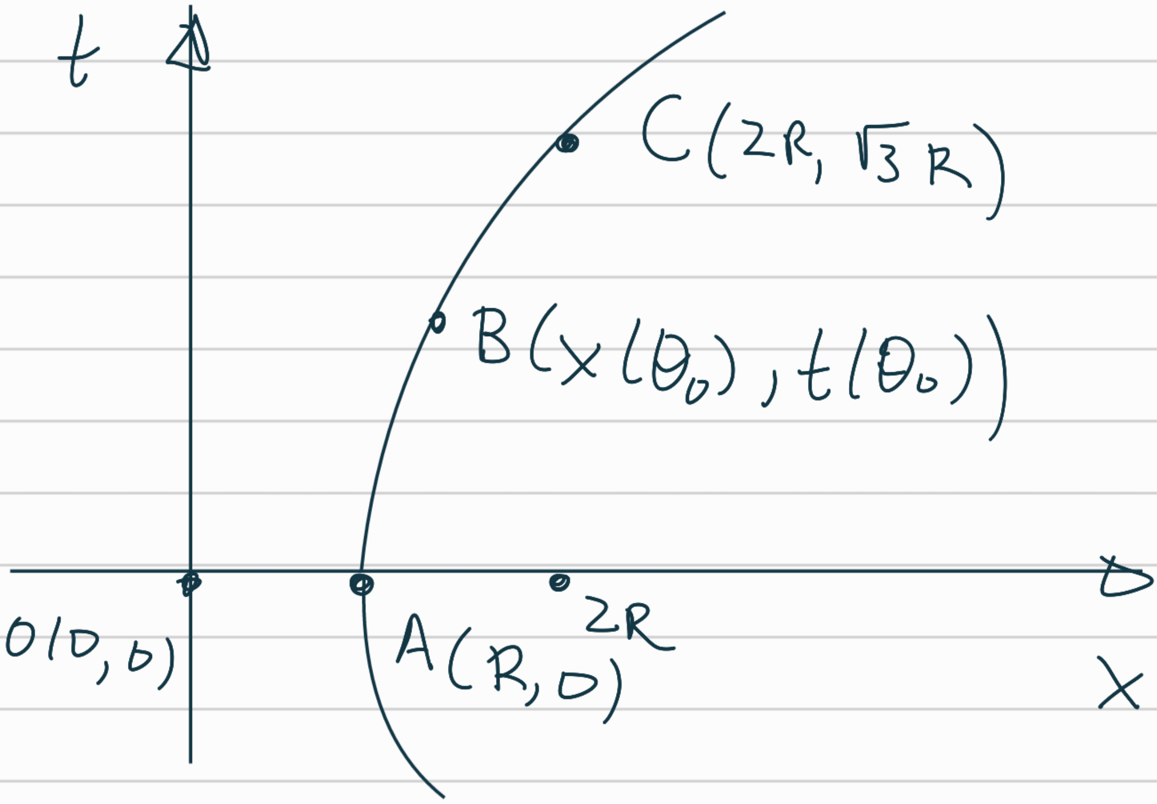
Χωροχρονος Minkowski



2. $\|OA\|^2 = -\Delta t_A^2 + \Delta x_A^2 = -0 + R^2 = R^2$

$\|OB\|^2 = -t(\theta_0)^2 + x^2(\theta_0)$

$= -R^2 \sinh^2\theta + R^2\cosh^2\theta_0 = R^2$

$\|OC\|^2 = -3R^2 + 4R^2 = R^2$

$\to \|OA\| = \|OB\| = \|OC\| = R$

spacelike separated!

3. $\frac{dx}{d\theta} = R\sinh\theta \qquad \frac{dt}{d\theta} = R\cosh\theta$

$$ds^2 = -dt^2 + dx^2 = -R^2\cosh^2\theta\, d\theta^2 + R^2\sinh^2\theta\, d\theta^2$$

