Fourth Aegean Summer School: Black Holes
Mytilene, Island of Lesvos
September 18, 2007

Central extensions in flat spacetimes
Duality & Thermodynamics of BH dyons

New classical central extension in asymptotically flat spacetimes at null infinity

with G. Compère, CQG 24 (2007) F15

Thermodynamics of black hole dyons with duality invariant extended double potential formalism

with A. Gomberoff **arXiv:0705.0632** [hep-th]

So, thanks very much for the opportunity to speak at this school. Today I would like to present two loosely connected results. They both involve some of the more subtle aspects of symmetries in gauge theories.

One is the question of classical central charges in gravitational theories and whether they only arise in asymptotically ADS spacetimes. The result I would like to report on in this context is the existence of a new central extension in 3D flat spacetime at null infinity. This work has been done with a student of mine, Geoffrey Compère.

The other topic I would like to talk about is black holes which carry both electric and magnetic charge and how to discuss their thermodynamics. In order to do so, it is extremely convenient to extend the construction of a manifestly duality invariant formulation of Maxwell's theory, the so-called double potential formalism. In fact, this formulation provides a full-fledged alternative to Dirac's theory of magnetic poles and does not involve Dirac strings in the case of fixed sources. It is based on work done in collaboration with Andy Gomberoff.

Central extensions: Surface charges & generalized Killing vectors

Generalized conserved charges:

$$\begin{cases} d\omega^{n-p} \approx 0 \\ \omega^{n-p} \sim \omega^{n-p} + d\eta^{n-p-1} + t^{n-p}, \quad t^{n-p} \approx 0 \end{cases}$$

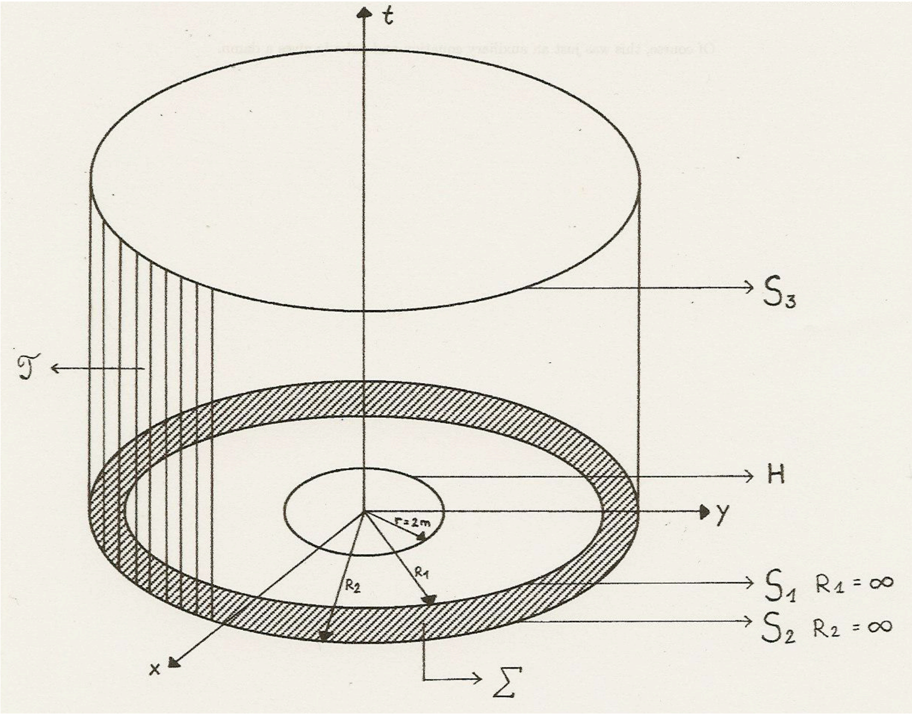
$p = 1:$ conserved currents associated with global symmetries

irreducible gauge theories (no 2,3-forms):

$$H^{n-p}_{char}(d) = 0 \ for \ p \geqslant 3$$

$$H^{n-2}_{char}(d) \longleftrightarrow f^\alpha \ such \ that \ R^i_\alpha(f^\alpha) = 0$$

charges: $$Q_f[\phi^s] = \oint_{S^{n-2}} k^{n-2}_f[\phi^s]$$



Generalized conserved charges are associated with differentials forms that are closed when the equations of motion hold. In this context, one should define as trivial forms which are automatically closed, either because they are exact, or because they vanish no-shell. This defines the so-called characteristic cohomology associated with given equations of motions.

In form degree n-1 one finds, the usual conserved currents of Noether's first theorem. They are associated with global symmetries.

For irreducible gauge theories, i.e., for gauge theories that do not contain 2,3-forms, one can show that the only other group that may be non trivial is in degree n-2 and that there is a 1-1 correspondence of elements of this group with gauge parameters that define trivial gauge transformations, so called reducibility parameters.

The associated charges are to be integrated over closed n-2 dimensional surfaces, for example spheres at constant t and r. By applying Stokes' theorem, they do not depend on t nor on r.

Central extensions: Surface charges & generalized Killing vectors

Examples

semi-simple YM theory: $\delta_\epsilon A^a_\mu = D_\mu \epsilon^a = 0 \Longrightarrow \epsilon^a = 0$

EM: $\delta_\epsilon A_\mu = \partial_\mu \epsilon = 0 \Longrightarrow \epsilon = cte \quad \longleftrightarrow \quad k^{n-2} = {}^*F$

electric charge $\quad Q = \oint_{S^{n-2}} {}^*F$

GR: $\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = 0 \Longrightarrow \xi^\mu = 0$

linearized gravity: $\delta_\xi h_{\mu\nu} = \mathcal{L}_\xi \bar{g}_{\mu\nu} = 0 \Longrightarrow \xi^\mu$ Killing vector of $\bar{g}_{\mu\nu}$

$$k_\xi[h;\bar{g}] = \frac{1}{16\pi}(d^{n-2}x)_{\mu\nu}\sqrt{-\bar{g}}\Big(\bar{\xi}^\nu \bar{D}^\mu h + \bar{\xi}^\mu \bar{D}_\sigma h^{\sigma\nu} + \bar{\xi}_\sigma \bar{D}^\nu h^{\sigma\mu} + \frac{1}{2} h \bar{D}^\nu \bar{\xi}^\mu + \frac{1}{2} h^{\mu\sigma} \bar{D}_\sigma \bar{\xi}^\nu + \frac{1}{2} h^{\nu\sigma} \bar{D}^\mu \bar{\xi}_\sigma - (\mu \longleftrightarrow \nu)\Big),$$

$$Q_\xi = \oint_{S^{n-2}} k_\xi[h,\bar{g}]$$

Abbott & Deser Nucl. Phys. B195 (1982) 76

For example, in YM theory they would be defined by gauge parameters whose covariant derivatives vanish, for all choices of potentials. There are no such parameters for semi-simple gauge groups, however in the U(1) case, for electromagnetism, there is precisely 1 solution, a constant gauge parameter. The associated conserved n-2 form turns out to be precisely the dual of of the field strength F, so that the charge is of course the electric charge, as determined by Gauss's law.

