A Systematic Treatment of Black Holes via Boundary Value Problems

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Introduction

Black holes in the textbooks

MTW, Gravitation (1972)

"…To calculate the external fields of a black hole, one can extremize the 'action integral' [...] for interacting gravitational and electromagnetic fields (see Chapter 21) subject to the anchored-down imprints of M, Q, and S at radial infinity, and subject to the existence of a physically nonsingular horizon (no infinite curvature at horizon!). Extremizing the action is equivalent to solving the coupled Einstein-Maxwell field equations subject to the constraints [...] and the existence of the horizon. The derivation of the solution and the proof of its uniqueness are much too complex to be given here …"

Hawking & Ellis, The large scale structure of space-time (1991)

"…The [Kerr] solutions can be given in Boyer and Lindquist coordinates (r, θ, ϕ, t) in which the metric takes the form …"

d'Inverno, Introducing Einstein's Relativity (1992)

"…It turns out to be a rather long process to solve Einstein's vacuum equations directly for the Kerr solution. We shall, instead, describe a 'trick' of Newman and Janis for obtaining the Kerr solution from the Schwarzschild solution …"

Comment:

- There is no “physical” derivation of black hole solutions in the textbooks (to the best of my knowledge). For the term “physical” compare textbooks on electrodynamics (“Jackson”) with chapters on initial/boundary problems.
- There exists an extensive literature about black hole uniqueness proofs (see M. Heusler 1996).
- Physicists need constructive methods (analytical or numerical) $\longrightarrow$ black hole binaries etc. Here: *“Inverse scattering”* for stationary black holes. Starting point must be the characteristic property of black holes: *the event horizon*.
  $\longrightarrow$ boundary problems for horizons

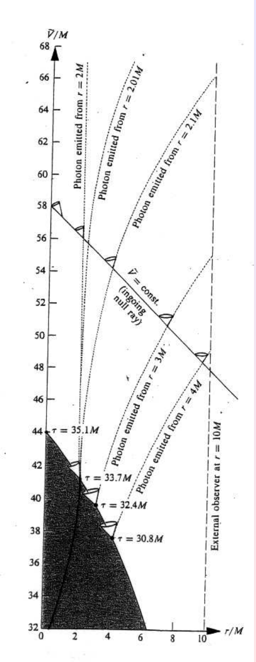

Figure: Oppenheimer-Snyder collapse in modified Eddington-Finkelstein coordinates (adapted from MTW). The diagram depicts a series of photons emitted radially from the surface of the collapsing star and received by an observer at $r = r_0 = 10M$. Any photon emitted radially at the Schwarzschild radius $r = 2M$ stays at $r = 2M$ forever. This external *event horizon* is the continuation of the internal *event horizon* (full curve in the shaded interior region of the star).

Consequences

- For an external observer (at $r > 2M$, here at $r = 10M$), the space-time domain beyond the event horizon is a “black hole” (no information [carried by photons and massive particles] can leave this domain to attain the observer)

- The "event horizon" is a global concept
  $\longrightarrow$ difficult mathematics (see black hole binaries!)
- There is a "local" characterization of the external horizon ($r = 2M$, stationary vacuum region with the Killing vector $\xi^i = \delta^i_4$)

$$r = 2M: \quad N = \xi^i \xi_i = g_{44} = -\left(1 - \frac{2M}{r}\right) = 0, \; N_{,i}N^{,i} = 0,$$

the time-like Killing vector ξ becomes a null vector on the event horizon, which is a null hypersurface.

Hawking's *strong rigidity theorem* relates the global concept to the local notion of "Killing horizons.

Killing horizon (Def. M. Heusler 1998):
Consider a Killing field ξ and the set of points with $N \equiv \xi^i \xi_i = 0$. A connected component of this set, which is a null hypersurface, $N_{,i} N^{,i} = 0$, is called a *Killing horizon* $\mathcal{H}(\xi) = \mathcal{H}$.

Strong rigidity theorem (Hawking 1972, Hawking & Ellis 1973):
The event horizon of a *stationary* black hole space-time is a Killing horizon.

Implication: Stationary black hole space-times are either non-rotating or axisymmetric.

Israel (1967, 1968): Static (non-rotating) vacuum and electrovac black hole space-times are spherically symmetric (and therefore axisymmetric, too).

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Physical problem of this talk

- Stationary black hole configurations (vacuum and electrovac space-times)

Mathematical task

- Solve boundary value problems (BVPs)for Killing horizons $\mathcal{H}$ in axisymmetric and stationary space-times
- Axisymmetry: azimuthal Killing vector $\eta,\ \eta^2 > 0$ (closed orbits)
- Stationarity: time-like Killing vector $\xi,\ \xi^2 < 0$

$\rightarrow G_2$ (*2-dimensional group of motions*)

We start with vacuum space-times.

- *Field equations:*

$$R_{ik} = 0\,; \qquad \eta^i = \delta^i_\varphi\,, \quad \xi^i = \delta^i_t\,:$$

$$\mathrm{d}s^2 = \mathrm{e}^{-2U}\left[\mathrm{e}^{2k}(\mathrm{d}\rho^2 + \mathrm{d}\zeta^2) + W^2\mathrm{d}\varphi^2\right] - \mathrm{e}^{2U}(\mathrm{d}t + a\mathrm{d}\varphi)^2$$

(Weyl-Lewis-Papapetrou form)

$$U = U(\rho,\zeta), \quad a = a(\rho,\zeta), \quad k = k(\rho,\zeta), \quad W = W(\rho,\zeta)$$

  - Invariant definition of the metric coefficients:

$$\mathrm{e}^{2U} := -\xi^i\xi_i,\; a := -\mathrm{e}^{-2U}\xi_i\eta^i,$$

$$W^2 := (\eta_i\xi^i)^2 - (\eta_i\eta^i)(\xi_k\xi^k), \quad k = k\{U, a, W\}$$

  - Note: A possible gauge is $W = \rho,\; \rho \geq 0$ everywhere

- *Boundary values:*
  1. Space-like infinity: Minkowski space; $a = 0,\; k = 0,\; U = 0$
  2. Axis of symmetry: $W = 0$, $a = 0$ ($\eta = 0$ on the axis!), $k = 0$ (elementary flatness!)
  3. Horizons: specific data

