Primordial Gravitational Waves

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Motivation

- Gravitational waves can be a **powerful probe** of the early universe:
  - Produced during inflation
  - Weak interactions with matter and radiation
  - May enconde information about the history of the universe

Classical Tensor Perturbations

- Flat Friedmann-Robertson-Walker background:

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$$

- Metric perturbations (*conformal time coordinate*): $g_{\mu\nu} = a^2(\eta_{\mu\nu} + h_{\mu\nu})$

- Tensor perturbations are **transverse** and **traceless**;

- Linearised Einstein equations (*synchronous gauge*):

$$\ddot{h}_{ij} + 2\frac{\dot{a}}{a}\dot{h}_{ij} - \nabla^2 h_{ij} = \boxed{16\pi G a^2 \Theta_{ij}} \quad , \quad \Theta_{ij} = T^i_j - p\delta^i_j$$

⬇

Evolution similar to scalar field case

Classical Tensor Perturbations

- Fourier expansion:

$$h_{ij}(x) = \sqrt{16\pi G}\sum_r \int \frac{d^3k}{(2\pi)^3}\epsilon^r_{ij}(\mathbf{k})h^r_{\mathbf{k}}(\tau)e^{i\mathbf{k}\cdot\mathbf{x}}$$

where the symmetric polarisation tensor is transverse and traceless and is normalised as

$$\sum_{ij}\epsilon^r_{ij}(\mathbf{k})\epsilon^s_{ij}(\mathbf{k})^* = 2\delta^{rs}$$

- Equation for the mode **k:**

$$\ddot{h}^r_{\mathbf{k}} + 2\frac{\dot{a}}{a}\dot{h}^r_{\mathbf{k}} + k^2 h^r_{\mathbf{k}} = 0$$

- Power law expansion: $a(\tau) = \alpha\tau^n$ $\quad n = \dfrac{2}{1+3\omega}$ $\quad \omega = p/\rho$

- General solution expressed in terms of Bessel functions:

$$\ddot{h}_k + \frac{2n}{\tau}\dot{h}_k + k^2 h_k = 0 \quad \Rightarrow \quad h_k(\tau) = \tau^{1-n}\left(Aj_{\nu-1/2}(k\tau) + By_{\nu-1/2}(k\tau)\right)$$

$$\nu^2 = n(n-1) + \frac{1}{4}$$

Quantisation and Power Spectrum

- In linear theory, one can use the analogy with the scalar field case to construct the quantum theory associated with the free tensor modes in a curved spacetime:

$$h_{ij}(\mathbf{x},\tau) = \sum_r \sqrt{16\pi G}\int \frac{d^3k}{(2\pi)^3}\left[\epsilon^r_{ij}(\mathbf{k})h_k(\tau)a^r_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} + \epsilon^r_{ij}(\mathbf{k})^* h_k(\tau)^* a^{r\dagger}_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}}\right]$$

physical time-dependent operator

➢ Wronskian Normalisation Condition:

$$h_k\dot{h}_k^* - h_k^*\dot{h}_k = \frac{i}{a^2}$$

Quantisation and Power Spectrum

- Power Spectrum: $$\langle 0|h_{ij}(\mathbf{x},\tau)h_{ij}(\mathbf{y},\tau)|0\rangle \equiv \int \frac{d^3k}{(2\pi)^3}\frac{2\pi^2}{k^3}\Delta_T^2(k,\tau)e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$$

$$\Delta_T^2(k,\tau) = 64\pi G\frac{k^3}{2\pi^2}|h_k(\tau)|^2$$

- Energy density:

$$T^{\mu\nu}_{GW} = -\frac{2}{\sqrt{\bar{g}}}\frac{\delta S_{GW}}{\delta \bar{g}_{\mu\nu}}$$

$$\rho_{GW} = T_{GW}{}^0{}_0 = \bar{g}_{00}T^{00}_{GW}$$

$$\Omega_{GW}(k,\tau) \equiv \frac{1}{\rho_c(\tau)}\frac{d\langle 0|\rho_{GW}|0\rangle}{d(\ln k)} = \frac{8\pi G}{3H(\tau)^2}\frac{k^3}{2\pi^2a^2(\tau)}\left(|\dot{h}_k(\tau)|^2 + k^2|h_k|^2\right)$$