In GR, one would have to find Killing vectors for a generic metric, which do not exist, so there are again no non trivial surface charges. For linearized gravity however, the gauge transformations of the metric perturbations involve the Lie derivative of the background metric, which might very well have Killing vectors, for instance in the flat case, the Kvf represent the Poincare algebra. The associated surface charges can then be constructed. In a flat background, they describe the ADM energy-momentum and angular momentum. In a curved background, they coincide with expressions derived originally in the AdS context by Abott and Deser.

Central extensions: Algebra & asymptotics

global symmetry $\mathcal{L}_\xi \bar{g}_{\mu\nu} = 0 \Longrightarrow \quad \delta^1_\xi h_{\mu\nu} = \mathcal{L}_\xi h_{\mu\nu}$

$\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ Poincaré invariance of Pauli-Fierz theory

Algebra: Surface charges form a representation of the algebra of Killing vectors

$$\{Q_{\xi_1}, Q_{\xi_2}\} := \delta^1_{\xi_1} Q_{\xi_2} = Q_{[\xi_1,\xi_2]}$$

full GR, asymptotics

$g_{\mu\nu} = \bar{g}_{\mu\nu} + O(\frac{1}{r^{\chi_{\mu\nu}}})$ at boundary $r \longrightarrow \infty$

replace $h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$ charges $Q_\xi = \oint_{S^\infty} k_\xi[g - \bar{g}, \bar{g}]$

In linearized gravity, if xi is a Killing vector of the background, then the Lie derivative of the metric perturbation defines a global symmetry of the linearized theory, to wit the Poincaré invariance in the case of a flat background. One can then show that the surface charges under the action of this global symmetry, form a representation of the Lie algebra of Killing vectors. The question is then how to use this analysis of the linearized theory in full GR.

One application is in the case where the metric approaches a background metric with some appropriate fall-off conditions. What one does is replace the metric perturbations by the difference between between the full metric and the background. In the asymptotic region, where the linearized EOM are supposed to be valid due to the fall-off conditions, one then still has conservation in time, but not in r because the linearized approximation will fail as one goes into the bulk. One way to think about this is that the linearized surface integrals contain all the sources, including those due to the self-interactions of the fields, only in the asymptotic region.

Central extensions: Algebra & asymptotics

new feature: asymptotic Killing vectors $\mathcal{L}_\xi \bar{g}_{\mu\nu} \to 0$ to leading order

that preserve the fall-off conditions $\mathcal{L}_\xi g_{\mu\nu} = O(\frac{1}{r^{\chi_{\mu\nu}}})$

suitable tuning of fall-off conditions on metrics and asymptotic Killing vectors:

centrally extended charge representation of algebra of asymptotic Killing vectors

$$\{Q_{\xi_1}, Q_{\xi_2}\} := \delta_{\xi_1} Q_{\xi_2} = Q_{[\xi_1,\xi_2]} + K_{\xi_1,\xi_2} \qquad K_{\xi_1,\xi_2} = \oint_{S^\infty} k_{\xi_2}[\mathcal{L}_{\xi_1}\bar{g}, \bar{g}]$$

NB: central extension vanishes for exact symmetries of the background

The new feature is then that one will only need approximate Killing vectors in order to guarantee conservation of charges. One has to require that the associated large gauge transformations leave the space of asymptotically background metrics invariant. Under suitable fall-off conditions on metrics and gauge parameters, one then arrives at the following algebra of conserved charges, which represents the algebra of asymptotic reducibility parameters. The central extension vanishes for exact Killing vector fields of the background metric.

Central extensions: Asymptotically ADS spacetimes

non trivial asymptotic Kvf= conformal Kvf of flat boundary metric

$$d\bar{s}^2 \equiv \bar{g}_{\mu\nu} dx^\mu dx^\nu = -(1+\frac{r^2}{l^2})d\tau^2 + +(1+\frac{r^2}{l^2})^{-1}dr^2 + r^2 \sum_A f_A (dy^A)^2,$$

n>3: $\mathfrak{so}(n-1,2)$ only exact Killing vectors of AdS, no central extension

n=3: pseudo-conformal algebra in 2 dimensions, 2 copies of Wit algebra

charges algebra: 2 copies of Virasoro

$$i\{\mathcal{L}^\pm_m, \mathcal{L}^\pm_n\} = (m-n)\mathcal{L}^\pm_{m+n} + \frac{c}{12}m(m^2-1)\delta_{n+m},$$

$$\{\mathcal{L}^\pm_m, \mathcal{L}^\mp_n\} = 0,$$

where $c = \frac{3l}{2G}$ is the central charge for the anti-de Sitter case.

Brown & Henneaux CMP 104 (1986) 207

similar results in de Sitter spacetimes at timelike infinity

In the example of asymptotically AdS spacetimes, one then finds that the non trivial asymptotic Killing vectors are given by the conformal Killing vectors of the flat metric induced on the boundary.

For n> 3, one finds so(n-1,2), the exact Kvf of the background and thus no central extension.

In three dimensions, one finds the pseudo-conformal algebra in 2 dimensions. For the charges one then gets 2 copies of the centrally extended Virasoro algebra, with a central charge that involves the gravitational coupling G and the cosmological constant l.

In the de Sitter case, essentially the same results hold, except that the boundary is now at timelike infinity instead of spatial infinity.