- Reformulation of the BVP in terms of the Ernst potential f:

$$f := \mathrm{e}^{2U} + \mathrm{i}b\,, \quad b = b\{a\}$$

$$a_{,\rho} = \rho \mathrm{e}^{-4U} b_{,\zeta}, \quad a_{,\zeta} = \rho \mathrm{e}^{-4U} b_{,\rho};$$

$$k_{,\rho} = \left[U_{,\rho}{}^2 - U_{,\zeta}{}^2 + \frac{1}{4} \mathrm{e}^{-4U} (b_{,\rho}{}^2 + b_{,\zeta}{}^2) \right],$$

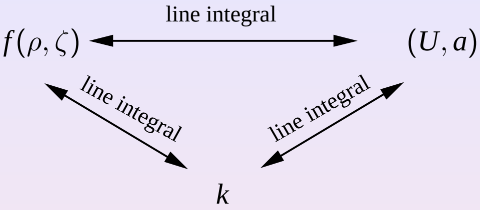
$$k_{,\zeta} = 2\rho \left(U_{,\rho} U_{,\zeta} + \frac{1}{4} \mathrm{e}^{-4U} b_{,\rho} b_{,\zeta} \right)$$

- *Field equations (Ernst equation):*

$$\Re f \triangle f = (\nabla f)^2\,, \quad f = f(\rho, \zeta), \quad \triangle\text{: Laplacian in cyl. coord.}$$

- *Boundary values for the Ernst equation:*
  1. Space-like infinity: $f = 1$
  2. Axis of symmetry: regularity of f
  3. Horizons: specific data

- *Metric:* Completely determined by the Ernst potential f



- *Note:* The Ernst equation holds likewise for any linear combination of the KVs, e.g. $\xi'^i = \xi^i + \Omega\eta^i, \;\; \eta'^i = \eta^i$ (Ω a constant)

$$\Re f' \triangle f' = (\nabla f')^2, \quad f' = -\xi'^i \xi'_i + \mathrm{i}b$$

Interpretation of Ω for $\xi'^i \xi'_i < 0$: $\xi'^i = \delta^i_4$, $t' = t$, $\varphi' = \varphi - \Omega t$

$\longrightarrow$ Ω: constant angular velocity,
$\{\rho, \zeta, t', \varphi'\}$: coordinate system rotating with respect to the asymptotic Minkowski space ("co-rotating system")

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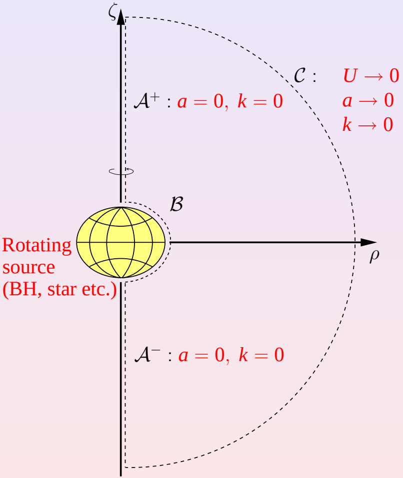
Boundary value problem



Field equations and Linear Problem

$$\Re f \triangle f = (\nabla f)^2 \text{ with } f = e^{2U} + ib$$

$$\Updownarrow$$

$$\begin{aligned}
\boldsymbol{\Phi}_{,z} &= \left\{ \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} + \lambda \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} \right\} \boldsymbol{\Phi}, \\
\boldsymbol{\Phi}_{,\bar{z}} &= \left\{ \begin{pmatrix} \bar{A} & 0 \\ 0 & \bar{B} \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} 0 & \bar{A} \\ \bar{B} & 0 \end{pmatrix} \right\} \boldsymbol{\Phi} \\
A &= \frac{f_{,z}}{f + \bar{f}},\ B = \frac{\bar{f}_{,z}}{f + \bar{f}}\ ,\ \lambda = \sqrt{\frac{K - i\bar{z}}{K + iz}} \\
z &= \rho + i\zeta \qquad K:\ \text{Spectral parameter}
\end{aligned}$$

Properties of the 2×2 matrix $\boldsymbol{\Phi}$ (from the LP)

- $\boldsymbol{\Phi}$ solution of LP $\longrightarrow \boldsymbol{\Phi}\boldsymbol{C}(K)$ solution of LP,
  $\boldsymbol{C}(K)$: 2×2 gauge matrix
- gauge freedom $\boldsymbol{C}(K)$ can be used to obtain the following standard form of $\boldsymbol{\Phi}$
  1. $\boldsymbol{\Phi}(-\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \boldsymbol{\Phi}(\lambda) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
     $\Leftrightarrow \boldsymbol{\Phi} = \begin{pmatrix} \psi(\rho,\zeta,\lambda) & \psi(\rho,\zeta,-\lambda) \\ \chi(\rho,\zeta,\lambda) & -\chi(\rho,\zeta,-\lambda) \end{pmatrix}$
  2. $\overline{\psi(\rho,\zeta,1/\overline{\lambda})} = \chi(\rho,\zeta,\lambda)$
  3. $K \to \infty, \lambda \to -1: \quad \psi(\rho,\zeta,-1) = \chi(\rho,\zeta,-1) = 1$
  4. $K \to \infty, \lambda \to +1: \quad \chi(\rho,\zeta,+1) = \Phi_{21} = \underline{f(\rho,\zeta)}$
     $\psi(\rho,\zeta,+1) = \Phi_{11} = \overline{f(\rho,\zeta)}$

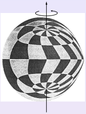
Note: Any solution f to the Ernst equation can be read off from $\boldsymbol{\Phi}$!

- Transition to a "co-rotating" system, i.e. $\xi^i, \eta^i \to \xi'^i = \xi^i + \Omega\eta^i$, $\eta'^i = \eta^i$, $(f \to f')$:

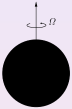
$$\boldsymbol{\Phi}' = \left[\begin{pmatrix} 1 + \Omega a - \Omega\rho \mathrm{e}^{-2U} & 0 \\ 0 & 1 + \Omega a + \Omega\rho \mathrm{e}^{-2U} \end{pmatrix} + \mathrm{i}(K + \mathrm{i}z)\Omega \mathrm{e}^{-2U} \begin{pmatrix} -1 & -\lambda \\ \lambda & 1 \end{pmatrix} \right] \boldsymbol{\Phi} \equiv \mathcal{L}\boldsymbol{\Phi}.$$

Henceforth a prime marks "co-rotating" quantities

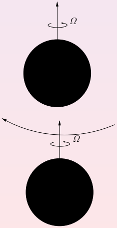
Does the Inverse method apply?

