The Quantum Dynamics of Boson Nebulae Formation

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Abstract

Soon after Kaup and Ruffini and Bonazzola derived asymptotically flat, spherically symmetric equilibrium solutions of Einstein-Klein-Gordon equations describing the so-called boson stars, these have been seen as candidates for non-baryonic dark matter in the universe, sources of gravitational waves, or the supermassive compact dark object which have been observed in the center of our Galaxy, producing similar spectra to a black hole. Since the the system of coupled equations describing fields interacting via gravity, have been treated mainly by numerical formalisms, the aim of the present paper is to derive approximating analytical solutions to the system of Klein-Gordon-Maxwell-Einstein equations, describing a minimally coupled charged boson to a spherically symmetric spacetime. The corresponding metric functions are used to compute the main observables of a boson nebula, in terms of the model parameters, which can accommodate a wide range of numerical estimations. Finally, within a first-order perturbative approach, we derive the effective potential and the current, employed in computing quantum transitions related to the gravitoelectric particle creation.

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1 Introduction

We term by a boson star a gravitationally bound, both globally $U(1)$ and spherically symmetric, compact equilibrium configurations of cold complex scalar fields. At almost 40 years, since Kaup [1] and Ruffini and Bonazzola [2] discovered them, they still arise as promising candidates for non-baryonic dark matter in the universe [3]. Recently, on serious theoretical grounds, it has been revealed that a boson star can supplant a black hole in the role of a compact object accreting matter since, for certain values of its parameters, they both produce similar spectra. Few years ago, by lensing experiments, it has been shown that the hypothesis of a central black hole was unconclusive and that a boson star could better explain the supermassive compact dark object Sagittarius A* which have been observed in the center of our Galaxy, [4].

Such configurations, if existed, are remnants of first-order gravitational phase transitions and their mass should be ruled by the epochs when they decoupled from the cosmological background.

These assumptions raise the question whether such configurations are dynamically stable [5] and the settling down of a boson star to a stable state has been investigated via numerical calculations [6].

In our approach, we are dealing with a boson nebula defined as the charged scalar cloud which finds itself in one of the spherically symmetric positive-frequency modes of radial wave number k, with $k^2 \to 0_+$, and pulsation $\omega_k = [m_0^2 + k^2]^{1/2}$. Such a configuration is obviously unstable [7] and the instabilities are expected to lead to the formation of a boson star from an initially smooth state [8]. Not very much is specifically known in this direction [7], although the reversed *stability to instability* passage has been extensively investigated [5, 6, 9]. Analytically, exactly solvable models for boson stars with large self-interaction [10] and for boson-fermion stars [11] have been worked out only in low dimensional gravity. In four dimensions, the bosonic or the mixed fermion-bosonic fields interacting via gravity have been investigated mainly by numerical calculations [1 − 7, 9, 12].

Nevertheless, using the Newtonian approximation, the whole analysis gets greatly simplified allowing interesting and inspiring investigations, as for example the process of gravitational-radiation emission from an excited boson star [13]. On the other hand, the role of a non-vanishing radial-momentum is important when dealing with quantum transitions since it is affecting both the effective potential and the current, leading to serious consequences regarding the actual dispersion, the continuity equation and the growth, and respectively decay, of the quantum mode-excitations. Therefore, a general-relativistic analytical study of the coupled field equations could be of interest for a better understanding of different stellar configurations as well as for a numerical-functional combined iterative treatment which describes the dynamics of charged boson nebulae.

Recently, besides the bosonic or the mixed fermion-bosonic configurations, the spherically symmetric stable quark stars with shells of hadronic composition has been also taken as candidates for the missing matter of any kind. For a total mass of few solar masses, these objects could be detected by gravitational microlensing of the galactic halo towards the Magellanic clouds [14].

2 The Klein–Gordon–Maxwell–Einstein System of Equations

For a spherically symmetric configuration, let us consider the metric, expressed in Schwarzschild coordinates, as being

$$ds^2 \,=\, e^{2f}\,(dr)^2 \,+\, r^2\,\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) - \,e^{2h}\,(dt)^2\,, \tag{1}$$

where f and h are functions of r and t. We introduce the pseudo-orthonormal tetradic frame $\{e_a\}_{a=\overline{1,4}}$, with the corresponding dual orthonormal base

$$\omega^1 = e^f\, dr\ ,\ \ \omega^2 = r\, d\theta\ ,\ \ \omega^3 = r\, \sin\theta\, d\varphi\ ,\ \ \omega^4 = e^h\, dt \tag{2}$$

and derive the non-vanishing connection coefficients

$$\begin{aligned} \Gamma_{12} &= -\frac{e^{-f}}{r}\,\omega^2\ ,\ \ \Gamma_{13} = -\frac{e^{-f}}{r}\,\omega^3\ ,\ \ \Gamma_{23} = -\frac{\cot\theta}{r}\,\omega^3\ , \\ \Gamma_{14} &= f_{|4}\,\omega^1 + h_{|1}\,\omega^4\ , \end{aligned} \tag{3}$$