$$= -R^2\, d\theta^2$$

$$S_{AB} = \int_0^{\theta_0} R\, d\theta = R\,\theta_0$$

4. $v^\mu = \left(\frac{dt}{d\theta}, \frac{dx}{d\theta}\right)$

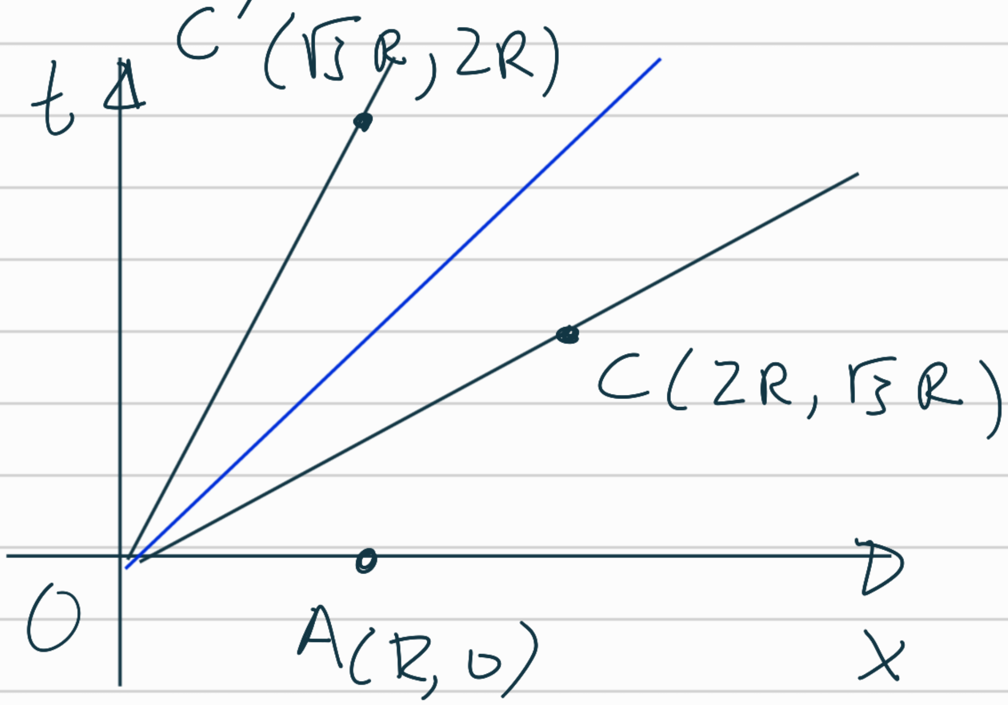
$$= (R\cosh\theta, R\sinh\theta)$$

$$= (x, t) = (v^t, v^x)$$

$$v^\mu_c = (x_c, t_c) = (2R, \sqrt{3}R)$$

$v^t_c = 2R \qquad v^x_c = \sqrt{3}R$

5.



Η κοσμική γραμμή του O' είναι η OC', άρα η ταχύτητά του σε σχέση με τον x–t είναι

$$v = \frac{\sqrt{3}R}{2R} = \frac{\sqrt{3}}{2}$$

$$\gamma = \frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-\frac{3}{4}}} = 2$$

$$t_A' = \gamma(t_A - v x_A) = 2\left(0 - \frac{\sqrt{3}}{2}R\right) = -\sqrt{3}R$$

Το C συμβαίνει για $t_C' = 0$, άρα

$$t'_{AC} = \sqrt{3}\, R$$

Καμπύλος χώρος

1. $d\theta = d\varphi = 0$, $r = 2M\lambda$

$$s = \int_{3M}^{4M} \frac{dr}{\left(1-\frac{2M}{r}\right)^{1/2}} = \int_{3/2}^{2} \frac{2M\, d\lambda}{\left(1-\frac{1}{\lambda}\right)^{1/2}}$$

$$= 2M\, I_1 \quad , \quad I_1 = \int_{3/2}^{2} \frac{d\lambda}{\left(1-\frac{1}{\lambda}\right)^{1/2}}$$

2. $g = \dfrac{1}{1-\frac{2M}{r}} \cdot r^2 \cdot r^2 \sin^2\theta \Rightarrow$

$$\sqrt{g} = \left(1-\frac{2M}{r}\right)^{-1/2} r^2 \sin\theta$$

$$V = \int \sqrt{g}\, dr\, d\theta\, d\varphi =$$

$$= \int_{3m}^{4m} \left(1 - \frac{2m}{r}\right)^{-1/2} \cdot r^2 dr \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi$$

$$= \int_{3/2}^{2} (2m)^3 \left(1 - \frac{1}{\tau}\right) \tau^2 d\tau \int_0^{\pi} \sin\theta \int_0^{2\pi} d\varphi$$

$$= (2m)^3\, I_2 \quad , \quad I_2 = \int_{3/2}^{2} \tau^2 \left(1 - \frac{1}{\tau}\right) d\tau \int_0^{\pi} \sin\theta \int_0^{2\pi} d\varphi$$

