dey, N. Ives, M. Stephanou, "Correlators of Operators with a Large R charge," JHEP 08 (2009) 083 [arXiv:0905.2273]