and the components of the Einstein tensor

$$\begin{aligned} G_{11} &= 2\,\frac{e^{-f}}{r}\,h_{|1} - \frac{1-e^{-2f}}{r^2}\ , \\ G_{22} &= G_{33} \\ &= h_{|11} + \left(h_{|1}\right)^2 - \left[f_{|44} + \left(f_{|4}\right)^2\right] + \frac{e^{-f}}{r}\,h_{|1} + \left[\left(\frac{e^{-f}}{r}\right)_{|1} + \frac{e^{-2f}}{r^2}\right], \\ G_{44} &= -2\left[\left(\frac{e^{-f}}{r}\right)_{|1} + \frac{e^{-2f}}{r^2}\right] + \frac{1-e^{-2f}}{r^2}\ , \\ G_{14} &= 2\,\frac{e^{-f}}{r}\,f_{|4}\ , \end{aligned} \tag{4}$$

where $(\,\cdot\,)_{|a} = e_a(\,\cdot\,)$.

In this configuration, a charged boson of mass m_0, coupled to the electromagnetic field, is described by the $SO(3,1)\times U(1)$-gauge invariant Lagrangean density

$$\mathcal{L} = \eta^{ab}\,\bar\phi_{;a}\phi_{;b} + m_0^2\,\bar\phi\phi + \frac{1}{4}\,F^{ab}F_{ab}\ , \tag{5}$$

where

$$\phi_{;a} = \phi_{|a} - iqA_a\phi\ ,\ \ \bar\phi_{;a} = \bar\phi_{|a} + iqA_a\bar\phi\ , \tag{6}$$

and the Maxwell tensor $F_{ab} = A_{b:a} - A_{a:b}$ is expressed in terms of the Levi–Civita covariant derivatives of the four-potential A_a, i.e. $A_{a:b} = A_{a|b} - A_c\Gamma^c_{ab}$.

By varying with respect to different fields, we come to the Klein–Gordon–Maxwell (KGM) equations

$$\Box\phi \,-\, m_0^2\,\phi \,=\, 2iq\,A^c\phi_{|c} \,+\, q^2A^cA_c\phi\;, \qquad \text{and its h.c.}\,, \tag{7}$$

$$F^{ab}{}_{:b} \,=\, -\,iq\,\eta^{ab}\left[\bar\phi\left(\phi_{|b} - iqA_b\phi\right) - \left(\bar\phi_{|b} + iqA_b\bar\phi\right)\phi\right] \tag{8}$$

and to the Einstein ones $G_{ab} \,=\, \kappa\,T_{ab}$, where G_{ab} have the explicit form (4) and the energy-momentum tensor T_{ab} is

$$T_{ab} \,=\, \bar\phi_{;a}\phi_{;b} \,+\, \bar\phi_{;b}\phi_{;a} \,+\, F_{ac}F_b{}^{\,c} \,-\, \eta_{ab}\mathcal{L}\,. \tag{9}$$

Working in the minimally symmetric ansatz $A_1 = A_1(r,t)$, $A_4 = A_4(r,t)$, $\phi = \phi(r,t)$, the KGM system (7–8) turns into

$$\begin{aligned}
&e^{-2f}\left\{\phi_{,rr} + \left[h_{,r} - f_{,r} + \frac{2}{r}\right]\phi_{,r}\right\} -\, e^{-2h}\left\{\phi_{,tt} + \left[f_{,t} - h_{,t}\right]\phi_{,t}\right\} -\, m_0^2\phi\\
&=\, 2iq\,A_1e^{-f}\phi_{,r} \,-2iqA_4e^{-h}\phi_{,t} \,+q^2\left[(A_1)^2 - (A_4)^2\right]\phi\;,\ \text{and its h.c.}
\end{aligned}\tag{10}$$

$$\begin{aligned}
&e^{-h}\left[e^{-f}\left(A_{4,r} + h_{,r}\,A_4\right) - e^{-h}\left(A_{1,t} + f_{,t}\,A_1\right)\right]_{,t}\\
&=\, iq\left[e^{-f}\left(\bar\phi\phi_{,r} - \bar\phi_{,r}\,\phi\right) - 2iqA_1\bar\phi\phi\right];
\end{aligned}\tag{11}$$

$$\begin{aligned}
&e^{-f}\left[e^{-f}\left(A_{4,r} + h_{,r}\,A_4\right) - e^{-h}\left(A_{1,t} + f_{,t}\,A_1\right)\right]_{,r}\\
&+2\,\frac{e^{-f}}{r}\left[e^{-f}\left(A_{4,r} + h_{,r}\,A_4\right) - e^{-h}\left(A_{1,t} + f_{,t}\,A_1\right)\right]\\
&=\, iq\left[e^{-h}\left(\bar\phi\phi_{,t} - \bar\phi_{,t}\,\phi\right) - 2iqA_4\bar\phi\phi\right],
\end{aligned}\tag{12}$$

where (,) stands for the usual derivatives, the Lorentz condition reads

$$e^{-f}\left[A_{1,r} + \left(h_{,r} + \frac{2}{r}\right)A_1\right] -\, e^{-h}\left[A_{4,t} + f_{,t}\,A_4\right] =\, 0\,, \tag{13}$$

