AdS black holes.

Plan : AdS spacetime : definition and basic properties.
slicings and spacetime diagrams.

static black holes : Basic properties, thermodynamics
and Hawking-Page phase transition.

Other example solutions : stationary black holes, black strings

Open problems : Weyl metrics with cosmological constant

hepth/9906040 Emparan
hepth/9903238 Emparan et al
Review. Emparan Reall
hepth/9803131 Witten
Math. Phys 83 Hawking & Page.

C. Charmousis Orsay LPT & Tours LMPT.

Spacetimes of constant curvature.

Consider $$I = \frac{1}{16\pi G}\int_{\mathcal{M}} d^dx \sqrt{-g}\,(R-2\Lambda) \qquad \dim \mathcal{M} = d.$$

Field equations: $$G_{AB} + \Lambda g_{AB} = 0$$

Consider spacetimes of constant curvature:

$$R_{ABCD} = \frac{R}{d(d-1)}(g_{AC}g_{BD} - g_{AD}g_{BC}) \quad , R \text{ constant}$$

then $$G_{AB} = -\frac{d-2}{2d} R\, g_{AB} = -\Lambda g_{AB}$$

therefore $$R = \frac{2d}{d-2}\Lambda \quad \begin{cases} \Lambda = 0 & \text{Mink.} \\ \Lambda > 0 & \text{de Sitter} \\ \Lambda < 0 & \text{anti de Sitter.} \end{cases}$$

For $\Lambda = 0$ we have (d translation) and d-dimensional Lorentz symmetry. ($\equiv$ Poincaré symmetry)

All in all $$d + \frac{d(d-1)}{2} = \frac{d(d+1)}{2} \quad \text{Killing vectors}$$

max # of Killing vector

↪ Flat st is maximally symmetric.

Maximally symmetric spaces. of dim $n = d-1$.

line element $\quad dK_n^2 = \dfrac{d\chi^2}{1-\kappa\chi^2} + \chi^2 \, d\Omega_{n-1}^2 \quad , \kappa = 0, 1, -1.$

where $\quad d\Omega_k^2 = d\theta_k^2 + \sin^2(\theta_k)\, d\Omega_{k-1}^2$ with $d\Omega_1 = d\theta_1$ and

$\forall k = 1, \dots, n-1, \quad \theta_k \in [0,\pi[, \dots, \theta_1 \in [0, 2\pi[$

Set $\chi = \sin\varphi$, $\kappa = 1$ $\quad d\Omega_n^2 = d\varphi^2 + \sin^2\varphi \, d\Omega_{n-1}^2 \quad \varphi \in [0,\pi[$

$\chi = \sinh\varphi$, $\kappa = -1$ $\quad dH_n^2 = d\psi^2 + \sinh^2\psi \, d\Omega_{n-1}^2 \quad \psi \in [0,+\infty[$

$d\Omega_n^2$ unit metric on $\mathcal{S}^n$

dH_n^2 unit metric on $\mathcal{H}^n$

AdS spacetime

Take $\Lambda \neq 0$. Basic idea is to embed cc spacetime in a d+1 dimensional flat spacetime.

de Sitter: $$-X_0^2 + \sum_{i=1}^{d} X_i^2 = a^2$$ where a is constant

Hyperboloid $\hookrightarrow$ in d+1 dim. flat spacetime.

$\uparrow$ time

Killing vectors $= \frac{d(d+1)}{2}$

symmetry group $SO(1,d)$

topology $\mathbb{R} \times S^{d-1}$!

anti de Sitter. $$-(X_0^2 + X_d^2) + \sum_{i=1}^{d-1} X_i^2 = -\ell^2 \,, \quad \ell \text{ constant.}$$

$$\hookrightarrow \quad ds^2 = -(dX_0^2 + dX_d^2) + \sum_{i=1}^{d-1} dX_i^2$$

Symmetry group is $SO(2, d-1)$ # Killing vectors $= \frac{d(d+1)}{2}$

Both spacetimes are maximally symmetric.

adS boundary: Consider $X_A \to \lambda X_A$ and take $\lambda \to \infty$

$$\Rightarrow \quad X_0^2 + X_d^2 - \sum_{i=1}^{d-1} X_i^2 = 0. \quad \text{at the boundary.}$$

For $X_0 \neq 0$ $\quad -X_d^2 + \sum_{i=1}^{d-1} X_i^2 = 1 \to$ de Sitter space. $\mathbb{R} \times S$

for $X_0 = 0$: $\sum_{i=1}^{d-1} X_i^2 = 1 \to$ d-2 sphere $\{\cdot\} \times S^{d-2}$

$S^1 \times S^{d-2}$

symmetries $= \frac{d(d-1)}{2} + d = \frac{d(d+1)}{2}$

Parametrisations of adS.