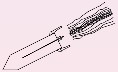


Rotating (Neutron) star — no! *(surface values of the star do not completely reflect its internal structure)*



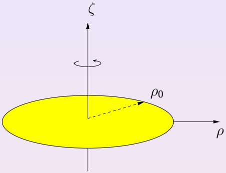
Rotating black hole — yes! $\longrightarrow$ *this talk*
(horizon determines the solution completely)





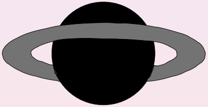
2 aligned rotating black holes — yes! $\longrightarrow$ *this talk*
(2 separated horizons; can spin-spin repulsion compensate gravitational attraction?)

Does the Inverse Method apply?



Rotating disk of dust (galaxies)

yes! *(golbal solution of a rotating body problem → 'galaxy' model, "testbed" for numerical calculations)*



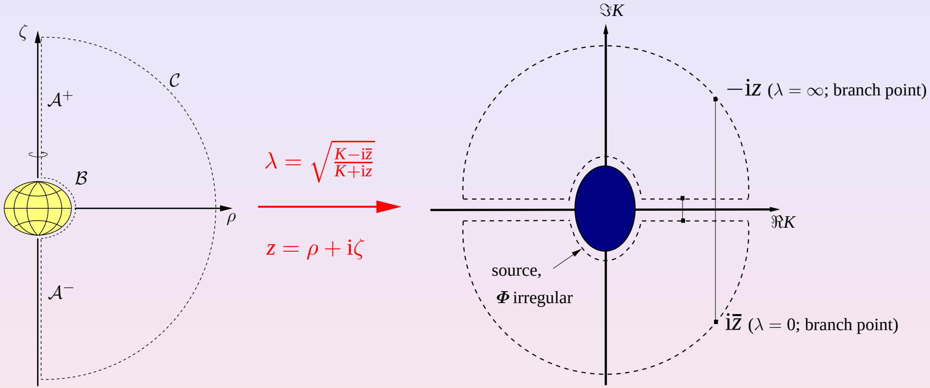
Black hole surrounded by a ring

should be possible *(AGN model: Galactic black hole)*

Idea of the Inverse Method

Find $\boldsymbol{\Phi} = \boldsymbol{\Phi}(\rho, \zeta; \lambda)$ by integrating the Linear Problem and get $f = f(\rho, \zeta)$ from $\boldsymbol{\Phi}$ $(f = \boldsymbol{\Phi}_{21}(\rho, \zeta; \lambda = 1))$.

The point made here is that $\boldsymbol{\Phi}$ is a holomorphic function of λ. Thus we can make use of the powerful theorems of the theory of holomorphic functions (concerning poles, zeros, Riemann surfaces etc.). In this way, we obtain the dependence on ρ, ζ in $\boldsymbol{\Phi}$ as a "byproduct". λ resp. K is a spectral parameter.

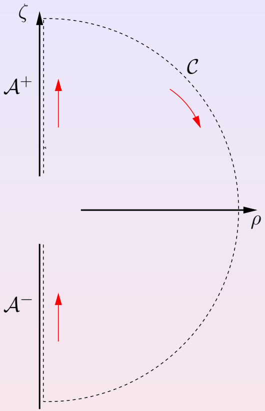


Spacetime representation
(t = constant, φ = constant)

Spectral representation
(two-sheeted Riemann surface)

- *Note:* We apply the Inverse Method to elliptic PDEs!
- *Programme:* Integrate the Linear Problem along $\mathcal{A}^+\mathcal{C}\mathcal{A}^-\mathcal{B}$ (dashed line) picking up the available information ($\mathcal{B}$: boundary values, $\mathcal{A}^\pm$: regularity, $\mathcal{C}$: $f = 1$, Minkowski space): *direct problem*
- *Result:*
  1. $\mathcal{A}^\pm$: $\boldsymbol{\Phi}(\zeta, K)$ as a holomorphic function in K ($\rightarrow$ zeros, poles, jumps etc.)
  2. Holomorphic structure allows continuation of $\boldsymbol{\Phi}(\zeta, K)$ to obtain $\boldsymbol{\Phi}(\rho, \zeta; K)$ resp. $\boldsymbol{\Phi}(\rho, \zeta; \lambda)$ $\quad (\lambda = \sqrt{(K - \mathrm{i}\bar{z})/(K + \mathrm{i}z)})$: *inverse problem*
  3. $\boldsymbol{\Phi}(\rho, \zeta; K) \longrightarrow f(\rho, \zeta) = \Phi_{21}(\rho, \zeta; \infty) \longrightarrow \mathrm{d}s^2$
- *Strategy*:
  1. $\boldsymbol{\Phi}(\mathcal{A}^+ - \mathcal{C} - \mathcal{A}^-)$: General solution
  2. $\boldsymbol{\Phi}(\mathcal{B})$: Particular solution corresponding to the physical situation

1. $\boldsymbol{\Phi}(\mathcal{A}^+ - \mathcal{C} - \mathcal{A}^-)$: General solution (holds for all [rigidly] rotating sources)



(a) Integration of the LP along $\mathcal{A}^+ - \mathcal{C} - \mathcal{A}^-$ to get $\boldsymbol{\Phi}$ there: ζ-K *separation*!

$$\begin{aligned}
\mathcal{A}^+ \quad &: \quad \boldsymbol{\Phi}_{\mathcal{A}^+} = \begin{pmatrix} \overline{f(\zeta)} & 1 \\ f(\zeta) & -1 \end{pmatrix} \begin{pmatrix} F(K) & 0 \\ G(K) & 1 \end{pmatrix} \quad (\zeta \in \mathcal{A}^+) \\
\mathcal{C} \quad &: \quad \boldsymbol{\Phi}_{\mathcal{C}} = \boldsymbol{\Phi}_0(K) \\
\mathcal{A}^- \quad &: \quad \boldsymbol{\Phi}_{\mathcal{A}^-} = \begin{pmatrix} \overline{f(\zeta)} & 1 \\ f(\zeta) & -1 \end{pmatrix} \begin{pmatrix} 1 & G(K) \\ 0 & F(K) \end{pmatrix} \quad (\zeta \in \mathcal{A}^-)
\end{aligned}$$

