Schwarzschild Black Holes

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{1 - \frac{2M}{r}}\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

- spherically symmetric

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- spherically symmetric

$$R = \partial_\varphi \qquad \rightarrow \text{easy to see: } \partial_\varphi g_{\mu\nu} = 0$$

$$S = \cos\varphi\, \partial_\theta - \cot\theta \sin\varphi\, \partial_\varphi$$

$$T = -\sin\varphi\, \partial_\theta - \cot\theta \cos\varphi\, \partial_\varphi$$

are Killing vector fields (KVF)

→ generate isometries

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  $[R, S] = T \qquad [S, T] = R \qquad [T, R] = S$

  $SO(3)$ algebra

- ∂_t is a KVF $\Leftarrow \partial_t g_{\mu\nu} = 0$

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$\boxed{\text{stationary!}}$

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The Schwarzschild metric is the unique spherically symmetric solution of the vacuum Einstein field equations

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- the metric inside a spherical cavity is Minkowski (flat!)

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time direction!

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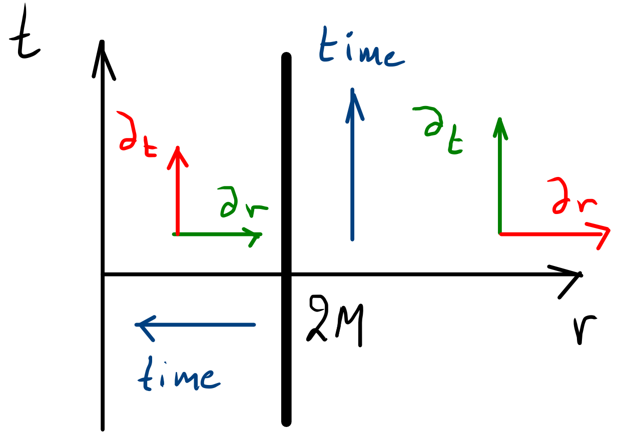

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- causal geodesics starting at $r_0 > 2M$ need infinite t to approach $r = 2M$ - seem to never cross it

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$$\Rightarrow t = \pm 2M\left[\frac{r}{2M} + \ln\left(\frac{r}{2M}-1\right)\right] + t_0$$

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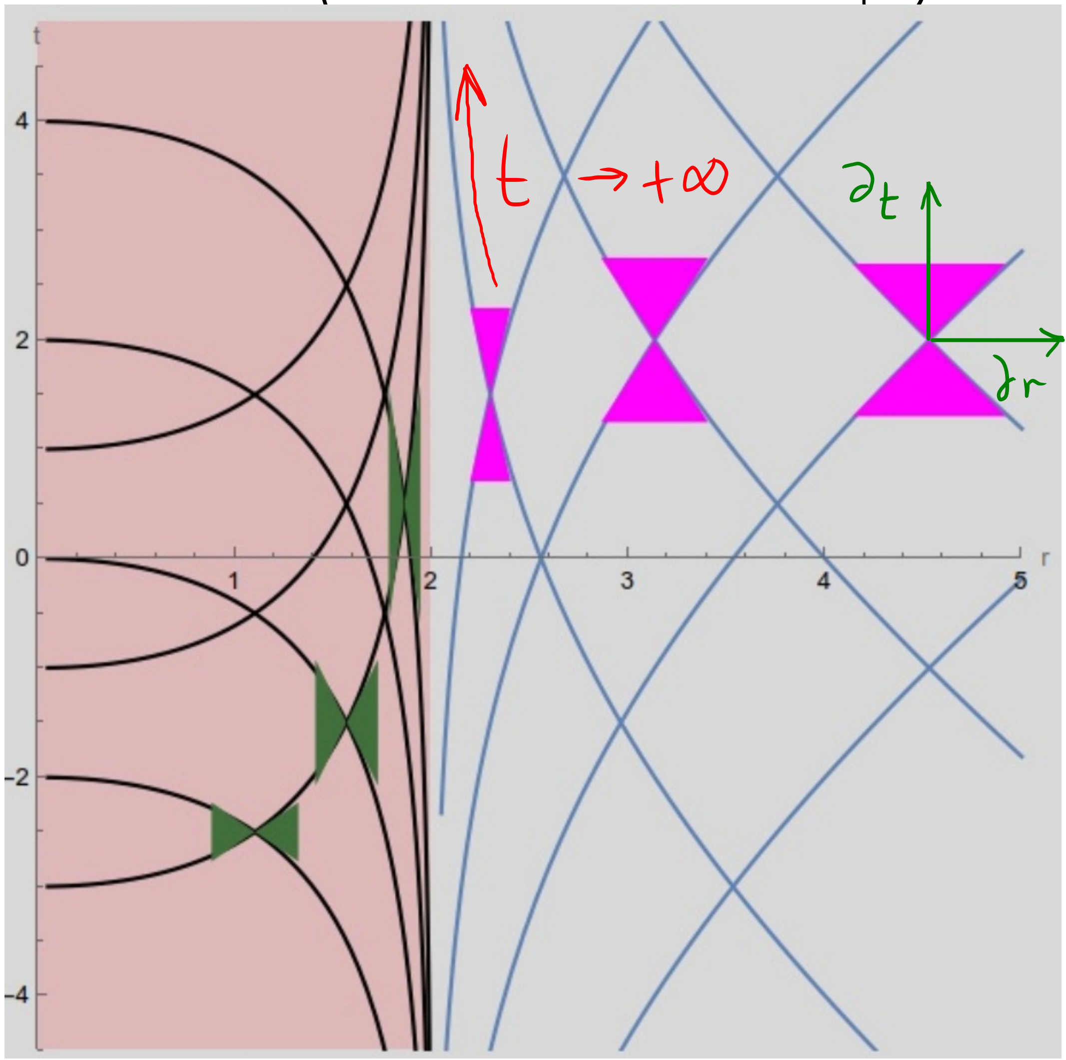

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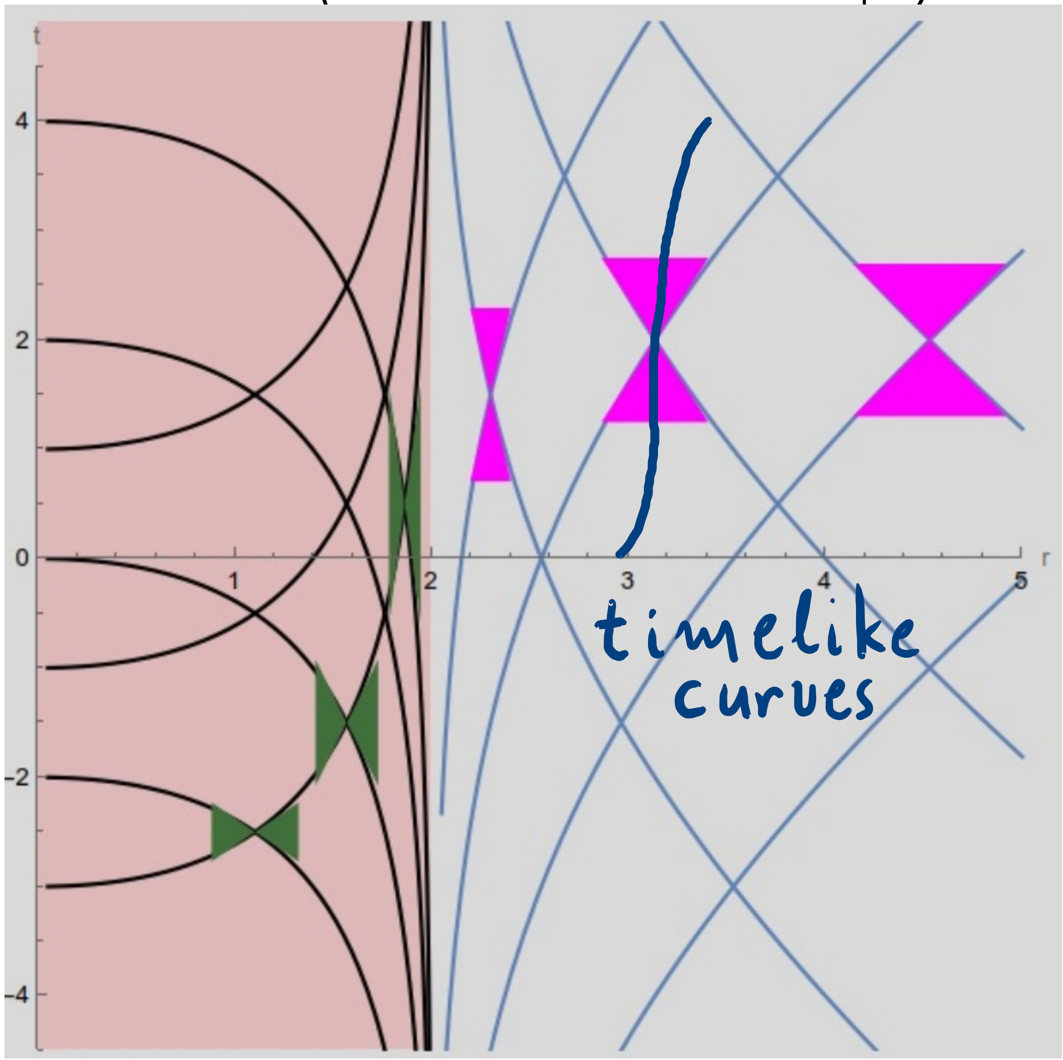

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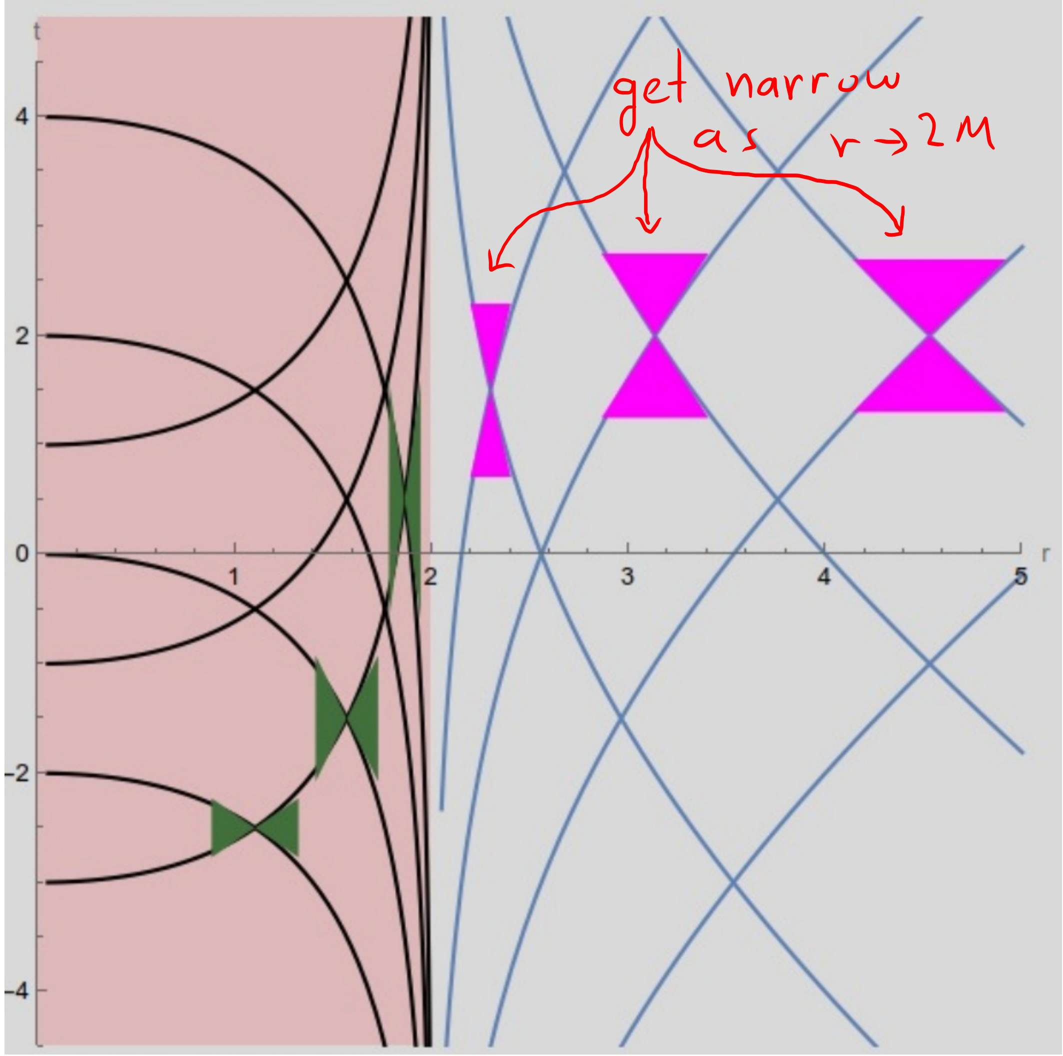

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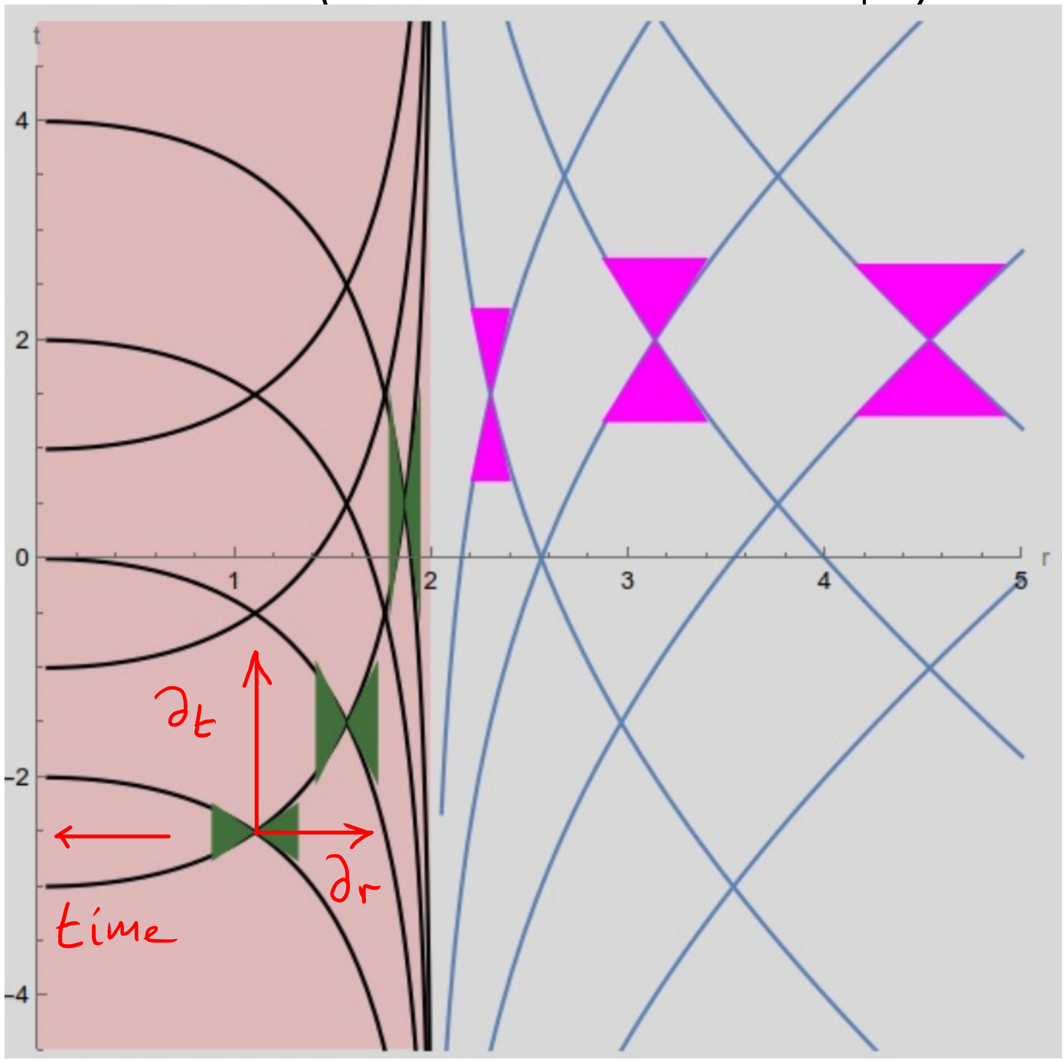

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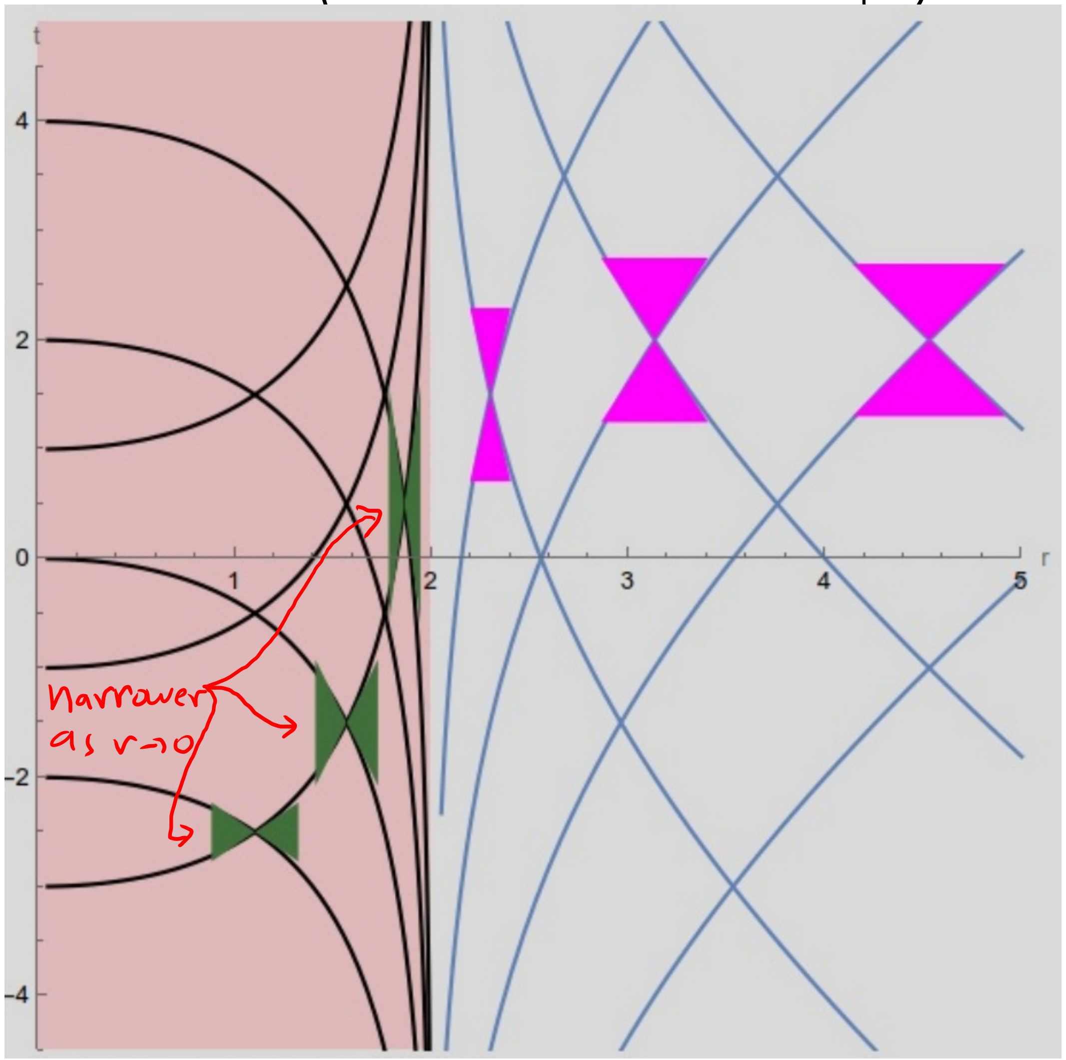

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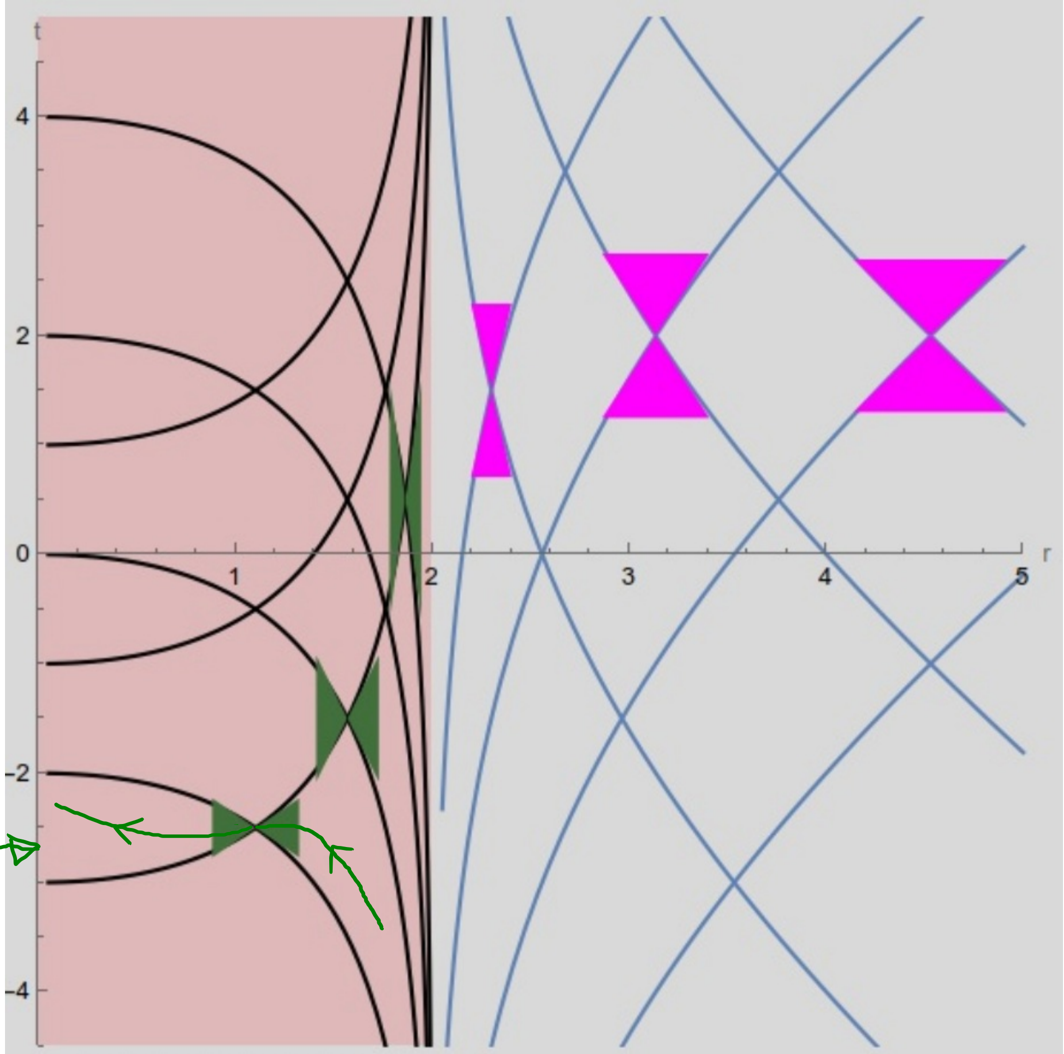

timelike curves must fall on $r=0$

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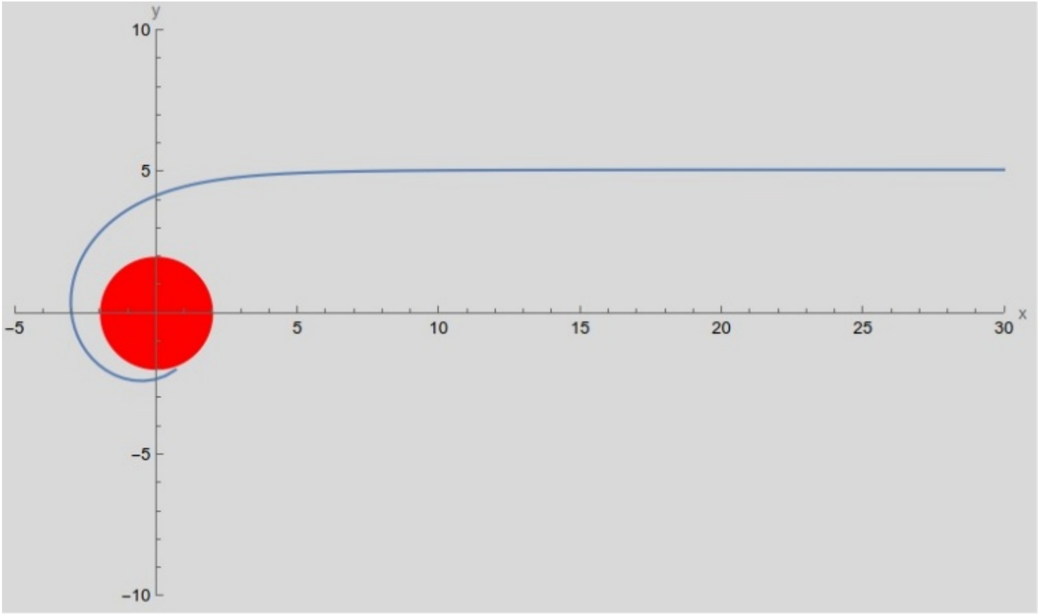
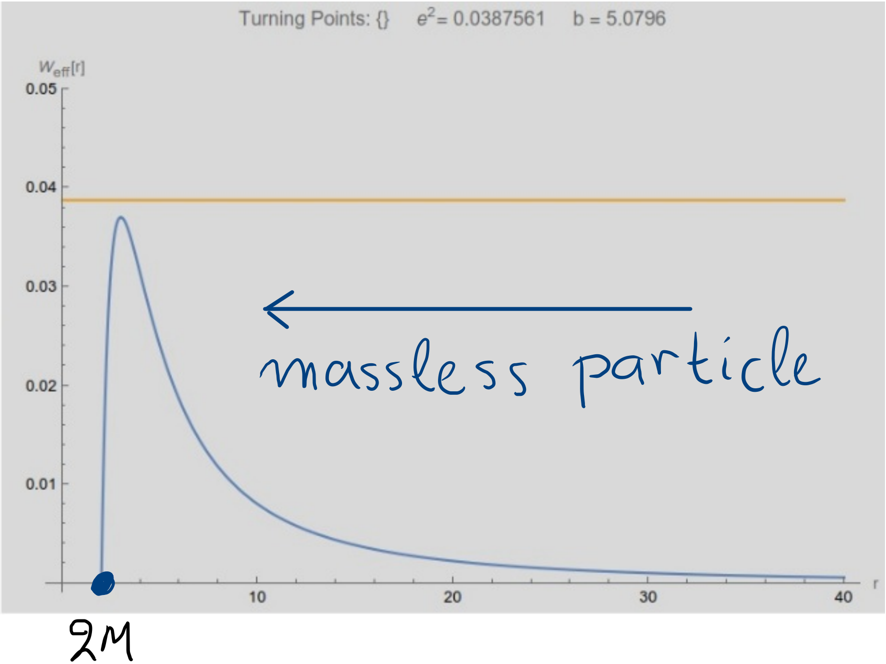
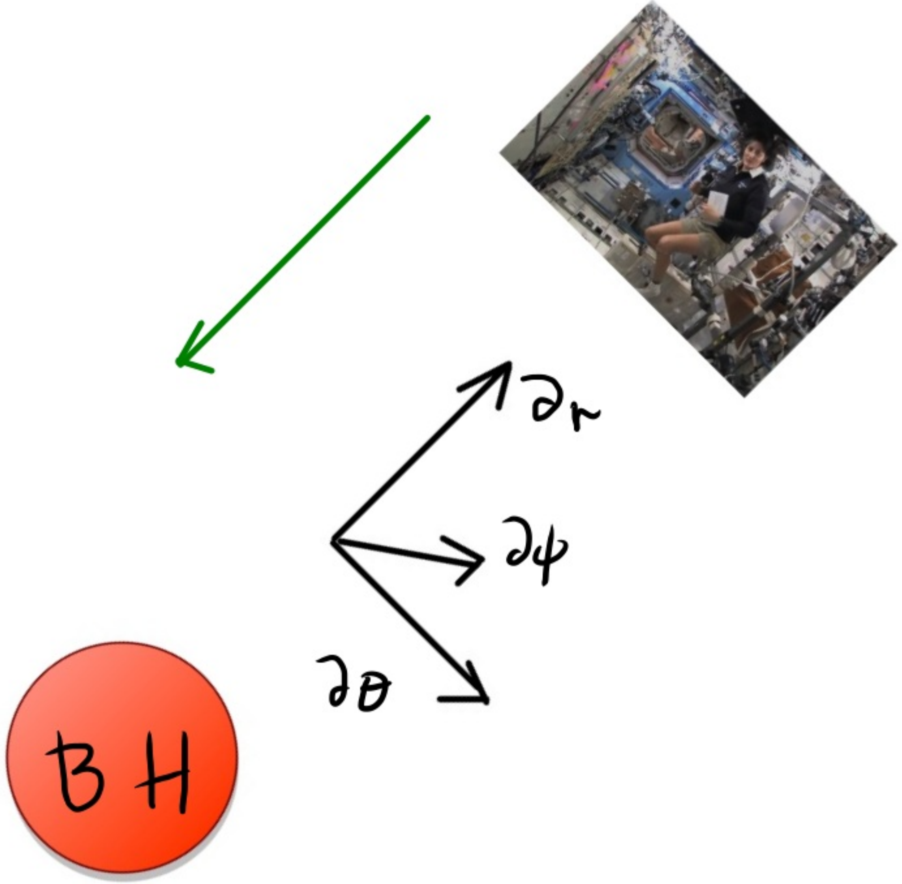




- we have seen particles (massless + massive) to get into the $r < 2M$ region in finite affine parameter
  $\Rightarrow$ must be a problem of the coordinate's choice!