Global slicing.

$$\begin{cases} X_0^2 + X_d^2 - \sum_{i=1}^{d-1} X_i^2 = \ell^2 \\ ds^2 = -(dX_0^2 + dX_d^2) + \sum_{i=1}^{d-1} dX_i^2 \end{cases}$$

take 2 spheres $X_0^2 + X_d^2 = r_1^2$ and $\sum_{i=1}^{d-1} X_i^2 = r_2^2$

therefore $r_1^2 - r_2^2 = \ell^2$

take $r_1 = \ell \cosh\left(\frac{u}{\ell}\right)$ $\quad u \in [0, +\infty[$

$r_2 = \ell \sinh\left(\frac{u}{\ell}\right)$

go back to the line element : $ds^2 = -(dr_1^2 + r_1^2 d\psi^2) + dr_2^2 + r_2^2 d\Omega_{d-2}^2$

$$\Rightarrow \boxed{ds^2 = -\ell^2 \cosh^2\left(\frac{u}{\ell}\right) d\psi^2 + du^2 + \ell^2 \sinh^2\left(\frac{u}{\ell}\right) d\Omega_{d-2}^2}$$

with $\psi \in [-\pi, \pi[$ periodic time! topology is $S^1 \times \mathbb{R}^{d-1}$

$$2\Lambda = -\frac{(d-1)(d-2)}{\ell^2}$$

take universal covering of adS where $t \in]-\infty, +\infty[$
and $t \equiv \psi$ for each 2π-interval.

topology is $\mathbb{R}^d$ and boundary $\mathbb{R} \times S^{d-2}$

Slicings:

$$ds^2 = -\left(k + \frac{r^2}{l^2}\right)dt^2 + \frac{dr^2}{k + \frac{r^2}{l^2}} + \frac{r^2}{l^2}\, dK_{n-1}^2$$

where $\kappa = 0, 1, -1$ and $dK_{n-1}^2 = \begin{cases} l^2 d\Omega_{n-1}^2 & \kappa = 1 \\ \sum dx_i^2 & \kappa = 0 \\ l^2 dH_{n-1}^2 & \kappa = -1 \end{cases}$

- $\kappa = 1$ spherical slicing, set $r = l \sinh \frac{u}{l}$ $\to$ global adS.

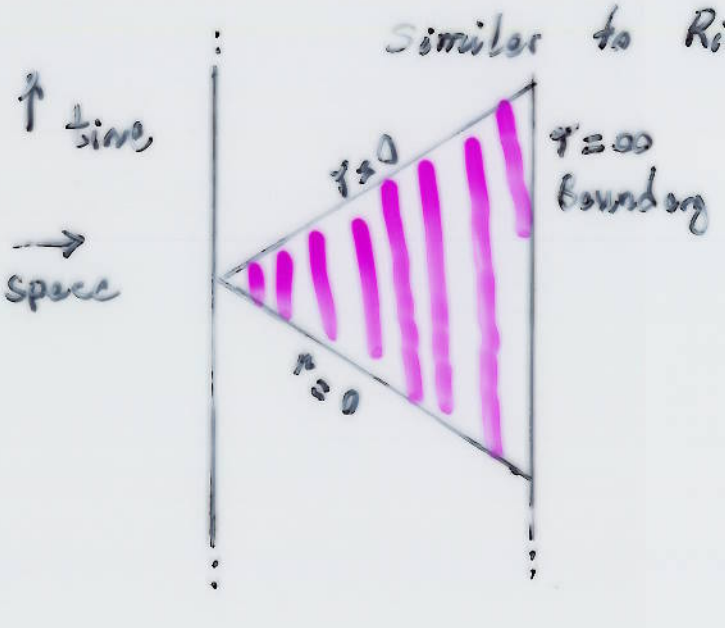
- $\kappa = 0$ planar or Poincaré slicing. Need $r > 0$ and $r < 0$ patch to cover all of adS.