(b) Uniqueness of $\boldsymbol{\Phi}$ in the branch points $K = \zeta$ $(K = \mathrm{i}\bar{z},\ K = -\mathrm{i}z;\ z = \mathrm{i}\zeta,\ \zeta \in \mathcal{A}^+)$:

$$\mathcal{A}^+ \ : \ F(\zeta) = \frac{2}{f(\zeta) + \overline{f(\zeta)}},\ G(\zeta) = \frac{f(\zeta) - \overline{f(\zeta)}}{f(\zeta) + \overline{f(\zeta)}}$$

Analytic continuation

$$F(K) = F(\zeta \to K),\ G(K) = G(\zeta \to K)$$

Result:

Axis values of $\boldsymbol{\Phi}$ are completely determined by the axis values of the Ernst potential $f(\zeta)$, $\zeta \in \mathcal{A}^\pm$, and vice versa.

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Construction of the stationary vacuum black hole solution

Boundary values on the Killing horizon $\mathcal{H}$

- *Co-rotating KV:* $\xi'^i = \xi^i + \Omega\eta^i,\ \eta'^i = \eta^i \quad (\xi^i = \delta^i_t,\ \eta^i = \delta^i_\varphi)$,
$$N = \xi'^i\xi'_i, \quad N = -\mathrm{e}^{2U'} \equiv -\mathrm{e}^{2V}$$
- *Definition of the Killing horizon $\mathcal{H}(\xi) = \mathcal{H}$:*
$\mathcal{H}: \quad N = \xi'^i\xi'_i = 0, \quad N_iN^i = 0$ (null hypersurface) $\rightarrow$
$\mathcal{H}: \quad W = W' = \rho = 0, \quad a = -\frac{1}{\Omega}$
Note: In Weyl-Lewis-Papapetrou coordinates horizon at $\rho = 0$

Proof (Carter 1973):

1. $W = W' = \rho = 0$: $\mathcal{H}$: $\xi'^i N_{,i} = 0, \quad \eta^i N_{,i} = 0 \quad (N_{,t} = N_{,\varphi} = 0)$
$\rightarrow N_{,i} = -2\kappa\xi'^i$ (two orthogonal null vectors are proportional),
$\rightarrow \eta^i \xi'_i = 0.$
$$\eta^i \xi'_i = 0 \,\&\, \xi'^i \xi'_i = 0 \rightarrow W^2 = (\eta_i \xi^i)^2 - \eta_i \eta^i \xi_k \xi^k = (\eta_i \xi'^i)^2 - \eta_i \eta^i \xi'^k \xi'_k = \rho^2 = 0$$
q.e.d.

2. $a = -\frac{1}{\Omega}$: $\quad e^{2U'} = e^{2V} = -(\xi^i + \Omega\eta^i)(\xi_i + \Omega\eta_i) = e^{2U}[(1+\Omega a)^2 - \rho^2 \Omega^2 e^{-4U}]$
$\mathcal{H}$: $e^{2U'} = 0 \;\&\; \rho = 0 \rightarrow 1 + \Omega a = 0$
q.e.d.

Bondary values for the Ernst equation on the horizon

$\mathcal{H}$: $\quad \rho = 0, \quad e^{2U'} = -\xi'^i \xi'_i = -|\xi + \Omega\eta|^2 = 0$

Comment on κ:

1. "surface gravity", using the field equations one can show that $\kappa =$ constant on $\mathcal{H}$ (Bardeen e al. 1973)
2. Extension of κ (everywhere outside the horizon):
   Def.: $\kappa^2 = -\frac{1}{2}\xi'_{i;k}\xi'^{i;k}, \quad \kappa\big|_{\mathcal{H}} = \text{constant}$
   $\mathcal{H}$: $\kappa = \text{constant}$
   $\mathcal{A}^{\pm}$: $\kappa^2 = \frac{1}{4}\mathrm{e}^{-2k}\left[(\mathrm{e}^{2U}{}_{,\zeta})^2 - (b - 2\Omega\zeta)^2_{,\zeta}\right]$

next transparency illustrates the horizon analysis and presents the results of the integration of the LP along $\mathcal{H}$.

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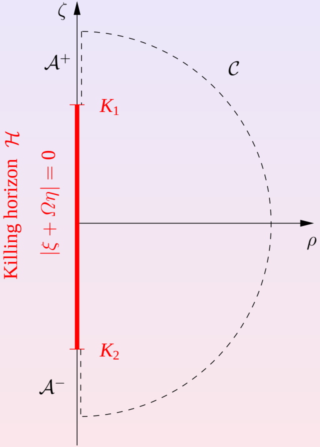
Integration along the horizon



(a) Result of the integration of LP along $\mathcal{H}$:

$$\mathcal{H}: \quad \boldsymbol{\Phi}_{\mathcal{H}} = \begin{pmatrix} \overline{f(\zeta)} & 1 \\ f(\zeta) & -1 \end{pmatrix} \begin{pmatrix} U(K) & V(K) \\ W(K) & X(K) \end{pmatrix}$$

$$\boldsymbol{\Phi}' = \mathcal{L}\boldsymbol{\Phi} \to \boldsymbol{\Phi}'_{\mathcal{H}} \text{ (co-rotating frame!)}$$

(b) Field equations hold in K_1, K_2, too:

$$\boldsymbol{\Phi}_{\mathcal{H}}(K_1) = \boldsymbol{\Phi}_{\mathcal{A}^+}(K_1),\ \boldsymbol{\Phi}'_{\mathcal{H}}(K_1) = \boldsymbol{\Phi}'_{\mathcal{A}^+}(K_1);$$

$$\boldsymbol{\Phi}_{\mathcal{H}}(K_2) = \boldsymbol{\Phi}_{\mathcal{A}^+}(K_2),\ \boldsymbol{\Phi}'_{\mathcal{H}}(K_2) = \boldsymbol{\Phi}'_{\mathcal{A}^+}(K_2)$$