- How does an observer falling radially into the BH "feel"? Does she feel anything special while crossing the horizon?



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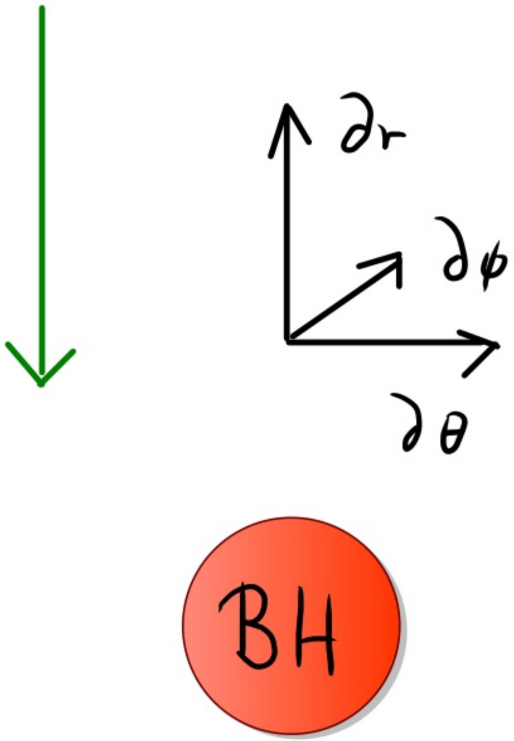

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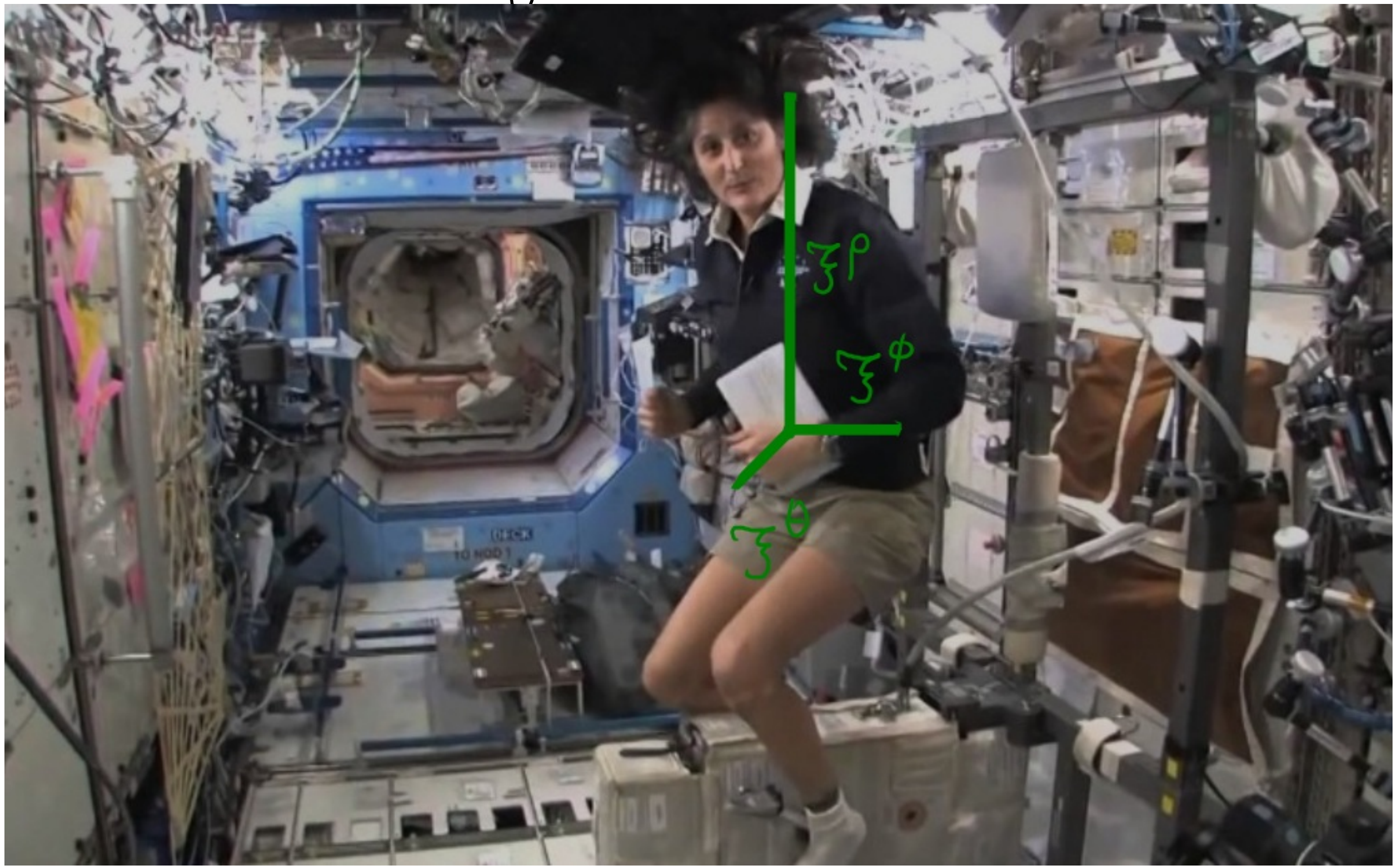

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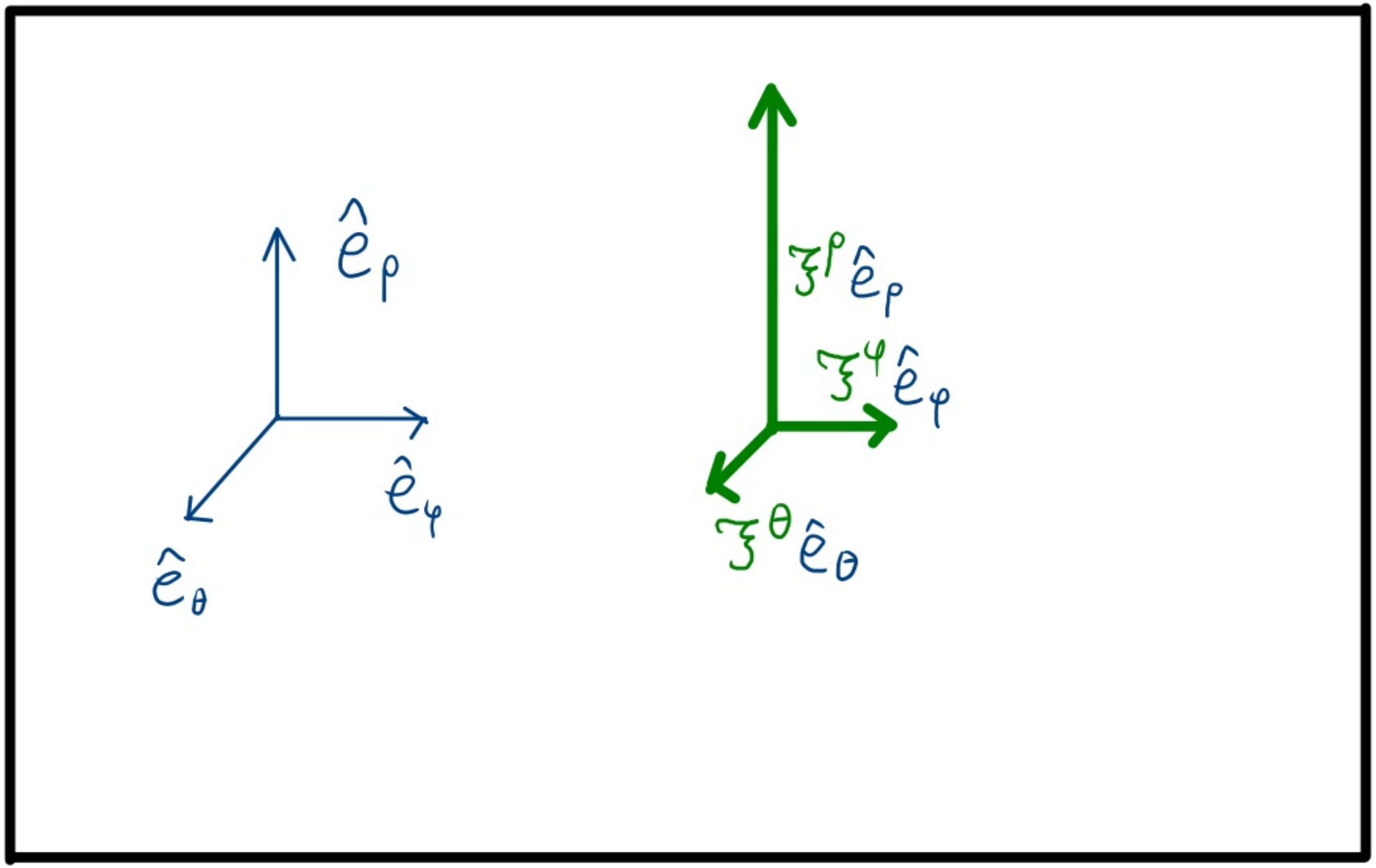

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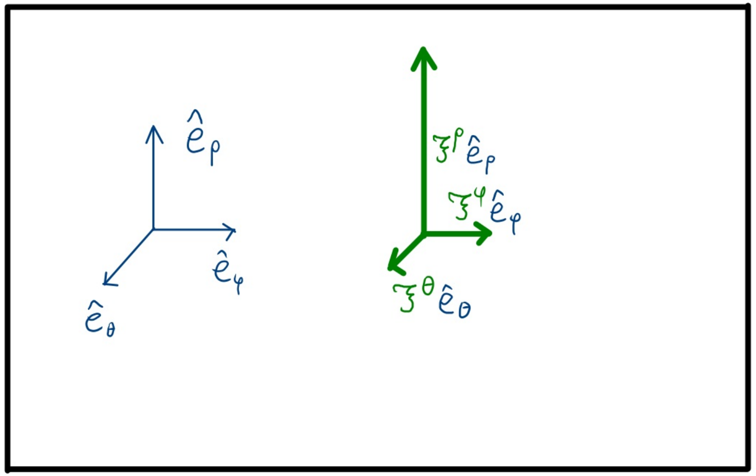
- Get into the ship:

  - orthonormal basis

$$\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$$

$$\hat{e}_\mu \cdot \hat{e}_\nu = \eta_{\mu\nu}$$



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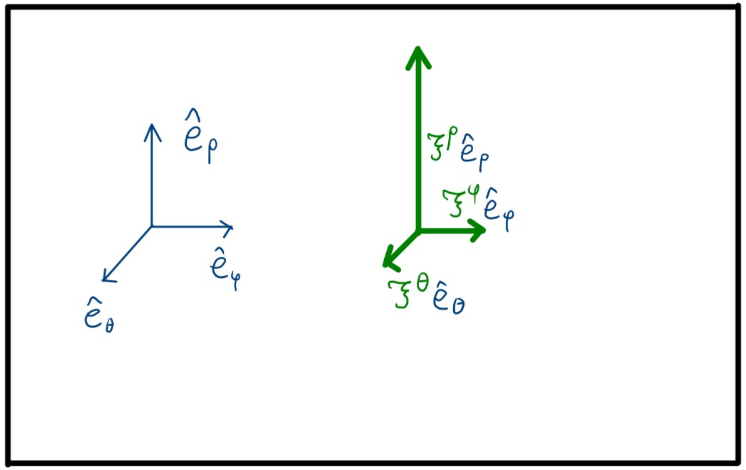

  - orthonormal basis

$$\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$$

$$\hat{e}_\tau \cdot \hat{e}_\tau = \eta_{00} = -1$$

$$\hat{e}_\rho \cdot \hat{e}_\rho = \hat{e}_\theta \cdot \hat{e}_\theta = \hat{e}_\phi \cdot \hat{e}_\phi = \eta_{ii} = +1$$

$$\hat{e}_i \cdot \hat{e}_j = \hat{e}_\tau \cdot \hat{e}_j = 0$$

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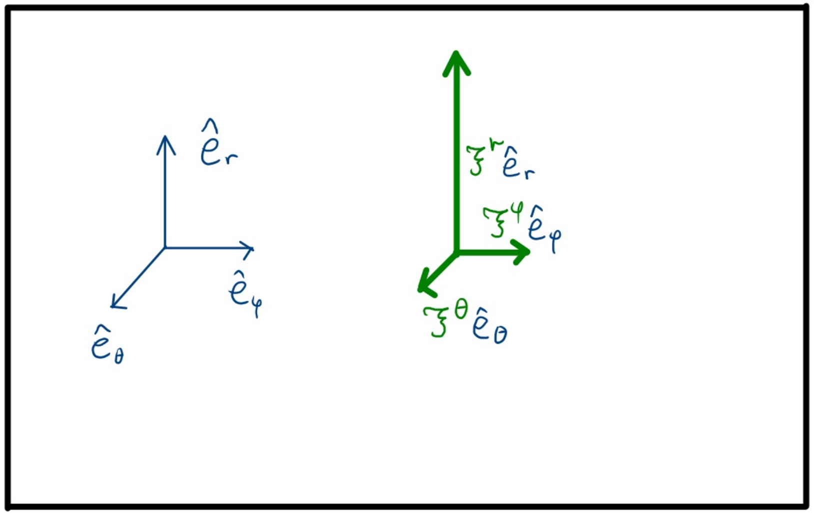
- Stationary Observer (first)



- orthonormal basis $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi\}$
- 4-velocity

$$u^\mu = [u^t, 0, 0, 0]$$

$$u^\mu u_\mu = g_{\mu\nu} u^\mu u^\nu = g_{00} u^t u^t = -\left(1 - \frac{2M}{r}\right)(u^t)^2$$

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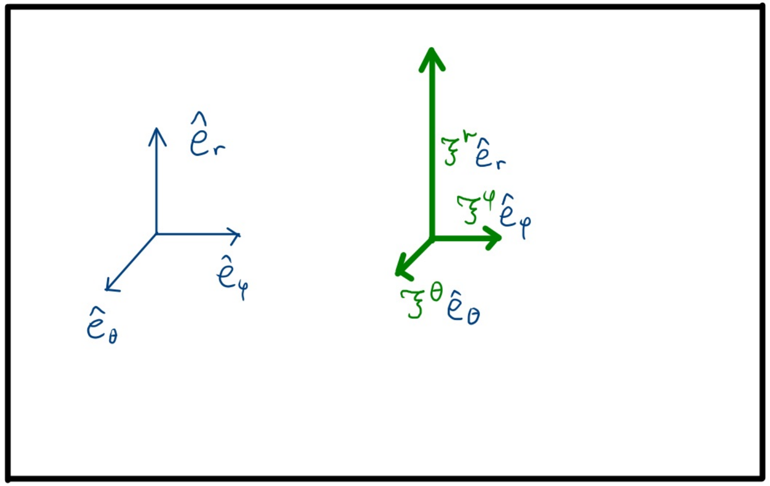

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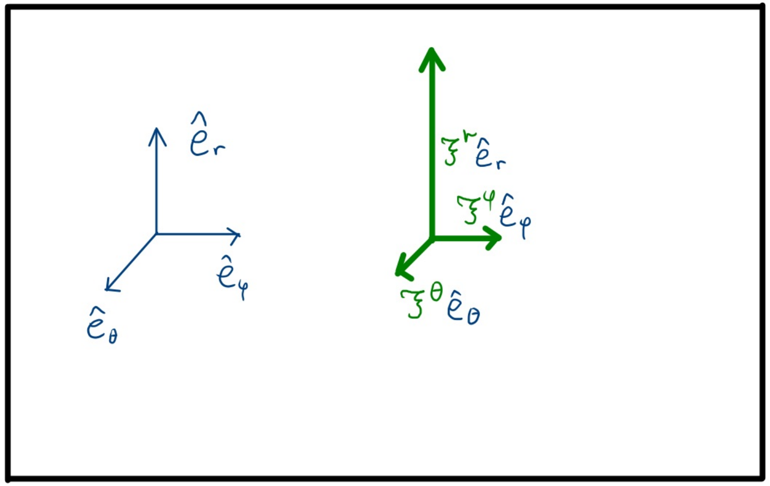

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$$u^\mu = [u^t, 0, 0, 0] = \left[\left(1-\frac{2M}{r}\right)^{-1/2}, 0, 0, 0\right] = \left(1-\frac{2M}{r}\right)^{-1/2} \partial_t$$

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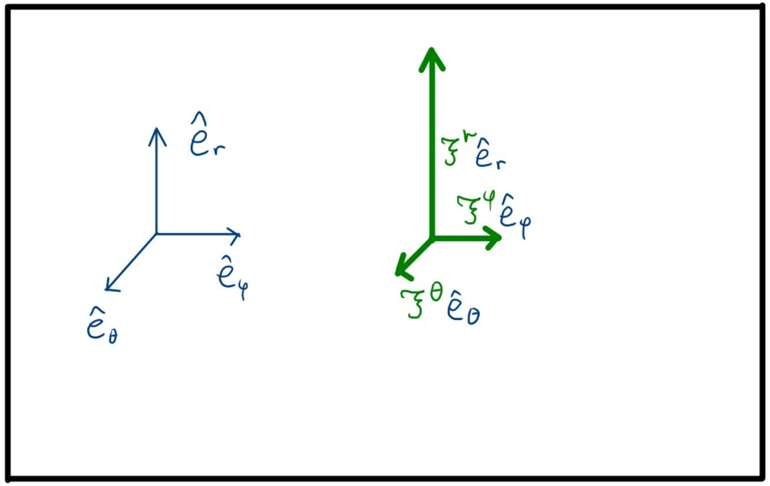

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$$\hat{e}_t = u = \left(1-\frac{2M}{r}\right)^{-1/2} \partial_t = \frac{1}{\sqrt{|g_{tt}|}} \partial_t \qquad r > 2M$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\, dt^2 + g_{rr}\, dr^2 + g_{\theta\theta}\, d\theta^2 + g_{\phi\phi}\, d\phi^2$$

will provide the orthonormal frame of the stationary observer

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\, dt^2 + g_{rr}\, dr^2 + g_{\theta\theta}\, d\theta^2 + g_{\phi\phi}\, d\phi^2$$

- coordinate basis

$$e_t = \partial_t \quad \Rightarrow \quad e_t{}^{\mu} = [1, 0, 0, 0]$$

$$e_r = \partial_r \quad \Rightarrow \quad e_r{}^{\mu} = [0, 1, 0, 0]$$

$$e_\theta = \partial_\theta \quad \Rightarrow \quad e_\theta{}^{\mu} = [0, 0, 1, 0]$$

$$e_\phi = \partial_\phi \quad \Rightarrow \quad e_\phi{}^{\mu} = [0, 0, 0, 1]$$

– Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$

- orthonormal basis: $(r > 2M)$

$$\hat{e}_\mu = \alpha\, e_\mu$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$

- orthonormal basis: $(r > 2M)$

$$\hat{e}_\mu = \alpha\, e_\mu \Rightarrow \hat{e}_\mu \cdot \hat{e}_\mu = \alpha^2\, e_\mu \cdot e_\mu$$

no summation if both indices are downstairs or upstairs

_ Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $(r > 2M)$

$$\hat{e}_\mu = \alpha\, e_\mu \Rightarrow \eta_{\mu\mu} = \alpha^2\, e_\mu \cdot e_\mu = \alpha^2\, g_{\mu\mu}$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$

- orthonormal basis: $(r > 2M)$

$$\hat{e}_\mu = \alpha\, e_\mu \Rightarrow \eta_{\mu\mu} = \alpha^2\, e_\mu \cdot e_\mu = \alpha^2\, g_{\mu\mu} \Rightarrow \alpha = |g_{\mu\mu}|^{-1/2}$$

same sign

– Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $(r > 2M)$

$$\hat{e}_\mu = \underbrace{|g_{\mu\mu}|^{-1/2}}_{\text{no-summation!}} e_\mu$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$

- orthonormal basis: $(r > 2M)$

$$\hat{e}_\mu = |g_{\mu\mu}|^{-1/2} e_\mu$$

$$\hat{e}_t = \left[|g_{tt}|^{-1/2}, 0, 0, 0 \right] = \left[\left|1 - \frac{2M}{r}\right|^{-1/2}, 0, 0, 0 \right]$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$

- orthonormal basis: $(r > 2M)$

$$\hat{e}_\mu = |g_{\mu\mu}|^{-1/2} e_\mu$$

$$\hat{e}_t = [\,|g_{tt}|^{-1/2}, 0, 0, 0\,] = [\,|1 - \tfrac{2M}{r}|^{-1/2}, 0, 0, 0\,]$$

$$\hat{e}_r = [\,0, |g_{rr}|^{-1/2}, 0, 0\,] = [\,0, |1 - \tfrac{2M}{r}|^{1/2}, 0, 0\,]$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis:

$$\hat{e}_\mu = |g_{\mu\mu}|^{-1/2} e_\mu$$

$$\hat{e}_t = [\,|g_{tt}|^{-1/2}, 0, 0, 0\,] = [\,|1 - \tfrac{2M}{r}|^{-1/2}, 0, 0, 0\,]$$

$$\hat{e}_r = [\,0, |g_{rr}|^{-1/2}, 0, 0\,] = [\,0, |1 - \tfrac{2M}{r}|^{1/2}, 0, 0\,]$$

$$\hat{e}_\theta = [\,0, 0, |g_{\theta\theta}|^{-1/2}, 0\,] = [\,0, 0, r^{-1}, 0\,]$$

_ Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$

- orthonormal basis:

$$\hat{e}_\mu = |g_{\mu\mu}|^{-1/2} e_\mu$$

$$\hat{e}_t = [\,|g_{tt}|^{-1/2}, 0, 0, 0\,] = [\,|1 - \tfrac{2M}{r}|^{-1/2}, 0, 0, 0\,]$$

$$\hat{e}_r = [\,0, |g_{rr}|^{-1/2}, 0, 0\,] = [\,0, |1 - \tfrac{2M}{r}|^{1/2}, 0, 0\,]$$

$$\hat{e}_\theta = [\,0, 0, |g_{\theta\theta}|^{-1/2}, 0\,] = [\,0, 0, r^{-1}, 0\,]$$

$$\hat{e}_\phi = [\,0, 0, 0, |g_{\phi\phi}|^{-1/2}\,] = [\,0, 0, 0, (r\sin\theta)^{-1}\,]$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $\hat{e}_\mu = |g_{\mu\mu}|^{-1/2}\, e_\mu$

vector: $v = v^\mu e_\mu = v^{\hat{\mu}}\, \hat{e}_\mu$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\, dt^2 + g_{rr}\, dr^2 + g_{\theta\theta}\, d\theta^2 + g_{\phi\phi}\, d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $\hat{e}_\mu = |g_{\mu\mu}|^{-1/2}\, e_\mu$

vector: $v = v^\mu e_\mu = v^{\hat{\mu}}\, \hat{e}_\mu \Rightarrow$

$$|g_{\mu\mu}|^{1/2} v^\mu\, \hat{e}_\mu = v^{\hat{\mu}}\, \hat{e}_\mu$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $\hat{e}_\mu = |g_{\mu\mu}|^{-1/2}\, e_\mu$

vector: $v = v^\mu e_\mu = v^{\hat{\mu}}\,\hat{e}_\mu \Rightarrow$

$$|g_{\mu\mu}|^{1/2} v^\mu\,\hat{e}_\mu = v^{\hat{\mu}}\,\hat{e}_\mu \Rightarrow v^{\hat{\mu}} = |g_{\mu\mu}|^{1/2} v^\mu$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $\hat{e}_\mu = |g_{\mu\mu}|^{-1/2}\, e_\mu$

$$v^{\hat{\mu}} = \underbrace{|g_{\mu\mu}|^{1/2}}_{\text{no summation!}}\, v^\mu$$

$$v^{\hat{t}} = |g_{tt}|^{1/2}\, v^t$$

$$v^{\hat{r}} = |g_{rr}|^{1/2}\, v^r$$

etc ...

Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $\hat{e}_\mu = |g_{\mu\mu}|^{-1/2}\, e_\mu$

$$v^{\hat{\mu}} = |g_{\mu\mu}|^{1/2}\, v^\mu$$

dual basis: $\hat{e}^\mu = |g_{\mu\mu}|^{1/2} e^\mu = |g_{\mu\mu}|\, dx^\mu$

indeed $\hat{e}^\mu(\hat{e}_\nu) = \hat{e}^\mu(|g_{\nu\nu}|^{-1/2} e_\nu) = |g_{\mu\mu}|^{1/2}|g_{\nu\nu}|^{-1/2} e^\mu(e_\nu)$

$$= |g_{\mu\mu}|^{1/2}|g_{\nu\nu}|^{-1/2}\delta^{\mu}{}_{\nu} = |g_{\mu\mu}|^{1/2}|g_{\nu\nu}|^{-1/2}\delta^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $\hat{e}_\mu = |g_{\mu\mu}|^{-1/2}\, e_\mu \qquad \hat{e}^\mu = |g_{\mu\mu}|^{1/2}\, e^\mu$

$$v^{\hat{\mu}} = |g_{\mu\mu}|^{1/2}\, v^\mu$$

one forms: $\omega = \omega_\mu e^\mu = \omega_{\hat{\mu}}\, \hat{e}^\mu$

_ Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $\hat{e}_\mu = |g_{\mu\mu}|^{-1/2}\, e_\mu$ $\quad \hat{e}^\mu = |g_{\mu\mu}|^{1/2}\, e^\mu$

$$v^{\hat{\mu}} = |g_{\mu\mu}|^{1/2}\, v^\mu$$

one forms: $\omega = \omega_\mu e^\mu = \omega_{\hat{\mu}} \hat{e}^\mu =$

$$|g_{\mu\mu}|^{-1/2} \omega_\mu \hat{e}^\mu = \omega_{\hat{\mu}} \hat{e}^\mu$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $\hat{e}_\mu = |g_{\mu\mu}|^{-1/2}\, e_\mu \qquad \hat{e}^\mu = |g_{\mu\mu}|^{1/2}\, e^\mu$

$$v^{\hat{\mu}} = |g_{\mu\mu}|^{1/2}\, v^\mu$$

one forms: $\omega = \omega_\mu e^\mu = \omega_{\hat{\mu}} \hat{e}^\mu =$

$$|g_{\mu\mu}|^{-1/2} \omega_\mu \hat{e}^\mu = \omega_{\hat{\mu}} \hat{e}^\mu \quad \Rightarrow \quad \omega_{\hat{\mu}} = |g_{\mu\mu}|^{-1/2}\, \omega_\mu$$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $\hat{e}_\mu = |g_{\mu\mu}|^{-1/2} e_\mu \qquad \hat{e}^\mu = |g_{\mu\mu}|^{1/2} e^\mu$

$$v^{\hat{\mu}} = |g_{\mu\mu}|^{1/2} v^\mu \qquad \omega_{\hat{\mu}} = |g_{\mu\mu}|^{-1/2} \omega_\mu$$

Tensors: $R^{\hat{\mu}}{}_{\hat{\nu}\hat{\rho}\hat{\lambda}} = |g_{\mu\mu}|^{1/2} |g_{\nu\nu}|^{-1/2} |g_{\rho\rho}|^{-1/2} |g_{\lambda\lambda}|^{-1/2} R^\mu{}_{\nu\rho\lambda}$

- Orthonormal basis for diagonal metric

$$ds^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 + g_{\theta\theta}\,d\theta^2 + g_{\phi\phi}\,d\phi^2$$

- coordinate basis $\{e_t, e_r, e_\theta, e_\phi\}$
- orthonormal basis: $\hat{e}_\mu = |g_{\mu\mu}|^{-1/2} e_\mu \qquad \hat{e}^\mu = |g_{\mu\mu}|^{1/2} e^\mu$

$$v^{\hat{\mu}} = |g_{\mu\mu}|^{1/2} v^\mu \qquad \omega_{\hat{\mu}} = |g_{\mu\mu}|^{-1/2} \omega_\mu$$

Tensors: $R^{\hat{\mu}}{}_{\hat{\nu}\hat{\rho}\hat{\lambda}} = |g_{\mu\mu}|^{1/2} |g_{\nu\nu}|^{-1/2} |g_{\rho\rho}|^{-1/2} |g_{\lambda\lambda}|^{-1/2} R^\mu{}_{\nu\rho\lambda}$

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = \eta_{\hat{\mu}\hat{\sigma}} R^{\hat{\sigma}}{}_{\hat{\nu}\hat{\rho}\hat{\lambda}}$$

$$R_{0101} = R_{trtr} = -\frac{2M}{r^3}$$

$$R_{0202} = R_{t\theta t\theta} = \frac{M}{r}\left(1 - \frac{2M}{r}\right)$$

$$R_{1212} = R_{r\theta r\theta} = -\frac{M}{r}\left(1 - \frac{2M}{r}\right)^{-1}$$

$$R_{0303} = R_{t\phi t\phi} = \frac{M}{r}\left(1 - \frac{2M}{r}\right)\sin^2\theta$$

$$R_{1313} = R_{r\phi r\phi} = -\frac{M}{r}\left(1 - \frac{2M}{r}\right)^{-1}\sin^2\theta$$

$$R_{2323} = R_{\theta\phi\theta\phi} = 2M\, r \sin^2\theta$$

$$R_{0101} = R_{trtr} = -\frac{2M}{r^3}$$

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$$R_{0303} = R_{t\phi t\phi} = \frac{M}{r}\left(1 - \frac{2M}{r}\right) \sin^2\theta$$

$$R_{1313} = R_{r\phi r\phi} = -\frac{M}{r}\left(1 - \frac{2M}{r}\right)^{-1} \sin^2\theta$$

$$R_{2323} = R_{\theta\phi\theta\phi} = 2M\, r \sin^2\theta$$

Notice singular behavior as $r \to 2M$!

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = | g_{\mu\mu} \, g_{\nu\nu} \, g_{\rho\rho} \, g_{\lambda\lambda} |^{-1/2} \, R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = | g_{tt} |^{-1} \, | g_{rr} |^{-1} \, R_{rtrt} = \left| 1 - \frac{2M}{r} \right|^{-1} \left| 1 - \frac{2M}{r} \right| \left(- \frac{2M}{r^3} \right)$$

$$= - \frac{2M}{r^3}$$

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = | g_{\mu\mu} \, g_{\nu\nu} \, g_{\rho\rho} \, g_{\lambda\lambda} |^{-1/2} \, R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -\frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = |g_{tt}|^{-1} |g_{\theta\theta}|^{-1} R_{\theta t \theta t} = \left|1 - \frac{2M}{r}\right|^{-1} (r^2)^{-1} \, \frac{M(r-2M)}{r^2}$$

$$= \left(1 - \frac{2M}{r}\right)^{-1} \frac{1}{r^2} \, \frac{Mr}{r^2} \left(1 - \frac{2M}{r}\right)$$

Notice $\left|1 - \frac{2M}{r}\right|^{-1} = \left(1 - \frac{2M}{r}\right)^{-1}$

for $r > 2M$

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -\frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = |g_{tt}|^{-1} |g_{\theta\theta}|^{-1} R_{\theta t\theta t} = \left|1-\frac{2M}{r}\right|^{-1} (r^2)^{-1} \frac{M(r-2M)}{r^2}$$

$$= \left(1-\frac{2M}{r}\right)^{-1} \frac{1}{r^2} \frac{Mr}{r^2} \left(1-\frac{2M}{r}\right)$$

$$= +\frac{M}{r^3}$$

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2}\, R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -\frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = +\frac{M}{r^3}$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = |g_{\theta\theta}|^{-1}\, |g_{\phi\phi}|^{-1}\, R_{\theta\phi\theta\phi} = r^{-2}\; r^{-2} \sin^{-2}\theta\;\; 2Mr \sin^2\theta$$

$$= +\frac{2M}{r^3}$$

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -\frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = +\frac{M}{r^3}$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = +\frac{2M}{r^3}$$

$$R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}} = |g_{tt}|^{-1} |g_{\phi\phi}|^{-1} R_{\phi t \phi t} = \left(1 - \frac{2M}{r}\right)^{-1} r^{-2} \sin^{-2}\theta \, \frac{M(r-2M)}{r^2} \sin^2\theta$$

$$= +\frac{M}{r^3}$$

$$R_{\hat\mu\hat\nu\hat\rho\hat\lambda} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat r\hat t\hat r\hat t} = -\frac{2M}{r^3}$$

$$R_{\hat\theta\hat t\hat\theta\hat t} = +\frac{M}{r^3}$$

$$R_{\hat\theta\hat\varphi\hat\theta\hat\varphi} = +\frac{2M}{r^3}$$

$$R_{\hat\varphi\hat t\hat\varphi\hat t} = +\frac{M}{r^3}$$

$$R_{\hat r\hat\theta\hat r\hat\theta} = |g_{rr}|^{-1}|g_{\theta\theta}|^{-1} R_{r\theta r\theta} = \left(1-\frac{2M}{r}\right) r^{-2} \frac{M}{2M-r} = -\frac{M}{r^3}$$

$$R_{\hat\mu\hat\nu\hat\rho\hat\lambda} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat r \hat t \hat r \hat t} = -\frac{2M}{r^3}$$

$$R_{\hat\theta\hat t\hat\theta\hat t} = +\frac{M}{r^3}$$

$$R_{\hat\theta\hat\phi\hat\theta\hat\phi} = +\frac{2M}{r^3}$$

$$R_{\hat\phi\hat t\hat\phi\hat t} = +\frac{M}{r^3}$$

$$R_{\hat r\hat\theta\hat r\hat\theta} = -\frac{M}{r^3}$$

$$R_{\hat r\hat\phi\hat r\hat\phi} = |g_{rr}|^{-1} |g_{\phi\phi}|^{-1} R_{r\phi r\phi} = \left(1-\frac{2M}{r}\right) r^{-2} \sin^{-2}\theta \, \frac{M \sin^2\theta}{2M-r} = -\frac{M}{r^3}$$

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -\frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = +\frac{M}{r^3}$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = +\frac{2M}{r^3}$$

$$R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}} = +\frac{M}{r^3}$$

$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = -\frac{M}{r^3}$$

$$R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = -\frac{M}{r^3}$$

$$R_{\hat\mu\hat\nu\hat\rho\hat\lambda} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat r\hat t\hat r\hat t} = -\frac{2M}{r^3}$$

$$R_{\hat\theta\hat t\hat\theta\hat t} = +\frac{M}{r^3}$$

$$R_{\hat\theta\hat\phi\hat\theta\hat\phi} = +\frac{2M}{r^3}$$

$$R_{\hat\phi\hat t\hat\phi\hat t} = +\frac{M}{r^3}$$

$$R_{\hat r\hat\theta\hat r\hat\theta} = -\frac{M}{r^3}$$

$$R_{\hat r\hat\phi\hat r\hat\phi} = -\frac{M}{r^3}$$

For $r < 2M$ $\quad e_t = \partial_t$ spacelike

$e_r = \partial_r$ timelike

so

$$\hat e_t = -|g_{rr}|^{-1/2} e_r$$

$$\hat e_r = |g_{tt}|^{-1/2} e_t \quad ,$$

and, e.g.

$$R_{\hat\theta\hat t\hat\theta\hat t} = |g_{\theta\theta}|^{-1} |g_{rr}|^{-1} R_{\theta r\theta r} \; !$$

<u>Exercise:</u> compute $R_{\hat\mu\hat\nu\hat\rho\hat\lambda}$, show they are the same!

$$R_{\hat\mu\hat\nu\hat\rho\hat\lambda} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2} R_{\mu\nu\rho\lambda} \qquad r > 2M$$

$$R_{\hat r\hat t\hat r\hat t} = -\frac{2M}{r^3}$$

$$R_{\hat\theta\hat t\hat\theta\hat t} = +\frac{M}{r^3}$$

$$R_{\hat\theta\hat\phi\hat\theta\hat\phi} = +\frac{2M}{r^3}$$

$$R_{\hat\phi\hat t\hat\phi\hat t} = +\frac{M}{r^3}$$

$$R_{\hat r\hat\theta\hat r\hat\theta} = -\frac{M}{r^3}$$

$$R_{\hat r\hat\phi\hat r\hat\phi} = -\frac{M}{r^3}$$

For $r < 2M$ $\quad e_t = \partial_t$ spacelike

$e_r = \partial_r$ timelike

so

time flows in $-e_r$ direction

$$\hat e_t = -|g_{rr}|^{-1/2}\, e_r$$

$$\hat e_r = |g_{tt}|^{-1/2}\, e_t \quad ,$$

2 (-) signs = (+)

and, e.g.

$$R_{\hat\theta\hat t\hat\theta\hat t} = |g_{\theta\theta}|^{-1}\, |g_{rr}|^{-1}\, R_{\theta r\theta r} \;!$$

<u>Exercise:</u> compute $R_{\hat\mu\hat\nu\hat\rho\hat\lambda}$, show they are the same!

$$R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}} = |g_{\mu\mu}\, g_{\nu\nu}\, g_{\rho\rho}\, g_{\lambda\lambda}|^{-1/2} R_{\mu\nu\rho\lambda}$$

$$R_{\hat{r}\hat{t}\hat{r}\hat{t}} = -\frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{t}\hat{\theta}\hat{t}} = +\frac{M}{r^3}$$

$$R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = +\frac{2M}{r^3}$$

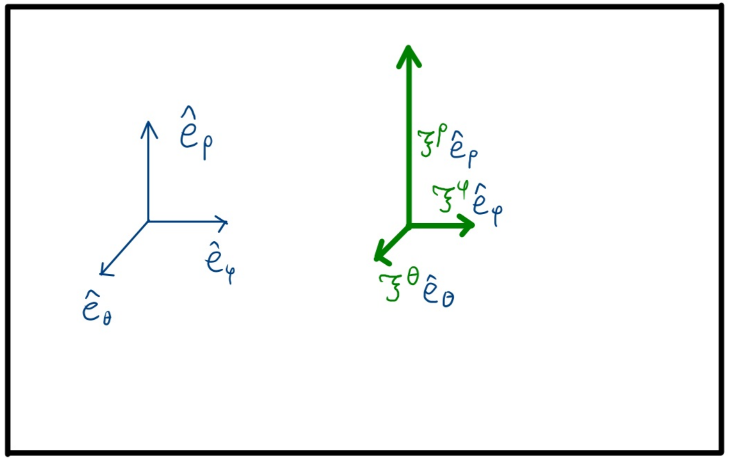
$$R_{\hat{\phi}\hat{t}\hat{\phi}\hat{t}} = +\frac{M}{r^3}$$

$$R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = -\frac{M}{r^3}$$

$$R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = -\frac{M}{r^3}$$

There is no singular behavior of $R_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\lambda}}$ at $r = 2M$!

- How does an observer falling radially into the BH "feel"? Does she feel anything special while crossing the horizon? Misner et al §32.6 p 860



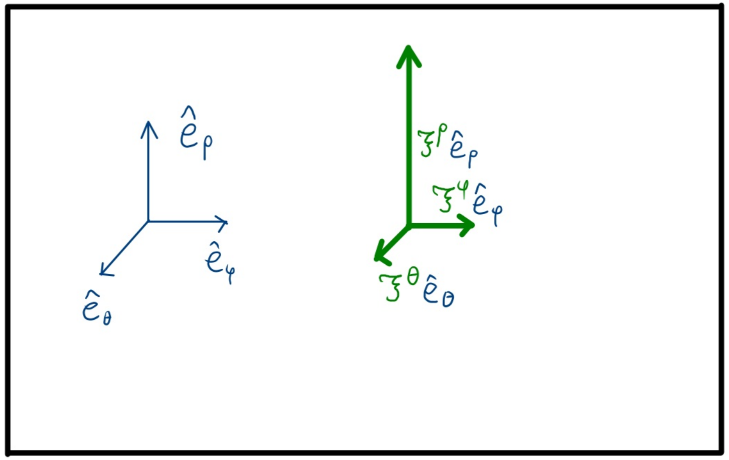
- Back to the ship:
  - orthonormal basis $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\varphi\}$
  - 4-velocity $u = \hat{e}_\tau = [1,0,0,0]$

not the same as stationary observer.

Now

$$u = u^t \partial_t + \underbrace{u^r \partial_r}_{\text{nonzero}}$$

- How does an observer falling radially into the BH "feel"? Does she feel anything special while crossing the horizon? Misner et al §32.6 p 860



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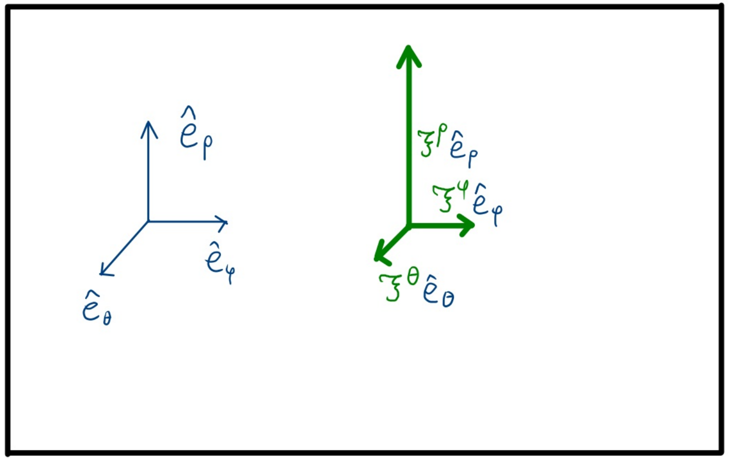
  All particles momentarily at rest in the ship's frame have the same 4 velocity

- How does an observer falling radially into the BH "feel"? Does she feel anything special while crossing the horizon? Misner et al §32.6 p 860



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  - Relative accelerations given by geodesic deviation equation:

$$\frac{D^2 \xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}} u^{\hat{\nu}} u^{\hat{\lambda}} \xi^{\hat{\sigma}}$$

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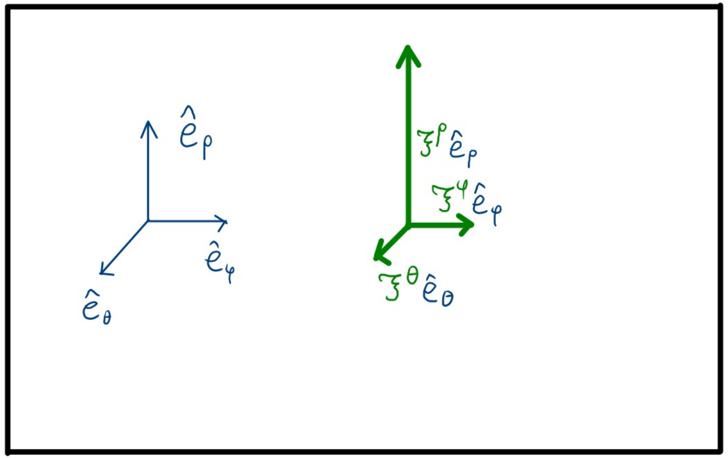

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$$\frac{D^2 \xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}}\, u^{\hat{\nu}} u^{\hat{\lambda}} \xi^{\hat{\sigma}}$$

only $\hat{\nu} = \hat{\lambda} = \hat{\tau}$ survive - only $u^{\hat{\tau}}$ is nonzero

- How does an observer falling radially into the BH "feel"? Does she feel anything special while crossing the horizon?

- Back to the ship:
  - orthonormal basis $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\varphi\}$
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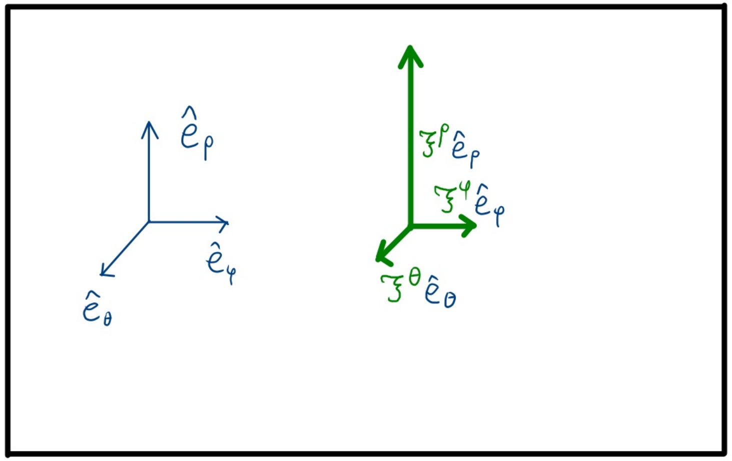

$$\frac{D^2 \xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}}\, u^{\hat{\nu}} u^{\hat{\lambda}} \xi^{\hat{\sigma}} = R^{\hat{\mu}}{}_{\hat{\tau}\hat{\tau}\hat{\sigma}}\, \xi^{\hat{\sigma}}$$

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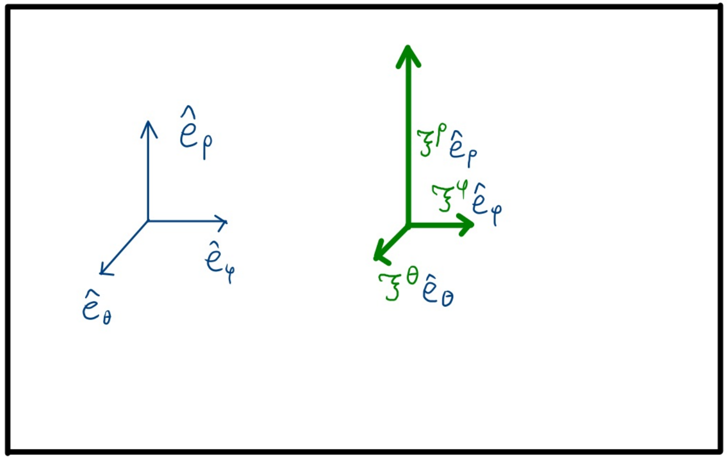

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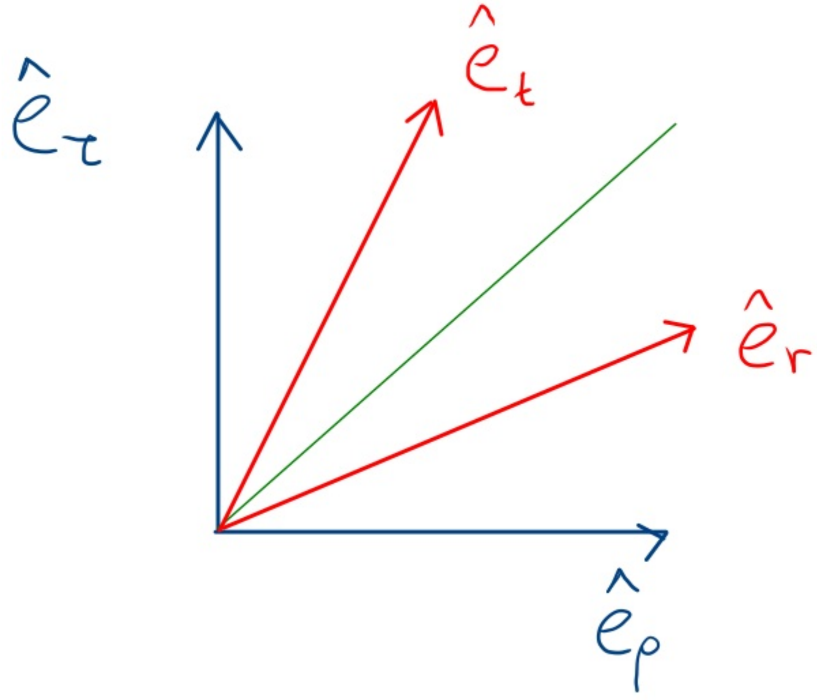
sum over $\hat{\sigma}$

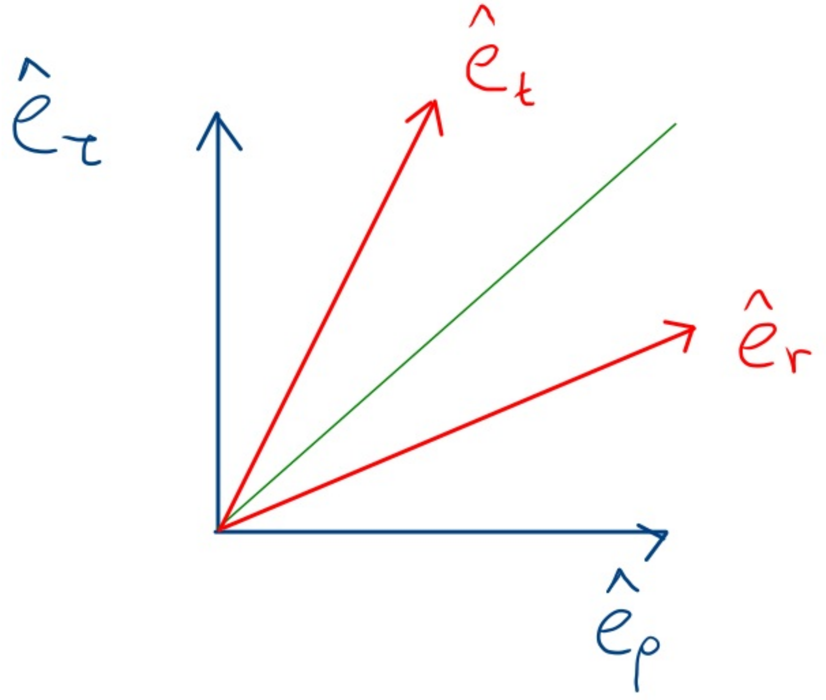
The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:

$$\hat{e}_\tau = \cosh\beta \; \hat{e}_t + \sinh\beta \; \hat{e}_r$$

$$\hat{e}_\rho = \sinh\beta \; \hat{e}_t + \cosh\beta \; \hat{e}_r$$



The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:



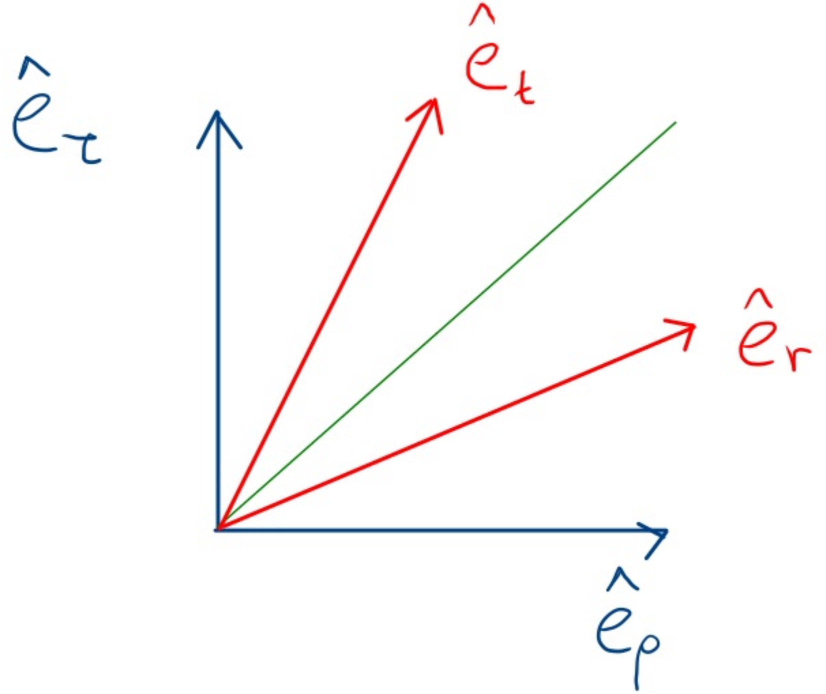
$$\hat{e}_\tau = \cosh\beta \; \hat{e}_t + \sinh\beta \; \hat{e}_r$$

$$\hat{e}_\rho = \sinh\beta \; \hat{e}_t + \cosh\beta \; \hat{e}_r$$

Indeed, the $\hat{e}_\tau = u = [u^t, u^r, 0, 0]$ (radial fall)

↳ components in (t, r, θ, ϕ)

The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:



$$\hat{e}_\tau = \cosh\beta \ \hat{e}_t + \sinh\beta \ \hat{e}_r$$

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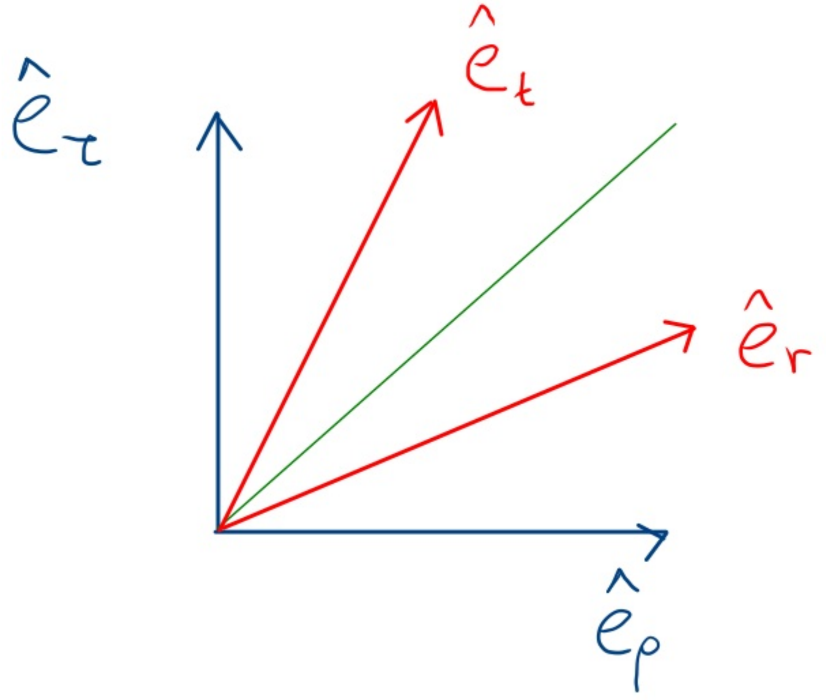
The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:



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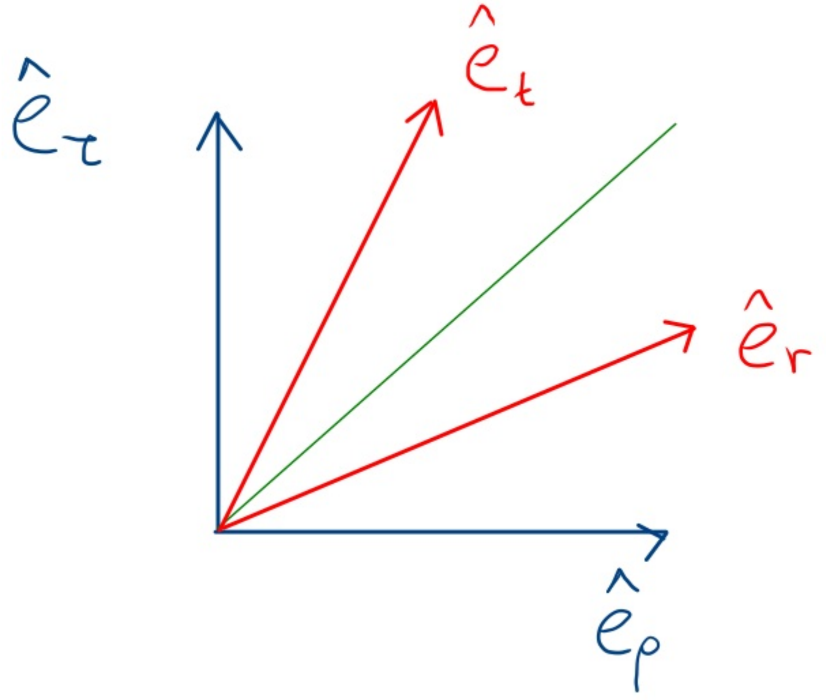
Indeed, the $\hat{e}_\tau = u = [u^t, u^r, 0, 0]$ (radial fall) $\Rightarrow$

$$u = u^t \partial_t + u^r \partial_r = \sqrt{|g_{tt}|}\, u^t \hat{e}_t + \sqrt{g_{rr}}\, u^r \hat{e}_r$$

The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:



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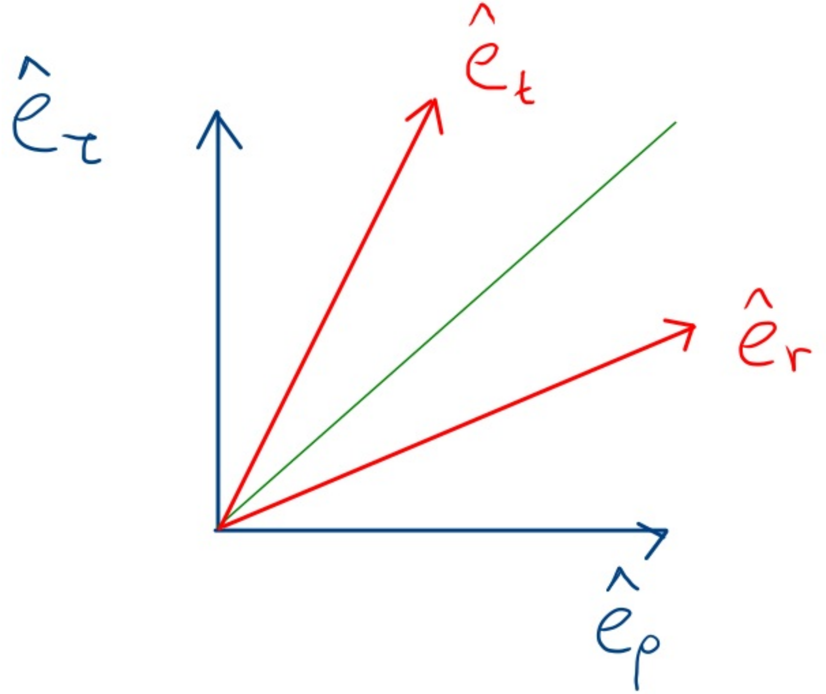
The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:



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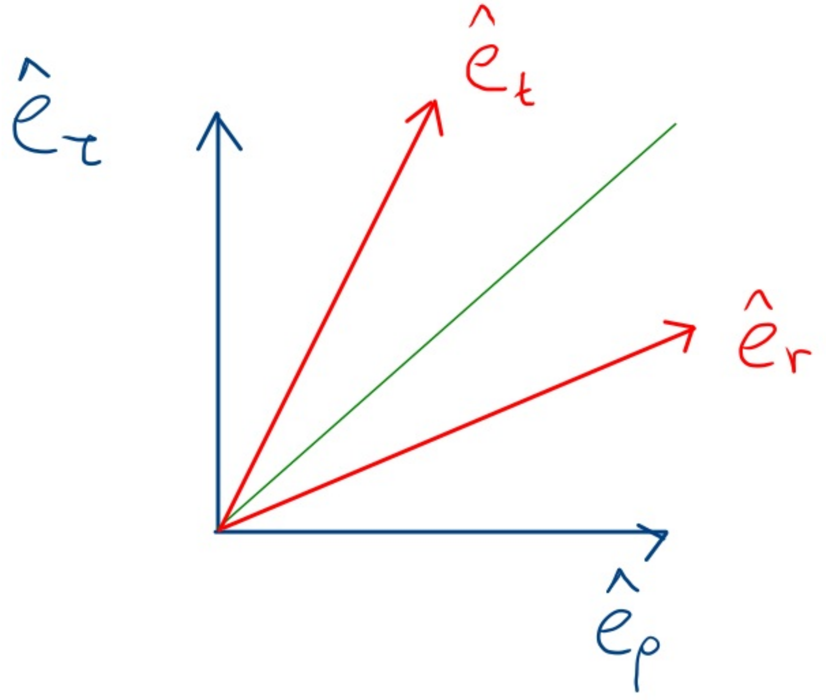
$$u = u^t \partial_t + u^r \partial_r = \sqrt{|g_{tt}|}\, u^t \hat{e}_t + \sqrt{g_{rr}}\, u^r \hat{e}_r$$
$$= u^{\hat{t}} \hat{e}_t + u^{\hat{r}} \hat{e}_r$$

Therefore $u^{\hat{t}} = \cosh\beta$ $u^{\hat{r}} = \sinh\beta$, since

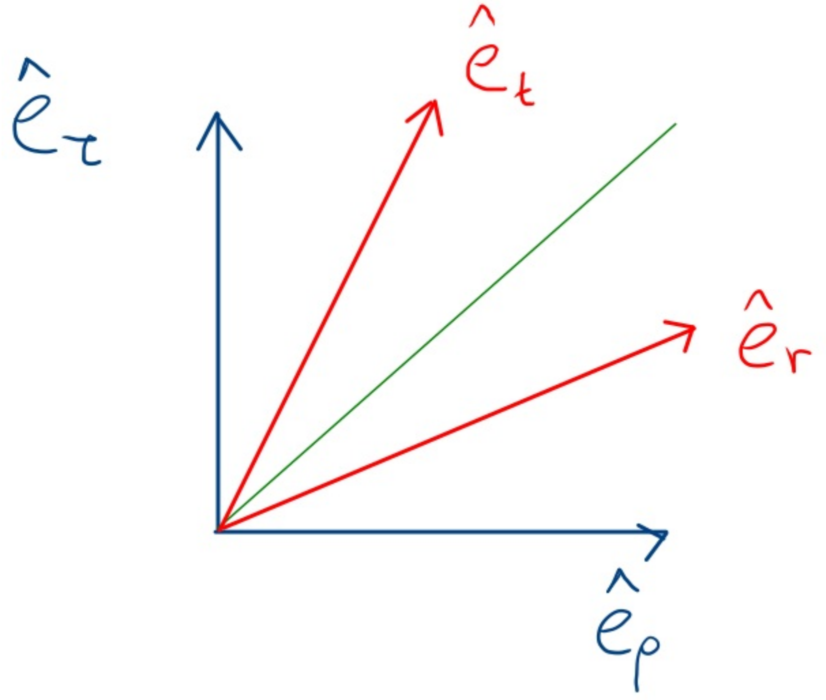
$$-1 = u \cdot u = (u^{\hat{t}})^2 \hat{e}_t \cdot \hat{e}_t + (u^{\hat{r}})^2 \hat{e}_r \cdot \hat{e}_r = -(u^{\hat{t}})^2 + (u^{\hat{r}}) = -\cosh^2\beta + \sinh^2\beta$$

The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:



$$\begin{pmatrix} \hat{e}_\tau \\ \hat{e}_\rho \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} \hat{e}_t \\ \hat{e}_r \end{pmatrix}$$

The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:

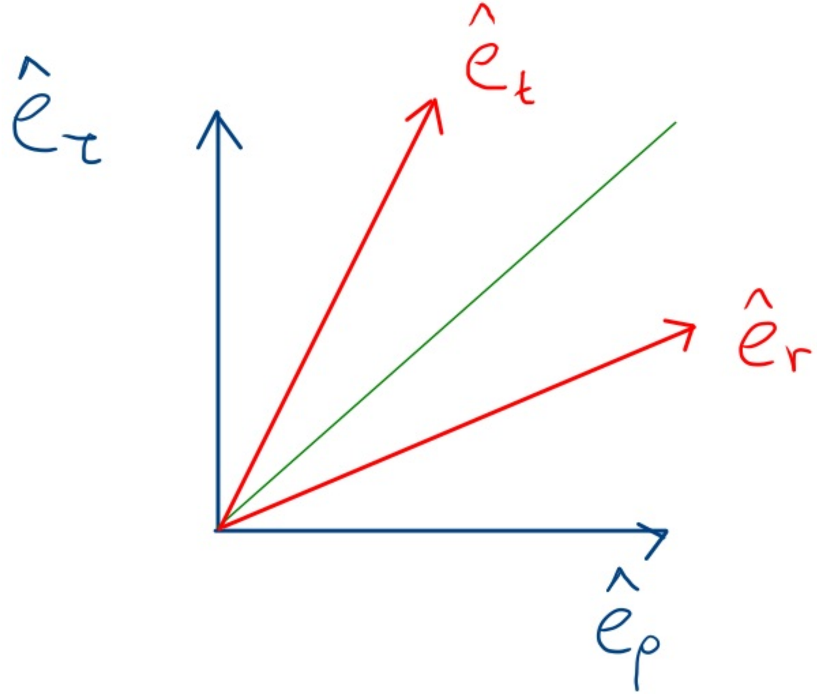


$$\begin{pmatrix} \hat{e}_\tau \\ \hat{e}_\rho \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} \hat{e}_t \\ \hat{e}_r \end{pmatrix}$$

$$\begin{pmatrix} v^{\hat{\tau}} \\ v^{\hat{\rho}} \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix}^{-1} \begin{pmatrix} v^{\hat{t}} \\ v^{\hat{r}} \end{pmatrix}$$

inverse matrix

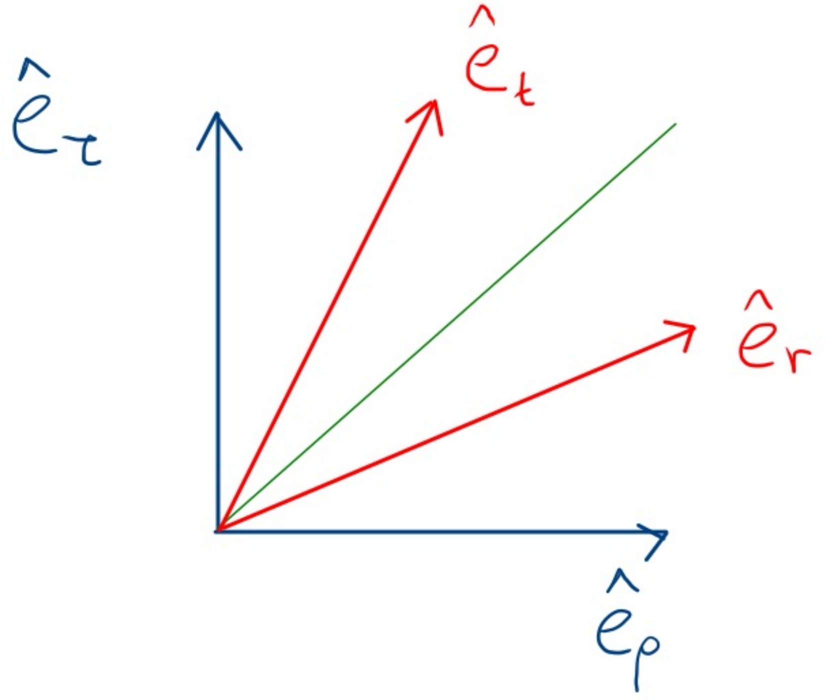
The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:



$$\begin{pmatrix} \hat{e}_\tau \\ \hat{e}_\rho \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} \hat{e}_t \\ \hat{e}_r \end{pmatrix}$$

$$\begin{pmatrix} V^{\hat{\tau}} \\ V^{\hat{\rho}} \end{pmatrix} = \begin{pmatrix} \cosh\beta & -\sinh\beta \\ -\sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} V^{\hat{t}} \\ V^{\hat{r}} \end{pmatrix}$$

The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:

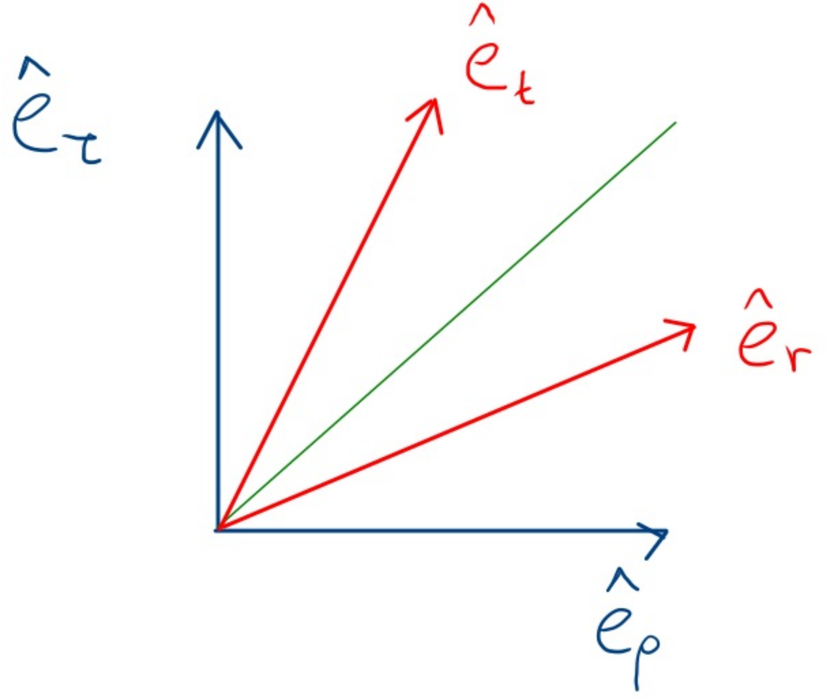


$$\begin{pmatrix} \hat{e}_\tau \\ \hat{e}_\rho \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} \hat{e}_t \\ \hat{e}_r \end{pmatrix}$$

$$\begin{pmatrix} V^{\hat{\tau}} \\ V^{\hat{\rho}} \end{pmatrix} = \begin{pmatrix} \cosh(-\beta) & \sinh(-\beta) \\ \sinh(-\beta) & \cosh(-\beta) \end{pmatrix} \begin{pmatrix} V^{\hat{t}} \\ V^{\hat{r}} \end{pmatrix}$$

$\hookrightarrow$ opposite velocity $\rightarrow$ inverse boost

The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:

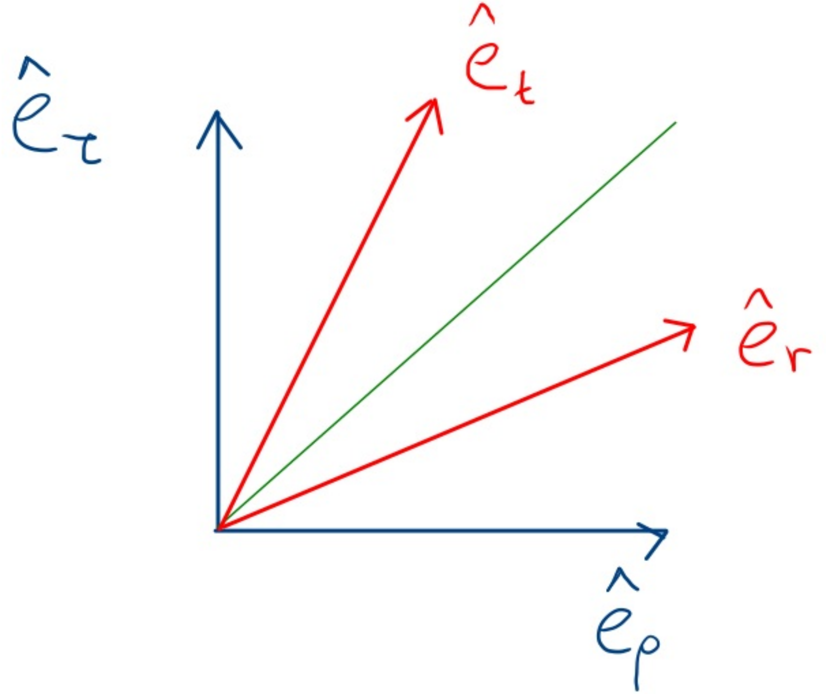


$$\begin{pmatrix} \hat{e}_\tau \\ \hat{e}_\rho \end{pmatrix} = \Lambda(\beta) \begin{pmatrix} \hat{e}_t \\ \hat{e}_r \end{pmatrix}$$

$$\begin{pmatrix} V^{\hat{\tau}} \\ V^{\hat{\rho}} \end{pmatrix} = \Lambda(-\beta) \begin{pmatrix} V^{\hat{t}} \\ V^{\hat{r}} \end{pmatrix}$$

$$\Lambda^{-1}(\beta) = \Lambda(-\beta)$$

The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:



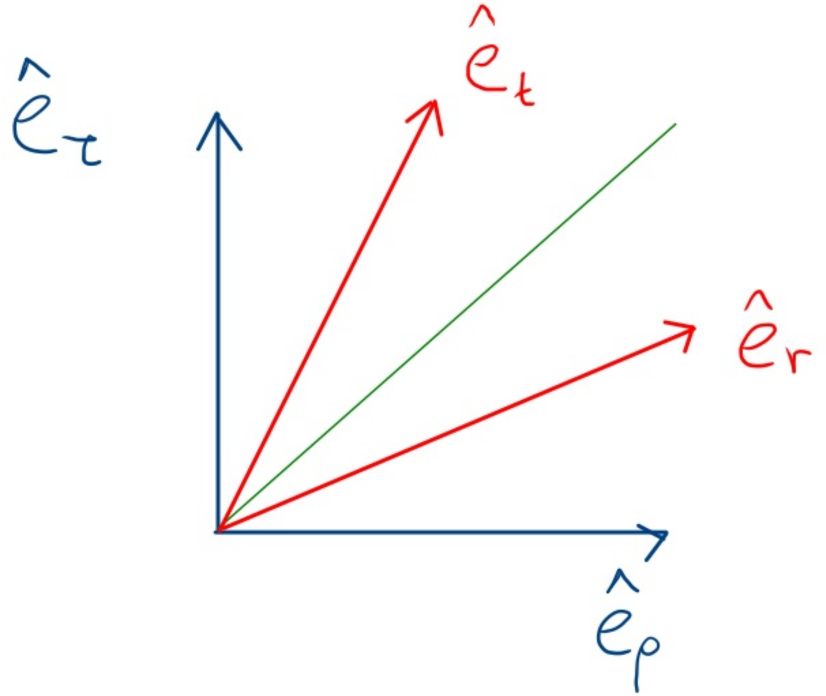
$$\begin{pmatrix} \omega_{\hat{\tau}} \\ \omega_{\hat{\rho}} \end{pmatrix} = \Lambda(\beta) \begin{pmatrix} \omega_{\hat{t}} \\ \omega_{\hat{r}} \end{pmatrix}$$

$$\begin{pmatrix} v^{\hat{\tau}} \\ v^{\hat{\rho}} \end{pmatrix} = \Lambda(-\beta) \begin{pmatrix} v^{\hat{t}} \\ v^{\hat{r}} \end{pmatrix}$$

components of one form ω

$$\Lambda^{-1}(\beta) = \Lambda(-\beta)$$

The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:



$$\begin{pmatrix}\omega_{\hat{\tau}}\\ \omega_{\hat{\rho}}\end{pmatrix} = \Lambda(\beta)\begin{pmatrix}\omega_{\hat{t}}\\ \omega_{\hat{r}}\end{pmatrix}$$

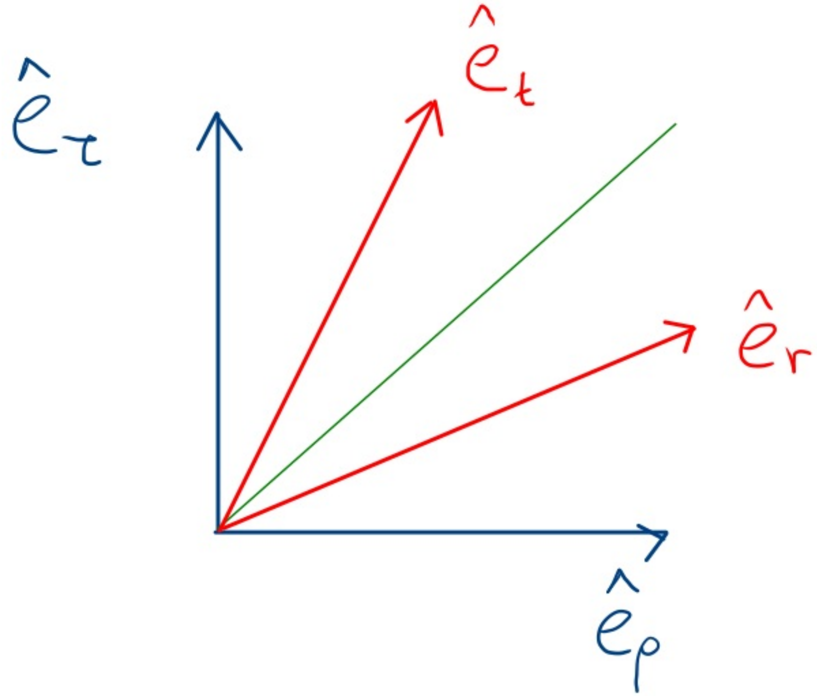
$$\omega_{\hat{\mu}'} = \Lambda_{\hat{\mu}'}{}^{\hat{\mu}}\,\omega_{\hat{\mu}}$$

$$\Lambda_{\hat{\tau}}{}^{\hat{t}} = \Lambda_{\hat{\rho}}{}^{\hat{r}} = \cosh\beta$$

$$\Lambda_{\hat{\rho}}{}^{\hat{t}} = \Lambda_{\hat{\tau}}{}^{\hat{r}} = \sinh\beta$$

$$\left.\begin{array}{l}\Lambda_{\hat{\theta}}{}^{\hat{\theta}} = \Lambda_{\hat{\phi}}{}^{\hat{\phi}} = 1\\ \text{and the rest are zero}\end{array}\right\}\ \text{not transformed}$$

The $\{\hat{e}_\tau, \hat{e}_\rho, \hat{e}_\theta, \hat{e}_\phi\}$ ship's basis related to $\{\hat{e}_t, \hat{e}_r, \hat{e}_\theta, e_\phi\}$ stationary frame by a Lorentz boost:

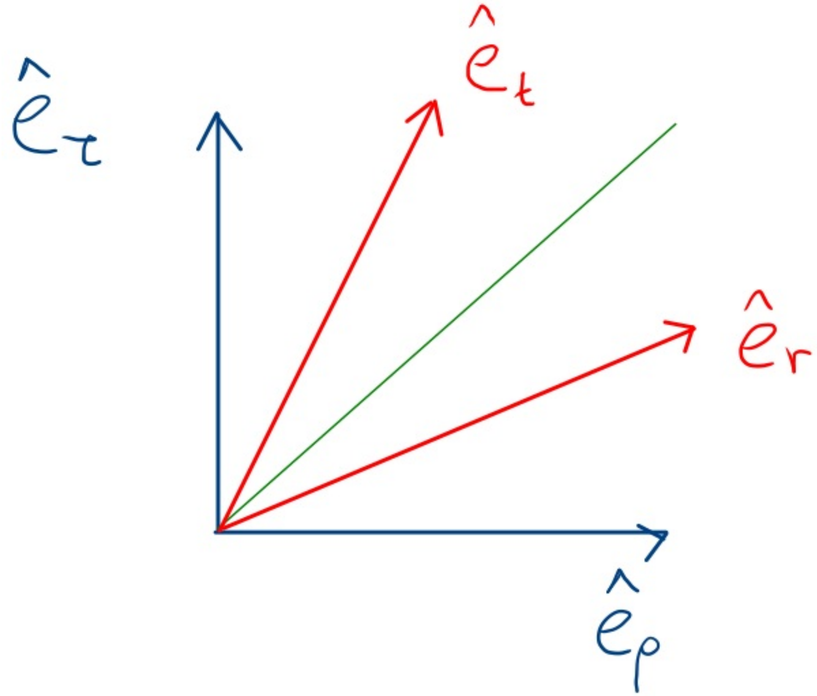


$$\begin{pmatrix} \omega_{\hat{\tau}} \\ \omega_{\hat{\rho}} \end{pmatrix} = \Lambda(\beta) \begin{pmatrix} \omega_{\hat{t}} \\ \omega_{\hat{r}} \end{pmatrix}$$

$$\omega_{\hat{\mu}'} = \Lambda_{\hat{\mu}'}{}^{\hat{\mu}} \, \omega_{\hat{\mu}}$$

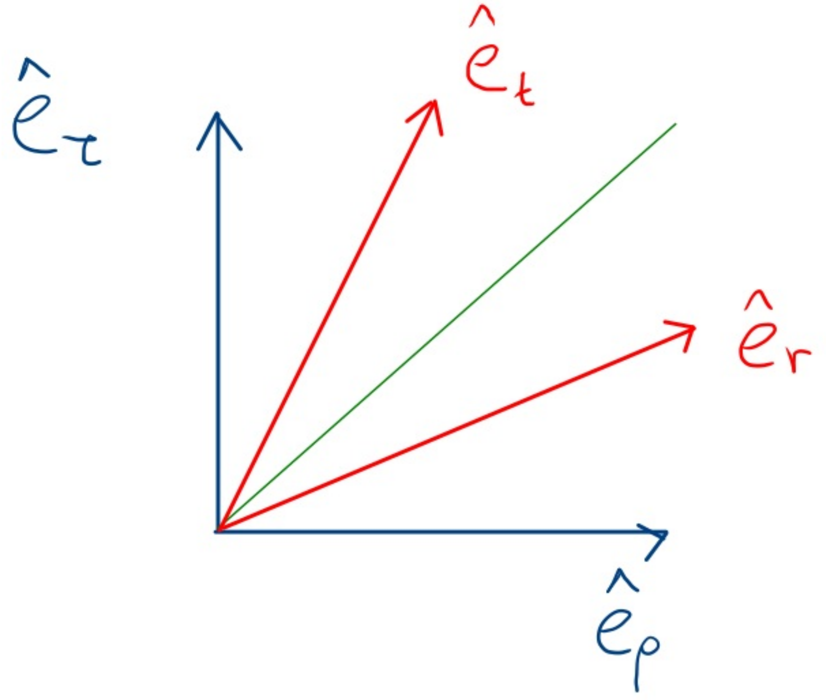
$$R_{\hat{\mu}'\hat{\nu}'\hat{\lambda}'\hat{\sigma}'} = \Lambda_{\hat{\mu}'}{}^{\hat{\mu}} \Lambda_{\hat{\nu}'}{}^{\hat{\nu}} \Lambda_{\hat{\lambda}'}{}^{\hat{\lambda}} \Lambda_{\hat{\sigma}'}{}^{\hat{\sigma}} R_{\hat{\mu}\hat{\nu}\hat{\lambda}\hat{\sigma}}$$