while the Einstein equations explicitly become

$$
\begin{aligned}
&2\,\frac{e^{-2f}}{r}\,h_{,r} - \frac{1-e^{-2f}}{r^2} \;=\; \kappa\left[\bar{\phi}_{;1}\phi_{;1} \,+\, \bar{\phi}_{;4}\phi_{;4} \,-\, m_0^2\,\bar{\phi}\phi \,-\, \frac{1}{2}\,(F_{14})^2\right];\\
&e^{-2f}\left[h_{,rr} + (h_{,r} - f_{,r})\,h_{,r} + \frac{1}{r}\,(h_{,r} - f_{,r})\right] - e^{-2h}\left[f_{,tt} + (f_{,t} - h_{,t})\,f_{,t}\right]\\
&= -\,\kappa\left[\bar{\phi}_{;1}\phi_{;1} \,-\, \bar{\phi}_{;4}\phi_{;4} \,+\, m_0^2\,\bar{\phi}\phi \,-\, \frac{1}{2}\,(F_{14})^2\right];\\
&2\,\frac{e^{-2f}}{r}\,f_{,r} + \frac{1-e^{-2f}}{r^2} \;=\; \kappa\left[\bar{\phi}_{;1}\phi_{;1} \,+\, \bar{\phi}_{;4}\phi_{;4} \,+\, m_0^2\,\bar{\phi}\phi \,+\, \frac{1}{2}\,(F_{14})^2\right];\\
&\frac{2}{r}\,e^{-(f+h)}f_{,t} \;=\; \kappa\left[\bar{\phi}_{;1}\phi_{;4} \,+\, \bar{\phi}_{;4}\phi_{;1}\right], \qquad (14)
\end{aligned}
$$

The system of equations (10–12, 14) generalizes the one derived in [12], for the equilibrium configuration when the metric functions are time-independent, the gauge field is chosen in such a way that one has only electric charges and a self-interacting term has been added in order to increase the limits on the critical mass and particle number.

3 Charge, Radius and Mass of Boson Nebulae

Since for bosonic fields interacting via gravity, only the zero-node solutions, corresponding to the lowest energy state, has been under investigation, and the interest has been focused mainly on numerical analyses of different stellar configurations, let us follow an analytical perturbative approach, in order to derive a solution to the system (10–14).

What we generally mean by a (charged) boson star is a charged scalar "cloud" which finds itself, at large radial coordinate distances, in one of the spherically symmetric positive-frequency modes, of the complex scalar field, of radial wave number k, with $k^2 \rightarrow 0_+$, and pulsation $\omega_k = [m_0^2 + k^2]^{1/2}$. Of course, such a configuration is obviously unstable as $\omega_k^2 - m_0^2 = k^2 \geq 0$, instead of $\omega < m_0$, i.e. $k^2 < 0$, as it is required for the possible stable (charged) boson star. Yet, we would like to derive first, within a first-order approximation by the $U(1)$-gauge coupling constant "q", the concretely expressed form, in terms of elementary functions, of the Klein–Gordon–Maxwell–Einstein equations. Their analytical, (r,t)-dependent, solutions might be serving well as an easily handling starting point in conceiving a combined numerical-functional iterative method which aims, by its higher-order solutions to the $U(1)$-gauge invariant KGME-system, to a better understanding of the dynamics of charged boson stars formation from the initially unstable stars. As it is mentioned in the Introduction, not very much is specifically known in this direction, although the reversed way — the stability to instability passage — has been extensively investigated.

So, we start with the physically reasonable assumption that the charged scalar field is the main source of both the electromagnetic and gravitational fields. Hence, at large radial distances and within the framework of a first-order approximation in the $U(1)$-gauge coupling q, the feedbacks of gravity and electromagnetism on the charged scalar source can be neglected and therefore, its equation of motion does simply become the one of an $l = 0$ (i.e. spherically symmetric) state on a Minkowskian background ($f = 0 = h$), i.e.

$$\phi_{,rr} + \frac{2}{r}\,\phi_{,r} - \phi_{,tt} - m_0^2\,\phi \;=\; 0\;, \quad \text{and its h.c.}\;, \tag{15}$$

with the positive-frequency mode solutions

$$\phi \;=\; \frac{\mathcal{N}}{r}\,e^{i(kr-\omega_k t)} \quad\Rightarrow\quad \bar{\phi} \;=\; \frac{\bar{\mathcal{N}}}{r}\,e^{-i(kr-\omega_k t)}\,, \tag{16}$$

where $\omega_k \;=\; [k^2 + m_0^2]^{1/2}$ and the dimensionless amplitude-factor $\mathcal{N}$ can be semiclassically related (see [7]), by its squared modulus, to the Minkowskian ($f = 0 = h$) estimated number

$$N_0 \;=\; \frac{Q}{q} \;=\; i\,4\pi \int_0^R \left[\bar{\phi}\phi_{,t} - \bar{\phi}_{,t}\,\phi\right] r^2\,dr \tag{17}$$

of charged spinless bosons, making up a Minkowskian averaged star of radius R, i.e.

