WANDs in higher-dimensional gravity

A new magic method?

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CMPP Formalism and WANDs

- This is a slightly non-standard approach to studying GR (with whatever exotic matter you happen to like).
- Idea is that for a few spacetimes (including several important ones), there is a choice of frame which simplifies things.
- Generalization of Petrov classification/Newman-Penrose formalism to higher dimensions.
- CMPP = Coley, Milson, Pravda, Pravdova (2004)

Possible Applications

Doing this in higher dimensions is a moderately new idea, some possible applications are:

- Finding new solutions to GR in higher dimensions.
- Classifying known solutions?
- Studying known solutions (e.g. asymptotics)
- Perturbations? - Teukolsky equation in 4D comes from Newman-Penrose approach.
- Numerical relativity?

Standard coordinate GR

- In standard undergraduate GR, everything is done in a *coordinate* basis.
- All complicated information about the curvature etc. is contained in the metric, with line element

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu \otimes dx^\nu .$$

- The D 1-forms dx^μ are a basis for the (dual) tangent bundle of the spacetime, obtained directly from the coordinates x^μ.
- Derivatives $d(dx^\mu) = 0$ trivial.

Arbitrary frames

- Instead, can define another basis $\mathbf{e}^a(x)$ for the tangent bundle, so that

$$ds^2 = \eta_{ab}\ \mathbf{e}^a(x) \otimes \mathbf{e}^b(x),$$

  where η is the Minkowski metric (used to raise/lower indices a,b,...).
- Here the complicated information is contained within $\mathbf{e}^a(x)$.
- Derivatives $d\mathbf{e}^a = -\gamma^a{}_b \wedge \mathbf{e}^b$ define the spin connection.
- These frames often called *tetrads* in four dimensions.

Null frames

- Usually in the frame formalism of the last page, take $\mathbf{e}^0$ to be timelike, others spacelike, with

$$\eta_{ab} = \mathbf{e}_a . \mathbf{e}_b = \mathrm{diag}(-1, 1, 1, ...).$$

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- In this talk we will work in a *null frame*, with $\mathbf{e}^0$, $\mathbf{e}^1$ null, and $\mathbf{e}^i$ spacelike ($i = 2, ..., D-1$). Have

$$\eta_{ab} = \mathbf{e}_a.\mathbf{e}_b = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

- Write $\mathbf{l} = \mathbf{e}_0 = \mathbf{e}^1$, $\mathbf{n} = \mathbf{e}_1 = \mathbf{e}^0$, $\mathbf{m_i} = \mathbf{e}_i = \mathbf{e}^i$.

Introduction to CMPP Formalism

[Coley, Milson, Pravda and Pravdova (2004)] started the development of a formalism for doing D-dimensional GR in null frames (generalizing Newman-Penrose formalism/Petrov classification in 4D).

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Let μ, ν,... be spacetime indices. We can expand any tensor T in a frame basis by defining

$$T_{ab...c} = e_a^\mu e_b^\nu ... e_c^\rho T_{\mu\nu...\rho}.$$

Spacetime scalars

Each object $T_{ab\ldots c}$ is now a spacetime *scalar* (no spacetime indices), so for example,

$$\nabla_\mu T_{ab\ldots c} = \partial_\mu T_{ab\ldots c}.$$

However, it does transform under the Lorentz group $SO(1, D-1)$. This corresponds to our freedom to choose different frame bases.

Action of the Lorentz group

We divide up the action of the Lorentz group on the basis vectors as follows:

Spins $SO(D-2)$ rotations of the spatial basis vectors $\mathbf{m_i}$.

Null Rotations Rotations of one of the null basis vectors about the other, for example a null rotation about $\mathbf{n}$ takes the form

$$\mathbf{l} \to \mathbf{l} - z_i\mathbf{n} - \frac{1}{2}z^2\mathbf{l}, \qquad \mathbf{n} \to \mathbf{n}, \qquad \mathbf{m_i} \to \mathbf{m_i} + z_i\mathbf{n}$$

for some z_i.

Boosts Under a local Lorentz boost we get

$$\mathbf{l} \to \lambda\mathbf{l}, \qquad \mathbf{n} \to \lambda^{-1}\mathbf{n}, \qquad \mathbf{m_i} \to \mathbf{m_i},$$

and we say that $\mathbf{l}$, $\mathbf{n}$ and $\mathbf{m_i}$ have *boost weights* $+1$, -1 and 0 respectively.