Central extensions: Asymptotically flat spacetimes

conformal boundary in asymptotically flat spacetimes: null infinity

Introducing the retarded time $u = t - r$, the luminosity distance r and angles θ^A on the $(n-2)$-sphere by $x^1 = r\cos\theta^1$, $x^A = r\sin\theta^1\cdots\sin\theta^{A-1}\cos\theta^A$, for $A = 2, \ldots, n-2$, and $x^{n-1} = r\sin\theta^1\cdots\sin\theta^{n-2}$, the Minkowski metric is given by

$$d\bar{s}^2 = -du^2 - 2\,du\,dr + r^2\sum_{A=1}^{n-2} s_A (d\theta^A)^2, \qquad (3.1)$$

where $s_1 = 1$, $s_A = \sin^2\theta^1\cdots\sin^2\theta^{A-1}$ for $2 \leqslant A \leqslant n-2$. The (future) null boundary is defined by $r = \text{constant} \to \infty$ with u, θ^A held fixed.

$$\mathfrak{bms}_n$$

$$\xi^u = T(\theta^A) + u\partial_1 Y^1(\theta^A) + o(r^0),$$
$$\xi^r = -r\partial_1 Y^1(\theta^A) + o(r),$$
$$\xi^A = Y^A(\theta^B) + o(r^0), \qquad A = 1, \ldots, n-2,$$

$Y^A(\theta^A)$ conformal Kvf of n-2 sphere

$T(\theta^A)$ “supertranslations”, arbitrary function on n-2 sphere

Witten suggested in 2001 that the appropriate boundary from a conformal point of view in asymptotically flat spacetimes is null infinity. If one introduces the retarded time u, future null infinity is still at r goes to infinity for fixed u and fixed angles.

The non trivial asymptotic Killing vectors turn out to be determined by functions T and Y which depend on the angles. The functions Y describe conformal Killing vectors of the n-2 sphere, while the functions T depend arbitrarily on the angles and are the so-called supertranslations.

Central extensions: Asymptotically flat spacetimes

$$\widehat{\xi} = [\xi, \xi']$$

$$\hat{T} = Y^A \partial_A T' + T \partial_1 Y'^1 - Y'^A \partial_A T - T' \partial_1 Y^1,$$
$$\hat{Y}^A = Y^B \partial_B Y'^A - Y'^B \partial_B Y^A.$$

algebra: semi-direct product with abelian ideal $\mathfrak{i}_{n-2}$

n>4: $\mathfrak{so}(n-1,1) \ltimes \mathfrak{i}_{n-2}$

n=4: conformal algebra in 2d $\ltimes \mathfrak{i}_2$

$\cup$

$\mathfrak{so}(3,1)$ Bondi-Metzner-Sachs (1962)

As an algebra, one finds the semi-direct product of the conformal Kvf's of the n-2 sphere with the abelian ideal of supertranslations.

In 4 dimensions, one finds the semi-direct product of the 2d conformal algebra with the supertranslations. The former contains the Lorentz algebra as a subalgebra. In the original 1962 derivation by Bondi Metzner & Sachs of the symmetry group of asymptotically flat spacetimes at null infinity, they required these trsf to be well defined on the 2-sphere, and found thus only the Lorentz algebra and as only symmetry enhancement with respect to the exact case the supertranslations, which contain the ordinary translations for particular choices of the function T. It would be interesting to study if there are central extensions in the representation by charges of bms4. We have not done so, though.

Central extensions: Asymptotically flat spacetimes

n=3: no restriction on $Y(\theta)$

$$J_n = \xi(T=0, Y=\exp(in\theta))$$

$$P_n \equiv \xi(T=\exp(in\theta), Y=0)$$

$$\mathrm{i}[J_m, J_n] = (m-n)J_{m+n},$$
$$\mathrm{i}[P_m, P_n] = 0,$$
$$\mathrm{i}[J_m, P_n] = (m-n)P_{m+n}.$$

1 copy of Wit algebra acting on $\mathfrak{i}_1$

$$\cup$$

$$\mathfrak{iso}(2,1)$$

Ashtekar et al. Phys. Rev. D55 (1997) 669

charge algebra:

$$\mathrm{i}\{\mathcal{J}_m, \mathcal{J}_n\} = (m-n)\mathcal{J}_{m+n},$$
$$\mathrm{i}\{\mathcal{P}_m, \mathcal{P}_n\} = 0,$$
$$\mathrm{i}\{\mathcal{J}_m, \mathcal{P}_n\} = (m-n)\mathcal{P}_{m+n} + \frac{1}{4G}m(m^2-1)\delta_{n+m}.$$

relation to AdS_3 similar to contraction between $\mathfrak{so}(2,2) \to \mathfrak{iso}(2,1)$

$$\mathrm{i}[J_m, J_n] = (m-n)J_{m+n},$$
$$\mathrm{i}[P_m, P_n] = \frac{1}{l^2}(m-n)J_{m+n},$$
$$\mathrm{i}[J_m, P_n] = (m-n)P_{m+n},$$

$$L^{\pm}_m = \frac{1}{2}(lP_{\pm m} \pm J_{\pm m}) \qquad l \to \infty$$

In 3 dimensions, the conformal Killing equation on the circle imposes no restrictions on the function Y(theta), so that the algebra is described by 2 arbitary functions on the circle. After Fourier analyzing, the algebra consists of 1 copy of the Wit algebra acting on the functions on the circle in a similar way than the Lorentz transformations act on the ordinary translations. In fact the bms3 algebra has been originally discussed in a paper by Ashtekar et al in 1997.

What was not done though was the computation of the charge algebra. It turns out to contain a non trivial central extension between the two factors. A posteriori, it is clear that this is the only place that the central extension as it cannot appear in the one copy of the Wit algebra on account of the missing dimensional parameter l in the flat case. The relation to the AdS3 Virasoro case is by a contraction similar to the one between so(2,2) and iso(2,1). More precisely, if one introduces a parameter of dimension length, there is an extension of the BMS algebra, and after redefining the generators, one finds both for the asymptotic symmetries and for the charges, including the central ones, the AdS3 results.

What would be interesting is to analyze in details what this classical central extension can teach us about quantum gravity in asymptotically flat 3d spacetimes. Indeed, let me recall that in the AdS3 case, the central extension was a crucial ingredient that allowed Strominger to use the Cardy formula to give a microscopic explanation of black entropy.

Magnetic charge as a surface charge ?