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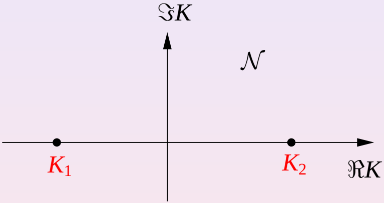
Result of the $\mathcal{A}^+$-$\mathcal{C}$-$\mathcal{A}^-$-$\mathcal{H}$ integration:

the *fundamental axis relation*

$$\mathcal{N} \equiv \begin{pmatrix} F & -G \\ G & (1-G^2)/F \end{pmatrix} = \left(\mathbf{1} + \frac{\mathbf{F}_1}{2\mathrm{i}\Omega(K-K_1)}\right)\left(\mathbf{1} - \frac{\mathbf{F}_2}{2\mathrm{i}\Omega(K-K_2)}\right)$$

$$\mathbf{F}_i = \begin{pmatrix} -f_i & 1 \\ -f_i^2 & f_i \end{pmatrix}, \quad \mathbf{F}^2 = 0; \quad f_i = f(\zeta = K_i)$$

- Ω is the constant angular velocity of the BH
- $f_i = -\overline{f_i} = \mathrm{i}b_i$ (e^{2U} vanishes at the horizon/axis points $\rightarrow$ "ergosphere")



(2 simple poles!)

- The axis matrix $\mathcal{N}$ summarizes the results of the integration along $\mathcal{A}^+$-$\mathcal{C}$-$\mathcal{A}^-$-$\mathcal{H}$ and yields
  1. the constraints $\mathcal{N}_{21} = -\mathcal{N}_{12}$
  2. the axis values of the Ernst potential (e.g. on $\mathcal{A}^+$)
     $\mathcal{A}^+$: $f(\zeta) = \frac{1-G(\zeta)}{F(\zeta)}$ (branch points!)

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Axis potential $f(\zeta)$ and BH thermodynamics

1. *Constraints* (w.l.g. symmetric position of $\mathcal{H}$, $K_1 = -K_2$):
   $f_1 = -f_2, \quad \Omega = \frac{\mathrm{i}f_1(1+f_1^2)}{2K_1(1-f_i^2)}$
2. *Axis potential* (using the constraints):
   $\mathcal{A}^+ \ (\zeta > K_1) : \quad f(\zeta) = \frac{(\zeta+K_1)(1+f_1^2)-2(1-f_1)}{(\zeta+K_1)(1+f_1^2)-2f_1(1-f_1)}$
3. *Asymptotics:*
   $e^{2U} = 1 - \frac{2M}{\zeta} \pm \ldots, \quad b(\zeta) = -\frac{2J}{\zeta^2} \pm \ldots$
   M: mass, J: (ζ-component of the) angular momentum
   identification:
   $f_1 - f_2 = 2f_1 = -4\mathrm{i}\Omega M, \quad K_1 - K_2 = 2K_1 = 2M - 4\Omega J$

$$2M\Omega = \frac{M^2}{J} \overset{-}{(+)} \sqrt{\frac{M^2}{J} - 1}, \quad \Omega = \Omega(M, J)$$

angular velocity as a function of mass and angular momentum
$0 \leq \frac{M^2}{J} \leq 1$, $0 \leq 2M\Omega \leq 1$; ($2\Omega M = 1$, $M^2 = J$: extrem case)

4. *Regularity of the metric on the axis:*

$a = 0$: (automatically satisfied, constraints!)

$k = 0$: k via a line intergral from f, one needs f off the axis
Is there an equivalent criterion in terms of $f(\zeta)$?
Yes → "surface gravity"

Constraints: $b_{,\zeta}\big|_{K=K_1,K_2} = 2\Omega \quad \rightarrow$

(straightforward verification) $\rightarrow \kappa^2 = \frac{1}{4}\mathrm{e}^{-2k}\left(\mathrm{e}^{2U}{}_{,\zeta}\right)^2\big|_{K=K_1,K_2}$

k-criterion: $k = 0$ on $\mathcal{A}^{\pm} \quad \leftrightarrow \quad \mathrm{e}^{2U}{}_{,\zeta}(K_1) = \pm\mathrm{e}^{2U}{}_{,\zeta}(K_2)$

Calculation of $\mathrm{e}^{2U}{}_{,\zeta}$:

$$\mathrm{e}^{2U}{}_{,\zeta}(K_1) = \frac{(1+f_1^2)^2}{2K_1(1-f_1^2)} = -\mathrm{e}^{2U}{}_{,\zeta}(K_2 = -K_1): \quad \textit{o.k.}$$

Surface gravity of the BH:

$$\kappa = \frac{\sqrt{M^4 - J^2}}{2M(M^2 + \sqrt{M^4 - J^2})}$$

5. *Black hole thermodynamics*

$\frac{\partial \kappa^{-1}}{\partial J} = -\frac{\partial \Omega \kappa^{-1}}{\partial M} \rightarrow$ there exists a "thermodynamic" potential

$$M^2 + \sqrt{M^4 - J^2} \equiv \frac{\mathcal{A}}{8\pi},$$

"first law of thermodynamics"

$\frac{1}{8\pi}\mathrm{d}\mathcal{A} = \frac{1}{\kappa}(\mathrm{d}M - \Omega \mathrm{d}J)$, $\mathcal{A}$: area of the horizon

(Hawking's area theorem: $\Delta\mathcal{A} \geq 0$ suggests that $\mathcal{A}\times$ positive constant is the BH entropy S ["second law of BH th."] and $\kappa =$ constant on $\mathcal{H}$ is proportional to the BH temperature ["zeroth law of BH thermodynamics"], Bardeen et al. 1973)

Summary:

The integration of the LP along the closed curve $\mathcal{A}^+$-$\mathcal{C}$-$\mathcal{A}^-$-$\mathcal{H}$ (direct problem) yields

- the Ernst potential $f(\zeta)$ on the axis of symmetry
- the first law of BH physics

The inverse problem consists in the construction of $f(\rho,\zeta)$ outside the horizon from the axis values $f(\zeta)$.