Inflationary perturbations

- *Slow-roll* inflation: energy density of the universe dominated by potential energy of a scalar field ϕ;

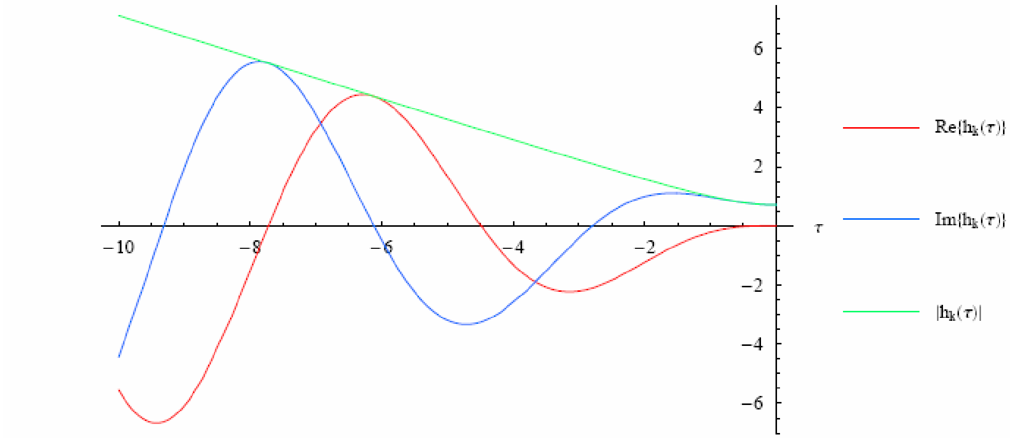
- Scale factor: $a(\tau) = -1/H\tau$

- *Slow-roll* parameters: $\epsilon \equiv \frac{1}{2}M_p^2\left(\frac{V'(\phi)}{V(\phi)}\right)^2$ $\eta \equiv M_p^2\left(\frac{V''(\phi)}{V(\phi)}\right)$

- Equation for tensor modes: $\ddot{\chi}_k - \frac{2}{\tau^2}\chi_k + k^2\chi_k = 0$ $\chi_k(\tau) \equiv h_k(\tau)/a(\tau)$

  - Solution: $$h_k(\tau) = -\frac{H}{\sqrt{2k}}\tau\left(1 - \frac{i}{k\tau}\right)e^{-ik\tau} = H\sqrt{\frac{k}{2}}\tau^2 h_1^{(2)}(k\tau)$$

Inflationary perturbations



Evolution of k=1 mode

- The solution exhibits two distinct behaviours:

  - *Subhorizon* - redshifted plane wave:

$$k \gg aH \qquad |k\tau| \gg 1 \quad \Rightarrow \quad h_k(\tau) \propto a^{-1}(\tau) e^{-ik\tau}$$

  - *Superhorizon* – frozen amplitude:

$$k \ll aH \qquad |k\tau| \ll 1 \quad \Rightarrow \quad |h_k(\tau)| = H/(k\sqrt{2k}) = 1/(a_*\sqrt{2k})$$

Horizon crossing:

$$k = a_* H$$

Inflationary perturbations

- Assume that at the end of inflation (τ=0) all modes of interest are well outside the horizon:
  - Power spectrum: $$\Delta_T^2(k,0) = 8\left(\frac{H}{2\pi M_p^2}\right)^2$$
  - Energy density: $$\Omega_{GW}(k,0) = \frac{H^2}{6\pi^2 M_p^2} = \frac{1}{12}\Delta_T^2(k,0)$$
  - Spectral index: $$n_T = \frac{d\ln\Delta_T^2}{d\ln k} = -2\epsilon_*$$

Slow-roll inflation produces a cosmic background of gravitational waves from quantum fluctuations with an almost scale invariant power spectrum

Post-inflationary behaviour

- Simplified model: Radiation + Matter with instantaneous transition

$$a(\tau) = \begin{cases} H_0\sqrt{\Omega_{r0}}\tau, & 0 \le \tau \le \tau_{eq} \\ a_{eq}\left(\frac{\tau}{\tau_{eq}}\right)^2, & \tau_{eq} < \tau \le \tau_0 \end{cases}$$

- General solutions:

  - Radiation: $h_k(\tau) = Aj_0(k\tau) + By_0(k\tau) \longrightarrow A = h_k(0),\ B = 0$