$$N_0 \;=\; 8\pi m_0\, R|\mathcal{N}|^2 \quad\Leftrightarrow\quad |\mathcal{N}| \;=\; \left[\frac{N_0}{8\pi m_0 R}\right]^{1/2} \;,$$

whenever $k^2 \to 0_+$. Similarly, concerning the employed approximation, the gravity feedback on the Maxwell sector can be neglected at first, as it effectively involves second-order contributions of the charged scalar ϕ, and, on the right hand side of the Maxwell equations, the source of the electromagnetic (actually, electric) field is only given — within the first-order contributions of q — by the globally $U(1)$-conserved current

$$j^a \;=\; -\,iq\,\eta^{ab}\left[\bar{\phi}\phi_{,b} - \bar{\phi}_{,b}\,\phi\right]\;,\;\; a \in \{1,4\}. \tag{18}$$

Thus, the Lorentz condition (13) and the Maxwell equations (11) and (12) do actually read

$$A_{1,r} \;+\; \frac{2}{r}\,A_1 \;-\; A_{4,t} \;=\; 0 \tag{19}$$

and

$$\begin{aligned} A_{1,rr} \;+\; \frac{2}{r}\,A_{1,r} \;-\; \frac{2}{r^2}\,A_1 \;-\; A_{1,tt} \;&=\; -\,2qk\,\frac{|\mathcal{N}|^2}{r^2}\,, \\ A_{4,rr} \;+\; \frac{2}{r}\,A_{4,r} \;-\; A_{4,tt} \;&=\; 2q\omega_k\,\frac{|\mathcal{N}|^2}{r^2}\,, \end{aligned} \tag{20}$$

with the (particular) solution(s)

$$A_1 \;=\; qk|\mathcal{N}|^2\,, \tag{21}$$

$$A_4 \;=\; 2q\omega_k|\mathcal{N}|^2\,\log\frac{r}{r_0} \;+\; 2qk\,\frac{|\mathcal{N}|^2}{r}\,t\,, \tag{22}$$

which correspond to the electric field (intensity)

$$E \;=\; F_{14} \;\approx\; A_{4,r} - A_{1,t} \;=\; 2q\omega_k\,\frac{|\mathcal{N}|^2}{r} - 2qk\,\frac{|\mathcal{N}|^2}{r^2}\,t\,. \tag{23}$$

In these assumptions, the components (9) of the energy-momentum tensor have the explicit form:

$$
\begin{aligned}
T_{11} &= 2k^2\,\frac{|\mathcal{N}|^2}{r^2} + \frac{|\mathcal{N}|^2}{r^4} - 2q^2\left(2k^2+m_0^2\right)\frac{|\mathcal{N}|^4}{r^2} + 4q^2\omega_k^2\,\frac{|\mathcal{N}|^4}{r^2}\log\frac{r}{r_0} \\
&\quad + 8q^2\omega_k k\,\frac{|\mathcal{N}|^4}{r^3}\,t - 2q^2k^2\,\frac{|\mathcal{N}|^4}{r^4}\,t^2\,, \\
T_{22} &= T_{33} \\
&= -\frac{|\mathcal{N}|^2}{r^4} + 2q^2\left(2k^2+m_0^2\right)\frac{|\mathcal{N}|^4}{r^2} + 4q^2\omega_k^2\,\frac{|\mathcal{N}|^4}{r^2}\log\frac{r}{r_0} + 2q^2k^2\,\frac{|\mathcal{N}|^4}{r^4}\,t^2\,, \\
T_{44} &= 2\omega_k^2\,\frac{|\mathcal{N}|^2}{r^2} + \frac{|\mathcal{N}|^2}{r^4} + 2q^2m_0^2\,\frac{|\mathcal{N}|^4}{r^2} + 4q^2\omega_k^2\,\frac{|\mathcal{N}|^4}{r^2}\log\frac{r}{r_0} + 2q^2k^2\,\frac{|\mathcal{N}|^4}{r^4}\,t^2\,, \\
T_{14} &= -2\omega_k k\,\frac{|\mathcal{N}|^2}{r^2} + 2q^2\omega_k k\,\frac{|\mathcal{N}|^4}{r^2} - 4q^2\omega_k k\,\frac{|\mathcal{N}|^4}{r^2}\log\frac{r}{r_0} - 4q^2k^2\,\frac{|\mathcal{N}|^4}{r^3}\,t\,.
\end{aligned}
\tag{24}
$$

Consequently, the system of Einstein equations, in the long range approximation and for $|\mathcal{N}|^2 = 1/q^2$, have the following solutions:

$$
\begin{aligned}
f(r,t) &= -\frac{1}{2}\log\left[1 + \frac{2\kappa(k^2-2\omega_k^2)}{q^2} + \frac{\kappa(1+2k^2t^2)}{q^2\,r^2} - \frac{C}{r}\right], \\
h(r,t) &= -\frac{C}{2r} + \frac{\kappa k^2 t^2}{q^2\,r^2}\,,
\end{aligned}
$$

where

$$
C = \frac{8\kappa q^2\omega_k^2}{4\kappa\omega_k^2 - q^2}\,.
$$

In the particular case $k = 0$ where the star is just above the passage to the possible stable excited states, its mode pulsation $\omega = m_0$ being located at the accumulation point of the eigenfrequencies of an excited boson star, the correspondingly linearized Einstein field equations become