So: • R $\hat\theta$ $\hat\phi$ $\hat\theta$ $\hat t$ unchanged



So: • $R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}}$ unchanged

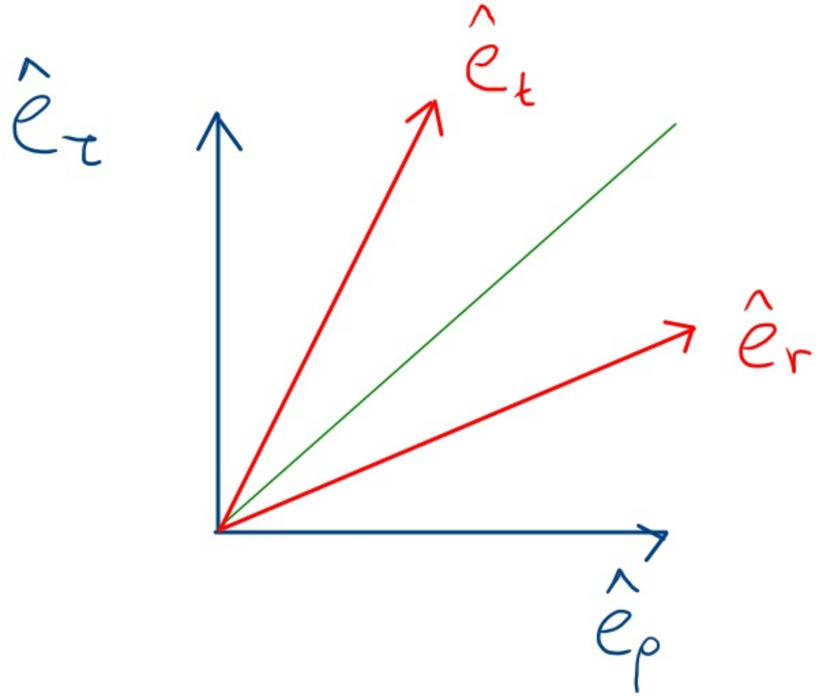
• $R_{\hat{\tau}\hat{\theta}\hat{\tau}\hat{\theta}} = \Lambda_{\hat{\tau}}{}^{\hat{\mu}} \Lambda_{\hat{\tau}}{}^{\hat{\nu}} R_{\hat{\mu}\hat{\theta}\hat{\nu}\hat{\theta}}$



So: • $R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}}$ unchanged

• $R_{\hat{\tau}\hat{\theta}\hat{\tau}\hat{\theta}} = \Lambda_{\hat{\tau}}{}^{\hat{\mu}} \Lambda_{\hat{\tau}}{}^{\hat{\nu}} R_{\hat{\mu}\hat{\theta}\hat{\nu}\hat{\theta}}$

$= \Lambda_{\hat{\tau}}{}^{\hat{t}} \Lambda_{\hat{\tau}}{}^{\hat{t}} R_{\hat{t}\hat{\theta}\hat{t}\hat{\theta}} + \Lambda_{\hat{\tau}}{}^{\hat{r}} \Lambda_{\hat{\tau}}{}^{\hat{r}} R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}}$



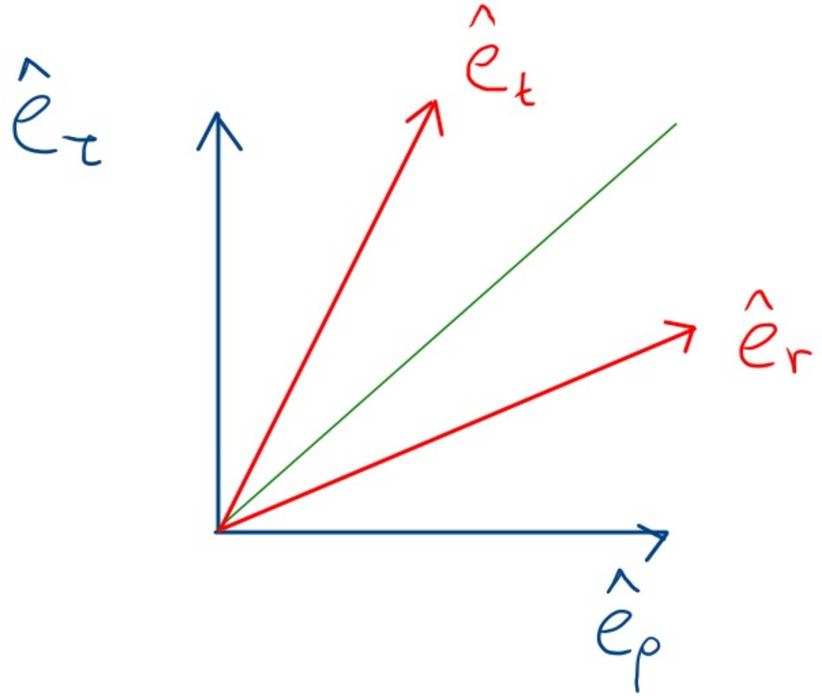
So: • $R_{\hat\theta\hat\phi\hat\theta\hat\phi}$ unchanged

• $R_{\hat\tau\hat\theta\hat\tau\hat\theta} = \Lambda_{\hat\tau}{}^{\hat\mu}\Lambda_{\hat\tau}{}^{\hat\nu} R_{\hat\mu\hat\theta\hat\nu\hat\theta}$

$$= \Lambda_{\hat\tau}{}^{\hat t}\Lambda_{\hat\tau}{}^{\hat t} R_{\hat t\hat\theta\hat t\hat\theta} + \Lambda_{\hat\tau}{}^{\hat r}\Lambda_{\hat\tau}{}^{\hat r} R_{\hat r\hat\theta\hat r\hat\theta}$$

$$= \cosh^2\beta \, \frac{M}{r^3} + \sinh^2\beta \left(-\frac{M}{r^3}\right)$$

notice the effect of opposite values



So: • $R_{\hat\theta\hat\phi\hat\theta\hat\phi}$ unchanged

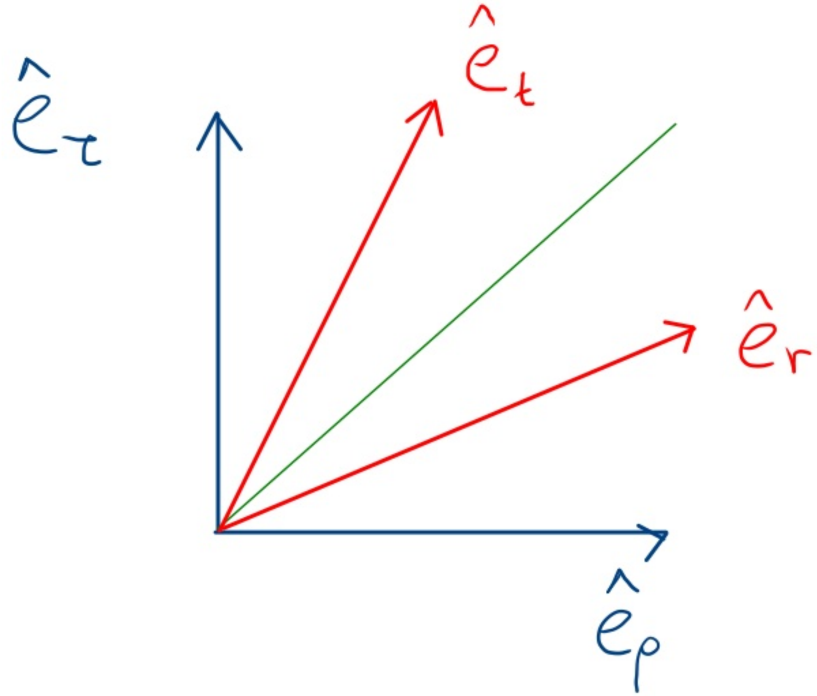
• $R_{\hat\tau\hat\theta\hat\tau\hat\theta} = \Lambda_{\hat\tau}{}^{\hat\mu}\Lambda_{\hat\tau}{}^{\hat\nu} R_{\hat\mu\hat\theta\hat\nu\hat\theta}$

$= \Lambda_{\hat\tau}{}^{\hat t}\Lambda_{\hat\tau}{}^{\hat t} R_{\hat t\hat\theta\hat t\hat\theta} + \Lambda_{\hat\tau}{}^{\hat r}\Lambda_{\hat\tau}{}^{\hat r} R_{\hat r\hat\theta\hat r\hat\theta}$

$= \cosh^2\beta \quad \frac{M}{r^3} \quad + \quad \sinh^2\beta \quad \left(-\frac{M}{r^3}\right)$

$= \left(\cosh^2\beta - \sinh^2\beta\right)\frac{M}{r^3} =$

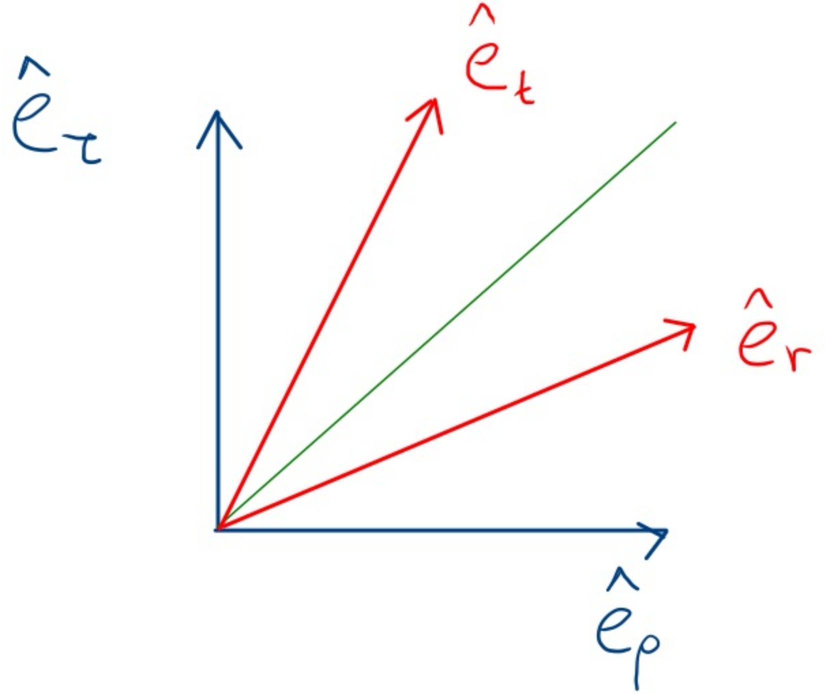
$= \frac{M}{r^3} = R_{\hat t\hat\theta\hat t\hat\theta}$



So: • $R_{\hat\theta\hat\phi\hat\theta\hat\phi}$ unchanged

• $R_{\hat\tau\hat\theta\hat\tau\hat\theta} = R_{\hat t\hat\theta\hat t\hat\theta}$

• $R_{\hat\rho\hat\theta\hat\rho\hat\theta} = \Lambda_{\hat\rho}{}^{\hat t}\Lambda_{\hat\rho}{}^{\hat t} R_{\hat t\hat\theta\hat t\hat\theta} + \Lambda_{\hat\rho}{}^{\hat r}\Lambda_{\hat\rho}{}^{\hat r} R_{\hat r\hat\theta\hat r\hat\theta}$

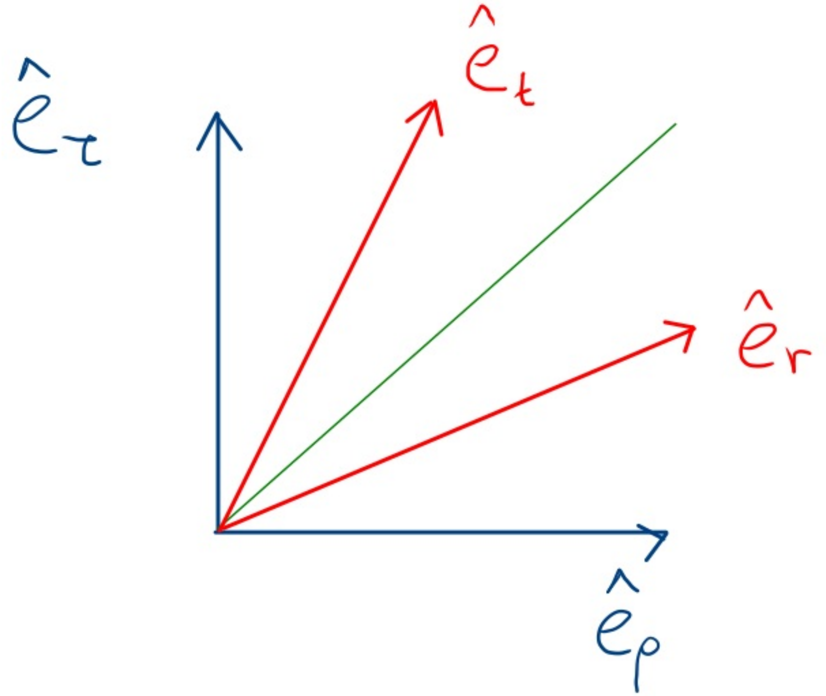


So: • $R_{\hat\theta\hat\phi\hat\theta\hat\phi}$ unchanged

• $R_{\hat\tau\hat\theta\hat\tau\hat\theta} = R_{\hat t\hat\theta\hat t\hat\theta}$

• $R_{\hat\rho\hat\theta\hat\rho\hat\theta} = \Lambda_{\hat\rho}{}^{\hat t}\Lambda_{\hat\rho}{}^{\hat t} R_{\hat t\hat\theta\hat t\hat\theta} + \Lambda_{\hat\rho}{}^{\hat r}\Lambda_{\hat\rho}{}^{\hat r} R_{\hat r\hat\theta\hat r\hat\theta}$

$$= \sinh^2\beta \, \frac{M}{r^3} + \cosh^2\beta \left(-\frac{M}{r^3}\right)$$



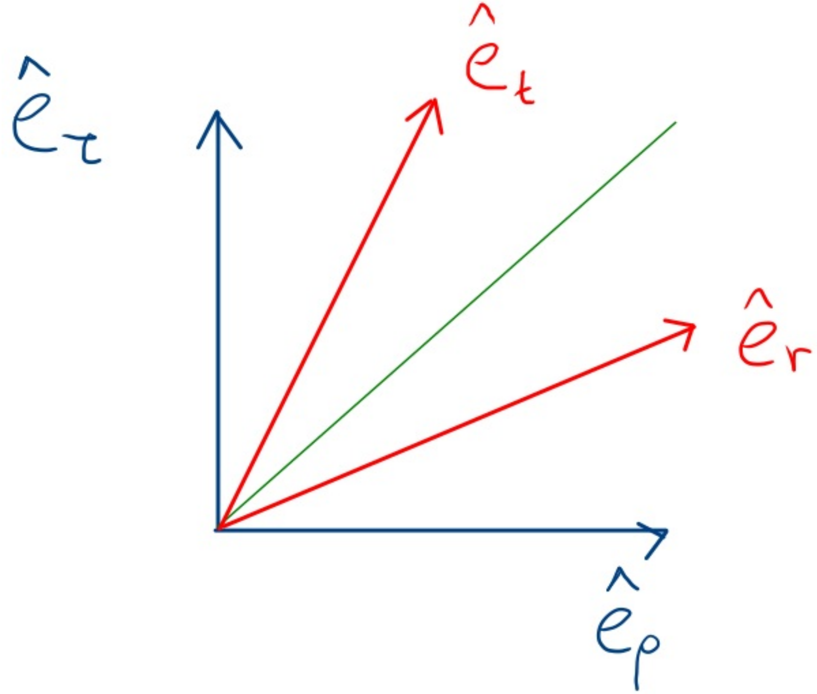
So: • $R_{\hat\theta\hat\phi\hat\theta\hat\phi}$ unchanged

• $R_{\hat\tau\hat\theta\hat\tau\hat\theta} = R_{\hat t\hat\theta\hat t\hat\theta}$

• $R_{\hat\rho\hat\theta\hat\rho\hat\theta} = \Lambda_{\hat\rho}{}^{\hat t}\Lambda_{\hat\rho}{}^{\hat t} R_{\hat t\hat\theta\hat t\hat\theta} + \Lambda_{\hat\rho}{}^{\hat r}\Lambda_{\hat\rho}{}^{\hat r} R_{\hat r\hat\theta\hat r\hat\theta}$

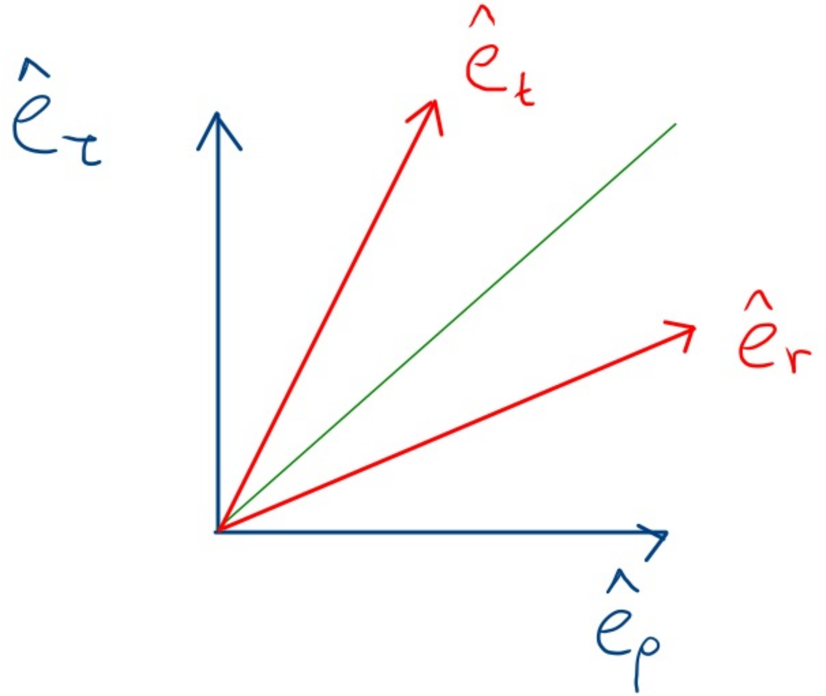
$$= \sinh^2\beta \; \frac{M}{r^3} + \cosh^2\beta \left(-\frac{M}{r^3}\right)$$

$$= -\frac{M}{r^3} = R_{\hat r\hat\theta\hat r\hat\theta}$$



So: • $R_{\hat\theta\hat\phi\hat\theta\hat\phi}$ unchanged

- $R_{\hat\tau\hat\theta\hat\tau\hat\theta} = R_{\hat t\hat\theta\hat t\hat\theta}$
- $R_{\hat\rho\hat\theta\hat\rho\hat\theta} = R_{\hat r\hat\theta\hat r\hat\theta}$
- $R_{\hat\rho\hat\phi\hat\rho\hat\phi} = R_{\hat r\hat\phi\hat r\hat\phi}$ (same as before!)



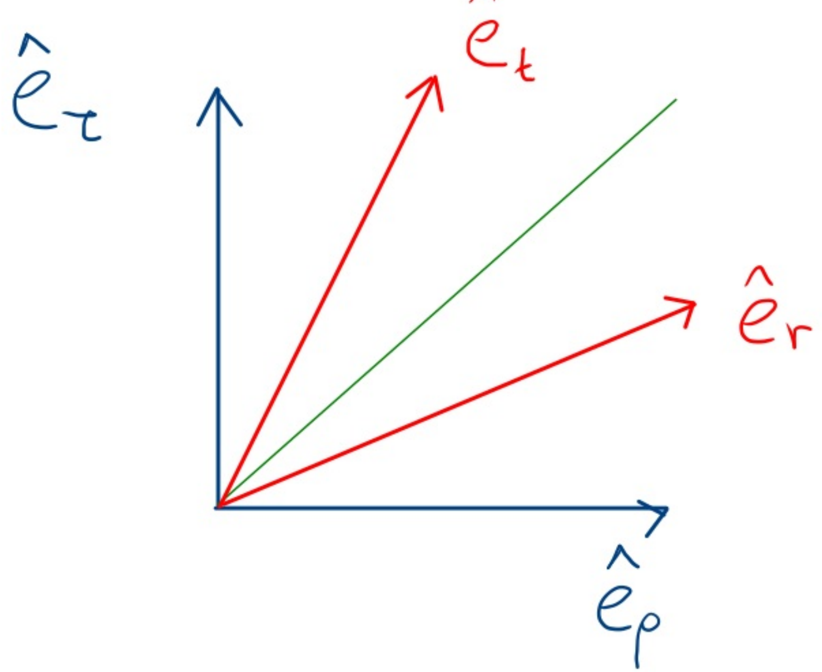
So: • $R_{\hat\theta\hat\phi\hat\theta\hat\phi}$ unchanged

• $R_{\hat\tau\hat\theta\hat\tau\hat\theta} = R_{\hat t\hat\theta\hat t\hat\theta}$

• $R_{\hat\rho\hat\theta\hat\rho\hat\theta} = R_{\hat r\hat\theta\hat r\hat\theta}$

• $R_{\hat\rho\hat\phi\hat\rho\hat\phi} = R_{\hat r\hat\phi\hat r\hat\phi}$

• $R_{\hat\tau\hat\rho\hat\tau\hat\rho} = \Lambda_{\hat\tau}{}^{\hat\mu}\Lambda_{\hat\rho}{}^{\hat\lambda}\Lambda_{\hat\tau}{}^{\hat\sigma}\Lambda_{\hat\rho}{}^{\hat\nu} R_{\hat\mu\hat\lambda\hat\sigma\hat\nu}$



So: • $R_{\hat\theta\hat\phi\hat\theta\hat\phi}$ unchanged

• $R_{\hat\tau\hat\theta\hat\tau\hat\theta} = R_{\hat t\hat\theta\hat t\hat\theta}$

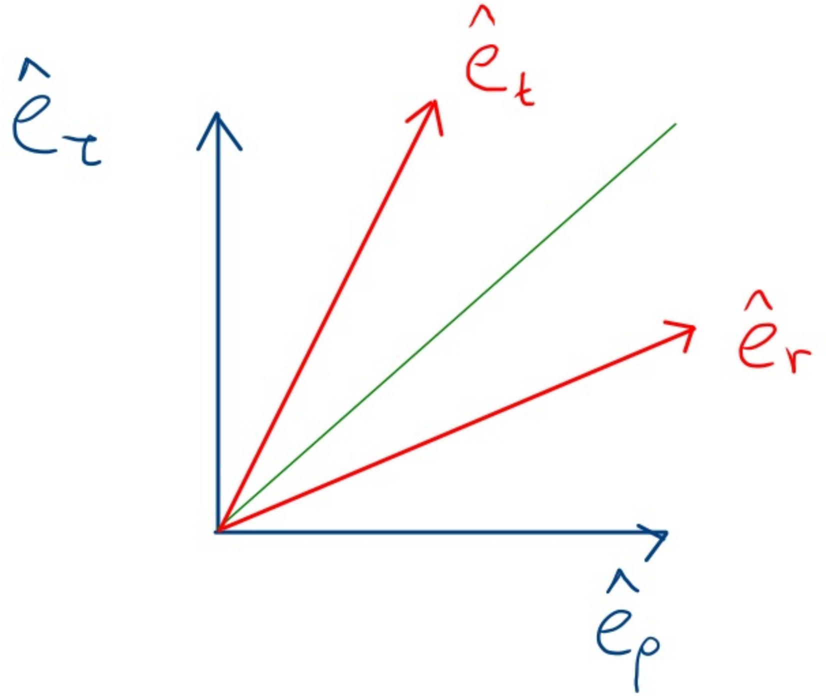
• $R_{\hat\rho\hat\theta\hat\rho\hat\theta} = R_{\hat r\hat\theta\hat r\hat\theta}$

• $R_{\hat\rho\hat\phi\hat\rho\hat\phi} = R_{\hat r\hat\phi\hat r\hat\phi}$

• $$R_{\hat\tau\hat\rho\hat\tau\hat\rho} = \Lambda_{\hat\tau}{}^{\hat\mu}\Lambda_{\hat\rho}{}^{\hat\lambda}\Lambda_{\hat\tau}{}^{\hat\sigma}\Lambda_{\hat\rho}{}^{\hat\nu} R_{\hat\mu\hat\lambda\hat\sigma\hat\nu}$$

$$\begin{aligned} &= + \Lambda_{\hat\tau}{}^{\hat t}\Lambda_{\hat\rho}{}^{\hat r}\Lambda_{\hat\tau}{}^{\hat t}\Lambda_{\hat\rho}{}^{\hat r} R_{\hat t\hat r\hat t\hat r} \\ &\quad + \Lambda_{\hat\tau}{}^{\hat t}\Lambda_{\hat\rho}{}^{\hat r}\Lambda_{\hat\tau}{}^{\hat r}\Lambda_{\hat\rho}{}^{\hat t} R_{\hat t\hat r\hat r\hat t} \\ &\quad + \Lambda_{\hat\tau}{}^{\hat r}\Lambda_{\hat\rho}{}^{\hat t}\Lambda_{\hat\tau}{}^{\hat t}\Lambda_{\hat\rho}{}^{\hat r} R_{\hat r\hat t\hat t\hat r} \\ &\quad + \Lambda_{\hat\tau}{}^{\hat r}\Lambda_{\hat\rho}{}^{\hat t}\Lambda_{\hat\tau}{}^{\hat r}\Lambda_{\hat\rho}{}^{\hat t} R_{\hat r\hat t\hat r\hat t} \end{aligned}$$

only nonzero terms (exercise!)