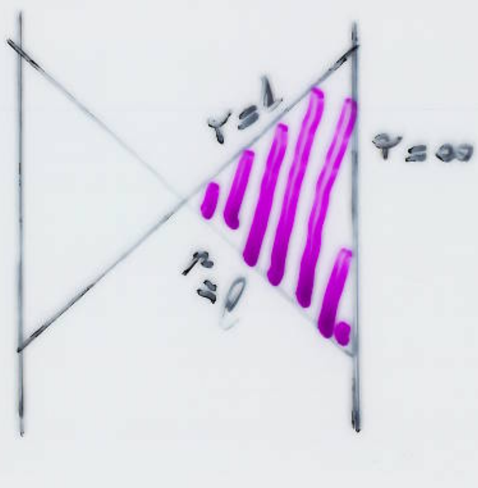
  For $r = 0$ we have coordinate singularity or degenerate Killing horizon. ie there is no temperature associated with horizon.

- $\kappa = -1$ hyperbolic slicing horizon at $r_+ = l$

  with $\beta = \left(\frac{1}{T}\right) = 2\pi l$

Similar to Rindler space!





Hawking & Ellis

Spacetime diagramm.

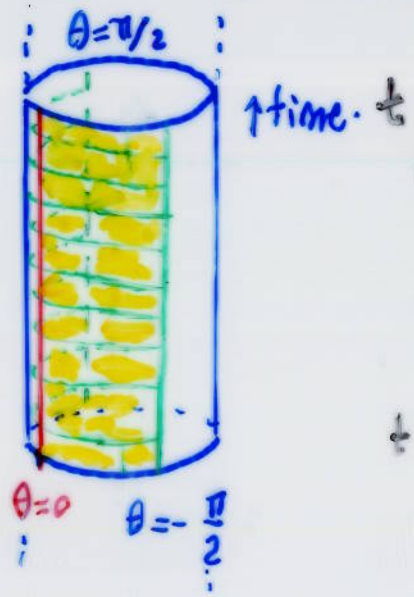
Start with global coordinate system. and apply conformal transformation to bring spacetime infinity to finite values

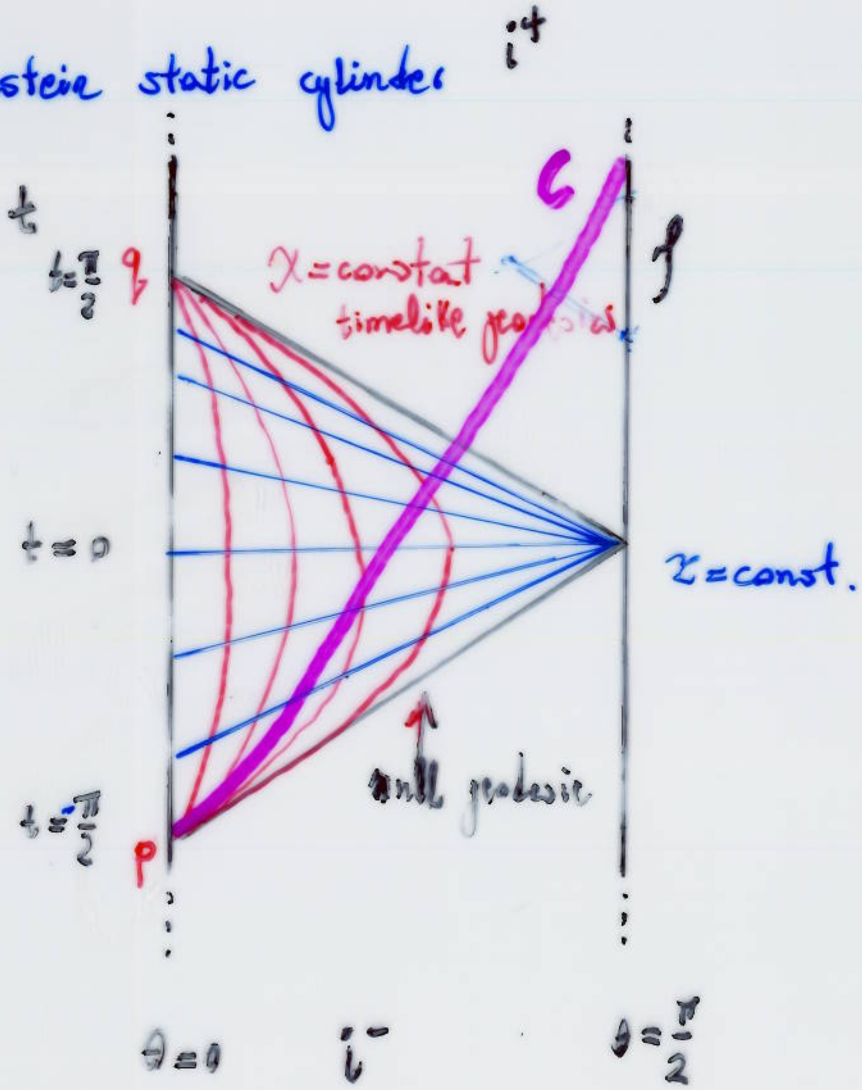
$$\text{set} \quad d\theta = \frac{du}{l\cosh\frac{u}{l}} \Rightarrow \tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right) = e^{u/l}$$