Classification by boost weight

We can use this to make some definitions that turn out to be useful:

- The idea is that we classify components of tensors by their *boost weights*.
- Most useful to apply this to the Weyl tensor $C_{\mu\nu\rho\sigma}$.

  (Recall: Weyl tensor is totally traceless part of Riemann curvature tensor).
- Just need to count the number of 0s and 1s in the indices to find boost weight of a component, as follows...

Classification of the Weyl tensor

(recall Weyl symmetries $C_{abcd} = C_{cdab} = C_{[ab]cd} = C_{ab[cd]}$ and $C_{a[bcd]} = 0$)

Boost Weight +2 C_{0i0j}

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Boost Weight 0 $C_{ijkl}, C_{01ij}, C_{0i1j}, C_{0101}$

Boost Weight -1 C_{1ijk}, C_{101i}

Boost Weight -2 C_{1i1j}

Recall Petrov classification in 4D

Those familiar with the Petrov classification will realise that all these components can be written in terms of complex scalars Ψ_A in 4 dimensions.

Boost Weight +2 $\Psi_0 \sim C_{0i0j}$

Boost Weight +1 $\Psi_1 \sim C_{0ijk}, C_{010i}$

Boost Weight 0 $\Psi_2 \sim C_{ijkl}, C_{01ij}, C_{0i1j}, C_{0101}$

Boost Weight -1 $\Psi_3 \sim C_{1ijk}, C_{101i}$

Boost Weight -2 $\Psi_4 \sim C_{1i1j}$.

We can't do this in general dimension.

Definition of a WAND

- Say that $\mathbf{l}$ is a *Weyl-aligned null direction (WAND)* iff all boost weight +2 components vanish.
  (In 4D this is equivalent to being a PND)
- Say that $\mathbf{l}$ is a *multiple WAND* iff all boost weight +2 and +1 components vanish.
  (In 4D this is equivalent to being a repeated PND)
- A spacetime admitting a (multiple) WAND is *algebraically special*.

Existence of WANDs

The natural first question is do WANDs always exist?

- In $D = 4$, yes. Any spacetime admits exactly 4 WANDs, some possibly degenerate.
- In $D > 4$, no. An arbitrary spacetime might admit no WANDs, a finite number of WANDs, or even a continuous family.
- Existence is a local property in general, but for analytic spacetimes can extend this globally (so in a smooth, non-analytic spacetime, everything I say is valid in some open neighbourhood of any point in a spacetime).

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Algebraic Types

Given a spacetime, we look to pick $\mathbf{l}$ so that as many high boost weight components of $\mathbf{l}$ vanish as possible. Different algebraic types are defined based on which components of C_{abcd} vanish in this chosen frame:

Type	b					
G	+2	+1	0	-1	-2	General
I		+1	0	-1	-2	Not really special

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N					-2	Even more special
I_i		+1	0	-1		$\mathbf{n}$ also WAND
D = $\text{II}_i i$			0			$\mathbf{n}$ also mWAND
II_i			0	-1		$\mathbf{n}$ also WAND
O						Conformally flat

Algebraic Types of various spacetimes

Generically, spacetimes are not algebraically special, but many important metrics are, for example:

- Schwarzchild: Type D
- Kerr/Myers-Perry-((A)dS): Type D
- C-metric (known in 4d only): Type D
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But:

- Singly-Spinning Black Ring:
  - Type D on the horizon
  - Type I_i elsewhere.

Constructing new solutions?

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Constructing new solutions?

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- Answer: Not yet...
- Lots more scope for trying.
- Think of this as a simplifying assumption...might or might not make things tractable.
- Potentially useful for AdS-CFT as studying solutions with AdS asymptotics often no more difficult than asymptotically flat (c.f. inverse scattering techniques where this is definitely not true).

Geodesity of WANDs

Goldberg-Sachs Theorem says that in 4D Einstein spacetimes, a null congruence is a multiple WAND iff it is geodesic and shearfree. (An Einstein spacetime is a solution of the vacuum Einstein equations with possible cosmological constant.)

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Theorem

An Einstein spacetime admits a multiple WAND if, and only if, it admits a geodesic multiple WAND. [M.N.D. and Reall (2009)]

The End

Axisymmetric Solutions

Another example of recent work is the following:

- In D dimensions, a spacetime is *axisymmetric* if it admits an $SO(D-3)$ isometry group.
- [Godazgar and Reall (2009)] constructed all algebraically special, axisymmetric solutions of the vacuum Einstein equations in arbitrary dimension.
- Nothing new found.
- No axisymmetric, alg. special C-metric in higher dimensions.

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