Magnetic charge: $$d^*F = 0 \longrightarrow {}^*F = dB$$

What about dyons?

Action principle?

We have seen that electric charge, in the same way as energy-momentum in GR, can be described as a surface charge. What about magnetic charge ? By introducing potentials for the dual of F instead of F, Magnetic charge can be described in a similar way as a surface charge. But then the problem arises for electric charge, or more generally, if you want to describe solutions that carry both electric and magnetic charge. What you want in order to do thermodynamics of dyonic black holes in the standard way is an action principle that is capable of describing through surface charges both electric and magnetic charges.

Dyons and duality: Thermodynamics for RN

RN dyon:

$$ds^2 = -N^2dt^2 + N^{-2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$
$$N = \sqrt{1 - \frac{2M}{r} + \frac{Q^2+P^2}{r^2}},$$
$$A = -\frac{Q}{r}dt + P(1-\cos\theta)d\phi,$$

infer thermodynamics for parameter variations from purely electric case using duality

$$\delta M = \frac{\kappa}{8\pi}\delta\mathcal{A} + \phi_H \delta Q + \psi_H \delta P,$$

$$\Delta = M^2 - (Q^2+P^2), \quad r_\pm = M \pm \sqrt{\Delta}$$
$$\kappa = \frac{r_+ - r_-}{2r_+^2}, \quad \mathcal{A} = 8\pi\left[M^2 - \frac{Q^2+P^2}{2} + M\sqrt{\Delta}\right],$$
$$\phi_H = \frac{Q}{r_+}, \quad \psi_H = \frac{P}{r_+},$$

Problem: excluded in action based derivations of 1st law for arbitrary variations because of string singularity and absence of magnetic potential

Consider for instance the Reissner-Nordstrom dyon. It is a black hole solution to the Einstein-Maxwell equations with both electric and magnetic charge. The thermodynamics for variations of parameters can easily be inferred from the well-known purely electric case by using duality. In a geometric derivation of the first law, for discussing uniqueness theorems and in Euclidean approaches one needs to derive the first law starting from an action principle. For stationary black hole solutions, for instance, the leading contribution to the path integral comes from the classical action which reduces to surface terms that are closely related to the surface charges we have discussed previously. The case with both electric and magnetic charges is usually excluded in these discussions because of the string singularity of the vector potential and the absence of the analog of A_0, whose value on the horizon plays the role of chemical potential for electric charge.

Dyons and duality: Double potential formalism

key: existence of 1st order formalism that makes invariance of action under duality rotations manifest

$$S = -\int d^4x[\vec{E}\cdot\dot{\vec{A}} + \tfrac{1}{2}(E^2 + 2\vec{B}\cdot\nabla\times\vec{A} - B^2) + A_0\nabla\cdot\vec{E}] \quad (2.1a)$$

$$= -\int d^4x\{\vec{E}\cdot\dot{\vec{A}} + \tfrac{1}{2}[E^2 + (\nabla\times\vec{A})^2] + A_0\nabla\cdot\vec{E}\} \quad (2.1b)$$

$$= -\int d^4x\{\vec{E}^T\cdot\dot{\vec{A}}^T + \tfrac{1}{2}[(\vec{E}^T)^2 + (\nabla\times\vec{A}^T)^2]\} .$$

We then define the finite rotation

$$\vec{E}' = \cos\beta\,\vec{E} + \sin\beta\,\nabla\times\vec{A}, \quad (2.2a)$$

$$\nabla\times\vec{A}' = \cos\beta\,\nabla\times\vec{A} - \sin\beta\,\vec{E} \quad (2.2b)$$

in the space of transverse (divergence-free) vector fields, so that the vanishing of the logitudinal components of $(\vec{E}, \vec{B})$ is automatically incorporated in (2.2). The variation of $\vec{A}^T$ is uniquely obtained from that of $\nabla\times\vec{A}^T$ by "inverting" with $-\nabla^{-2}\nabla\times$;

in curved space:

$$S = -\int d^4x\,[\mathcal{E}^i\dot{A}_i + \tfrac{1}{2}N({}^3g)^{-1/2}g_{ij}(\mathcal{E}^i\mathcal{E}^j + \mathcal{B}^i\mathcal{B}^j) - \epsilon_{ijk}N^i\mathcal{E}^j\mathcal{B}^k], \quad (2.8)$$

$$\mathcal{E}^i = \epsilon^{ijk}\partial_j Z_k \qquad \mathcal{B}^i = \epsilon^{ijk}\partial_j A_k$$

Deser & Teitelboim Phys. Rev. D 13 (1976) 1592

The idea is then to use a formulation where duality is manifestly a symmetry of the action. Such a formulation is well known in the Hamiltonian approach where the electric field is an independent variable, the momentum conjugate to the vector potential. A0 is the Lagrange multiplier for Gauss' law, which can be solved in the sourcefree case to yield an action principle purely in terms of the transverse fields.

In terms of these variables, the duality rotations mix transverse electric fields with vector potential and indeed leave the action invariant. Hence, in this formulation, duality is a symmetry of the action rather than just of the equations of motion.

By using the standard ADM decomposition of the metric, Deser and Teitelboim have generalized this duality invariant action to curved space and also shown that they have a theory for 2 vector potentials, since the divergence free electric field can be written in terms of the curl of a second vector potential.

independent rederivation :

$$S = -\frac{1}{2}\int d^4x \Big(B^{(\alpha)i}\mathcal{L}_{\alpha\beta}E_i^{(\beta)} + B^{(\alpha)i}B^{(\alpha)i} \Big),$$

$$\delta A_0^{(\alpha)} = \Psi^{(\alpha)}, \quad \delta A_i^{(\alpha)} = \partial_i \Lambda^{(\alpha)}.$$

$$E_i^{(\alpha)} = \partial_0 A_i^{(\alpha)} - \partial_i A_0^{(\alpha)}, \quad B^{(\alpha)i} = \epsilon^{ijk}\partial_j A_k^{(\alpha)} \quad 1 \le i,j,k \le 3$$

$$A_0^{(\alpha)} = 0. \qquad (2.5)$$

Since $A_0^{(\alpha)}$ only appears as part of a total derivative in the action, no equations of motion are lost. (This is to be contrasted with choosing $A_0 = 0$ gauge in the usual

Schwarz & Sen Nucl. Phys. B411 (1994) 35

In 1993, Schwarz and Sen gave an independent rederivation of these result. They wrote the first order action in the above form, collecting the 2 vector potentials in an SO(2) doublet and using only the two SO(2) invariant tensor deltaab and epsilonab to write the action. What is really interesting is that they also introduce a doublet of scalar potential. They are spurious in their action because electic and magnetic fields are transverse, so that the scalar potentials only appear in a total derivative. This is the reason why their gauge invariance is not 2 copies of the electromagnetic one and why these scalar potential are not really present in their theory at all.