So: • $R_{\hat\theta\hat\phi\hat\theta\hat\phi}$ unchanged

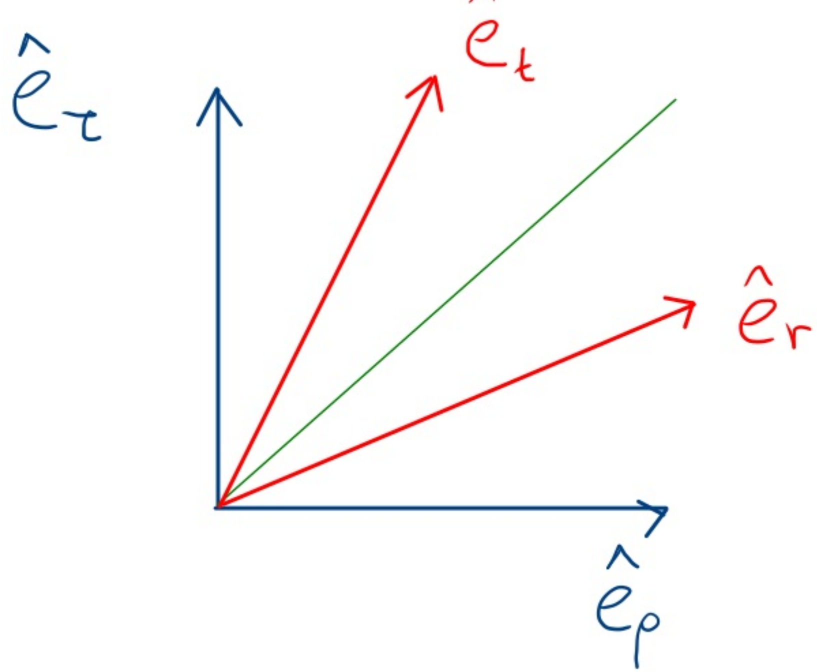
• $R_{\hat\tau\hat\theta\hat\tau\hat\theta} = R_{\hat t\hat\theta\hat t\hat\theta}$

• $R_{\hat\rho\hat\theta\hat\rho\hat\theta} = R_{\hat r\hat\theta\hat r\hat\theta}$

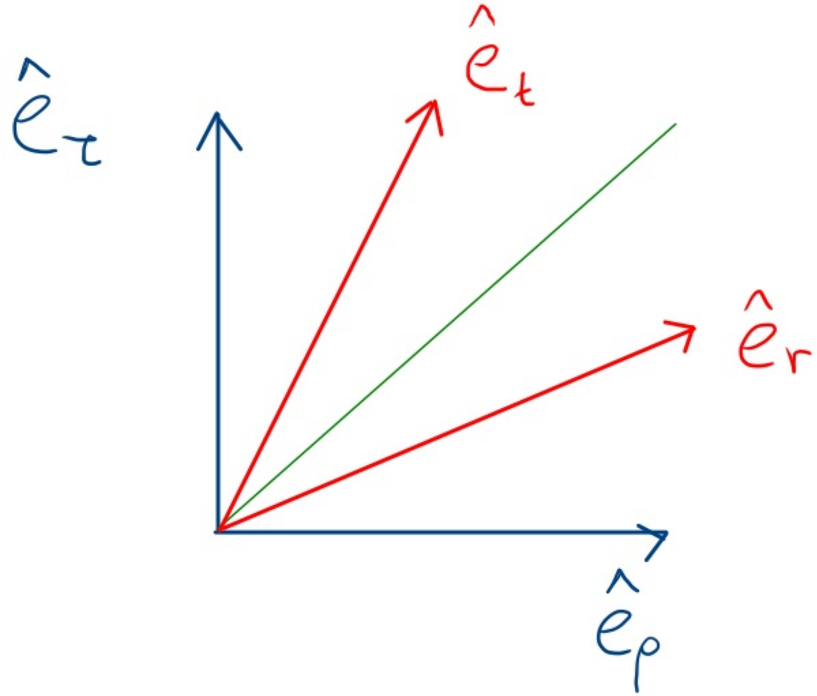
• $R_{\hat\rho\hat\phi\hat\rho\hat\phi} = R_{\hat r\hat\phi\hat r\hat\phi}$

• $R_{\hat\tau\hat\rho\hat\tau\hat\rho} = \Lambda_{\hat\tau}{}^{\hat\mu}\Lambda_{\hat\rho}{}^{\hat\lambda}\Lambda_{\hat\tau}{}^{\hat\sigma}\Lambda_{\hat\rho}{}^{\hat\nu} R_{\hat\mu\hat\lambda\hat\sigma\hat\nu}$

$$
\begin{aligned}
&= + \Lambda_{\hat\tau}{}^{\hat t}\Lambda_{\hat\rho}{}^{\hat r}\Lambda_{\hat\tau}{}^{\hat t}\Lambda_{\hat\rho}{}^{\hat r} R_{\hat t\hat r\hat t\hat r} \\
&\quad - \Lambda_{\hat\tau}{}^{\hat t}\Lambda_{\hat\rho}{}^{\hat r}\Lambda_{\hat\tau}{}^{\hat r}\Lambda_{\hat\rho}{}^{\hat t} R_{\hat t\hat r\hat t\hat r} \\
&\quad - \Lambda_{\hat\tau}{}^{\hat r}\Lambda_{\hat\rho}{}^{\hat t}\Lambda_{\hat\tau}{}^{\hat t}\Lambda_{\hat\rho}{}^{\hat r} R_{\hat t\hat r\hat t\hat r} \\
&\quad + \Lambda_{\hat\tau}{}^{\hat r}\Lambda_{\hat\rho}{}^{\hat t}\Lambda_{\hat\tau}{}^{\hat r}\Lambda_{\hat\rho}{}^{\hat t} R_{\hat t\hat r\hat t\hat r}
\end{aligned}
$$



So: • $R_{\hat\theta\hat\phi\hat\theta\hat\phi}$ unchanged



• $R_{\hat\tau\hat\theta\hat\tau\hat\theta} = R_{\hat t\hat\theta\hat t\hat\theta}$

• $R_{\hat\rho\hat\theta\hat\rho\hat\theta} = R_{\hat r\hat\theta\hat r\hat\theta}$

• $R_{\hat\rho\hat\phi\hat\rho\hat\phi} = R_{\hat r\hat\phi\hat r\hat\phi}$

• $R_{\hat\tau\hat\rho\hat\tau\hat\rho} = \Lambda_{\hat\tau}{}^{\hat\mu}\Lambda_{\hat\rho}{}^{\hat\lambda}\Lambda_{\hat\tau}{}^{\hat\sigma}\Lambda_{\hat\rho}{}^{\hat\nu} R_{\hat\mu\hat\lambda\hat\sigma\hat\nu}$

$$= + \cosh^4\beta \quad R_{\hat t\hat r\hat t\hat r}$$
$$- \cosh^2\beta \sinh^2\beta \quad R_{\hat t\hat r\hat t\hat r}$$
$$- \cosh^2\beta \sinh^2\beta \quad R_{\hat t\hat r\hat t\hat r}$$
$$+ \sinh^4\beta \quad R_{\hat t\hat r\hat t\hat r}$$

So: • $R_{\hat\theta\hat\phi\hat\theta\hat\phi}$ unchanged

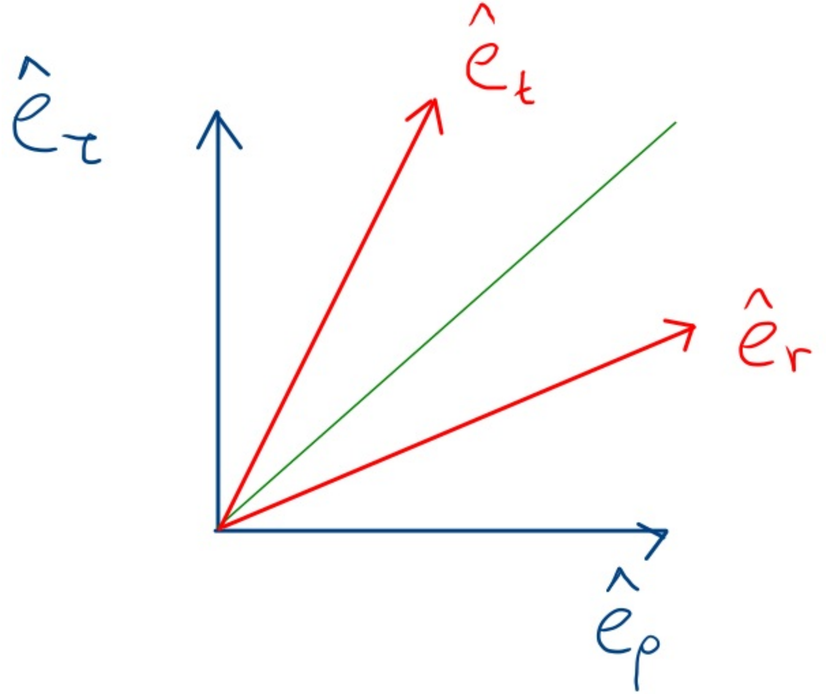
• $R_{\hat\tau\hat\theta\hat\tau\hat\theta} = R_{\hat t\hat\theta\hat t\hat\theta}$

• $R_{\hat\rho\hat\theta\hat\rho\hat\theta} = R_{\hat r\hat\theta\hat r\hat\theta}$

• $R_{\hat\rho\hat\phi\hat\rho\hat\phi} = R_{\hat r\hat\phi\hat r\hat\phi}$

• $R_{\hat\tau\hat\rho\hat\tau\hat\rho} = \Lambda_{\hat\tau}{}^{\hat\mu}\Lambda_{\hat\rho}{}^{\hat\lambda}\Lambda_{\hat\tau}{}^{\hat\sigma}\Lambda_{\hat\rho}{}^{\hat\nu} R_{\hat\mu\hat\lambda\hat\sigma\hat\nu}$

$$
\begin{aligned}
&= + \cosh^4\beta \; R_{\hat t\hat r\hat t\hat r} \\
&\quad - \cosh^2\beta \sinh^2\beta \; R_{\hat t\hat r\hat t\hat r} \\
&\quad - \cosh^2\beta \sinh^2\beta \; R_{\hat t\hat r\hat t\hat r} \quad = 1 \cdot R_{\hat t\hat r\hat t\hat r} \\
&\quad + \sinh^4\beta \; R_{\hat t\hat r\hat t\hat r}
\end{aligned}
$$



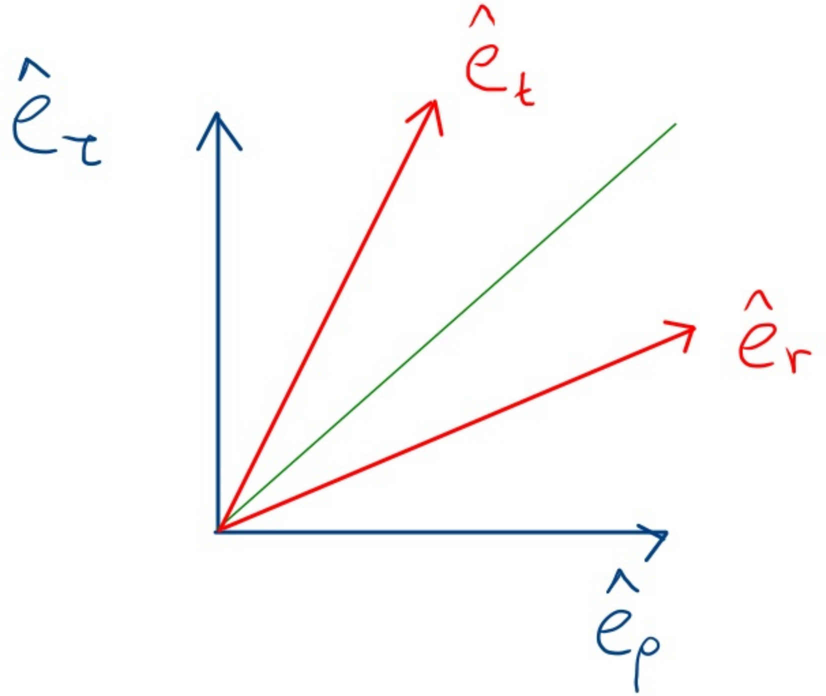
So: • $R_{\hat\theta\hat\phi\hat\theta\hat\phi}$ unchanged

• $R_{\hat\tau\hat\theta\hat\tau\hat\theta} = R_{\hat t\hat\theta\hat t\hat\theta}$

• $R_{\hat\rho\hat\theta\hat\rho\hat\theta} = R_{\hat r\hat\theta\hat r\hat\theta}$

• $R_{\hat\rho\hat\phi\hat\rho\hat\phi} = R_{\hat r\hat\phi\hat r\hat\phi}$

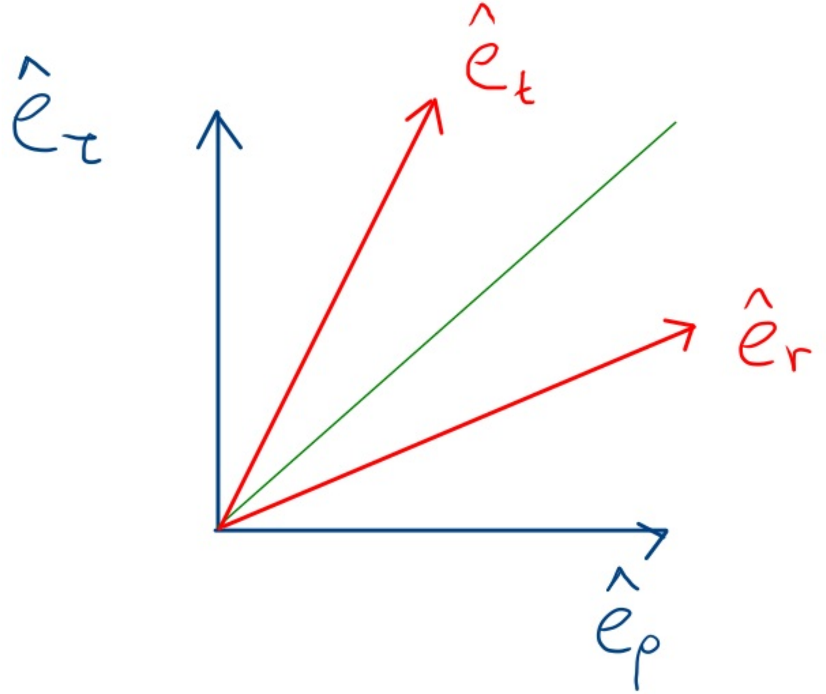
• $R_{\hat\tau\hat\rho\hat\tau\hat\rho} = R_{\hat t\hat r\hat t\hat r}$



• Components of R are invariant under this Lorentz boost!

• The freely falling observer measures the same R-components!

So: • $R_{\hat\theta\hat\phi\hat\theta\hat\phi}$ unchanged



• $R_{\hat\tau\hat\theta\hat\tau\hat\theta} = R_{\hat t\hat\theta\hat t\hat\theta}$

• $R_{\hat\rho\hat\theta\hat\rho\hat\theta} = R_{\hat r\hat\theta\hat r\hat\theta}$

• $R_{\hat\rho\hat\phi\hat\rho\hat\phi} = R_{\hat r\hat\phi\hat r\hat\phi}$

• $R_{\hat\tau\hat\rho\hat\tau\hat\rho} = R_{\hat t\hat r\hat t\hat r}$

• special for the Schwarzschild metric:

- structure of non zero $R_{\hat\mu\hat\nu\hat\lambda\hat\sigma}$ components
- values of components, e.g. we used $R_{\hat t\hat\theta\hat t\hat\theta} = -R_{\hat r\hat\theta\hat r\hat\theta}$, etc

use results from

$\{e_{\hat{t}}, e_{\hat{r}}, e_{\hat{\theta}}, e_{\hat{\phi}}\}$

calculation

$$\left\{\begin{aligned} -R_{\hat{\rho}\hat{\tau}\hat{\rho}\hat{\tau}} &= R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{2M}{r^3} \\ R_{\hat{\theta}\hat{\tau}\hat{\theta}\hat{\tau}} &= R_{\hat{\phi}\hat{\tau}\hat{\phi}\hat{\tau}} = \frac{M}{r^3} \\ R_{\hat{\rho}\hat{\theta}\hat{\rho}\hat{\theta}} &= R_{\hat{\rho}\hat{\phi}\hat{\rho}\hat{\phi}} = -\frac{M}{r^3} \end{aligned}\right.$$

$$\frac{D^2\xi^{\hat\tau}}{d\tau^2} = -\underbrace{R^{\hat\tau}{}_{\hat\tau\hat\sigma\hat\tau}}_{\text{no nonzero } R \text{ with } 3\ \hat t\text{s}}\xi^{\hat\sigma} = 0$$

$$-R_{\hat\rho\hat\tau\hat\rho\hat\tau} = R_{\hat\theta\hat\phi\hat\theta\hat\phi} = \frac{2M}{r^3}$$

$$R_{\hat\theta\hat\tau\hat\theta\hat\tau} = R_{\hat\phi\hat\tau\hat\phi\hat\tau} = \frac{M}{r^3}$$

$$R_{\hat\rho\hat\theta\hat\rho\hat\theta} = R_{\hat\rho\hat\phi\hat\rho\hat\phi} = -\frac{M}{r^3}$$

- Relative accelerations given by geodesic deviation equation:

$$\frac{D^2\xi^{\hat\mu}}{d\tau^2} = R^{\hat\mu}{}_{\hat\nu\hat\lambda\hat\sigma}u^{\hat\nu}u^{\hat\lambda}\xi^{\hat\sigma} = -R^{\hat\mu}{}_{\hat\tau\hat\sigma\hat\tau}\xi^{\hat\sigma}$$

antisymmetric change

$$\frac{D^2\xi^{\hat\tau}}{d\tau^2} = -R^{\hat\tau}{}_{\hat\tau\hat\sigma\hat\tau}\,\xi^{\hat\sigma} = 0$$

$$\frac{D^2\xi^{\hat\rho}}{d\tau^2} = -R^{\hat\rho}{}_{\hat\tau\hat\sigma\hat\tau}\,\xi^{\hat\sigma} = -R_{\hat\rho\hat\tau\hat\rho\hat\tau}\,\xi^{\hat r}$$

lower index with $\eta_{\hat\mu\hat\nu}$

$$-R_{\hat\rho\hat\tau\hat\rho\hat\tau} = R_{\hat\theta\hat\phi\hat\theta\hat\phi} = \frac{2M}{r^3}$$

$$R_{\hat\theta\hat\tau\hat\theta\hat\tau} = R_{\hat\phi\hat\tau\hat\phi\hat\tau} = \frac{M}{r^3}$$

$$R_{\hat\rho\hat\theta\hat\rho\hat\theta} = R_{\hat\rho\hat\phi\hat\rho\hat\phi} = -\frac{M}{r^3}$$

– Relative accelerations given by geodesic deviation equation:

$$\frac{D^2\xi^{\hat\mu}}{d\tau^2} = R^{\hat\mu}{}_{\hat\nu\hat\lambda\hat\sigma}\,u^{\hat\nu}u^{\hat\lambda}\xi^{\hat\sigma} = -R^{\hat\mu}{}_{\hat\tau\hat\sigma\hat\tau}\,\xi^{\hat\sigma}$$

$$\frac{D^2 \xi^{\hat{\tau}}}{d\tau^2} = -R^{\hat{\tau}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\, \xi^{\hat{\sigma}} = 0$$

$$\frac{D^2 \xi^{\hat{\rho}}}{d\tau^2} = -R^{\hat{\rho}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\, \xi^{\hat{\sigma}} = -R_{\hat{\rho}\hat{\tau}\hat{\rho}\hat{\tau}}\, \xi^{\hat{r}}$$

$$= -\left(-\frac{2M}{r^3}\right) \xi^{\hat{\rho}} = +\frac{2M}{r^3}\, \xi^{\hat{\rho}}$$

$$-R_{\hat{\rho}\hat{\tau}\hat{\rho}\hat{\tau}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{\tau}\hat{\theta}\hat{\tau}} = R_{\hat{\phi}\hat{\tau}\hat{\phi}\hat{\tau}} = \frac{M}{r^3}$$

$$R_{\hat{\rho}\hat{\theta}\hat{\rho}\hat{\theta}} = R_{\hat{\rho}\hat{\phi}\hat{\rho}\hat{\phi}} = -\frac{M}{r^3}$$

- Relative accelerations given by geodesic deviation equation:

$$\frac{D^2 \xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}}\, u^{\hat{\nu}} u^{\hat{\lambda}} \xi^{\hat{\sigma}} = -R^{\hat{\mu}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\, \xi^{\hat{\sigma}}$$

$$\frac{D^2 \xi^{\hat{\tau}}}{d\tau^2} = - R^{\hat{\tau}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\, \xi^{\hat{\sigma}} = 0$$

$$\frac{D^2 \xi^{\hat{\rho}}}{d\tau^2} = \frac{2M}{r^3}\, \xi^{\hat{\rho}}$$

↳ relative acceleration of head w.r.t. waist

$$-R_{\hat{\rho}\hat{\tau}\hat{\rho}\hat{\tau}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{\tau}\hat{\theta}\hat{\tau}} = R_{\hat{\phi}\hat{\tau}\hat{\phi}\hat{\tau}} = \frac{M}{r^3}$$

$$R_{\hat{\rho}\hat{\theta}\hat{\rho}\hat{\theta}} = R_{\hat{\rho}\hat{\phi}\hat{\rho}\hat{\phi}} = -\frac{M}{r^3}$$

- Relative accelerations given by geodesic deviation equation:

$$\frac{D^2 \xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}}\, u^{\hat{\nu}} u^{\hat{\lambda}} \xi^{\hat{\sigma}} = -R^{\hat{\mu}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\, \xi^{\hat{\sigma}}$$

$$\frac{D^2 \xi^{\hat{\tau}}}{d\tau^2} = -R^{\hat{\tau}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\, \xi^{\hat{\sigma}} = 0$$

$$\frac{D^2 \xi^{\hat{\rho}}}{d\tau^2} = \frac{2M}{r^3}\, \xi^{\hat{\rho}}$$

↳ relative acceleration of head w.r.t. waist

→ there is no $\left(1 - \frac{2M}{r}\right)$

$$-R_{\hat{\rho}\hat{\tau}\hat{\rho}\hat{\tau}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{\tau}\hat{\theta}\hat{\tau}} = R_{\hat{\phi}\hat{\tau}\hat{\phi}\hat{\tau}} = \frac{M}{r^3}$$

$$R_{\hat{\rho}\hat{\theta}\hat{\rho}\hat{\theta}} = R_{\hat{\rho}\hat{\phi}\hat{\rho}\hat{\phi}} = -\frac{M}{r^3}$$

- Relative accelerations given by geodesic deviation equation:

$$\frac{D^2 \xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}}\, u^{\hat{\nu}} u^{\hat{\lambda}} \xi^{\hat{\sigma}} = -R^{\hat{\mu}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\, \xi^{\hat{\sigma}}$$

$$\frac{D^2\xi^{\hat\tau}}{d\tau^2} = -R^{\hat\tau}{}_{\hat\tau\hat\sigma\hat\tau}\,\xi^{\hat\sigma} = 0$$

$$\frac{D^2\xi^{\hat\rho}}{d\tau^2} = \frac{2M}{r^3}\,\xi^{\hat\rho}$$

↳ relative acceleration of head w.r.t. waist

→ there is no $\left(1-\frac{2M}{r}\right)$

→ blows up as $r \to 0$

$$-R_{\hat\rho\hat\tau\hat\rho\hat\tau} = R_{\hat\theta\hat\phi\hat\theta\hat\phi} = \frac{2M}{r^3}$$

$$R_{\hat\theta\hat\tau\hat\theta\hat\tau} = R_{\hat\phi\hat\tau\hat\phi\hat\tau} = \frac{M}{r^3}$$

$$R_{\hat\rho\hat\theta\hat\rho\hat\theta} = R_{\hat\rho\hat\phi\hat\rho\hat\phi} = -\frac{M}{r^3}$$

- Relative accelerations given by geodesic deviation equation:

$$\frac{D^2\xi^{\hat\mu}}{d\tau^2} = R^{\hat\mu}{}_{\hat\nu\hat\lambda\hat\sigma}\,u^{\hat\nu}u^{\hat\lambda}\xi^{\hat\sigma} = -R^{\hat\mu}{}_{\hat\tau\hat\sigma\hat\tau}\,\xi^{\hat\sigma}$$

$$\frac{D^2 \xi^{\hat{\tau}}}{d\tau^2} = - R^{\hat{\tau}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}} \xi^{\hat{\sigma}} = 0$$

$$\frac{D^2 \xi^{\hat{\rho}}}{d\tau^2} = \frac{2M}{r^3} \xi^{\hat{\rho}}$$

↳ relative acceleration of head w.r.t. waist

→ there is no $\left(1 - \frac{2M}{r}\right)$

→ blows up as $r \to 0$

→ head moves away from feet

$$-R_{\hat{\rho}\hat{\tau}\hat{\rho}\hat{\tau}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{\tau}\hat{\theta}\hat{\tau}} = R_{\hat{\phi}\hat{\tau}\hat{\phi}\hat{\tau}} = \frac{M}{r^3}$$

$$R_{\hat{\rho}\hat{\theta}\hat{\rho}\hat{\theta}} = R_{\hat{\rho}\hat{\phi}\hat{\rho}\hat{\phi}} = -\frac{M}{r^3}$$

– Relative accelerations given by geodesic deviation equation:

$$\frac{D^2 \xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}} u^{\hat{\nu}} u^{\hat{\lambda}} \xi^{\hat{\sigma}} = -R^{\hat{\mu}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}} \xi^{\hat{\sigma}}$$

$$\frac{D^2 \xi^{\hat{\tau}}}{d\tau^2} = -R^{\hat{\tau}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\,\xi^{\hat{\sigma}} = 0$$

$$\frac{D^2 \xi^{\hat{\rho}}}{d\tau^2} = \frac{2M}{r^3}\,\xi^{\hat{\rho}}$$

$$\frac{D^2 \xi^{\hat{\theta}}}{d\tau^2} = -R^{\hat{\theta}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\,\xi^{\hat{\sigma}} = -R_{\hat{\theta}\hat{\tau}\hat{\theta}\hat{\tau}}\,\xi^{\hat{\theta}}$$

$$= -\frac{M}{r^3}\,\xi^{\hat{\theta}}$$

$$-R_{\hat{\rho}\hat{\tau}\hat{\rho}\hat{\tau}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{\tau}\hat{\theta}\hat{\tau}} = R_{\hat{\phi}\hat{\tau}\hat{\phi}\hat{\tau}} = \frac{M}{r^3}$$

$$R_{\hat{\rho}\hat{\theta}\hat{\rho}\hat{\theta}} = R_{\hat{\rho}\hat{\phi}\hat{\rho}\hat{\phi}} = -\frac{M}{r^3}$$

- Relative accelerations given by geodesic deviation equation:

$$\frac{D^2 \xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}}\,u^{\hat{\nu}}u^{\hat{\lambda}}\,\xi^{\hat{\sigma}} = -R^{\hat{\mu}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\,\xi^{\hat{\sigma}}$$

$$\frac{D^2\xi^{\hat\tau}}{d\tau^2} = -R^{\hat\tau}{}_{\hat\tau\hat\sigma\hat\tau}\,\xi^{\hat\sigma} = 0$$

$$\frac{D^2\xi^{\hat\rho}}{d\tau^2} = \frac{2M}{r^3}\,\xi^{\hat\rho}$$

$$\frac{D^2\xi^{\hat\theta}}{d\tau^2} = -\frac{M}{r^3}\,\xi^{\hat\theta}$$

$$-R_{\hat\rho\hat\tau\hat\rho\hat\tau} = R_{\hat\theta\hat\phi\hat\theta\hat\phi} = \frac{2M}{r^3}$$

$$R_{\hat\theta\hat\tau\hat\theta\hat\tau} = R_{\hat\phi\hat\tau\hat\phi\hat\tau} = \frac{M}{r^3}$$

$$R_{\hat\rho\hat\theta\hat\rho\hat\theta} = R_{\hat\rho\hat\phi\hat\rho\hat\phi} = -\frac{M}{r^3}$$

- Relative accelerations given by geodesic deviation equation:

$$\frac{D^2\xi^{\hat\mu}}{d\tau^2} = R^{\hat\mu}{}_{\hat\nu\hat\lambda\hat\sigma}\,u^{\hat\nu}u^{\hat\lambda}\xi^{\hat\sigma} = -R^{\hat\mu}{}_{\hat\tau\hat\sigma\hat\tau}\,\xi^{\hat\sigma}$$

$$\frac{D^2 \xi^{\hat{\tau}}}{d\tau^2} = - R^{\hat{\tau}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\, \xi^{\hat{\sigma}} = 0$$

$$\frac{D^2 \xi^{\hat{\rho}}}{d\tau^2} = \frac{2M}{r^3}\, \xi^{\hat{\rho}}$$

$$\frac{D^2 \xi^{\hat{\theta}}}{d\tau^2} = -\frac{M}{r^3}\, \xi^{\hat{\theta}}$$

→ nothing wrong at $r=2M$

→ left + right hand approach squeezed as $r \to 0$

$$-R_{\hat{\rho}\hat{\tau}\hat{\rho}\hat{\tau}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{\tau}\hat{\theta}\hat{\tau}} = R_{\hat{\phi}\hat{\tau}\hat{\phi}\hat{\tau}} = \frac{M}{r^3}$$

$$R_{\hat{\rho}\hat{\theta}\hat{\rho}\hat{\theta}} = R_{\hat{\rho}\hat{\phi}\hat{\rho}\hat{\phi}} = -\frac{M}{r^3}$$

– Relative accelerations given by geodesic deviation equation:

$$\frac{D^2 \xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}}\, u^{\hat{\nu}} u^{\hat{\lambda}} \xi^{\hat{\sigma}} = -R^{\hat{\mu}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\, \xi^{\hat{\sigma}}$$

$$\frac{D^2 \xi^{\hat{\tau}}}{d\tau^2} = -R^{\hat{\tau}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\,\xi^{\hat{\sigma}} = 0$$

$$\frac{D^2 \xi^{\hat{\rho}}}{d\tau^2} = \frac{2M}{r^3}\,\xi^{\hat{\rho}}$$

$$\frac{D^2 \xi^{\hat{\theta}}}{d\tau^2} = -\frac{M}{r^3}\,\xi^{\hat{\theta}}$$

$$\frac{D^2 \xi^{\hat{\phi}}}{d\tau^2} = -\frac{M}{r^3}\,\xi^{\hat{\phi}}$$

$$-R_{\hat{\rho}\hat{\tau}\hat{\rho}\hat{\tau}} = R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = \frac{2M}{r^3}$$

$$R_{\hat{\theta}\hat{\tau}\hat{\theta}\hat{\tau}} = R_{\hat{\phi}\hat{\tau}\hat{\phi}\hat{\tau}} = \frac{M}{r^3}$$

$$R_{\hat{\rho}\hat{\theta}\hat{\rho}\hat{\theta}} = R_{\hat{\rho}\hat{\phi}\hat{\rho}\hat{\phi}} = -\frac{M}{r^3}$$