3. $\|\partial_r\|^2 = g(\partial_r, \partial_r) = g_{rr} = \dfrac{1}{1 - \frac{2m}{r}}$

$\|\partial_\theta\|^2 = r^2 \qquad \|\partial_\varphi\|^2 = r^2 \sin^2\theta$

4. Η μετρική είναι διαγώνια, οπότε

$$e_r = \frac{\partial r}{\|\partial r\|} = \left(1 - \frac{2M}{r}\right)^{1/2} \partial r$$

$$e_\theta = \frac{1}{r}\partial\theta \qquad e_\phi = \frac{1}{r\sin\phi}\partial_\phi$$

5. $v^\mu = (v^r, v^\theta, v^\phi)$

$$v^r = \frac{dr}{d\lambda} = -\frac{2M}{\lambda^2}$$

$$v_\theta = \lambda \qquad v_\phi = \lambda^2$$

6. $S = \int |g_{\mu\nu} dx^\mu dx^\nu|^{1/2}$

$$dr = -\frac{2M}{\lambda^2} d\lambda \qquad d\theta = \lambda d\lambda \quad d\phi = \lambda^2 d\lambda$$

$$g_{\mu\nu} dx^\mu dx^\nu = g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\phi\phi} d\phi^2$$

$$= \frac{1}{1 - \frac{2M}{r}} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

$$= \frac{1}{1 - \frac{2M}{\frac{2M}{\lambda}}} \left(- \frac{2M}{\lambda^2} d\lambda \right)^2 + \left(\frac{2M}{\lambda} \right)^2 \lambda^2 d\lambda^2$$

$$+ \left(\frac{2M}{\lambda} \right)^2 \sin^2\left(\frac{\lambda^2}{2} \right) \lambda^6 d\lambda^2$$

$$= 4M^2 \left\{ \frac{d\lambda^2}{\lambda^2(1-\lambda)} + d\lambda^2 + \sin^2\left(\frac{\lambda^2}{2}\right) \lambda^4 d\lambda^2 \right\}$$

$$= 4M^2 f^2(\lambda)\, d\lambda^2$$

$$f(\lambda) = \left[1 + \frac{1}{\lambda^2(1-\lambda)} + \sin^2\left(\frac{\lambda^2}{2}\right) \lambda^4 \right]^{1/2}$$

$$\Rightarrow ds = 2M f(\lambda)\, d\lambda$$

$$s = \int_{1/4}^{1/2} 2M f(\lambda)\, d\lambda = 2M I_3$$

$$I_3 = \int_{1/4}^{1/2} f(\lambda)\, d\lambda$$

Τανυστής Ενέργειας - Ορμής

$$u_\nu u^\nu = -1 \Rightarrow (\partial_\mu u_\nu) u^\nu + u_\nu (\partial_\mu u^\nu) = 0$$

$$\Rightarrow u_\nu \partial_\mu u^\nu = 0 \quad (1)$$

1.

$$0 = u_\nu \partial_\mu T^{\mu\nu} =$$

$$= u_\nu \Big\{ [\partial_\mu (\rho + p)] u^\mu u^\nu + (\rho + p)(\partial_\mu u^\mu) u^\nu$$

$$+ (\rho + p) u^\mu \partial_\mu u^\nu + \partial_\mu p \, \eta^{\mu\nu} \Big\}$$

$$= \partial_\mu (\rho + p) u^\mu \underbrace{(u_\nu u^\nu)}_{=-1} + (\rho + p)(\partial_\mu u^\mu) \underbrace{(u_\nu u^\nu)}_{=-1}$$

$$+ (\rho + p) u^\mu \underbrace{(u_\nu \partial_\mu u^\nu)}_{=0} + \partial_\mu p \, u_\nu \eta^{\mu\nu}$$

$$= -(\partial_\mu \rho) u^\mu - \cancel{(\partial_\mu p) u^\mu} - (\rho + p)\, \partial_\mu u^\mu$$

$$+ \cancel{(\partial_\mu p) u^\mu}$$

$$= -\left\{ \partial_\mu (\rho u^\mu) + p\, \partial_\mu u^\mu \right\}$$

2
=

$$P^\mu{}_\sigma P^\sigma{}_\nu = (\delta^\mu{}_\sigma + u^\mu u_\sigma)(\delta^\sigma{}_\nu + u^\sigma u_\nu)$$