where $\theta \in [0, \frac{\pi}{2}[$

$$ds^2 = \frac{l^2}{\cos^2\theta}\left(-dt^2 + d\theta^2 + \sin^2\theta \, d\Omega^2_{n-1}\right)$$

Half of the Einstein static cylinder





$$-d\tau^2 + l^2\cos^2\tau \, dH^2_{n-1}$$

Static black holes.

$d = n+1$

$$ds^2 = -V(r)\,dt^2 + \frac{dr^2}{V(r)} + \frac{r^2}{l^2}\,dK_{n-1}^2$$

$$V(r) = \kappa + \frac{r^2}{l^2} - \frac{\mu}{r^{n-2}}$$

- Unlike $\Lambda = 0$ or $\Lambda > 0$ we can have black holes with differing topologies which are asymptotically locally adS.

- For $\kappa = 1$ we asymptote global adS

  For $\kappa = 0, -1$ we asymptote a patch of adS.

- For $\kappa = 1$, $l \to \infty$ we approach Schw. solution

Denote with $r = r_+$ the outermost zero of V

$$V(r_+) = 0 \implies \kappa + \frac{r_+^2}{l^2} - \frac{\mu}{r_+^{n-2}} = 0.$$

Event horizon of the static black hole.

$$\mu = \kappa r_+^{n-2} + \frac{r_+^n}{l^2} \quad \nearrow \text{ in } r_+$$

Temperature: We Wick rotate to Euclidean signature for $r > r_+$

$t = i\tau$ periodic time coordinate.

then $\rho_+ = \sqrt{\dfrac{2(r-r_+)}{|V'(r_+)|}}$ gives

$$ds^2 \sim \frac{1}{2} V'(r_+)^2 \rho^2 d\theta^2 + d\rho^2 + \dots \quad \text{for } r \sim r_+$$

Generically we have a conical singularity and the Euclidean quantum field propagator will be ill defined

Setting period $\beta = \dfrac{4\pi}{|V'(r_+)|}$ propagator describes a canonical ensemble in thermal equilibrium with heat bath of temperature $T = \beta^{-1}$

We get $$\beta = \frac{4\pi \ell^2 r_+}{n r_+^2 + \kappa (n-2) \ell^2}$$

or $$r_+ = \frac{2\pi \ell^2}{n\beta}\left[1 \pm \sqrt{1 - \kappa \frac{n(n-2)\beta^2}{4\pi^2 \ell^2}}\right]$$

For $\kappa = 1$ we have two branches.

with $\mu = r_+^{n-2} + \dfrac{r_+^n}{\ell^2}$ ↑ increasing μ with r_+

Hyperbolic black hole

$$ds^2 = -V\,dt^2 + \frac{dr^2}{V} + r^2 d\mathcal{H}_2^2$$

$$-u^2 + x^2 + y^2 = 1$$

$$ds^2 = -du^2 + dx^2 + dy^2$$

$$V = -1 + \frac{r^2}{l^2} - \frac{\mu}{r}$$

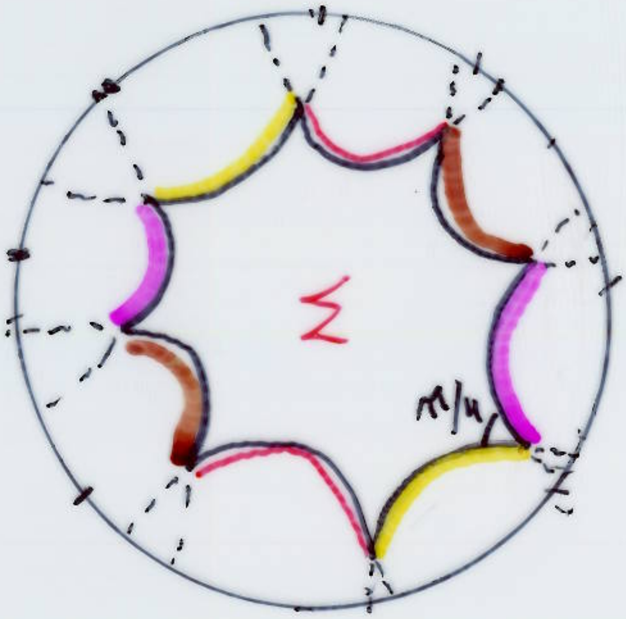
Horizon surface $\Sigma = \mathcal{H}_2/\Gamma$ where $\Gamma \leq SO(1,2)$

Σ compact surface with $g > 1$

Σ: polygon bounded by geodesics with sum of angles 2π.