Contents

Ernst potential f everywhere

$\mathcal{A}^+$:
$f(\zeta) = \frac{\zeta - M - \mathrm{i}A}{\zeta + M + \mathrm{i}A}, \quad A = J/M$

$\longrightarrow$

$f(\rho, \zeta)$ everywhere
(incl. $\mathcal{A}^-$, $\mathcal{C}$)

$\downarrow$ $\uparrow$

$$\boldsymbol{\Phi} = \begin{pmatrix} \overline{f(\zeta)} & 1 \\ f(\zeta) & -1 \end{pmatrix} \begin{pmatrix} F(K) & 0 \\ G(K) & 1 \end{pmatrix}$$

$\longrightarrow$

$\boldsymbol{\Phi}(\rho, \zeta, \lambda)$,
$f(\rho, \zeta) = \Phi_{21}(\rho, \zeta, \lambda = 1)$

"inverse problem"

Def.: $\mathcal{M} = \boldsymbol{\Phi} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \boldsymbol{\Phi}^{-1}$; "monodromy matrix"

$$\rho \to 0, \zeta \to \infty: \quad \mathcal{M} = \mathcal{M}_0 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathcal{N} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Compare $\mathcal{M}$ and its limit $\mathcal{M}_0 = \frac{P_2(K)}{(K-K_1)(K-K_2)}$, P_2 matrix polynomial in K, and make use of the identity

$$(K-K_1)(K-K_2) = \frac{(\lambda^2-\lambda_1^2)(\lambda^2-\lambda_2^2)(K+\mathrm{i}z)^2}{(1-\lambda_1^2)(1-\lambda_2^2)}, \quad \lambda_i = \sqrt{\frac{K_i-\mathrm{i}\bar{z}}{K_i+\mathrm{i}z}}:$$

1. $\boldsymbol{\Phi}$ must be proportional to a matrix polynomial of second degree in λ
2. $\det \boldsymbol{\Phi} = s(\rho, \zeta, \lambda)(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2)$, $\quad s$ a scalar $\to$
3. $\boldsymbol{\Phi} = \frac{K+\mathrm{i}z}{K}(\mathbf{Y}_0 + \mathbf{Y}_1\lambda + \mathbf{Y}_2\lambda^2)$

Calculation of the Ys

(1) $\det \boldsymbol{\Phi}(\lambda_i) = 0 \leftrightarrow \boldsymbol{\Phi}(\rho, \zeta, \lambda_i) \begin{pmatrix} \alpha_i + 1 \\ \alpha_i - 1 \end{pmatrix} = 0$

LP: $\alpha_i =$ constant, $\alpha_i \overline{\alpha_i} = 1$

(2) Normalization (see Chapter 2): $\boldsymbol{\Phi}(\rho, \zeta, \lambda = -1) = \begin{pmatrix} \overline{f} & 1 \\ f & -1 \end{pmatrix}$

The linear algebraic system (1), (2) determines the **Y**s and f uniquely:

$f(\rho, \zeta) = \frac{\alpha_1 r_1 - \alpha_2 r_2 - 2K_1}{\alpha_1 r_1 - \alpha_2 r_2 + 2K_1}, \quad r_i = \left| \sqrt{(K_i - \mathrm{i}\bar{z})(K_i + \mathrm{i}z)} \right|$

$f(\rho, \zeta)$ satisfies the Ernst equation (elegant proof via discussion of $\boldsymbol{\Phi}_{,z} \boldsymbol{\Phi}^{-1}$ as a function of λ!)

Axis value of the Ernst potential: $\mathcal{A}^+ \colon f(\zeta) = \frac{(\alpha_1 - \alpha_2)\zeta - K_1(\alpha_1 + \alpha_2 - 2)}{(\alpha_1 - \alpha_2)\zeta - K_1(\alpha_1 + \alpha_2 + 2)}$

This axis potential $f(\zeta)$ must be identified with the result of the $\mathcal{A}^+$-$\mathcal{C}$-$\mathcal{A}^-$-$\mathcal{H}$ integration to determine the constants α_i:

$$M\alpha_1 = \sqrt{M^2 - A^2} + \mathrm{i}A, \quad M\alpha_2 = -\sqrt{M^2 - A^2} + \mathrm{i}A, \quad (A = J/M)$$

Result:

The stationary black hole solution in Weyl-Lewis-Papapetrou coordinates,

$$f(\rho,\zeta) = \frac{\sqrt{M^2-A^2}(r_1+r_2-2M)+\mathrm{i}A(r_1-r_2)}{\sqrt{M^2-A^2}(r_1+r_2+2M)+\mathrm{i}A(r_1-r_2)},$$

$$r_i = \left|\sqrt{(K_i - \mathrm{i}\bar{z})(K_i + \mathrm{i}z)}\right|, \quad K_1 = -K_2 = \sqrt{M^2 - A^2}$$

Comment:

- This Kerr solution (Roy Kerr 1963) was found in the context of algebraically special gravitational fields (Type D solution). Interpretation as a rotating BH later on
- For the extensive discussion of uniqueness proofs for this and other black hole solutions see M. Heusler's monopgarph (1993)
- 2 parameter solution $(M, J) \longrightarrow$ "no hairs",
- Our construction is based on necessary conclusions and implies *uniqueness*

Combining the calculation of the metric coefficients a, k with the coordinate transformation $\rho = \sqrt{r^2 - 2Mr + A^2}\sin\vartheta$, $\zeta = (r - M)\cos\vartheta$ one obtains the Kerr solution in Boyer-Lindquist coordinates r, ϑ,

$$ds^2 = e^{-2U}\left[\left(r^2 - 2Mr + A^2\cos^2\vartheta\right)\left(d\vartheta^2 + \frac{dr^2}{r^2 - 2Mr + A^2}\right) + \rho^2 d\varphi^2\right] - e^{2U}(dt + a\,d\varphi)^2$$

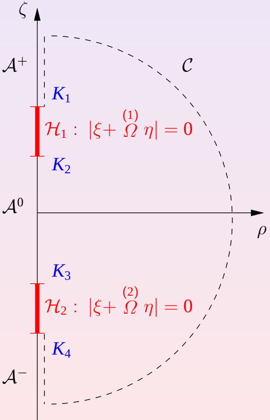
$$e^{2U} = 1 - \frac{2Mr}{r^2 + A^2\cos^2\vartheta}, \quad a = \frac{2MAr\sin^2\vartheta}{r^2 - 2Mr + A^2\cos^2\vartheta}$$

- the stationary BH solution is analytic outside and on the horizon
- *interior:*
  - maximal analytic extension via the Kruskal procedure
  - parametric collapse of a disk of dust
- “interior” Kerr?