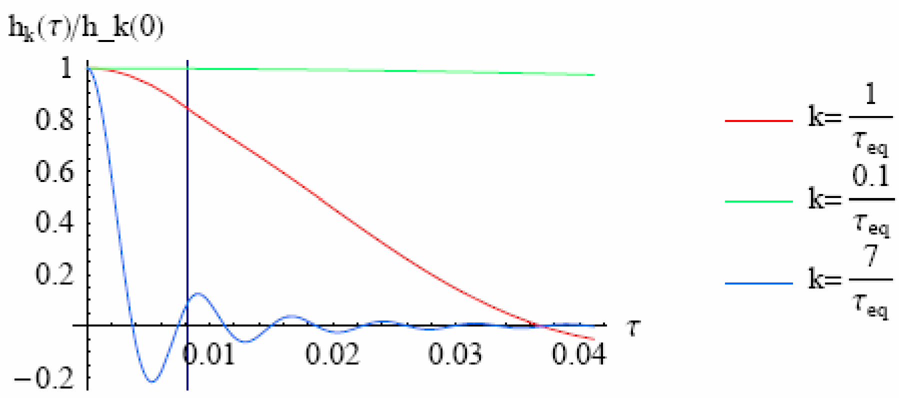
  - Matter: $h_k(\tau) = A_k\left(\frac{3j_1(k\tau)}{k\tau}\right) + B_k\left(\frac{3y_1(k\tau)}{k\tau}\right),$

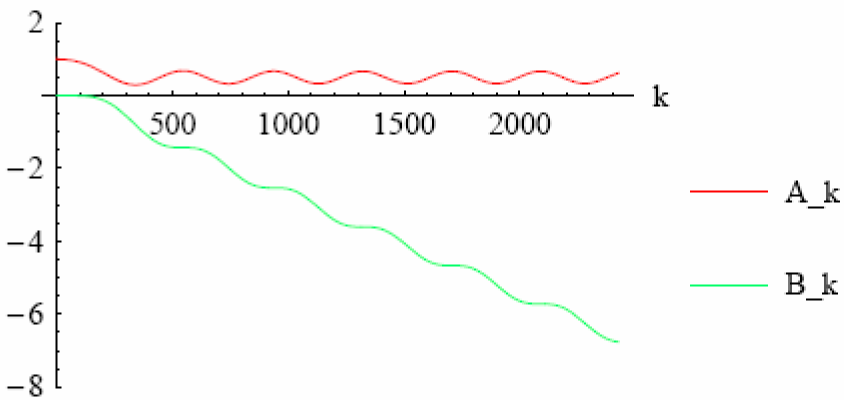
Post-inflationary behaviour

- Transfer function coefficients:

$$A_k = h_k(0)\frac{3k\tau_{eq} - k\tau_{eq}\cos(2k\tau_{eq}) + 2\sin(2k\tau_{eq})}{6k\tau_{eq}}$$

$$B_k = h_k(0)\frac{2 - 2k^2\tau_{eq}^2 - 2\cos(2k\tau_{eq}) - k\tau_{eq}\sin(2k\tau_{eq})}{6k\tau_{eq}}$$





Modes reenter the Hubble horizon during the radiation era or the matter era

Post-inflationary behaviour

- Smooth radiation-matter transition: $$a(\tau) = \frac{1}{4}\Omega_{m0}H_0^2\tau^2 + \sqrt{a_{eq}}\sqrt{\Omega_{m0}}H_o\tau$$

  - Rescaled variables: $x \equiv (\sqrt{2}-1)\tau/\tau_{eq}$ $y \equiv k/(k_{eq}(\sqrt{2}-1))$