- Relative accelerations given by geodesic deviation equation:

$$\frac{D^2 \xi^{\hat{\mu}}}{d\tau^2} = R^{\hat{\mu}}{}_{\hat{\nu}\hat{\lambda}\hat{\sigma}}\,u^{\hat{\nu}} u^{\hat{\lambda}} \xi^{\hat{\sigma}} = -R^{\hat{\mu}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\,\xi^{\hat{\sigma}}$$

$$\frac{D^2 \xi^{\hat{\tau}}}{d\tau^2} = - R^{\hat{\tau}}{}_{\hat{\tau}\hat{\sigma}\hat{\tau}}\, \xi^{\hat{\sigma}} = 0$$

$$\frac{D^2 \xi^{\hat{\rho}}}{d\tau^2} = \frac{2M}{r^3}\, \xi^{\hat{\rho}}$$

$$\frac{D^2 \xi^{\hat{\theta}}}{d\tau^2} = -\frac{M}{r^3}\, \xi^{\hat{\theta}}$$

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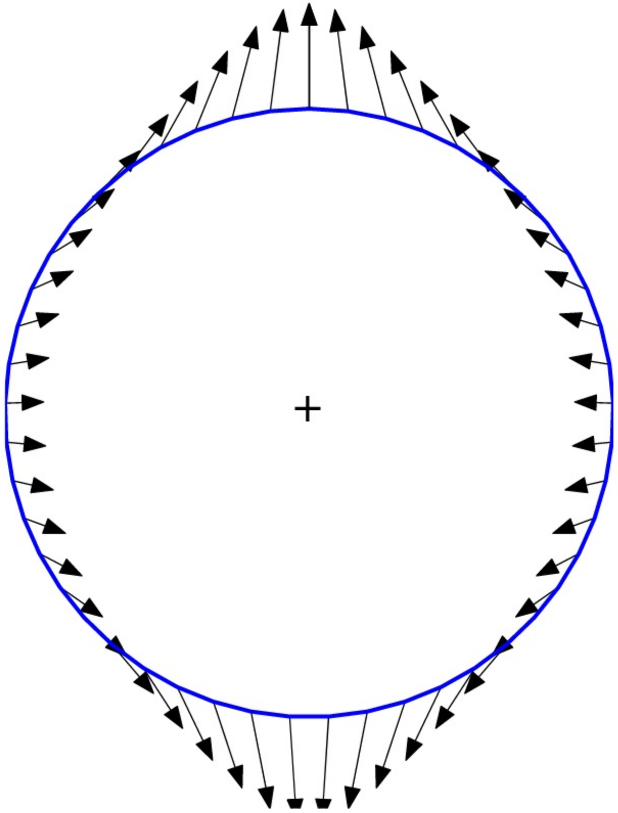
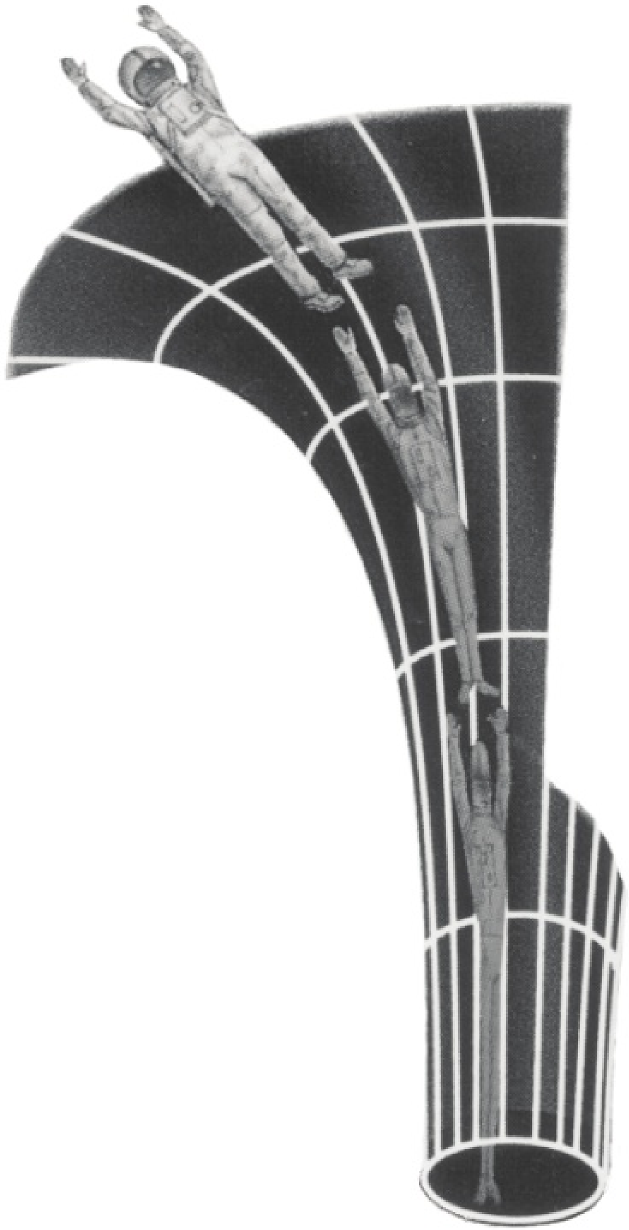
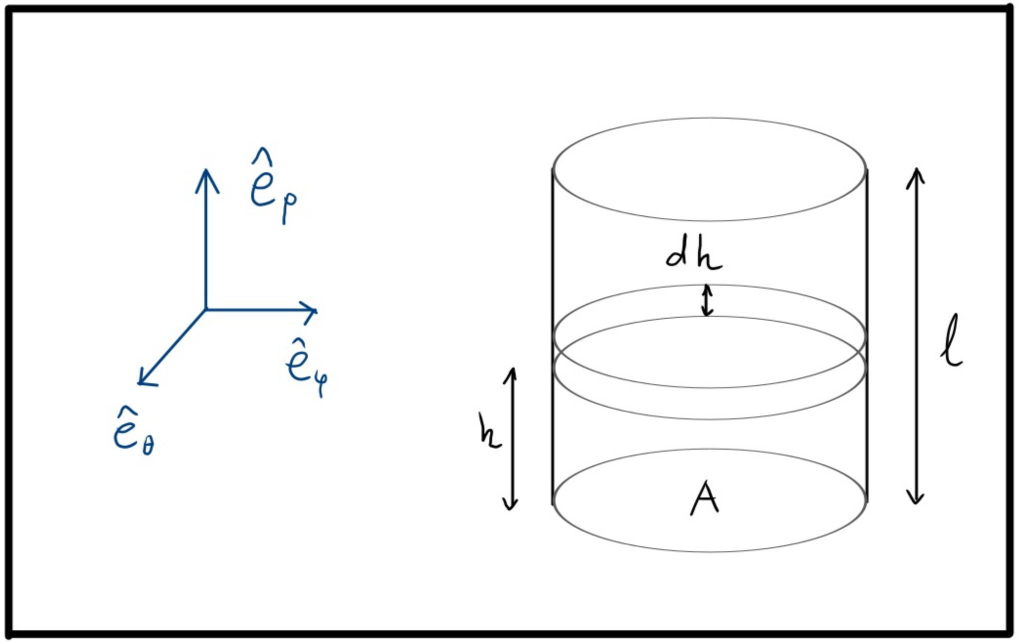




spaghettification ...

$$\frac{D^2 \xi^{\hat{\rho}}}{d\tau^2} = \frac{2M}{r^3}\, \xi^{\hat{\rho}}$$



Relative acceleration of disks at distance h:

$$a = \frac{2M}{r^3} \cdot h$$

$$\frac{D^2 \xi^{\hat{\rho}}}{d\tau^2} \qquad \xi^{\hat{\rho}}$$

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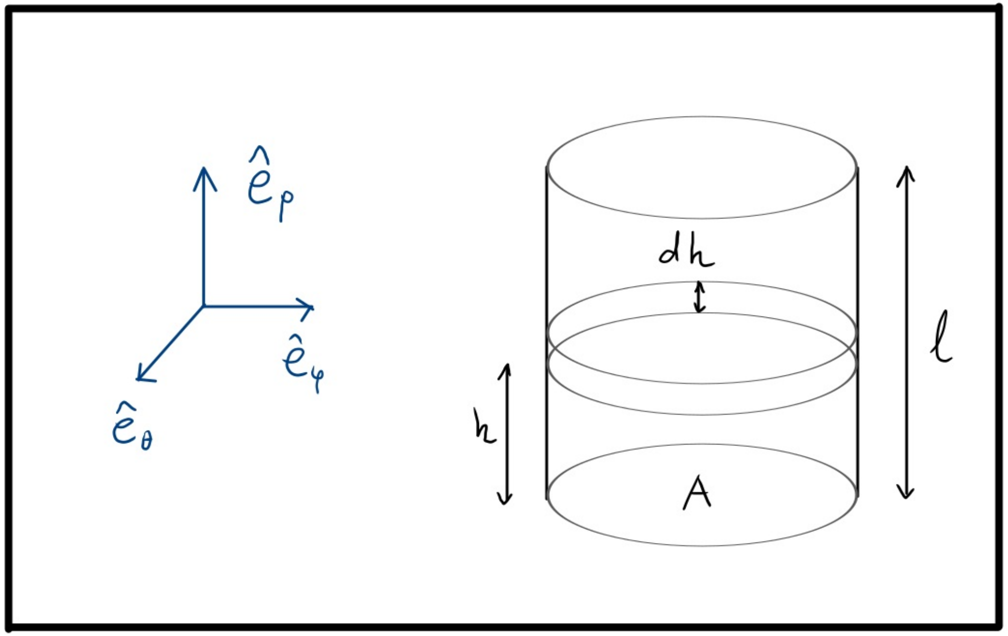

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Astronaut of height l:

$$\left(\begin{matrix} \text{Pressure on} \\ \text{waist} \end{matrix} \right) = \frac{1}{A} \int_0^{l/2} dF = \frac{1}{A} \int_0^{l/2} a \, dm$$

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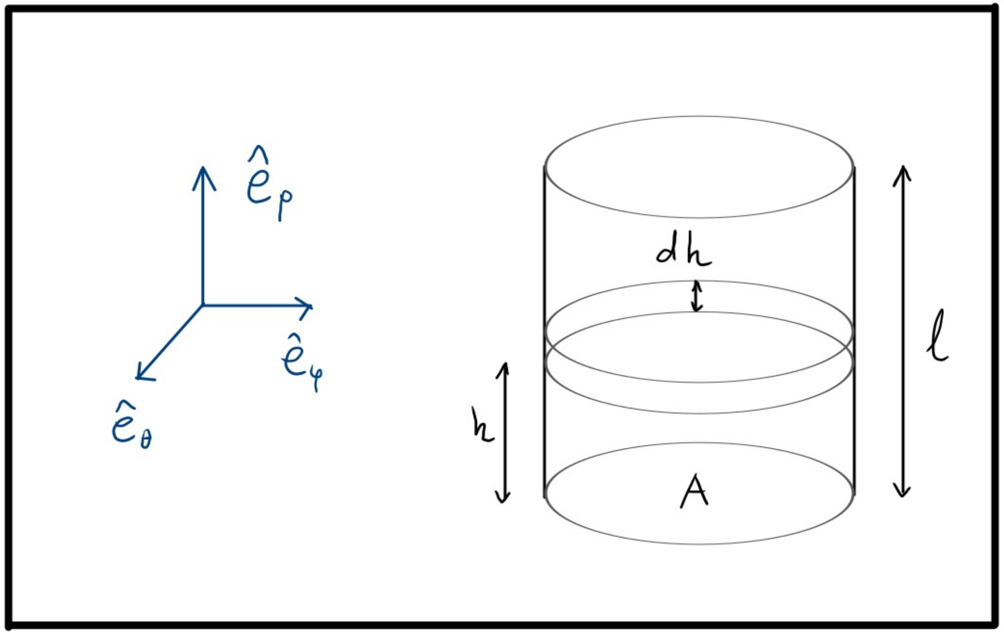

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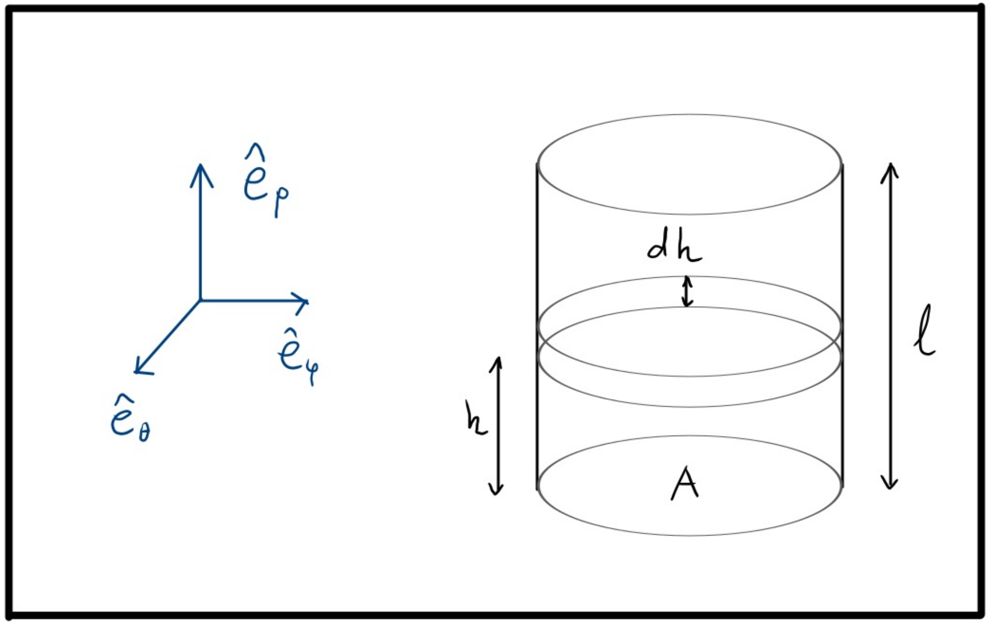

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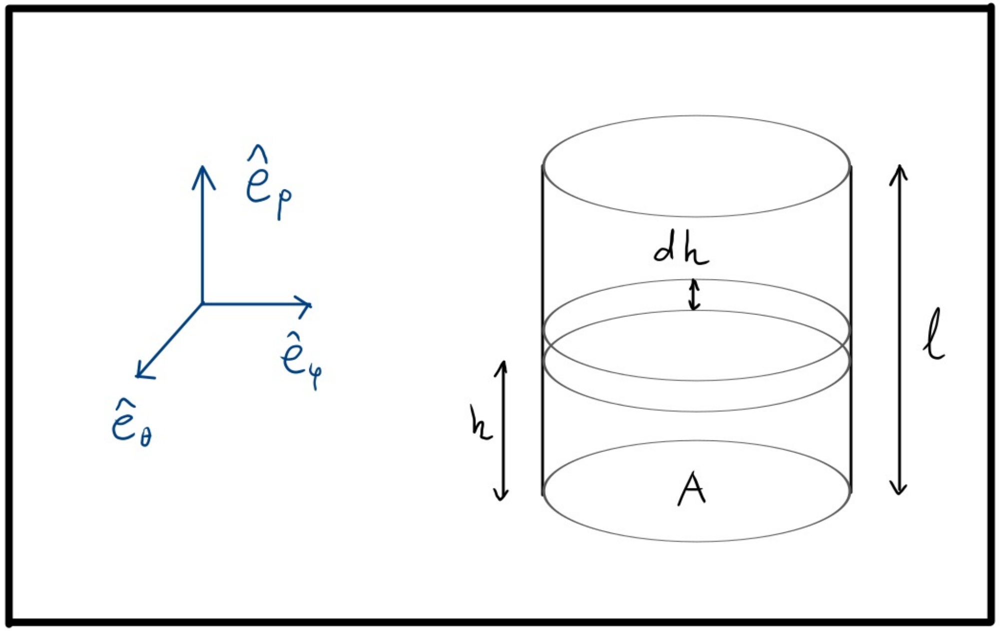
$$= \frac{1}{4}\,\frac{mM\ell}{A r^3}$$

$m = 75\,\text{kg} \quad A = (0.2\,\text{m})^2 \quad \ell = 1.8\,\text{m}$

$\Rightarrow$

$$(\text{Pressure}) \approx 1.1 \times 10^9 \, \frac{(M/M_\odot)}{(r/1\,\text{km})^3} \, \text{Atm}$$

Misner § 32.6



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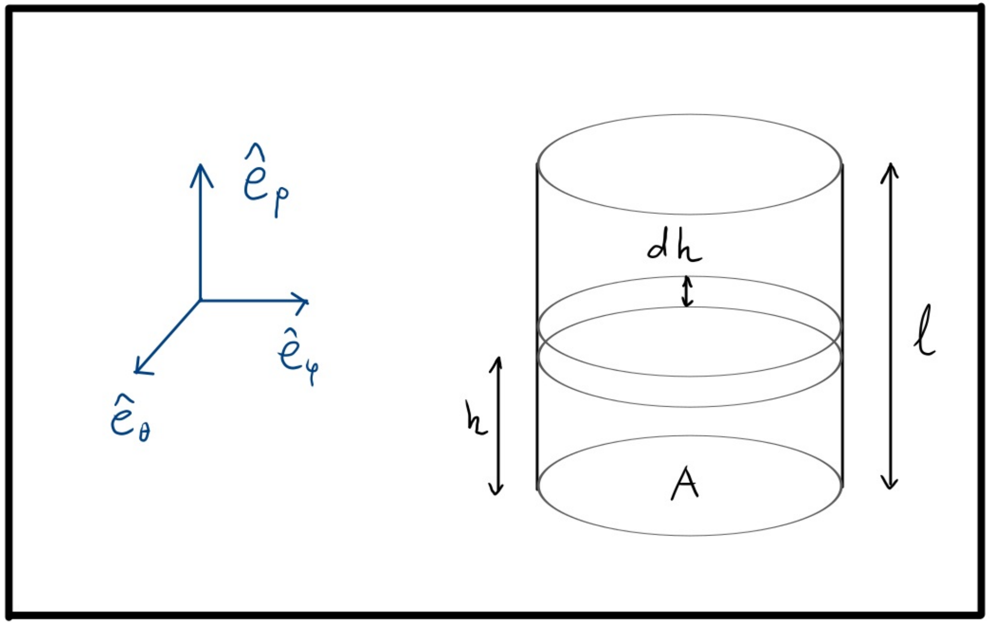

For a stellar black hole

$M \approx M_\odot \approx 1.5\,km \qquad r_s \approx 3.0\,km$

so at $r = r_s$ $\quad (\text{pressure}) \approx 10^9 \frac{1}{3^3}\ \text{Atm} \approx 10^7\ \text{Atm}$

Human body may withstand $\approx 10^2$ Atm

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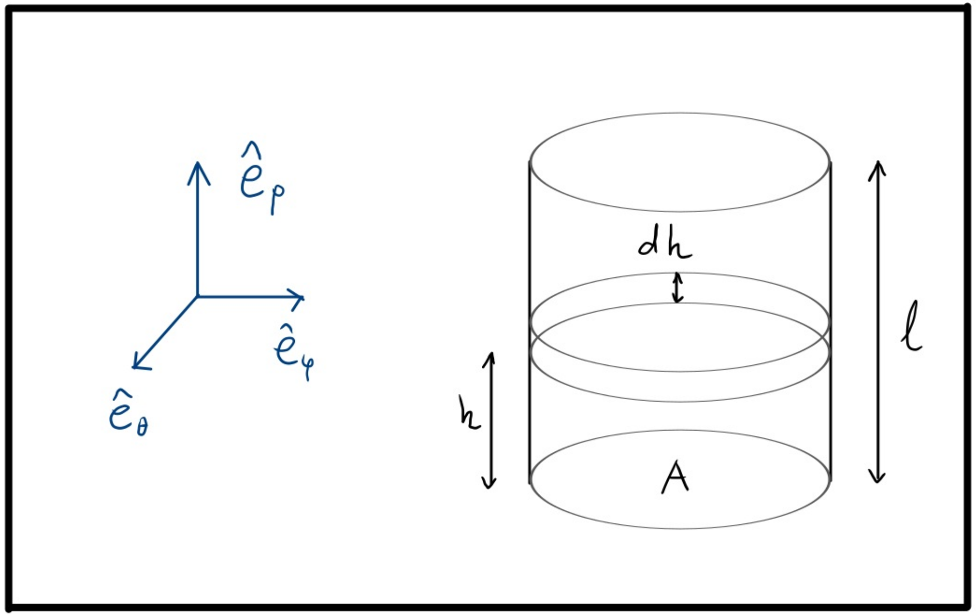


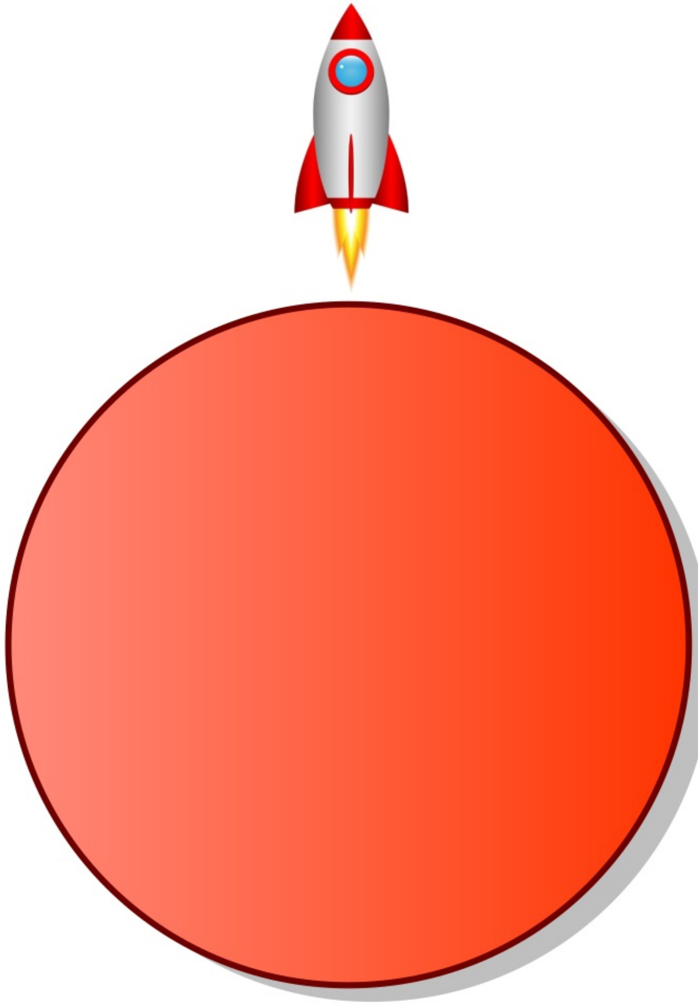
For supermassive black hole

$M \approx 10^9 M_\odot \approx 10^9\,\mathrm{km}$ $\qquad r_s \approx 10^9\,\mathrm{km}$

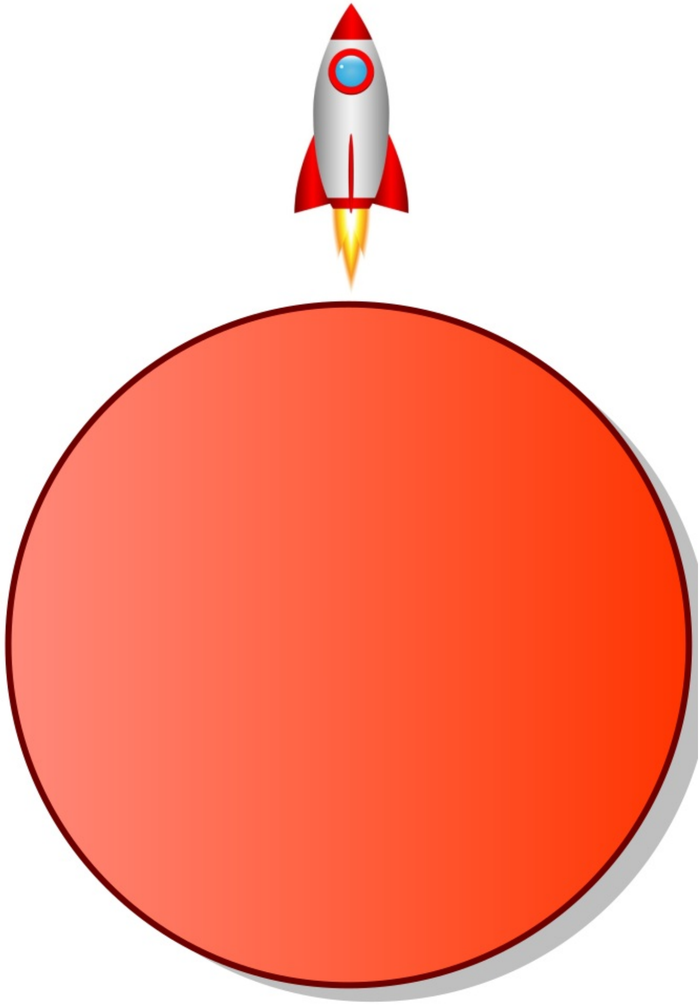
so at $r = r_s$ $\quad (\text{pressure}) \approx 10^9 \dfrac{10^9}{(10^9)^3} \text{ Atm} \approx 10^{18-27} \text{ Atm} \lessapprox 10^{-9} \text{ Atm}$

Human body may withstand $\approx 10^2$ Atm

- Freely falling observer sees nothing special crossing the horizon

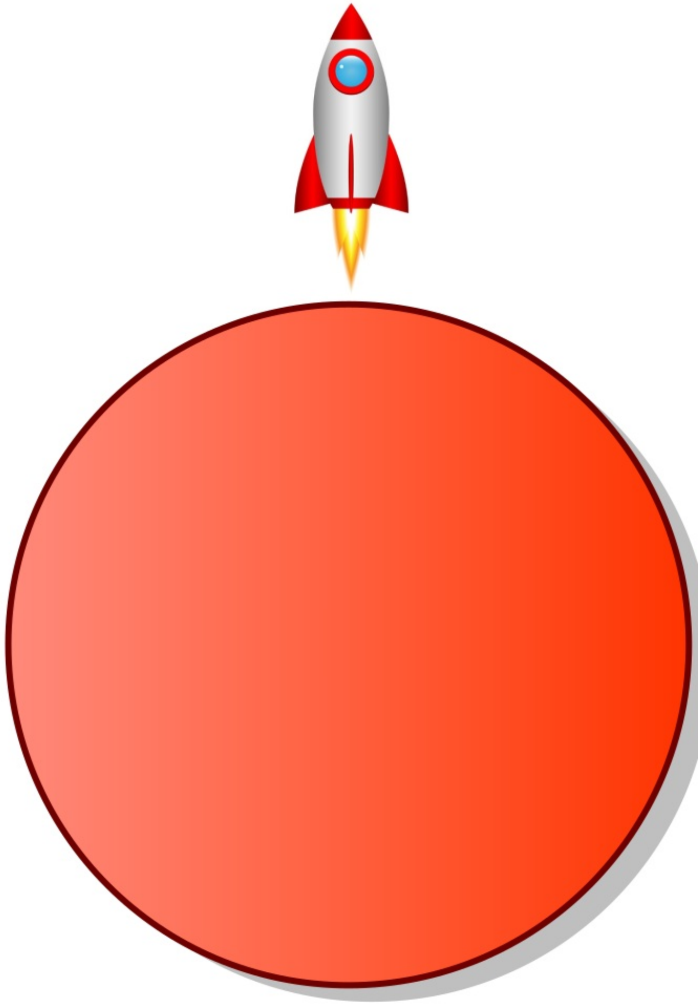


- Freely falling observer sees nothing special crossing the horizon
- But an (accelerated) observer struggles infinitely hard to remain stationary infinitesimally close to the horizon