$$
\begin{aligned}
&\frac{2}{r}h,_r - \frac{2}{r^2}f = \kappa\left[\frac{|\mathcal{N}|^2}{r^4} - 2q^2m_0^2\,\frac{|\mathcal{N}|^4}{r^2} + 4q^2m_0^2\,\frac{|\mathcal{N}|^4}{r^2}\,\log\frac{r}{r_0}\right];\\
&h,_{rr} + \frac{1}{r}\,(h,_r - f,_r) = \kappa\left[-\frac{|\mathcal{N}|^2}{r^4} + 2q^2m_0^2\,\frac{|\mathcal{N}|^4}{r^2} + 4q^2m_0^2\,\frac{|\mathcal{N}|^4}{r^2}\,\log\frac{r}{r_0}\right];\\
&\frac{2}{r}f,_r + \frac{2}{r^2}f = \kappa\left[2m_0^2\,\frac{|\mathcal{N}|^2}{r^2} + \frac{|\mathcal{N}|^2}{r^4} + 2\,q^2m_0^2\,\frac{|\mathcal{N}|^4}{r^2} + 4q^2m_0^2\,\frac{|\mathcal{N}|^4}{r^2}\log\frac{r}{r_0}\right].
\end{aligned}
\tag{25}
$$

The solutions

$$
f(r) \;=\; \frac{C_1}{r} - \frac{b}{2r^2} + 2c\,\log\frac{r}{r_0} + (a-c)\,, \tag{26}
$$

$$
h(r) \;=\; -\frac{C_1}{r} + C_2 + (a-2c)\log\frac{r}{r_0} + 2c\left(\log\frac{r}{r_0}\right)^2, \tag{27}
$$

where

$$
a \;=\; \kappa m_0^2\,|\mathcal{N}|^2\;,\quad b \;=\; \kappa|\mathcal{N}|^2\;,\quad c \;=\; \kappa q^2m_0^2\,|\mathcal{N}|^4 \tag{28}
$$

and C_1, C_2 are two integration constants — the former getting the clear significance of a Schwarzschild mass M — generalize the asymptotic relation in [12]. By imposing to have the ordinary Minkowski metric at asymptotia, we get $C_2 = 0$ and $a = 2c$ or $|\mathcal{N}|^2 = 1/(2q^2)$ and thus, in the next calculations, we may use the simplified h function

$$
h(r) \;=\; -\frac{C_1}{r}\,. \tag{29}
$$

With these metric functions, solutions of Klein–Gordon–Maxwell–Einstein equations (in a first order approximation), one is able to compute the total charge, particle number, radius and mass of the analyzed configuration. Thus, by integrating the time component of the conserved current, one gets the total charge of the boson star,

$$
Q \;=\; \int e^{f-h}\,j_4\,4\pi r^2dr \;=\; 4\pi\,\frac{m_0}{q}\left(\frac{b}{2r_0^2}\right)^c\sqrt{\frac{b}{2}}\,e^z\;Intq\,, \tag{30}
$$

where $z = 2C_1^2/b$ and

$$Intq = \int_{-\sqrt{z}}^{\infty} \frac{e^{-y^2}}{(y+\sqrt{z})^{2c+2}}\, dy = \frac{\pi}{2} e^{-z} \frac{1}{\Gamma\left(\frac{3}{2}+c\right)} \times \left\{ -\frac{1}{\cos(c\pi)}\, {}_1F_1\left(-\frac{1}{2}-c, \frac{1}{2}, z\right) - \frac{\sqrt{\pi z}}{\sin(c\pi)}\, L\left(c, \frac{1}{2}, z\right) \right\} \tag{31}$$

is expressed in terms of the Kummer confluent hypergeometric function ${}_1F_1(\alpha;\beta;z)$ and of the *LaguerreL* polynomial.

The radius of the Bose star, defined via the particle number density $N = Q/q$ [15], is

$$R = \frac{1}{qN} \int r\, e^{f-h}\, j_4\, 4\pi r^2 dr = \frac{1}{Q}\, 4\pi\, \frac{m_0}{q} \left(\frac{b}{2r_0^2}\right)^c \frac{b}{2}\, e^z\, Intr\,, \tag{32}$$

where

$$Intr = \int_{-\sqrt{z}}^{\infty} \frac{e^{-y^2}}{(y+\sqrt{z})^{2c+3}}\, dy = \frac{1}{2} e^{-z} \tag{33}$$
$$\times \left\{ \Gamma(-1-c)\, {}_1F_1\left(-1-c, \frac{1}{2}, z\right) + 2\sqrt{z}\, \Gamma\left(-\frac{1}{2}-c\right)\, {}_1F_1\left(-\frac{1}{2}-c, \frac{3}{2}, z\right) \right\}.$$

Finally, the total gravitational mass of the Bose star, given by the Tolman's relation [16], which in our conventions reads

$$\mathcal{M} = \int T_{44}\, e^{f+h}\, 4\pi r^2 dr\,, \tag{34}$$

gets the expression

$$\mathcal{M} = \frac{8\pi^2}{\kappa} \sqrt{2b} \left(\frac{b}{2}\right)^c \left\{ \Gamma\left(\frac{1}{2}-c\right) + c\,\Gamma\left(-\frac{1}{2}-c\right) \left[3 + \log\frac{b}{2} - PolyGamma\left(0, -\frac{1}{2}-c\right) \right] \right\}, \tag{35}$$

depending on the mass of the quanta and on the value of the $U(1)$ coupling constant, included in the c and b parameters.