Dyons and duality: Applications to black hole duality

"complicated" proof of duality between electric and magnetic BH

partition function through semi-classical evaluation of Euclidean path integral

$Z(\beta, P)$ vs $Z(\beta, \phi_H)$

Hawking & Ross, Phys. Rev. D52 (1995) 5685

additional Legendre transformation needed to compare

In the BH context Hawking and Ross have shown that there is a duality between electric and magnetic BH. Their proof is somewhat involved in the standard formulation, where duality is not manifest. Indeed, they compute the partition function through a semi-classical approximation of the Euclidean path integral. In the magnetic case, this is done at fixed charge, but in the electric case at fixed potential, so they need an additional Legendre transformation in order to be able to compare both results and show that they are duality symmetric.

Dyons and duality: Applications to black hole duality

duality symmetric formulation:

$$B^i = \epsilon^{ijk}\partial_j A_k + B^i_S \tag{9}$$

where B^i_S is a fixed field that carries the magnetic flux,

$$\oint_{S^2_\infty} B^i_S dS_i = 4\pi\mu, \tag{10}$$

and where $B^i_T = \epsilon^{ijk}\partial_j A_k$ is the transverse part of B^i,

$$\partial_i B^i_T = 0, \oint_{S^2_\infty} B^i_T dS_i = 0. \tag{11}$$

In order to discuss duality, it is convenient to treat the nondynamical components of E^i and B^i symmetrically. To that end, one may either redefine B^i_S by adding to it an appropriate transverse part so that it shares the spherical symmetry of E^i_L, or one may redefine E^i_L by adding to it an appropriate transverse part so that it is entirely localized on the string. Both choices (or, actually, any other intermediate choice) are acceptable here. For concreteness we may take the first choice; the fields then have no string singularity.

$$E^i = E^i_T + E^i_L, \tag{12}$$

where the longitudinal part carries all the electric flux

$$\oint_{S^2_\infty} E^i_L dS_i = 4\pi e, \tag{13}$$

and the transverse field obeys

$$\partial_i E^i_T = 0, \oint_{S^2_\infty} E^i_T dS_i = 0 \tag{14}$$

and can thus again be written as $E^i_T = \epsilon^{ijk}\partial_j Z_k$ for some Z_k. Given the electric charge e, the longitudinal electric field is completely determined if we impose in addition, say, that it be spherically symmetric. As we have done above, we shall work with a variational principle in which we have solved Gauss's law and in which the competing histories have a fixed electric flux $\oint_{S^2_\infty} E^i dS_i$ at infinity. This means that the longitudinal electric field is completely frozen and that only the tranverse components E^i_T or Z^i are dynamical, as for the magnetic field.

$$B^i_L = \mu V^i, \; E^i_L = e V^i,$$

The derivation of this result has been streamlined and generalized in the manifestly duality invariant formulation. Charges have been taken into account symmetrically by including non-dynamical longitudinal spherically symmetric electric and magnetic fields.

Dyons and duality: Applications to black hole duality

action principle appropriate for fixed charges

$$I_M^{e,\mu}[\mathbf{E}_T,\mathbf{A}] = -\int d^4x \Big[E_T^i \dot{A}_i + \frac{1}{2} N g^{-1/2} g_{ij}(E^i E^j + B^i B^j) - \epsilon_{ijk} N^i E^j B^k \Big]. \quad (15)$$

$$I_M^{e,\mu}[\mathbf{E}_T,\mathbf{A}_T] = I_M^{e',\mu'}[\mathbf{E}'_T,\mathbf{A}'_T],$$

Deser, Henneaux, Teitelboim Phys. Rev. D 55 (1997) 826

duality:

$$\begin{pmatrix} \mathbf{Z}' \\ \mathbf{A}' \end{pmatrix} = R \begin{pmatrix} \mathbf{Z} \\ \mathbf{A} \end{pmatrix} \equiv \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{Z} \\ \mathbf{A} \end{pmatrix},$$

$$\begin{pmatrix} e' \\ \mu' \end{pmatrix} = R \begin{pmatrix} e \\ \mu \end{pmatrix},$$

BH duality:

$$I_M^{e,0}[\mathbf{0},\mathbf{0}] = I_M^{0,e}[\mathbf{0},\mathbf{0}],$$

The action principle at fixed charges is then the same as before except for the presence of the non-dynamical longitudinal fields carrying the charges. The duality of this action then holds if the charges are rotated together with the dynamical components of the fields and Hawking and Ross's results of electric and magnetic BH duality boils down to the statement that electromagnetic action with zero transverse fields with fixed electric charge and 0 magnetic one is equal to the one with zero electric charge and the same magnetic one.

Dyons and duality: Extended formulation

New construction: dynamical longitudinal fields and non spurious scalar potentials

$$A^a_\mu \equiv (A_\mu, Z_\mu) \qquad C^a \equiv (C, Y) \qquad \vec{B}^a \equiv (\vec{B}, \vec{E}) \qquad \vec{B}^a = \vec{\nabla} \times \vec{A}^a + \vec{\nabla} C^a$$

external sources: $\partial_\mu j^{a\mu} = 0$

action principle:

$$I_M[A^a_\mu, C^a; j^{a\mu}] = \frac{1}{2}\int d^4x \Big[\epsilon_{ab}\big(\vec{B}^a \cdot (\partial_0 \vec{A}^b - \vec{\nabla} A^b_0) - -\vec{\nabla} C^a \cdot \vec{\nabla} A^b_0 + \partial_0 C^a \vec{\nabla} \cdot \vec{A}^b\big) - \vec{B}^a \cdot \vec{B}_a + 2\epsilon_{ab} A^a_\mu j^{b\mu}\Big],$$