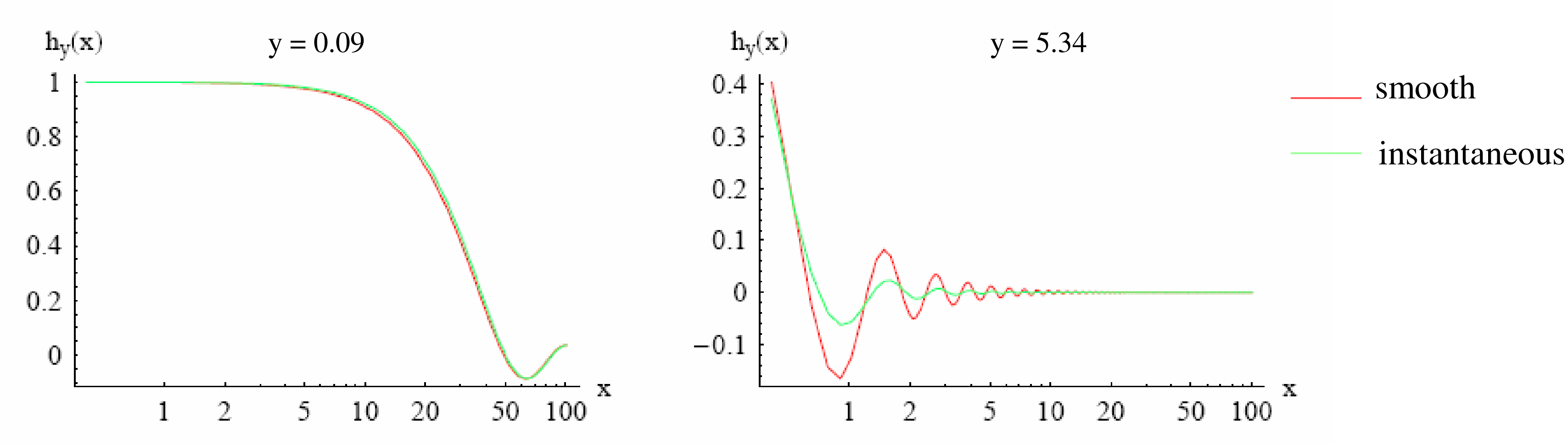
  - Scale factor: $a(x) = a_{eq}x(x+2)$

  - Tensor modes equation: $$h_y'' + 4\frac{x+1}{x(x+2)}h_y' + y^2 h_y = 0$$

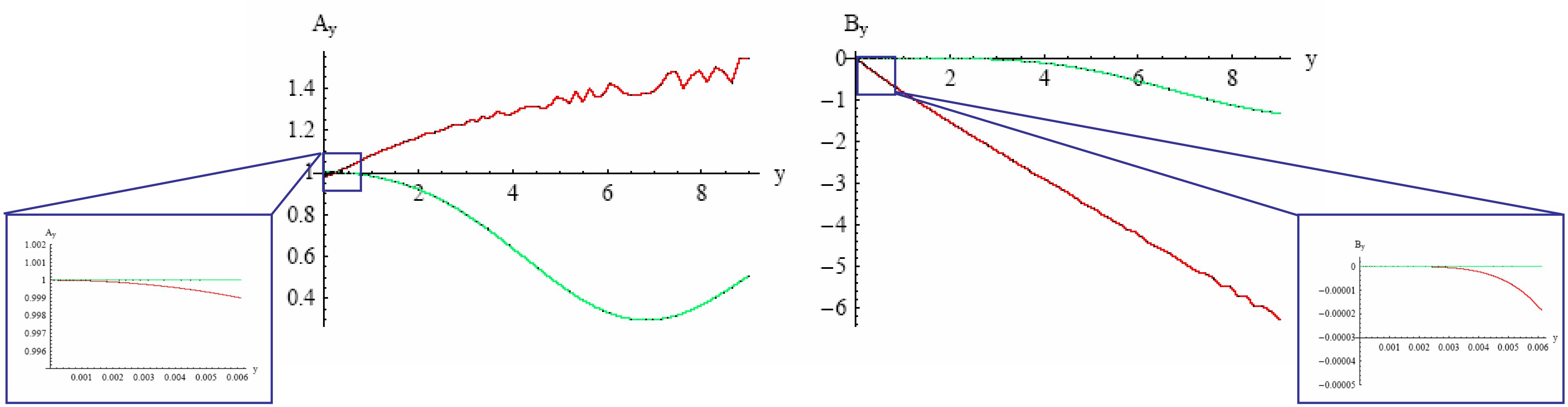
  - Solve numerically with initial conditions (at the end of inflation):

$$h_y(0) = 1 \qquad h_y'(0) = 0$$

Post-inflationary behaviour



- Transfer function coefficients:



Post-inflationary behaviour

- If we neglect phase shift induced by radiation era:

$$h_k(\tau) = h_k(0)\underbrace{T(k)}\left(3\frac{j_1(k\tau)}{k\tau}\right)$$

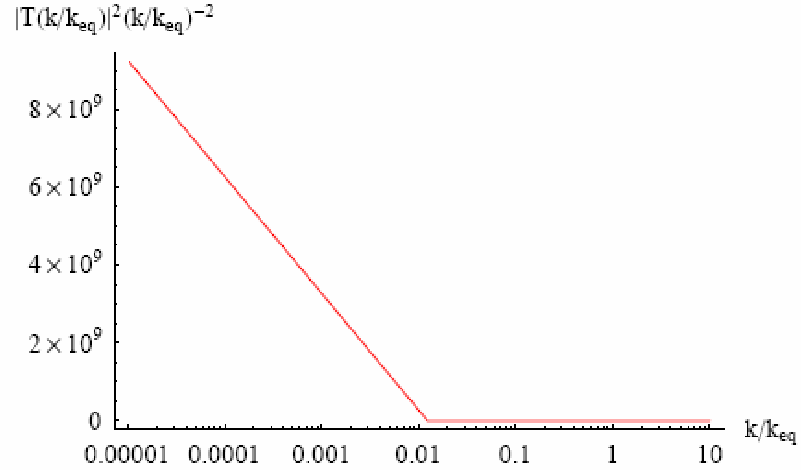
Transfer function

  - Fit to numerical data:

$$T(s) = \left(1 + 1.4s + 2.16s^2\right)^{1/2} \qquad s = k/k_{eq}$$

- Energy density
  (*averaged over several periods*)

$$\Omega_{GW0} = \frac{V_k}{16\pi^2 M_p^4}|T(k)|^2(k\tau_0)^{-2}$$



Post-inflationary behaviour

- Effect of phase transition at $\tau = \tau_*$ (radiation era):
  - Number of relativistic d.o.f. changes from g_*^i to $g_*^f < g_*^i$;
  - Energy density of radiation fluid:

$$\rho_r = \frac{\pi^2}{30} g_* T^4 = \frac{\pi^2}{30} \frac{s^{4/3} g_*^{-1/3}}{a^4}$$

$$\boxed{s = a^3 g_* T^3}$$ → Conservation of entropy

$$\rho_r = \begin{cases} \frac{\rho_{r0} r}{a^4}, & a \leq a_* \\ \frac{\rho_{r0}}{a^4}, & a > a_* \end{cases}$$

$$\rho_{r0} = (\pi^2/30) s^{4/3} (g_*^f)^{-1/3}$$
$$r \equiv (g_*^i / g_*^f)^{-1/3}$$

  - Scale factor evolution (*instantaneous transitions*):

$$\boxed{a(\tau) = \begin{cases} H_0 \sqrt{\Omega_{r0} r}\, \tau, & 0 < \tau \leq \tau_* \\ H_0 \sqrt{\Omega_{r0}} (\tau + (\sqrt{r} - 1)\tau_*), & \tau_* < \tau \leq \tau_{eq} \\ a_{eq} \left(\frac{\tau}{\tau_{eq}}\right)^2, & \tau > \tau_{eq} \end{cases}}$$

Post-inflationary behaviour

- Transfer function coefficients for phase transition:

  - New time variable: $\bar{\tau} \equiv \tau + (\sqrt{r} - 1)\tau_*$
  - General solution for second radiation-domination period:

$$h_k(\tau) = A_k j_0(k\bar{\tau}) + B_k y_0(k\bar{\tau})$$

  - From continuity:

$$A_k = h_k(0)(k\tau_*)^2 r \left[\frac{\cos(k\tau_*(\sqrt{r} - 1))}{(k\tau_*)^2\sqrt{r}} - \frac{\sin(k\tau_*)\cos(\sqrt{r}k\tau_*)}{(k\tau_*)^3} \left(\frac{\sqrt{r} - 1}{r} \right) \right]$$
$$B_k = h_k(0)(k\tau_*)^2 r \left[\frac{\sin(k\tau_*(\sqrt{r} - 1))}{(k\tau_*)^2\sqrt{r}} - \frac{\sin(k\tau_*)\sin(\sqrt{r}k\tau_*)}{(k\tau_*)^3} \left(\frac{\sqrt{r} - 1}{r} \right) \right]$$

  - Compute coefficients C_k and D_k after matter-radiation transition

Post-inflationary behaviour

- Example: QCD Phase Transition (Laine, 2001)