Of course, the results we have analytically derived are quite general and a numerical estimation would be of a real help in a better understanding of the analyzed configurations. In this respect, we use the asymptotically flat simplified solutions

$$\begin{aligned} f(r) &= \frac{C_1}{r} - \frac{b}{2r^2}\,, \\ h(r) &= -\frac{C_1}{r}\,, \end{aligned} \tag{36}$$

where, in the expression of f, the term in $a \sim (m_0/M_P)^2$ can be neglected at energy scales much below the Planck one. Here,

$$|\mathcal{N}|^2 = \frac{1}{8\pi\alpha}\,,\ b = \kappa\,|\mathcal{N}|^2 = \frac{1}{\alpha M_P^2}\,,\ C_1 = \frac{M_{ADM}}{M_P^2}\,, \tag{37}$$

where $\alpha = q^2/(4\pi)$ is the fine structure constant, M_P denotes the Planck mass and

$$M_{ADM} = -\frac{1}{8\pi}\int_{S^2_\infty} *\,dK \tag{38}$$

is the Arnowitz-Desser-Misner mass computed at infinity, using the timelike Killing 1-form field $K = -\,e^h\,\omega^4$. The dimensionfull parameter r_0 is a *pivot* length scale characterizing the nebula radial extension with respect to the central Coulomb-like singularity in the source field $\phi(r,t)$.

Using the proper energy density

$$T_{44} = \frac{\alpha^{-1}m_0^2}{8\pi r^2}\left[3 + \frac{m_0^{-2}}{r^2} + 2\ln\left(\frac{r}{r_0}\right)\right]$$

and the metric functions (36) the Tolman formula (34) for the boson nebula mass comprised within a sphere of radius R, does concretely become (in dimensionless variables $z = m_0 r$, $z_0 = m_0 r_0$)

$$\mathcal{M} = \frac{\alpha^{-1}}{2}\,m_0\int_0^{m_0 R}\left[\frac{1}{z^2} + 2\ln\left(\frac{z}{z_0}\right) + 3\right]e^{-\frac{c}{z^2}}\,dz\,,\ \ \text{with}\ \ c = \frac{\alpha^{-1}}{2}\left(\frac{m_0}{M_P}\right)^2 \tag{39}$$

and exhibits an intriguing *self-similarity-based resonance* which uniquely sets the pivot length scale, the radius of the *becoming* star and the maximum mass of the *seeds*. A complete analysis of this point is too long to be given here. Nevertheless, we notice that, making the *similarity* assumption

$$R = m_0^{-1}\, s\ , \quad r_0 = m_0^{-1} \cdot s \cdot e^{3/2}\ , \quad \text{i.e.}\ \ R = e^{-3/2} r_0 \tag{40}$$

the mass function $\mathcal{M}$ turns into the form

$$\mathcal{M} = \frac{\alpha^{-1}}{2}\, m_0 \int_0^s \left[\frac{1}{z^2} + 2\ln\left(\frac{z}{s}\right)\right] e^{-\frac{c}{z^2}}\, dz\, , \tag{41}$$

whose approximative value is:

$$\mathcal{M} \;\approx\; \sqrt{\frac{\pi}{8\alpha}}\, M_P \left[1 - \mathrm{Erf}\left(\frac{\sqrt{c}}{s}\right)\right]$$

and the extremum, $\mathcal{M}(s_*)$, of (41) is located at the solution of the highly transcendental equation

$$\left.\frac{d\mathcal{M}}{ds}\right|_{s_*} = 0 \quad \Leftrightarrow \quad \int_0^{s_*} e^{-\frac{c}{z^2}}\, dz = \frac{1}{2s_*}\, e^{-\frac{c}{s_*^2}}\, . \tag{42}$$

Fortunately, at scales $m_0 \ll M_P$, $c \to 0$ *rapidly* and e^{-c/s^2} (for $s \neq 0$) does basically equate 1. At $s = 0$ there is no singularity — on the contrary, it stands for the trivial solution — and so, the equation turns extremely simple

$$s_* = \frac{1}{2s_*} \quad \Leftrightarrow \quad 2s_*^2 - 1 = 0\, . \tag{43}$$

This gets the positive root $s_* = \sqrt{2}/2$, which clearly fixes the length-scale at $r_0 \approx 3.17\, m_0^{-1}$ and the radius at $R_* \approx 0.707\, m_0^{-1}$, for the region of the nebula which is going to become a Bose star. A numerical estimation (nevertheless imprecise because of the exponentially entering coefficient $c \sim 10^{-36}$, at $m_0 \sim 1\ GeV$) yields the maximum mass $\mathcal{M}_* \triangleq \mathcal{M}(s_*)$ of the *seeding* region somewhere around $10\, M_P$, being, somewhat intriguingly, quite insensitive to the realistic sub-Planckian values of the scalar quanta mass, m_0. Also, it can be shown that this *similarity-induced* mass-resonance $\mathcal{M}_*$ is indeed a global maximum over the rest of the other formations with radii $R \neq R_* \sim m_0^{-1}$ and therefore, just alike the case of Bose stars, it can be identified with M_{ADM}.