Maxwell's equations:

$$\begin{cases} A^a_0: \quad \vec{\nabla} \cdot \vec{B}^a \equiv \nabla^2 C^a = j^{0a} \qquad C^a: \ \nabla^2 C_a = \epsilon_{ab}(\vec{\nabla} \cdot \partial_0 \vec{A}^b - \nabla^2 A^b_0) \\ \vec{A}^a: \ -\epsilon_{ab}\partial_0 \vec{B}^b + \vec{\nabla} \times \vec{B}_a = \epsilon_{ab} \vec{j}^b. \end{cases}$$

What has not been done so far, and what we would like for the application to black holes, is the following new construction, involving dynamical longitudinal fields and non spurious scalar potentials, which make the scalar potentials non spurious. In the case of conserved external sources, the correct action principle is the following. Indeed, variation with respect to the scalar potential gives the electric and magnetic Gauss law. Variation with respect to the potentials for longitudinal fields shows that they are auxiliary and relates them to the scalar potentials. Finally, variation with respect to the vector potentials gives Maxwell's equations for electric and magnetic fields.

Dyons and duality: Extended formulation

point particle dyon at origin: $j^{a\mu}(x) = 4\pi Q^a \delta^\mu_0 \delta^3(x)$ $A^a = -\frac{\epsilon^{ab}Q_b}{r}dt,\ C^a = -\frac{Q^a}{r}$

gauge invariance : $\delta_\epsilon A^a_\mu = \partial_\mu \epsilon^a, \qquad \delta_\epsilon C^a = 0$ no string singularity !

spectrum: additional pure gauge degrees of freedom "quartet"
besides longitudinal and temporal photon

2 reducibility parameters: magnetic charge is surface integral, no longer a topologically charge

For a point particle dyon at the origin, the solution of the EOM is described by two spherically symmetric Coulomb type fields.

This action has twice the gauge invariance of the usual Maxwell action. Without sources, the associated action is of course equivalent to the standard action because variation with respect to the Lagrange multipliers gives two Gauss type constraints forcing the longitudinal fields to be zero. Quantum mechanically, when one analyses the spectrum, one sees that all one has done is introduce additional pure gauge degrees of freedom, in BRST language, an additional quartet.

We also see that there are 2 reducibility parameters. In other words, magnetic charge now appears not as a topological charge but as a surface charge on a par with electric charge.

Dyons and duality: Extended formulation

curved space: $$I_M[A^a_\mu, C^a, g_{ij}, N, N^i] = \frac{1}{8\pi}\int d^4x \Big[(\mathcal{B}^{ai} + \frac{\sqrt{g}}{N}\partial^i C^a)\epsilon_{ab}(\partial_0 A^b_i - \partial_i A^b_0) - \frac{N}{\sqrt{g}}\mathcal{B}^i_a\mathcal{B}^a_i + \epsilon_{ab}[ijk]N^i\mathcal{B}^{aj}\mathcal{B}^{bk}\Big]$$

$$\mathcal{B}^{ai} = [ijk]\partial_j A^a_k + \frac{\sqrt{g}}{N}\partial^i C^a$$

RN dyon: $$ds^2 = -N^2dt^2 + N^{-2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$
$$N = \sqrt{1 - \frac{2M}{r} + \frac{Q^2+P^2}{r^2}}, \qquad A^a = -\frac{\epsilon^{ab}Q_b}{r}dt, \ C^a = -\frac{Q^a}{r}$$

BH thermodynamics in grand canonical ensemble

variables: $z^A \equiv (g_{ij}, \pi^{ij}, Z_i, A_i, Y, C) \qquad \lambda^a \equiv (N, N^i, A_0, Z_0) \qquad \epsilon^a \equiv (\epsilon^\perp, \epsilon^i, \lambda, \mu)$

surface integrals: $$\delta(\gamma_\alpha \epsilon^\alpha) = \delta z^A \frac{\delta(\gamma_\alpha\epsilon^\alpha)}{\delta z^A} - \partial_i k^{T[0i]}_\epsilon[\delta z^A]$$

Regge & Teitelboim, Ann. Phys. 88 (1974) 286

reducibility parameters for RN: $\epsilon^\alpha = \lambda^\alpha = (N, 0, 0, A^a_0)$

Hamiltonian EOM: $$\begin{cases} \dot z^A = \sigma^{AB}\frac{\delta(\gamma_\alpha\lambda^\alpha)}{\delta z^B} \\ \gamma_\alpha = 0 \end{cases} \Longrightarrow \quad \partial_i k^{T[0i]}_\lambda = 0$$

In curved space, we finally have all the ingredients to discuss the thermodynamics of BH dyons in the grand canonical ensemble, i.e., with fixed potentials of both kinds instead of fixed charges. The RN Nordstrom dyon for instance now appears as a solution without string like singularity but 2 non vanishing spherically symmetric scalar potentials that provide the correct longitudinal magnetic and electric fields.

The thermodynamics can then be discussed in your favourite approach. One can for instance work out the conserved surface integrals in the Hamiltonian framework. The canonical variables are the spatial metric, their momenta, the vector potentials and the potentials for the longitudinal fields. The Lagrange multipliers, are, besides lapse and shift, also the scalar potentials. Besides the diffeomorphism vectors, there are also electric and magnetic gauge parameters. In the Hamiltonian formalism, the time-space components of the surface integrals can be worked out from the boundary terms that arise in the variation of the constraints. This is the Regge-Teitelboim approach. Reducibility parameters can be shown to be given by the Lagrange multipliers themselves. That the corresponding time-space component of the n-2 form is conserved in space follows from the general theory. It can also be seen directly in the Hamiltonian approach as follows. Because the RN dyon is stationary, the EL derivatives of the constraints contracted with the multipliers vanish. If furthermore, the variation satisfies the linearized EOM, which is the case in particular for a variation of the parameters, the LHS in the definition of the boundary term vanishes as well.