Horizon geometry.

Unit ball.



Regular octagon
Opposite sides are identified via Γ

$g = 2$

Aminneborg et al.

gr-qc/9604005.

Hyperbolic black hole and background temperature.

Temperature of black hole is fixed. Hence need arbitrary temperature for background in order to obtain the partition function for a canonical ensemble.

Consider a hyperbolic black hole.

then pure adS (for $\mu=0$) has $\beta = 2\pi l$. otherwise we have conical singularity

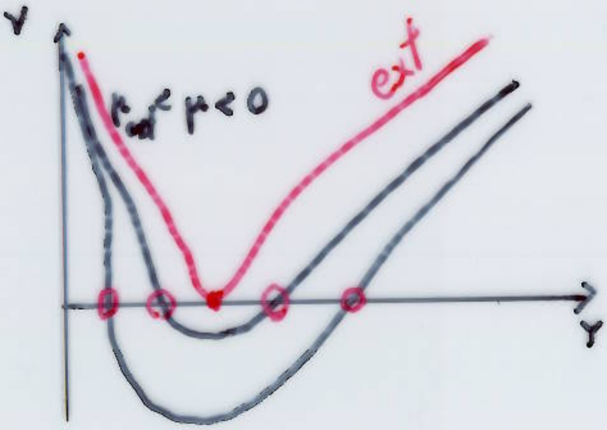
For $\kappa=-1$ we can have negative μ

In fact $\mu_{ext} < \mu < 0$ where $\mu_{ext} = -\frac{2}{n-2}\left(\frac{n-2}{n}\right)^{n/2} l^{n-}$

$$r_{ext} = \sqrt{\frac{n-2}{n}}\, l$$

Black hole has inner horizon like RNAdS black ho

For extremal case no conical singularity



Q What is the correct background?

Hawking & Page, Witten.

Note that Vol is infinite for adS ($\mu=0$) and for the black hole.

Introduce cut-off R such that $r<R$

and subtract the two volumes in order to get rid of divergence.

$$\mathrm{Vol}_1(R) = \int_0^{\beta_0} d\tau \int_0^R dr \int_{S^{d-2}} d\Omega \; r^{d-2}$$

$$\mathrm{Vol}_2(R) = \int_0^{\beta} d\tau \int_{r_+}^R dr \int_{S^{d-2}} d\Omega \; r^{d-2}$$

Notice that volumes are different
however for canonical ensemble have to fix temperature at arbitrary surface $r=R$.

$$T = \frac{1}{\beta}$$

therefore $$\beta_0 = \frac{\beta\sqrt{1+\frac{R^2}{l^2}-\frac{\mu}{r^{d-3}}}}{\sqrt{1+\frac{R^2}{l^2}}}$$

We get $$I = -\log Z = \frac{l^2 r_+^{d-2} - r_+^{d+2}}{4G\left[(d-1)r_+^2+(d-3)l^2\right]}\,\mathrm{Vol}(S^{d-2})$$

Stick to global case with $\kappa = 1$.