4 First-Order Perturbative Approach

In this section, we are going further and analyze the feedback of gravity and electric field, respectively expressed by the metric functions (36) and the four-potential (21-22), on the Klein–Gordon equations.

Within a first-order perturbative approach, we write down the wave function describing the charged scalar field as

$$\Phi(r,\theta,\varphi,t) \;=\; \phi(r,t) + \chi(r,\theta,\varphi,t)e^{i(kr-\omega t)}\,, \tag{44}$$

where ϕ is the Minkowskian background solution (16), with $|\mathcal{N}| = 1/\sqrt{2q^2}$, namely

$$\phi(r,\theta,\varphi,t) \;=\; \frac{1}{\sqrt{2}qr}\, e^{i(kr-\omega t)}\,. \tag{45}$$

While dealing with the Klein–Gordon equation (10), we employ the following approximations:

- $|\chi| \ll 1/(qr)$,
- in the metric function $f(r)$ in (36), the term br^{-2} can be discarded compared to $2C_1 r^{-1}$, once r exceeds few tens of Planck distances.
- We apply the long range approximation where

$$e^{\pm\frac{\lambda C_1}{r}} \approx 1 \pm \frac{\lambda C_1}{r} + \mathcal{O}\left[\left(\pm\frac{\lambda C_1}{r}\right)^n\right]_{n\geq 2}\,, \quad \text{for} \;\; \lambda = 1,2\,. \tag{46}$$

Also, while grouping together the terms

$$\frac{C_1}{r} + \ln\left(\frac{r}{r_0}\right) + \ln^2\left(\frac{r}{r_0}\right) ,$$

we notice that, at any distance r, at least few units above $2r_0 \approx 6.3\, m_0^{-1} \approx 9R_*$, the leading contribution comes solely from the term $\ln^2(r/r_0)$.

Finally, trading $C_1 = \mathcal{M}_*/M_P^2$ for the more common notation M, the approximating expression of the basic Klein–Gordon equation can be put in the standard form employed in Perturbation Theory, $\mathrm{D}\chi = \hat{V}\chi + \mathcal{J}$, as:

$$\frac{\partial^2\chi}{\partial r^2} + \left[\frac{2}{r} + ik\right]\frac{\partial\chi}{\partial r} - \frac{\partial^2\chi}{\partial t^2} + \frac{1}{r^2}\tilde{\triangle}\chi = \hat{V}\chi + \mathcal{J} . \tag{47}$$

The operators

$$\hat{V}(r,t) = \frac{2M}{r}\left[\frac{\partial^2}{\partial r^2} + \left(\frac{3ik}{2} + \frac{1}{r}\right)\frac{\partial}{\partial r} + \frac{\partial^2}{\partial t^2}\right] - 2iG(r,t)\left(1 + \frac{M}{r}\right)\frac{\partial}{\partial t}, \tag{48}$$

$$\mathcal{J}(r,t) = -\frac{1}{\sqrt{2q r}}\,G(r,t)^2 - \frac{k^2}{\sqrt{2q r}}\left(\frac{3}{4} + \frac{M}{r}\right) - \frac{ik}{\sqrt{2q}r^2}\left(1 + \frac{M}{r}\right), \tag{49}$$

where

$$G(r,t) = \frac{kt}{r} + \omega\ln\left(\frac{r}{r_0}\right) , \tag{50}$$

are respectively describing the (perturbed) effective potential and the current. In order to develop a first-order perturbative approach, we start with the 0-th order equation

$$\frac{\partial^2\chi}{\partial r^2} + \left[\frac{2}{r} + ik\right]\frac{\partial\chi}{\partial r} - \frac{\partial^2\chi}{\partial t^2} - \frac{\ell(\ell+1)}{r^2}\chi = 0 \tag{51}$$

and we perform the variables separation

$$\chi = e^{-i\alpha t}\, r^\ell \exp\left[-\frac{i}{2}(k+\Omega)r\right]\eta(r)\, Y_\ell^m(\theta,\varphi) , \tag{52}$$

with $\Omega = \sqrt{k^2 + 4\alpha^2}$. We come to the following differential equation,

$$z\,\frac{d^2\eta}{dz^2} \,+\, \left[2\,(\ell+1) - z\right]\frac{d\eta}{dz} - \left[\frac{k}{\Omega} + \ell + 1\right]\eta \,=\, 0\,,\;\; z = i\Omega r\,, \tag{53}$$

which is satisfied by the hypergeometric function $U(\alpha;\gamma;z)$, with $\alpha = k/\Omega + \ell + 1$ and $\gamma = 2\ell + 2$. Finally, one is able to compute the first-order transition amplitudes:

$$\mathcal{A}^{\omega' l' m'}_{\omega l m} = \int \chi^*_{\omega' l' m'}(x)\left(\hat{V}\chi_{\omega l m}(x)\right) r^2 dr d\Omega dt\,, \tag{54}$$

where $(x) = (r,\theta,\varphi,t)$ and $\hat{V}$ is given by (48), or to use (49) to study the spontaneous creation of charged bosons in the presence of the electric field potential A_4. It can be noticed that the radial wave number k destroys by the Coulomb term $2ikr^{-1}$ the hermiticity of the radial operator in (51), affects as well the effective potential by the gravity-based dipole contribution $3ikMr^{-2}$ and drives the "current" $\mathcal{J}$ complex. These lead to serious consequences regarding the actual dispersion, the continuity equation and the growth, and respectively decay, of the quantum mode-excitations.

The case $k = 0$ deserves a closer attention and it has been extensively investigated in previous papers [8]. Now, the 0-th order equation,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\frac{\partial h}{\partial r}\right) + \frac{1}{r^2}\,\tilde{\Delta} h - \frac{\partial^2 h}{\partial t^2} \,=\, 0\,, \tag{55}$$

provides the complete orthonormal set of positive-frequency modes (in terms of spherical Hankel functions)

$$h_{\omega l m}(r,\theta,\varphi,t) \,=\, \frac{1}{2\sqrt{r}}\,H^{(1)}_{l+\frac{1}{2}}(\omega r) Y^m_l(\theta,\varphi) e^{-i\omega t}\,. \tag{56}$$

This has been employed, in [8], to compute the first-order transition amplitudes between the initial and final states as:

$$\mathcal{A}^{\omega' l' m'}_{\omega l m} = \int h^*_{\omega' l' m'}(x)\left(\hat{V} h_{\omega l m}(x)\right) r^2 dr d\Omega dt\,, \tag{57}$$

where $(x) = (r, \theta, \varphi, t)$ and $\hat{V}$ is given by (48), with $k = 0$, namely

$$\hat{V}(r,t) = \frac{2M}{r}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial t^2}\right] - 2i\omega \ln\left(\frac{r}{r_0}\right)\left(1 + \frac{M}{r}\right)\frac{\partial}{\partial t}\,. \quad (58)$$

In what it concerns the current operator (49), whose form becomes

$$\mathcal{J}(r,t) = -\frac{1}{\sqrt{2}qr}\,\omega^2 \ln^2\left(\frac{r}{r_0}\right), \quad (59)$$

it enables us to study the possibility of spontaneous creation of charged bosons in the presence of the electric field potential A_4. This process is described by the transition amplitude

$$\begin{aligned}\mathcal{A}_{\mathcal{J}} &= \int \sqrt{\omega}\, h^*_{\omega lm}(x)\mathcal{J}(r,t)\, r^2 dr d\Omega dt \\ &= -2\pi\delta(\omega - m_0)\sqrt{2\pi\omega}\,\frac{m_0^2}{2q}\int_0^\infty \sqrt{r}\, H^{(2)}_{1/2}(\omega r)\ln^2\left(\frac{r}{r_0}\right) dr \quad (60)\end{aligned}$$

which comes to the closed form expression

$$\mathcal{A}_{\mathcal{J}} = 2\pi\delta(\omega - m_0)\,\frac{2m_0^2}{q\omega}\,\mathcal{I}\,, \quad (61)$$

where

$$\mathcal{I} = -\frac{1}{2}\left[X^2 + \mathrm{i}\pi X - \frac{\pi^2}{12}\right], \quad (62)$$

with

$$X = \gamma + \ln[\omega r_0]. \quad (63)$$

Finally, using

$$\frac{d}{dt}\mathcal{P}_+(m_0;\mathcal{J}) = 2\pi\delta(\omega - m_0)\left(\frac{2m_0^2}{q\omega}\right)^2 |\mathcal{I}|^2\,, \quad (64)$$

we come to the coherent source-field regeneration rate and the power involved in this process as being

$$\begin{aligned}\Gamma &= \int_0^\infty \frac{d\mathcal{P}_+}{dt}\frac{d\omega}{\omega} = 2\,\frac{m_0 c^2}{\alpha\hbar}\,|\mathcal{I}|^2\,, \quad \tau = \Gamma^{-1}\,, \\ P &= 2\,\frac{m_0^2 c^4}{\alpha\hbar}\,|\mathcal{I}|^2\,, \quad \text{where } |\mathcal{I}|^2 = 5.35\times 10^{-3}\,. \quad (65)\end{aligned}$$

At $m_0 \sim 1\,GeV$ level, one gets the following numerical results: $\Gamma \sim (2 \div 3) \times 10^{24}\ s^{-1}$, $\tau \sim (3 \div 5) \times 10^{-25}\ s$ and $P \sim 4 \times 10^{14}\ W$ respectively. The extremely short time-constant (τ) seems to confirm the main conclusion drawn in [13] that intense gravitational bursts are accompanying the boson star formation. Nevertheless, in the case of nebulae, on their way to becoming stars, the released power (P) in the initiating phase is 24 orders of magnitude smaller than the one emitted in the final stage.

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