Dyons and duality: Extended formulation

Stokes' theorem:

$$\frac{1}{16\pi}\oint_{S_\infty} d^{n-1}x_i\, k_\lambda^{T[0i]} = \frac{1}{16\pi}\oint_{S_{r_+}} d^{n-1}x_i\, k_\lambda^{T[0i]}.$$

explicitly:

$$k_\epsilon^{T[0i]}[\delta z^A] = k_\epsilon^{grav[0i]}[\delta g_{ij}, \delta\pi^{ij}] + k_\epsilon^{mat[0i]}[\delta z^A].$$

matter part:

$$k_\epsilon^{mat[0i]}[\delta z^A] = 4\Big(\frac{\epsilon^\perp}{\sqrt{g}}[ijk](\mathcal{E}_j\delta Z_k + \mathcal{B}_j\delta A_k) - \frac{\epsilon^\perp}{N}(\mathcal{E}^i\delta Y + \mathcal{B}^i\delta C) -$$

$$-\epsilon^i(\mathcal{B}^k\delta Z_k - \mathcal{E}^k\delta A_k) + \mathcal{B}^i\epsilon^k\delta Z_k - \mathcal{E}^i\epsilon^k\delta A_k - \frac{\sqrt{g}g^{il}}{N}[ljk]\epsilon^j(\mathcal{B}^k\delta Y - \mathcal{E}^k\delta C) -$$

$$-\frac{1}{2}\frac{\sqrt{g}}{N}(\lambda\partial^i\delta Y - \partial^i\lambda\delta Y - \mu\partial^i\delta C + \partial^i\mu\delta C)\Big).$$

both magnetic and electric contributions !

first law:

$$\delta M = \frac{\kappa}{8\pi}\delta\mathcal{A} + \phi_H\delta Q + \psi_H\delta P$$

As a consequence of Stokes' theorem, we thus find that the surface integral is independent of r and can be evaluated at infinity or at the horizon.

Explicitly, the surface integrals split into the standard purely gravitational part and a matter part. The matter part of the surface integrals involves both electric and magnetic contributions. For instance, the first law for RN dyons now follows directly direct from the equality of the surface integral at infinity or at the horizon.

Conclusion

new classical central extension in asymptotically flat 3D spacetimes at null infinity

contraction of the ADS case

construction of an explicitly duality invariant version of electromagnetism through addition of pure gauge degrees of freedom

enhanced gauge invariance

static dyon described by Coulomb fields without string singularities

electric and magnetic charges are surface integrals

applications in the context of thermodynamics of BH dyons

To conclude, there is a new classical central extension at null infinity in asymptotically flat 3D spacetimes, which corresponds to a contraction of the ADS situation. Then, by introducing additional pure gauge degrees of freedom, we have the constructed of a local, explicitly duality invariant formulation of electromagnetism for which the field of a static dyon is described by Coulomb type fields without Dirac-type string singularities. Furthermore, both electric and magnetic charges appear as surface integrals. Finally I have tried to show how this formulation can for instance be useful in the context of the thermodynamics of black hole dyons.

Dyons and duality: Dynamical sources & Dirac strings

duality requires both electric and magnetic sources

action principle for charged point particles: Dirac string

monopole field described by singular potential

Consider once more the pure monopole field

(9.21) $$\vec{B} = \frac{g_M}{4\pi r^3}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

It can be derived from the gauge potential (cf.(7.91)):

(9.22) $$q\vec{A} = -\kappa(\cos\theta + 1)\vec{\nabla}\varphi = \kappa\begin{bmatrix} \frac{y}{r(r-z)} \\ -\frac{x}{r(r-z)} \\ 0 \end{bmatrix} \quad \text{with } \kappa = \frac{qg_M}{4\pi}$$

However, $\vec{A}$ is singular at

replaced by thin solenoid with no magnetic charge

$$q\vec{A} \to q\vec{A}_\varepsilon = \kappa\begin{bmatrix} \frac{y}{R(R-z)} \\ -\frac{x}{R(R-z)} \\ 0 \end{bmatrix} \quad \text{with } R = \sqrt{r^2+\varepsilon^2} \; .$$

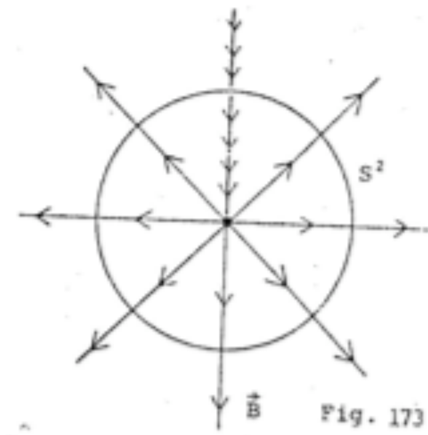


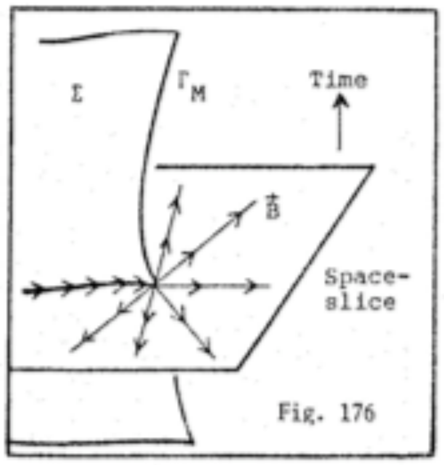
Fig. 173

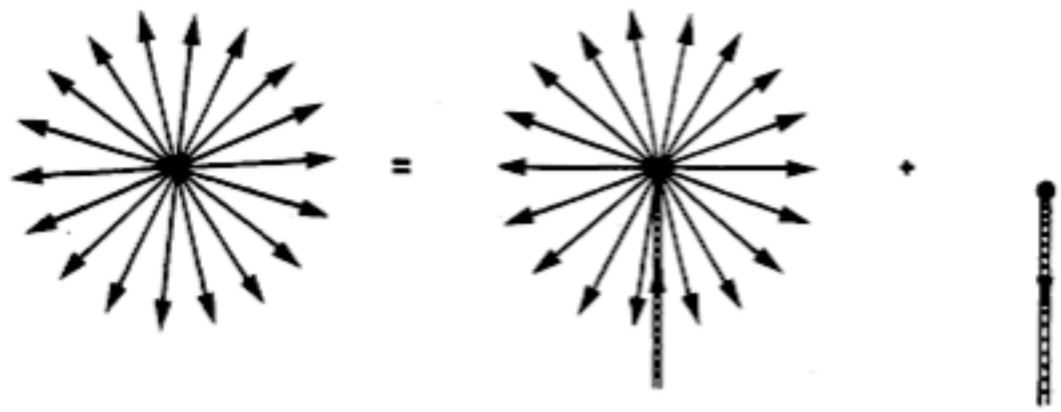
An action principle for point particles carrying electric or magnetic charges has been devised by Dirac. To generate the monopole field, he first replaces the singular potential by a thin solenoid with no net magnetic charge.