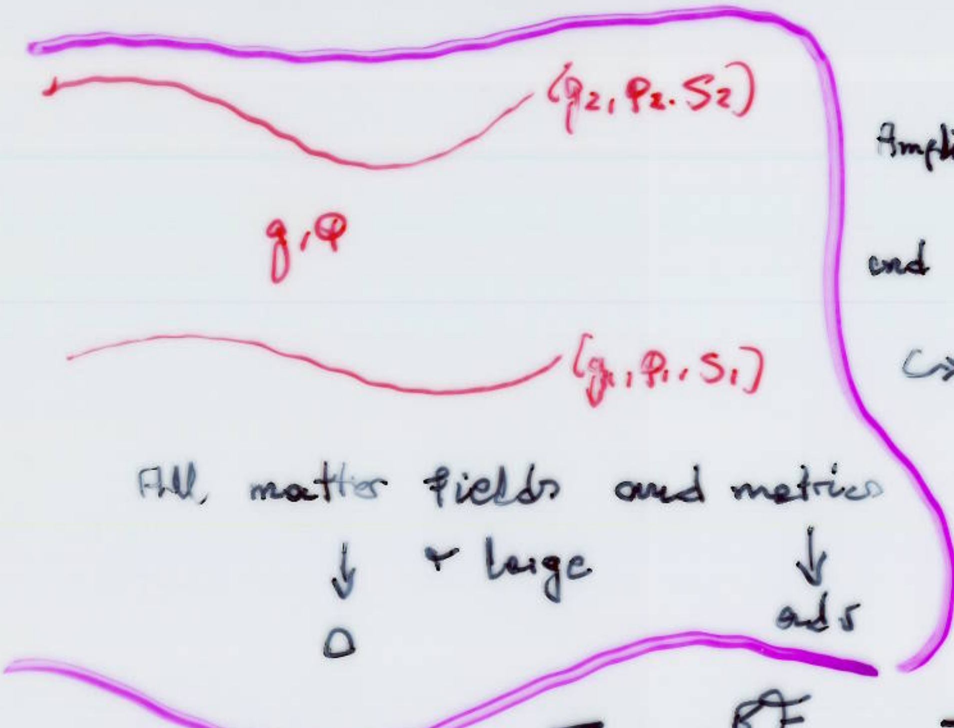
AdS has a confining gravitational potential

$$V \sim \frac{r^2}{\ell^2} \text{ for large } r.$$

Non-zero rest mass particles are confined by gravitational well and prevented from escaping to infinity

Hence the canonical ensemble where the black hole is in equilibrium heat bath of temperature T is well defined for adS

Need to evaluate partition function for the ensemble.



Amplitude $= \int \mathcal{D}(g, \varphi) \exp[-\hat{I}(g,\varphi)]$

and saddle point approx.

$\hookrightarrow$ Regular classical solutions give dominant contributions to the partition f $\overset{an}{=}$ $|\Lambda| < m_{Pl}^2$

$$Z = e^{\beta F} \sim e^{-I_{sol}}$$

$$I_{sol} = -\frac{1}{16\pi G} \int d^d x \sqrt{g} \, (R - 2\Lambda) + B.T$$

$$= \frac{d-1}{8\pi G} \int d^d x \sqrt{g} = \frac{d-1}{8\pi G} \mathrm{Vol}$$

Heuristically note that $I > 0$ for $r_+ < l$

$\rightarrow$ tunneling from ads $\rightarrow$ BH is exp. suppressed

whereas the opposite holds for $r_+ > l$ $\quad \Gamma \sim e^{-I}$

mean energy of Ensemble $\quad E = \sum_{\text{states}} E_i \, p_i \;, \; p_i = \frac{1}{Z} e^{-\beta E_i}$

$$E = -\frac{\partial}{\partial \beta} \log Z = \underbrace{\frac{d-2}{16\pi G} \mathrm{Vol}(S^{d-2}) \left(l^{-2} r_+^{d-1} - r_+^{d-3} \right)}$$

$= M$, mass of B.H.

Ashtekar & Magnon

Ashtekar & Das

Grav. def. of mass in ads.

Entropy of ensemble $\quad \left(S = \sum_{\text{states}} P_i \ln P_i \right)$

$$S = \beta E - I = \frac{1}{4G} r_+^{d-2} \mathrm{Vol}(S^{d-2}) = \frac{\mathrm{Area}^+}{4G}$$

heat capacity $\quad C = \frac{\partial E}{\partial T}$

Sign of C determines thermodynamic stability of the system.

Free energy $\quad \mathcal{F} = I T$

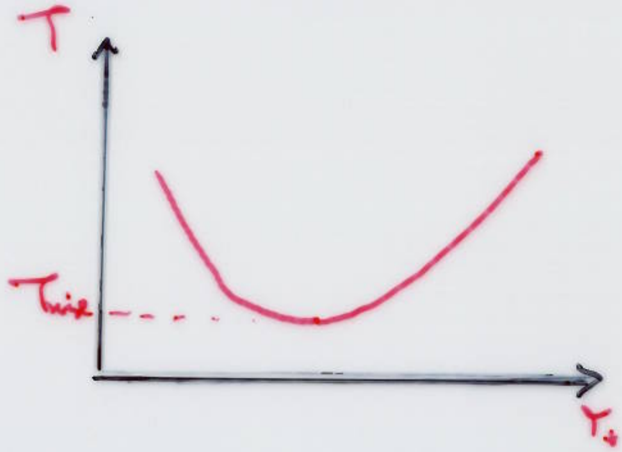
of the ensemble

Phase structure for gravity at adS.