Contents

Problem: Can the spin-spin repulsion of two aligned black holes compensate their gravitational attraction (see post-Newtonian approximation)

Solution: by discussing the fundamental matrix $\mathcal{N}$ and the surface gravity in $K_1, K_2; K_3, K_4$ (“ends” of horizons $\mathcal{H}_1, \mathcal{H}_2$)



$$\Omega_1 = \Omega_2 = \overset{(1)}{\Omega}, \ \Omega_3 = \Omega_4 = \overset{(2)}{\Omega}$$

$$\mathcal{N}(K) = \begin{pmatrix} F & -G \\ G & (1-G^2)/F \end{pmatrix} = \prod_{i=1}^{4} \left(\mathbf{1} - \frac{(-1)^i \mathbf{F}_i}{2\mathrm{i}\Omega_i (K - K_i)} \right)$$

(1) $f(\zeta) = \frac{1-G(\zeta)}{F(\zeta)}, \ \mathrm{e}^{2U(\zeta)} = \frac{1}{F(\zeta)}$ on $\mathcal{A}^+$

(2) $\mathcal{N}_{12} = -\mathcal{N}_{12}$

(3) $\kappa(\zeta) = \pm\frac{1}{2}\mathrm{e}^{2U}{}_{,\zeta};$
$\kappa(K_1) = \kappa(K_2), \quad \kappa(K_3) = \kappa(K_4)$

Discussion

9 BV parameters (f_1, f_2, f_3, f_4; $\overset{(1)}{\Omega}, \overset{(2)}{\Omega}$; K_1, K_2, K_3, K_4 can be fixed)
Eq. (2), $\mathcal{N}_{12} = -\mathcal{N}_{12}$: 4 equations
Eq. (3), $\kappa(K_1) = \kappa(K_2)$, implies $\kappa(K_3) = \kappa(K_4)$: 1 equation

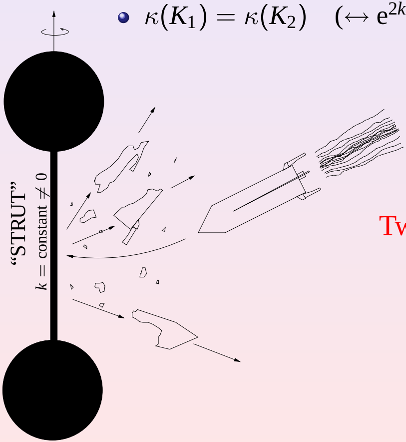
$\rightarrow$ 5 equations for 9 parameters: 4 free parameters, $(\overset{(1)}{\Omega}, K_1, K_2, K_3)$, say, determine $\overset{(2)}{\Omega}$: positions of the 2 BHs and their angular velocities cannot be prescribed idependently.

State of the art:

1. $\overset{(1)}{\Omega} \,/\, \overset{(2)}{\Omega} > 0$ (no counterrotation)
2. The only candiates to describe aligned balanced BHs are polynomial solutions (“Bäcklund generated”, N. and Krenzer 1999)
3. Two identical BHs cannot be balanced (Dietz and Hoenselaers 1985 for the Bäcklund class, N. and Krenzer 1999 by discussing Eqs. (2) and (3))
4. Bäcklund generated solutions with positive *Komar masses* do not exist (Manko and Ruiz 2001)

Conclusion:

- general problem not yet solved
- *expectation:*
  - Eq. (2), $\mathcal{N}_{21} = -\mathcal{N}_{12}$ solvabel (cf. superposition of non-rotating BHs, Papapetrou-Majumdar solution)
  - $\kappa(K_1) = \kappa(K_2) \quad (\leftrightarrow e^{2k} = 1 \text{ on } \mathcal{A}^{\pm}, \mathcal{A}^{0})$ unsolvabel



Two identical black holes

Contents

Stationary axisymmetric Einstein-Maxwell eqs.

$$(\Re f + \bar{g}g) \triangle f = \nabla f(\nabla f + 2\bar{g}\nabla g)$$

$$(\Re f + \bar{g}g) \triangle g = \nabla g(\nabla f + 2\bar{g}\nabla g)$$

$\Leftrightarrow$

Linear Problem

$$\boldsymbol{\Phi}_{,z} = \left[\begin{pmatrix} B & 0 & E \\ 0 & A & 0 \\ -C & 0 & \frac{A+B}{2} \end{pmatrix} + \lambda \begin{pmatrix} 0 & B & 0 \\ A & 0 & -E \\ 0 & -C & 0 \end{pmatrix}\right] \boldsymbol{\Phi}$$

$$\boldsymbol{\Phi}_{,\bar{z}} = \left[\begin{pmatrix} \bar{A} & 0 & -\bar{C} \\ 0 & \bar{B} & 0 \\ \bar{E} & 0 & \frac{\bar{A}+\bar{B}}{2} \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} 0 & \bar{A} & 0 \\ \bar{B} & 0 & \bar{C} \\ 0 & \bar{E} & 0 \end{pmatrix}\right] \boldsymbol{\Phi}$$

$f(\rho, \zeta)$: Ernst potential

$g(\rho, \zeta)$: em potential

(elstatic +i×magstatic)

$e^{2U} = \Re f + g\bar{g}$,

Weyl-Papapetrou coordinates

$G/H =$ $\mathrm{SU}(2,1)/\mathrm{SU}(1,1) \times \mathrm{U}(1)$ coset space

$\mathrm{SU}(2,1)$ symmetry group

$$A = \tfrac{1}{2} e^{-2U}(f_{,z} + 2\bar{g}g_{,z}),$$

$$B = \tfrac{1}{2} e^{-2U}(\bar{f}_{,z} + 2g\bar{g}_{,z}),$$

$$E = i e^{-2U} g_{,z},$$

$$C = i e^{-2U} \bar{g}_{,z}$$

Note: Notation!