- Freely falling observer sees nothing special crossing the horizon
- But an (accelerated) observer struggles infinitely hard to remain stationary infinitesimally close to the horizon
- Indeed, 4-force per unit mass needed is:

$$f^{\mu} = \frac{Du^{\mu}}{d\tau} = u^{\nu}\nabla_{\nu}u^{\mu} = \frac{du^{\mu}}{d\tau} + \Gamma^{\mu}{}_{\nu\rho}u^{\nu}u^{\rho}$$



$$\Gamma^0{}_{10} = \frac{M}{r(r-2M)} \qquad \Gamma^1{}_{00} = \frac{M(r-2M)}{r^3} \qquad \Gamma^1{}_{11} = \frac{M}{2Mr - r^2}$$

$$\Gamma^1{}_{22} = 2M - r \qquad \Gamma^1{}_{33} = (2M-r)\sin^2\theta \qquad \Gamma^2{}_{21} = \frac{1}{r}$$

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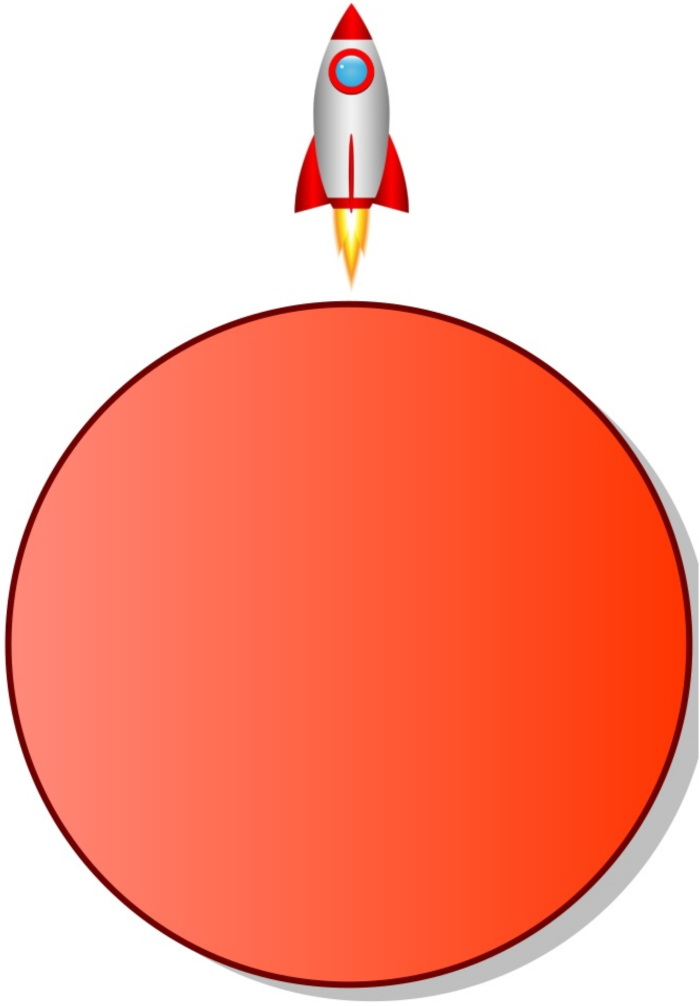

$$f^\mu = \frac{Du^\mu}{d\tau} = u^\nu \nabla_\nu u^\mu = \frac{du^\mu}{d\tau} + \Gamma^\mu{}_{\nu\rho} u^\nu u^\rho$$

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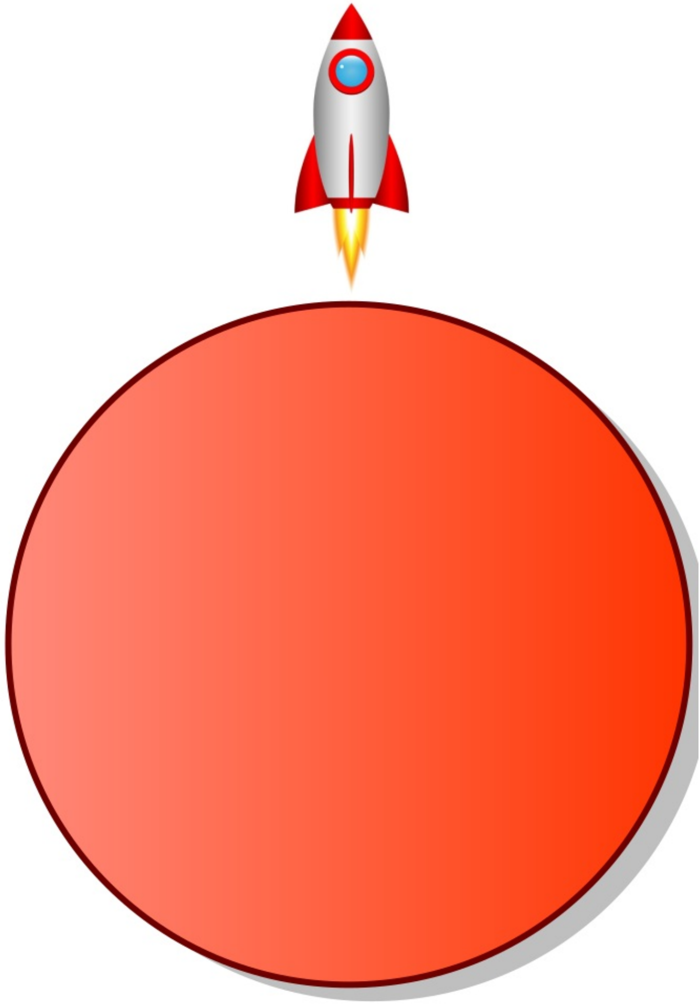
$$U^\mu = [U^0, 0, 0, 0]$$

$$U_\mu U^\mu = -1 \Rightarrow -\left(1-\frac{2M}{r}\right)(U^0)^2 = -1 \Rightarrow U^0 = \left(1-\frac{2M}{r}\right)^{-1/2}$$

$$f^\mu = \frac{DU^\mu}{d\tau} = U^\nu \nabla_\nu U^\mu = \frac{dU^\mu}{d\tau} + \Gamma^\mu{}_{\nu\rho} U^\nu U^\rho$$



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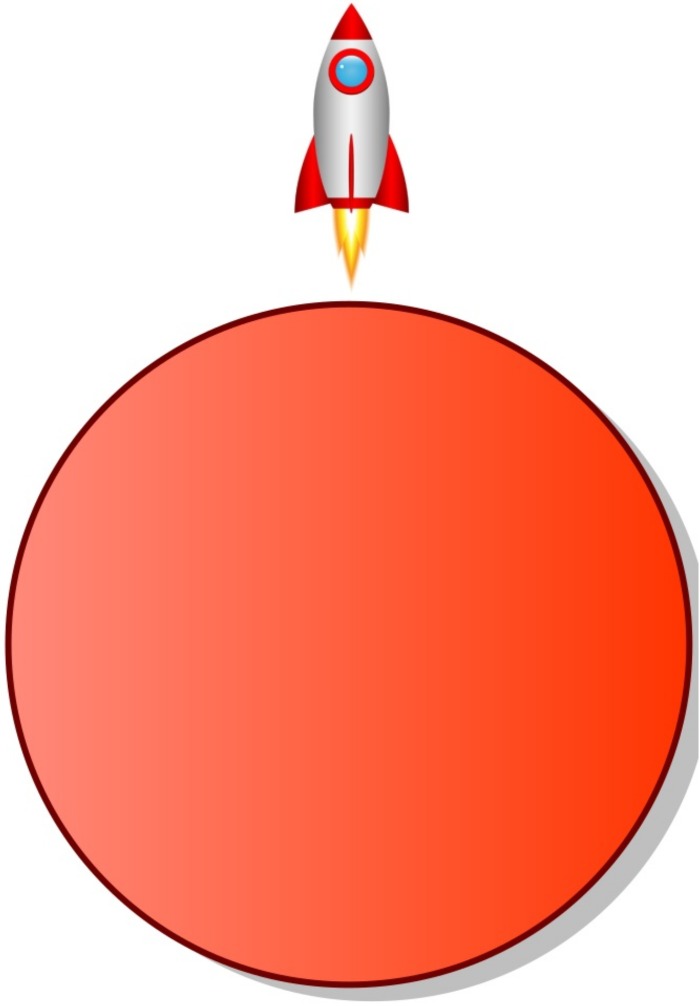

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Since she stays at fixed $r = R$

$$\frac{du^\mu}{d\tau} = 0$$

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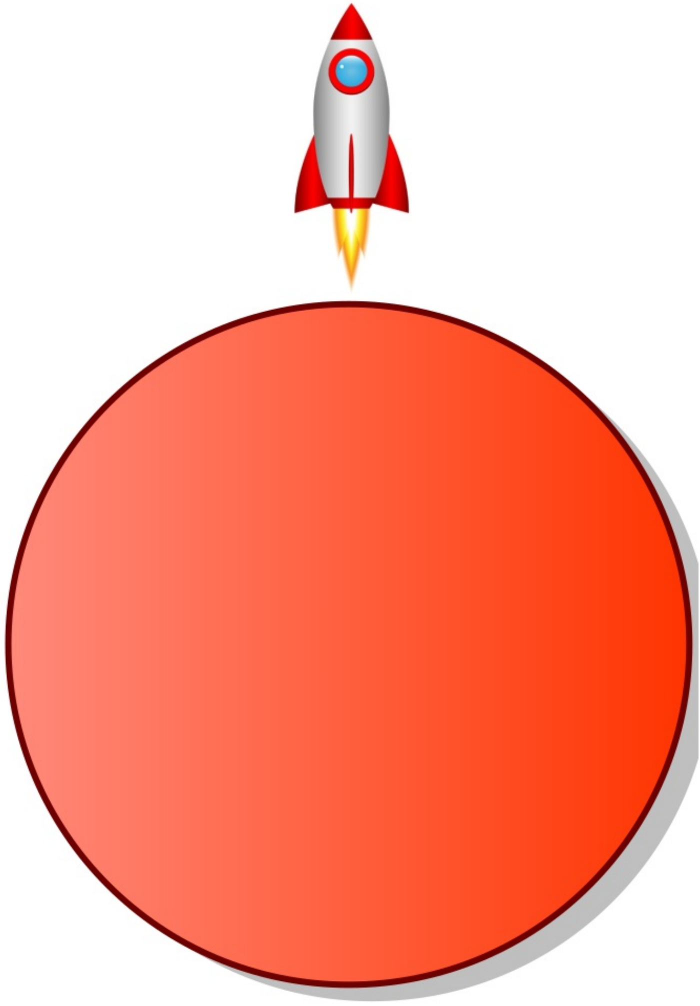

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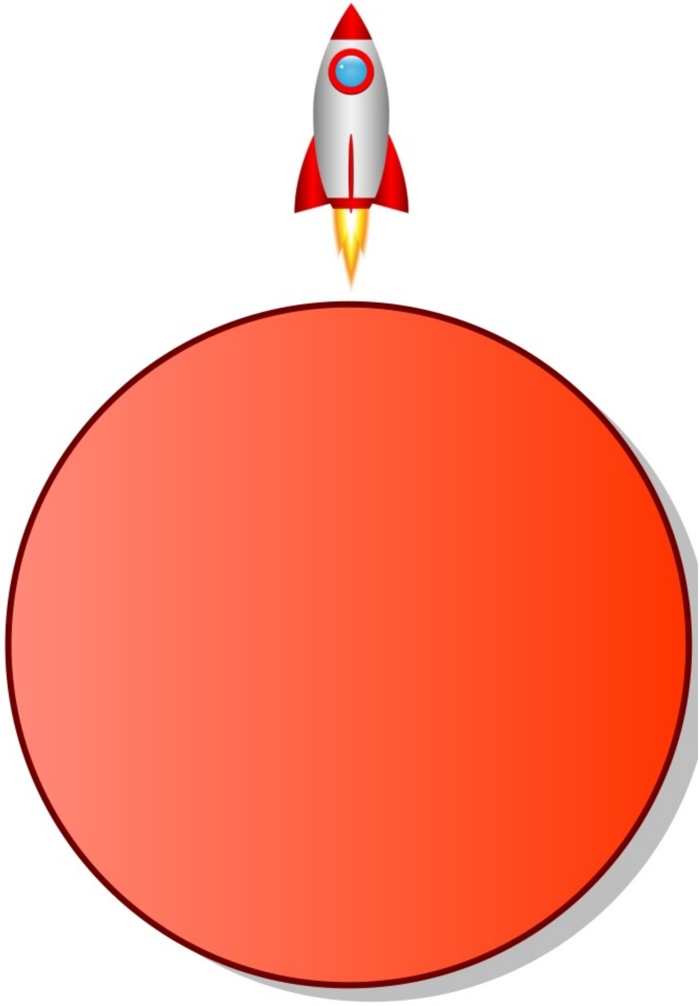
Since she stays at fixed $r = R$

$$\frac{du^\mu}{d\tau} = 0 \quad \Rightarrow$$

all other components $= 0$

$$\frac{Du^\mu}{d\tau} = 0 + \Gamma^\mu{}_{00}\, u^0 u^0$$



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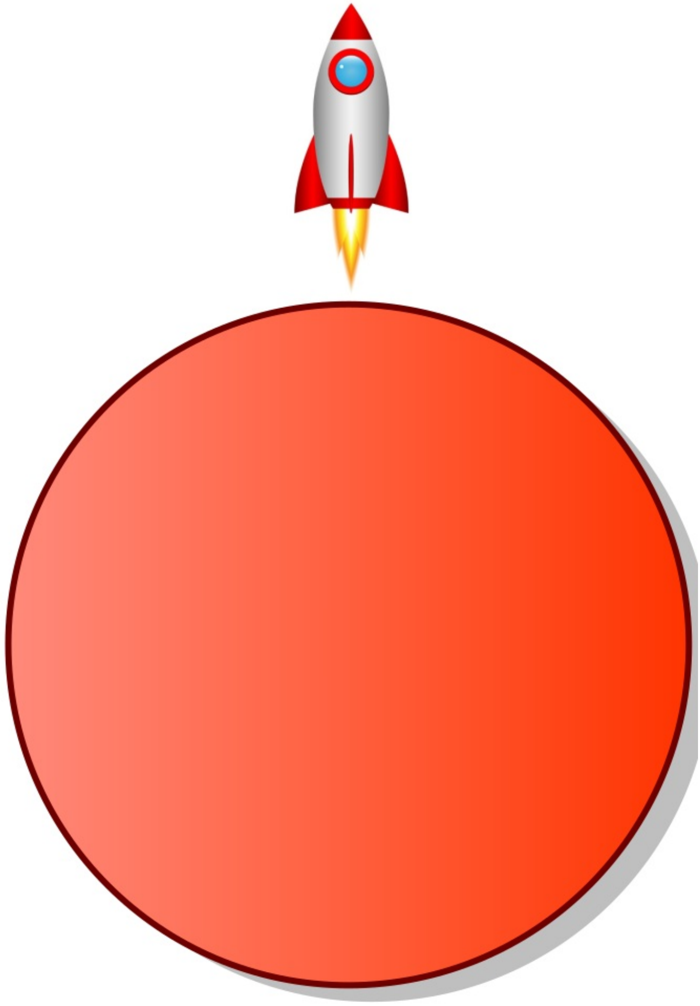
Only $\Gamma^1{}_{00} = \frac{M}{r^2}\left(1 - \frac{2M}{r}\right) \neq 0$



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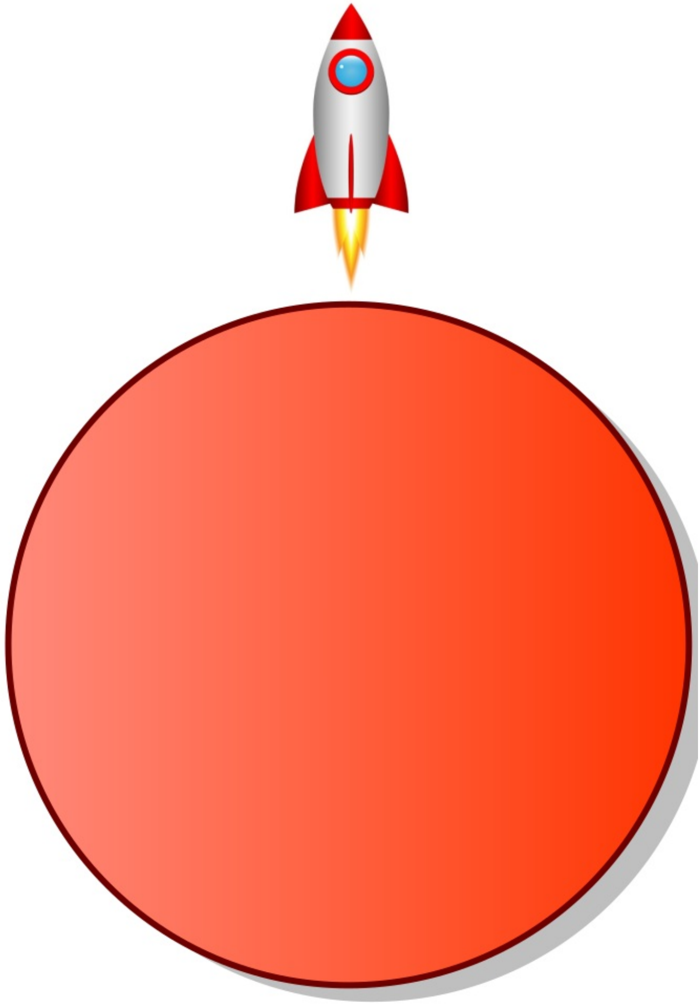

$$u^\mu = \left[\left(1-\frac{2M}{R}\right)^{-1/2}, 0, 0, 0\right]$$

Since she stays at fixed $r = R$

$$\frac{du^\mu}{d\tau} = 0 \quad \Rightarrow$$

$$\frac{Du^1}{d\tau} = \Gamma^1{}_{00}\, u^0 u^0 = \frac{M}{r^2}\cancel{\left(1-\frac{2M}{r}\right)}\left[\cancel{\left(1-\frac{2M}{r}\right)}^{-\frac{1}{2}}\right]^2$$

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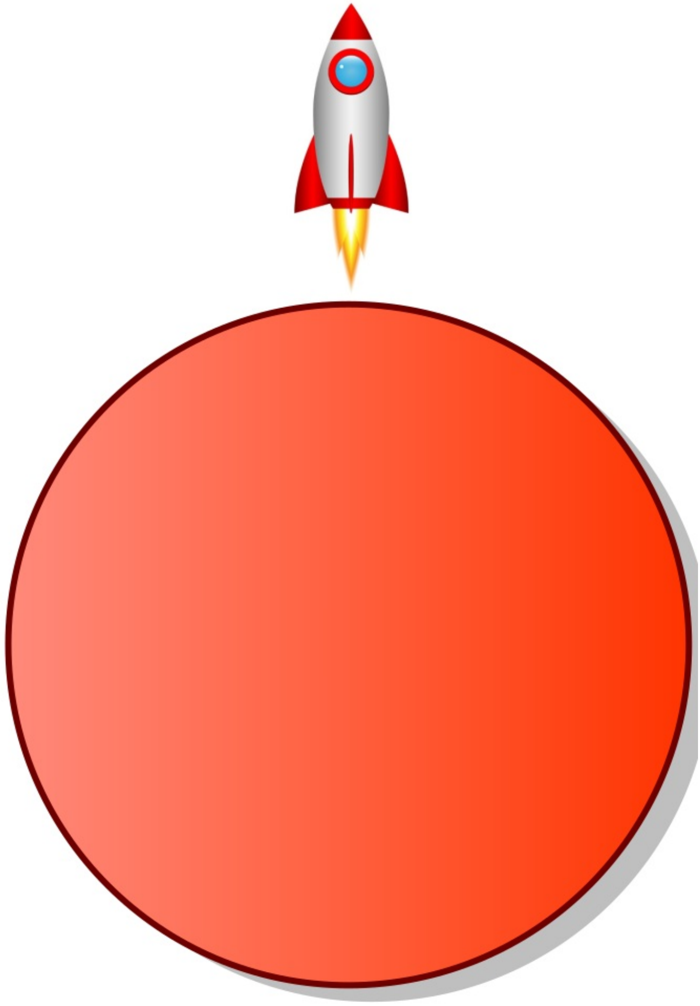
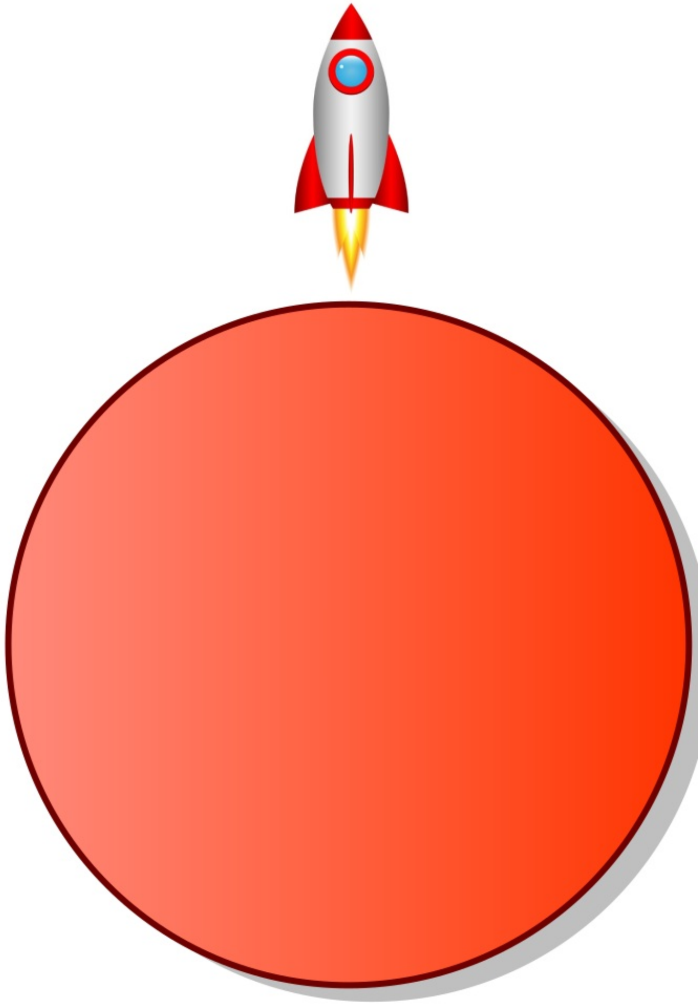
$$f^r = \frac{M}{r^2}$$

$$f^\mu = \left[0, \frac{M}{R^2}, 0, 0\right]$$



But the observer measures force in her local inertial frame $\{\hat{e}_0, \hat{e}_r, \hat{e}_\theta, \hat{e}_\phi\}$



$$\Rightarrow f^{\hat{r}} = |g_{rr}|^{1/2} f^{r}$$

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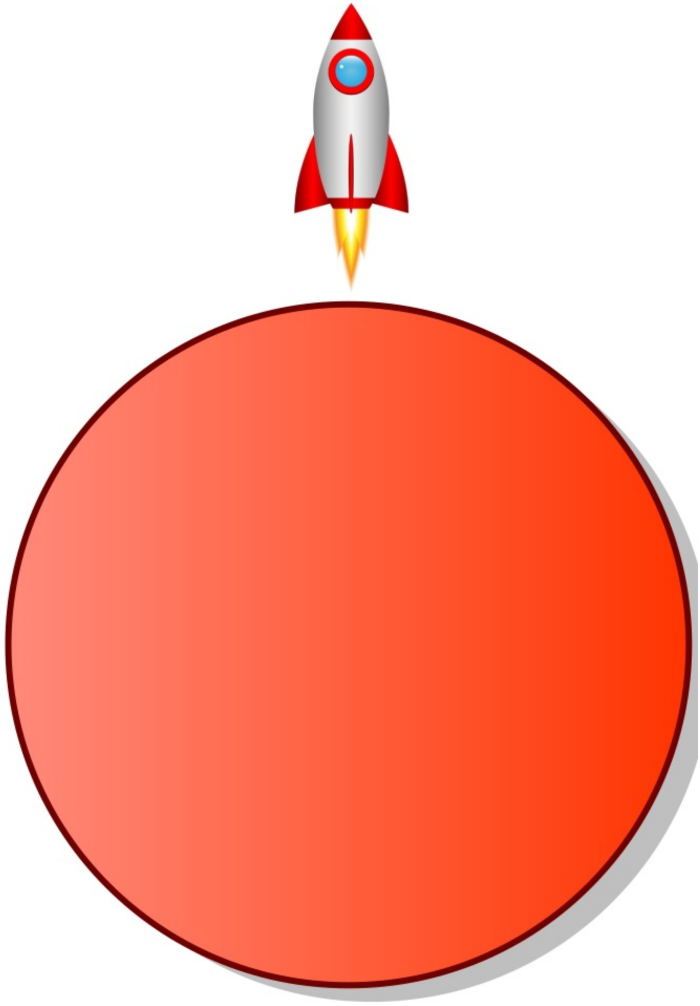
Blows up as $R \to 2M$!



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- Although it takes an infinite t for an observer to fall radially to $r=2M$, her proper time is finite. Even worse, after a finite τ, she will crush on $r=0$ (her time ends! - think about it carefully ...)

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- Consider an observer that starts from rest, relative to a stationary observer at $r = 10\,M$. How much time will elapse on the observer's clock before hitting the singularity?

Hartle, problem 5, ch. 12

- $R = 10M$
- $\ell = 0$ (radial motion)
- stationary's observer 4-velocity:

$$\left.\begin{array}{l} U^\mu = [U^0, 0, 0, 0] \\ U^\mu U_\mu = -1 \end{array}\right\} \Rightarrow U^\mu = \left[\left(1 - \frac{2M}{R}\right)^{-1/2}, 0, 0, 0\right]$$

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- $\frac{e^2 - 1}{2} = \mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 \underbrace{- \frac{M}{r}}_{}$

↳ $V_{eff}(r)$ for $\ell = 0$

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- $e = \left(1 - \frac{2M}{R}\right) \cdot u^0 = \left(1 - \frac{2M}{R}\right)\left(1 - \frac{2M}{R}\right)^{-\frac{1}{2}} = \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} = \frac{2}{\sqrt{5}}$
- $\frac{e^2 - 1}{2} = \mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} \Rightarrow \left(\frac{dr}{d\tau}\right)^2 = e^2 - 1 + \frac{2M}{r} = \frac{2M}{r} - \frac{1}{5}$

- $R = 10M$
- $\ell = 0$ (radial motion)
- stationary's observer 4-velocity: $U^\mu = [\sqrt{5}, 0, 0, 0]$
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↳ r decreases

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$$\Rightarrow \quad \frac{\tau}{M} = 5\sqrt{5}\,\pi$$

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Eddington - Finkelstein Coordinates

$$(t, r, \theta, \phi) \rightarrow (v, r, \theta, \phi)$$

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$$dt = dv - dr - 2M \frac{\frac{dr}{2M}}{\frac{r}{2M} - 1} = dv - dr\left[1 + \frac{1}{\frac{r}{2M} - 1}\right]$$

$$= dv - dr \frac{\cancel{\frac{r}{2M}}}{\cancel{\frac{r}{2M}}\left(1 - \frac{2M}{r}\right)} = dv - \left(1 - \frac{2M}{r}\right)^{-1} dr$$

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- $r > 2M$ and $r < 2M$ are smoothly connected
- $r \to \infty \Rightarrow ds^2$ flat, $t = v - r$

$$t = v - r - 2M \ln\left|\frac{r}{2M} - 1\right|$$

$$\left(r \to \infty \quad \frac{r}{2M} \gg \ln \frac{r}{2M}\right)$$

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

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$g_{rv} = g_{vr} = 1$

Careful: $2\,dv\,dr$ is $(dv\,dr + dr\,dv)$

- $g_{\mu\nu}$ non-diagonal

$$(g_{\mu\nu}) = \begin{pmatrix} -\left(1-\frac{2M}{r}\right) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \qquad (g^{\mu\nu}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & \left(1-\frac{2M}{r}\right) & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2}\sin^{-2}\theta \end{pmatrix}$$

Radial null geodesics

$d\theta = d\varphi = 0$

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$$r = 2M \quad (dr = 0) \qquad (\gamma)$$

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$$r = 2M \quad (dr = 0) \qquad (\gamma)$$

$$\Rightarrow \begin{cases} v = \text{const} & (\alpha) \\ v = 2\left(r + 2M \ln\left|\frac{r}{2M} - 1\right|\right) + c & (\beta) \\ r = 2M & (\gamma) \end{cases}$$