Hawking & Page
Witten

$T_{min} \sim \ell/2\pi$

For $T < T_{min}$: thermal radiation no black hole



For $T > T_{min}$ we have two black holes $M_- < M_+$

M_- black hole always has $C_- < 0$ and $\mathcal{F} > 0$.
it can therefore decay to pure thermal radiati[on]
or M_+ with $C_+ > 0$

$$M_- \to \text{rad.}, \quad M_- \to M_+$$

For $T = T_{HP}$ Free energy of M_+ is lower than pure radiation in adS therefore black hole production is energetically favored.

This is a phenomenon encountered in adS space

Tunneling probability $\Gamma \sim e^{-(I_{bh} - I_{AdS})}$

Stationary black holes.

D=4 We have local timelike and axial Killing vector but they are no longer orthogonal
Solution found by Carter (M, J) Kerr adS metric.

D=5 In higher dimensions more Killing vectors and therefore more planes of rotation. In D=5 (M, J_1, J_2) Hawking, Hunter & Taylor

In higher dimension general solution found recently by Gibbons, Lu, Page & Pope.

d=4
$$ds^2 = -\frac{\Delta_r}{\rho^2}\left[dt - \frac{a}{\Xi}\sin^2\theta\, d\varphi\right]^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 +$$
$$+ \frac{\sin^2\theta\,\Delta_\theta}{\rho^2}\left[a\,dt - \frac{r^2+a^2}{\Xi}d\varphi\right]^2$$

$\rho^2 = r^2 + a^2\cos^2\theta$ Need $a^2 \le \frac{1}{\ell^2}$

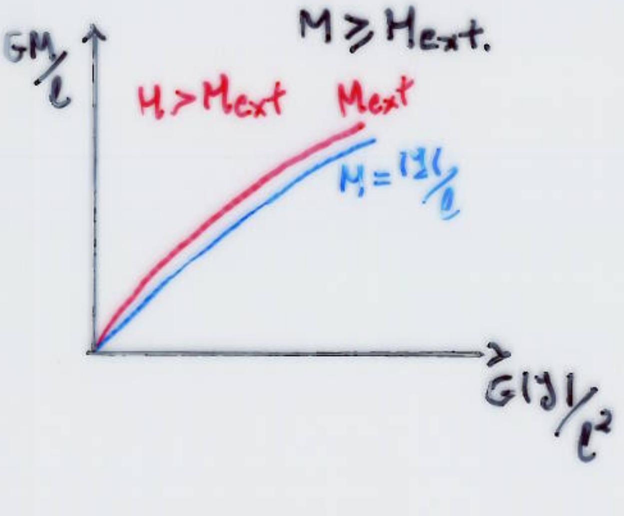
$\Delta_r = (r^2+a^2)\left(1+\frac{r^2}{\ell^2}\right) - 2Mr \rightarrow$ Event horizon $r = r_+$

$\Delta_r(r_+) = 0.$

$\Delta_\theta = 1 - \frac{a^2}{\ell^2}\cos^2\theta$

$M \geq M_{ext}.$

$\Xi = 1 - \frac{a^2}{\ell^2}$



$\Lambda = 0 \rightarrow$ Kerr

$a = 0 \rightarrow$ AdS S-hw

$d=5$ 3 parameter family (M, J_1, J_2) (Hawking, Hunter, Taylor)
$\downarrow \quad \downarrow$
$a \quad b$

For $a=b=0$ we get adS-Schw.
Inversely, regular & stationary perturb of adS-Schw lead to ΛMP solution.

If other black holes exist such as black rings then they are not continuously connected to Schw adS

Also difference with $\Lambda=0$ solutions.

$\partial/\partial t$ timelike

χ spacelike.

$$\chi = \frac{\partial}{\partial t} + \Omega_i \frac{\partial}{\partial \varphi_i} \text{ null.}$$

ergosphere : Energy extraction becomes possible where $E_{initial} < E_{scatt}$.

Superradiance effect may point towards a classical instability for adS.

Superradiant modes would be reflected back due to the adS potential barrier reamplified and scattered back etc.

For $\Omega_i l \leq 1$ χ remains timelike and matter corotates with the black hole.

$\Rightarrow$ energy cannot be extracted from black hole !!!

Agrees with thermodynamics!