Remarks:

- Charged stationary black hole solution (Kerr-Newman 1965) by SU(2,1) transformation of the stationary vacuum black hole (Kerr 1963)
- Solution of the electrovac boundary problem for charged black holes along the lines indicated in this talk
- Conformastatic black holes in equilibrium (electromagnetic forces compensate gravitational forses, $|e| = M$ [big mass])

Contents

Parametric collapse of rotating perfect fluid bodies in equilibrium

- Do cosmic collapse processes (galaxies, stars) lead inevitably to the formation of (stationary) black holes?
- No dynamical (analytic) models
- Some insight by the discussion of the parametric collapse of rotating perfect fluid bodies in equilibrium?

$$V_0 \to -\infty, \quad \text{finite baryonic mass} \quad \Rightarrow \quad M = 2\Omega J$$

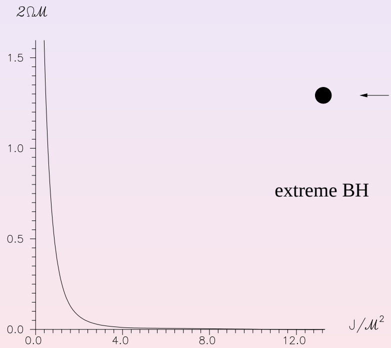
black hole limit?

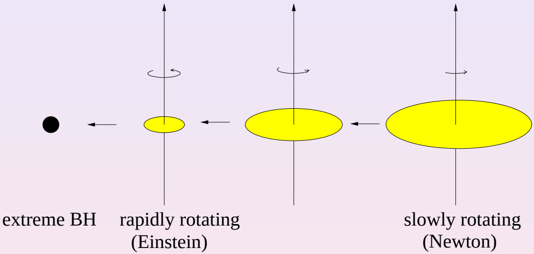
$$\Omega = \Omega_H = \frac{J}{2M^2[M + \sqrt{M^2 - (J/M)^2}]} \quad \Rightarrow \quad J = \pm M^2$$

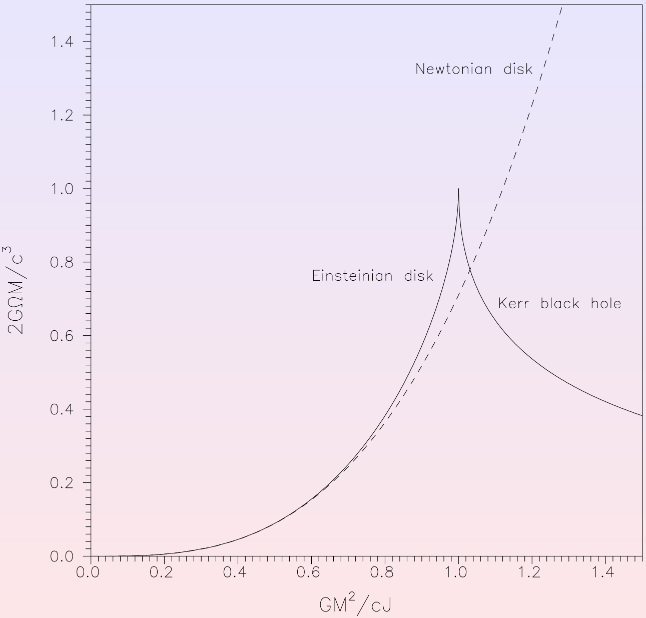
extreme Kerr black hole!

R. Meinel, grqc/0405074

Parametric collapse of a rotating disk

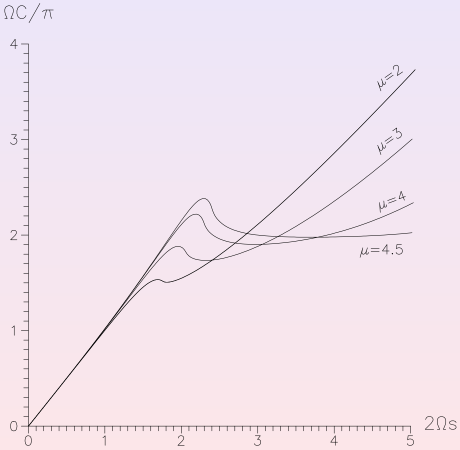




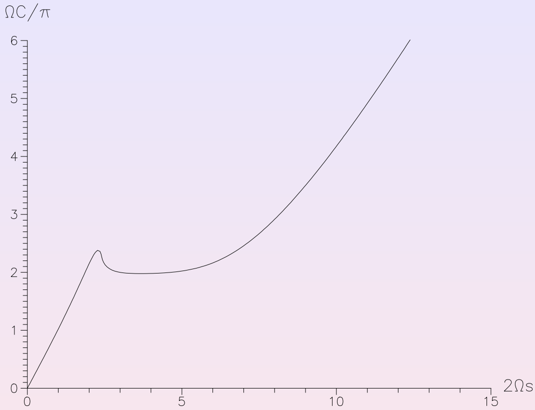

Newtonian disk
Einsteinian disk
Kerr black hole
$2G\Omega M/c^3$
GM^2/cJ
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4

Example

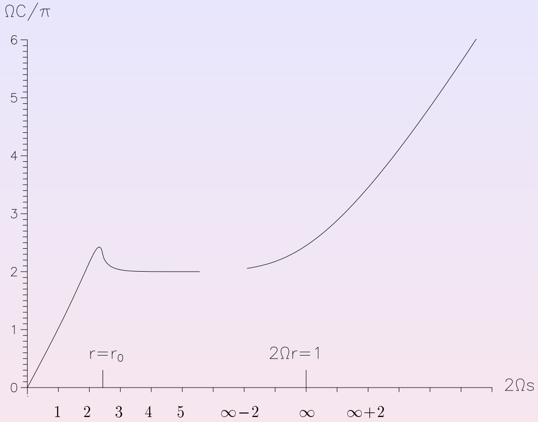
The geometry of the phase transition disk/black hole:
separation of the ‘interior world’ from the ‘exterior world’ (extreme Kerr)



Geometry in the disk plane. Depicted is the circumferential diameter C/π of a circle around the centre of the disk vs the real distances s from the center for increasing values of $\mu = 2\Omega^2\rho_0^2 e^{2-V_0}$, ρ_0: coordinate radius of the disk. (Here $\Omega C/\pi$ and Ωs are dimensionless quantities, $c = 1$).



For ultrarelativistic values of μ (here $\mu = 4.5$), the 'interior region' around the disk (around the local maximum on the left hand side) is far from the 'exterior region' (right ascending branch of the curve), which becomes more and more Kerr-like.



In the limit $\mu = \mu_0 = 4.63\ldots$, the 'disk world' (left branch) and the 'world of the extreme Kerr black hole' (right branch) are separated from each other. The point labeled ∞ on the abscissa corresponds to a coordinate radius $r = 1/2\Omega$. Points to the 'Kerr world' (right branch) are at infinite spatial distance from the disk (in the left branch).

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