$$= \delta^\mu{}_\sigma \delta^\sigma{}_\nu + \delta^\mu{}_\sigma u^\sigma u_\nu + u^\mu u_\sigma \delta^\sigma{}_\nu$$

$$+ u^\mu \underbrace{u_\sigma u^\sigma}_{-1} u_\nu$$

$$= \delta^\mu{}_\nu + u^\mu u_\nu + u^\mu u_\nu - u^\mu u_\nu$$

$$= \delta^\mu{}_\nu + u^\mu u_\nu = P^\mu{}_\nu$$

$\underline{\underline{3}}$

$$\partial_\mu T^{\mu\nu} = \partial_\mu(\rho+p)\, u^\mu u^\nu + (\rho+p)\left((\partial_\mu u^\mu) u^\nu + u^\mu \partial_\mu u^\nu\right) + \partial_\mu p\, \eta^{\mu\nu}$$

$$P^\sigma{}_\nu\, \partial_\mu T^{\mu\nu} = \left(\delta^\sigma{}_\nu + u^\sigma u_\nu\right)\partial_\mu T^{\mu\nu}$$

$$= \partial_\mu T^{\mu\sigma} + u^\sigma u_\nu \partial_\mu T^{\mu\nu}$$

$$= \partial_\mu(\rho+p)\, u^\mu u^\sigma + (\rho+p)\left[(\partial_\mu u^\mu) u^\sigma + u^\mu \partial_\mu u^\sigma\right]$$

$$+ \partial^\sigma p$$

$$+ \partial_\mu(\rho+p)\, u^\mu \overbrace{u^\nu\, u^\sigma}^{-1} u_\nu$$

$$+ (\rho+p)\left[(\partial_\alpha u^\alpha)\, \underbrace{u^\nu + \cancel{u^\alpha \partial_\alpha} \overbrace{u^\nu]\, u^\sigma}^{0} u_\nu}_{-1}$$

$$+ \partial^\nu p\;\; u^\sigma u_\nu$$

$$= \cancel{\partial_\mu(\rho+p)\, u^\mu u^\sigma} + \cancel{(\rho+p)(\partial_\mu u^\mu)\, u^\sigma}$$

$$+(\rho+p)\, u^\mu \partial_\mu u^\sigma \quad + \partial^\sigma p$$

$$-\partial_\mu \cancel{(\rho+p)}\, u^\mu u^\sigma \quad -(\rho+p) \cancel{(\partial_\mu u^\mu)}\, u^\sigma$$

$$+\partial^\nu p\, u^\sigma u_\nu$$

$$=(\rho+p)\, u^\mu \partial_\mu u^\sigma + \partial^\sigma p + u^\sigma u^\mu \partial_\mu p$$

Καμπυλότητα

1. Δείτε διαφάνεια διάλεξης 6

3. Δείτε διαφάνεια διάλεξης 6, σελ. 101

2. Από τη σχέση (7) έχουμε:

$$C^{\lambda}{}_{\mu\lambda\nu} = R_{\mu\nu}$$

$$-\frac{1}{2}\Big(\underbrace{g^{\lambda}{}_{\lambda}}_{=4} R_{\nu\mu} - g^{\lambda}{}_{\nu} R_{\lambda\mu} - g_{\mu}{}^{\lambda} R_{\nu\lambda} + g_{\mu\nu} R\Big)$$

$$+\frac{1}{6}\Big(\underbrace{g^{\lambda}{}_{\lambda}}_{=4} g_{\nu\mu} - g^{\lambda}{}_{\nu} g_{\lambda\mu}\Big) R$$

$$= R_{\mu\nu}$$

$$-\frac{1}{2}\Big(4 R_{\nu\mu} - R_{\nu\mu} - R_{\nu\mu} + g_{\mu\nu} R\Big)$$

$$+\frac{1}{6}\Big(4 g_{\nu\mu} - g_{\nu\mu}\Big) R$$

$$= \cancel{R_{\mu\nu}}$$

$$-\frac{1}{2}\Big(2 \cancel{R_{\mu\nu}} + \cancel{g_{\mu\nu}} R\Big)$$

$$+\frac{1}{6}\, 3\, \cancel{g_{\mu\nu}} R = 0$$