Hawking & Reall
hep-th/9908109

Exotic black holes.

start with flat adS slicing

$$ds^2 = \frac{l^2}{z^2}\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right)$$

idea is to replace flat 4d metric with solution of the vacuum 4 dim equations $g_{\mu\nu}$ s.t. $R^{(4)}_{\mu\nu}(g)=0$

⇒ Black string

$$ds^2 = \frac{l^2}{z^2}\left\{-U(r)dt^2 + \frac{dr^2}{U(r)} + r^2 d\Omega^2_{II}\right) + dz^2\Big]$$

topology at horizon is now $\mathbb{R}\times S^2$

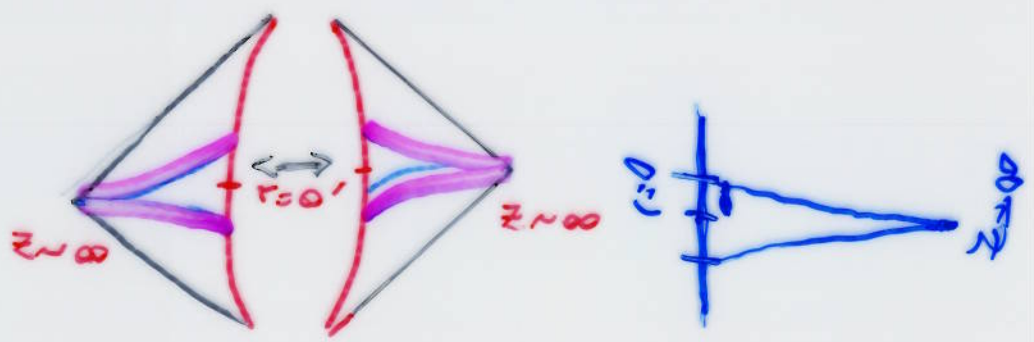
not allowed in 4 dim $\underline{\underline{\text{ons}}}$. Works with differing slicings

Black strings suffer from Gregory-Laflamme instabili[ty]
(long wavelength perturbations)

for adS they can be avoided by fixing l of adS.

$$Riem^2 = \frac{1}{l^4}\left(40 + \frac{48M^2z^4}{r^6}\right) \quad \text{sing at } \underset{r\to 0}{z\to\infty}$$

Chamblin
Hawking,
Reall



Open problems - Integrability in ads.

Problems integrable for $\Lambda = 0$ are no longer so for $\Lambda \neq 0$. (or in higher dimensions)

Examples: Black Rings

C-metric braneworld black holes.

Weyl Metrics. $R_{ab} = 0$ solutions which are locally static and axially symmetric.

$$ds^2 = e^{2\nu(r,z)} \alpha(r,z)^{-\frac{1}{2}} (dr^2 + dz^2) + \alpha(r,z) \left[-e^{\frac{\Omega(r,z)}{2}} dt^2 + e^{-\frac{\Omega(r,z)}{2}} d\varphi^2 \right]$$

variables are $\alpha = \alpha(r,z)$

$\Omega = \Omega(r,z)$ and $\nu = \nu(r,z)$

Field Equations are:

$$\Delta \alpha = 0$$

$$\frac{1}{\alpha} \vec{\nabla} \cdot (\alpha \vec{\nabla} \Omega) = 0.$$

$$2\nu_{,\zeta} \frac{\alpha_{,\zeta}}{\alpha} - \frac{\alpha_{,\zeta\zeta}}{\alpha} = \frac{1}{8} \Omega_{,\zeta}^2 \qquad \zeta \leftrightarrow \bar{\zeta} \quad \zeta = r + iz.$$

α is harmonic in 2d. Choose $\alpha = r$

then $\frac{d^2\Omega}{dr^2} + \frac{d^2\Omega}{dz^2} + \frac{1}{r}\frac{d\Omega}{dr} = 0$ Laplace eq. in 3 dim cylindrical coords

$\hookrightarrow$ Find ν directly. PROBLEM IS INTEGRABL[E]

For $\Lambda \neq 0$ integrability is destroyed.

$$\Delta \alpha = -2\Lambda \alpha^{1/2} e^{2\nu} + \kappa(\ldots)$$

$$\frac{1}{\alpha} \vec{\nabla} \cdot (\alpha \vec{\nabla} \Omega) = 0$$

$$2\nu_{,J} \frac{\alpha_{,J}}{\alpha} - \frac{\alpha_{,JJ}}{\alpha} = \frac{1}{8} \Omega_{,J}^{2}$$

Can no longer easily define a suitable coordinate system to adequately integrate.

Same happens in higher dimensions where curvature scalar are added to the problem