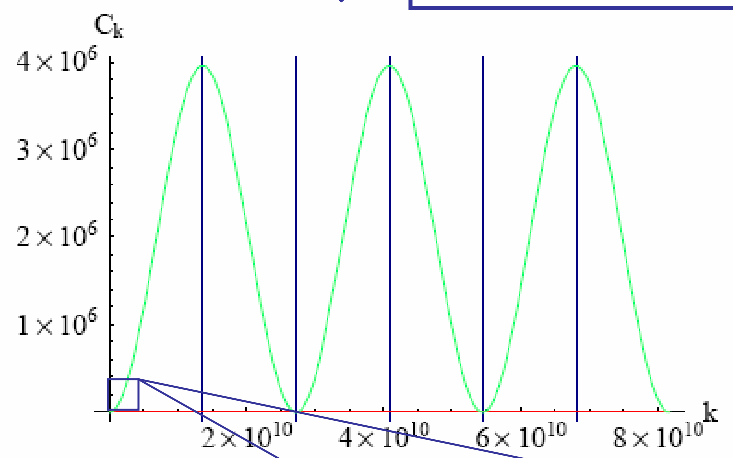
  - $\tau_* = 1.4 \times 10^{-8}\, \tau_{eq}$ (T = 170 MeV)
  - $g_*^i = 51.25$ (quark-gluon plasma) → $r = 0.6956$
  - $g_*^f = 17.25$ (hadrons)
  - Relevant scales:

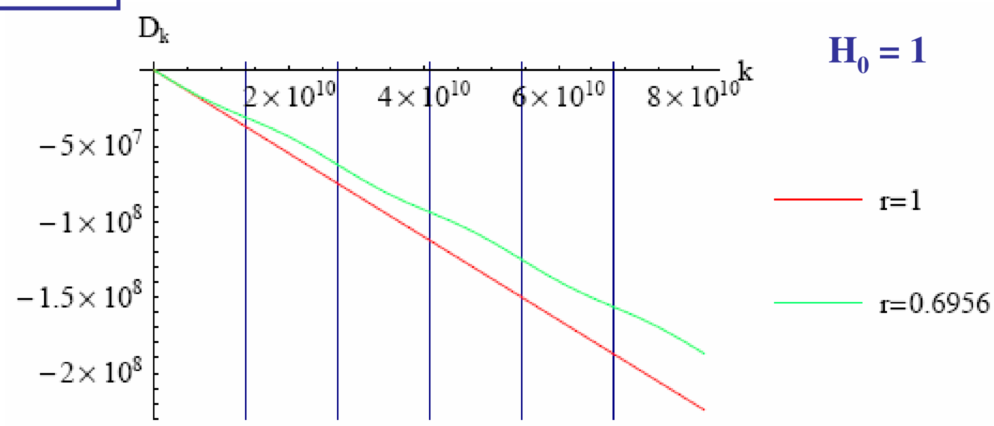
    $k_* = 7.1 \times 10^7\, k_{eq}$ → $f \sim 10^{-8}$ Hz

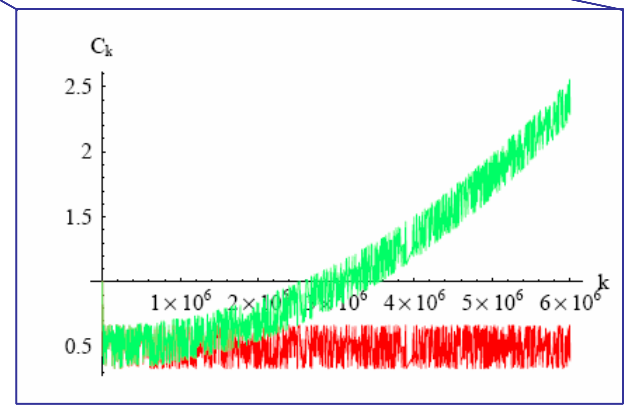
Post-inflationary behaviour

- Results:

Oscillations due to QCD phase transition







Oscillations due to matter-radiation transition

Conclusions

- Tensor perturbations are powerful tools for understanding the evolution of our universe;

- Studying the cosmic background of gravitational waves may provide important information about the mechanism behind inflation;

- The tensor transfer function may encode information about the radiation-matter transition and other possible phase transitions where relativistic d.o.f. are lost;