Dyons and duality: Dynamical sources & Dirac strings

+ dynamical string

(9.25) $\overset{*}{S}^{\alpha\beta}(x) = -g_M \int_\Sigma \frac{1}{\sqrt{-g}} \delta^4(x-\tilde{x}(P))\, d\tilde{x}^\alpha \wedge d\tilde{x}^\beta$





Felsager

Goddard & Olive

action principle:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - S_{\mu\nu} .$$

consists of three pieces:

(9.32) $$S = S_{PARTICLES} + S_{INTERACTION} + S_{FIELD}$$

where

$$S_{PARTICLES} = -m_e \int_{\Gamma_q} \sqrt{g_{\alpha\beta} \frac{d\tilde{x}_e^\alpha}{d\lambda} \frac{d\tilde{x}_e^\beta}{d\lambda}}\, d\lambda - m_m \int_{\Gamma_m} \sqrt{-g_{\alpha\beta} \frac{d\tilde{x}_m^\alpha}{d\lambda} \frac{d\tilde{x}_m^\beta}{d\lambda}}\, d\lambda$$

$$S_{INTERACTION} = q \int_{\Gamma_q} A_\alpha \frac{d\tilde{x}_e^\alpha}{d\lambda}\, d\lambda = \int_{\Gamma_q} A_\alpha(x) J^\alpha(x) \sqrt{-g}\, d^4x$$

$$S_{FIELD} = -\frac{1}{4} \int F_{\mu\nu} F^{\mu\nu} \sqrt{-g}\, d^4x$$

NB: asymmetric treatment of both types of sources

He then adds a dynamical string into the theory which brings the magnetic charge and enters through a modified field strength in the action. The final action principle then describes the electrodynamics of both particles with either type of charge. The Dirac treatment of both types of sources is asymmetric, so duality invariance is again not manifest.

dynamical dyons with Dirac strings:

$$\begin{aligned} q_n^a &= (g_n, e_n), \\ \mathbf{A}^a &= (\mathbf{A}, \mathbf{Z}), \\ \mathbf{B}^a &= (\mathbf{B}, -\boldsymbol{\pi}), \end{aligned}$$

$$\mathbf{B}^a = \nabla \times \mathbf{A}^a + \boldsymbol{\beta}^a,$$

$$\boldsymbol{\beta}^a = \sum_n q_n^a \int dy_n^0 \wedge d\mathbf{y}_n \, \delta^4(x - y_n)$$

$$\alpha_i^a = \frac{1}{2}\epsilon_{ijk} \sum_n q_n^a \int dy_n^j \wedge dy_n^k$$

$$I = \int d^4x \left[\frac{1}{2}\epsilon_{ab}\mathbf{B}^a \cdot (\dot{\mathbf{A}}^b + \boldsymbol{\alpha}^b) - \mathcal{H}\right] + I_C + I_P$$

$$I_C = \frac{1}{2}\sum_n \epsilon_{ab} q_n^b \int \mathbf{A}^a(z_n) \cdot d\mathbf{z}_n.$$

generalizes Dirac's action

Deser, Henneaux, Teitelboim, Gomberoff Nucl. Phys. B520 (1998) 179

To complete the history of the subject, these considerations have been extended to a duality invariant theory with dynamical point charges carrying both type charges. Instead of the longitudinal fields there are now dynamical strings for both types of charges and the resulting action, with the approriate interaction term between vector potentials and generalizes Dirac's action to the case of dyons.

Dyons and duality: Extended formulation

dynamical point particle dyons need strings for Lorentz force law:

$$j^{a\mu}(x) = \sum_n q_n^a \int_{\Gamma_n} \delta^4(x - z_n) dz_n^\mu \qquad I_P[z_n^\mu] = -\sum_n \int_{\Gamma_n} \sqrt{-dz_n^\mu dz_{n\mu}}$$

total action: $I'_M[A_\mu^a, C^a, y_n^\mu] + I'_I[A_\mu^a, z_n^\mu] + I_P[z_n^\mu]$

$$I'_M[A_\mu^a, C^a, y_n^\mu] = \frac{1}{2}\int d^4x \Big\{\epsilon_{ab}\big[(\vec{B}^a + \vec{\nabla}C^a)(\partial_0\vec{A}^b - \vec{\nabla}A_0^b + \vec{\alpha}^b)$$
$$-\vec{A}^a\partial_0\vec{\beta}^b - \vec{\beta}^a\vec{\alpha}^b - \vec{\beta}^a\nabla^{-2}\vec{\nabla}\times\partial_0\vec{\beta}^b\big] - \vec{B}^a\vec{B}_a\Big\},$$

$$I'_I[A_\mu^a, z_n^\mu] = \int d^4x\, \epsilon_{ab}\Big(A_0^a j^{0b} + \frac{1}{2}\vec{A}^a\cdot\vec{j}^b\Big).$$

$$y_n^\mu(\sigma_n, \tau_n), \quad y_n^\mu(0, \tau_n) = z_n^\mu(\tau_n)$$

$$\vec{\nabla}\cdot\vec{\beta}^a = j^{a0}, \quad \vec{\nabla}\times\vec{\alpha}^a - \partial_0\vec{\beta}^a = \vec{j}^a.$$

$$\vec{\alpha}^a = \sum_n q_n^a \int_{\Sigma_n} \delta^4(x - y_n)\frac{1}{2}d\vec{y}_n \times \wedge d\vec{y}_n, \quad \vec{\beta}^a = \sum_n q_n^a \int_{\Sigma_n} \delta^4(x - y_n) dy_n^0 \wedge d\vec{y}_n,$$

variation with respect to z_n^μ gives Lorentz force law

standard veto: string attached to dyon n cannot cross any other dyon

leads to standard quantization condition: $\bar{e}g - e\bar{g} = 2\pi n\hbar$

In the case of dynamical dyons, appropriate string terms are needed in order to get the Lorentz force law correctly from a variation of the position of dyons and the strings. This is established using the standard Dirac veto and leads to the standard quantization condition for dyons.