Hartle
ch 7

5. Consider the two-dimensional spacetime spanned by coordinates (v, x) with the line element

$$ds^2 = -x\, dv^2 + 2\, dv\, dx.$$

(a) Calculate the light cone at a point (v, x).
(b) Draw a (v, x) spacetime diagram showing how the light cones change with x.
(c) Show that a particle can cross from positive x to negative x but cannot cross from negative x to positive x.

(*Comment*: The light cone structure of this model spacetime is in many ways analogous to that of black-hole spacetimes to be considered in Chapter 12, in particular in having a surface such as $x = 0$, out from which you cannot get.)

18. Consider the three-dimensional space with the line element

$$dS^2 = \frac{dr^2}{(1 - 2M/r)} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

(a) Calculate the radial distance between the sphere $r = 2M$ and the sphere $r = 3M$.
(b) Calculate the spatial volume between the two spheres in part (a).

19. The surface of a sphere of radius R in four flat Euclidean dimensions is given by

$$X^2 + Y^2 + Z^2 + W^2 = R^2.$$

(a) Show that points on the sphere may be located by coordinates (χ, θ, ϕ), where

$$X = R \sin\chi \sin\theta \cos\phi, \qquad Z = R \sin\chi \cos\theta,$$
$$Y = R \sin\chi \sin\theta \sin\phi, \qquad W = R \cos\chi.$$

(b) Find the metric describing the geometry on the surface of the sphere in these coordinates.

20. *Make the cover* Consider the two-dimensional geometry with the line element

$$d\Sigma^2 = \frac{dr^2}{(1 - 2M/r)} + r^2 d\phi^2.$$

Find a two-dimensional surface in three-dimensional flat space that has the *same* intrinsic geometry as this slice. Sketch a picture of your surface. (*Comment*: This is a slice of the Schwarzschild black-hole geometry to be discussed in Chapter 12. It is also the surface on the cover of this book.)

Carroll 3.4

$$x = uv\cos\phi \qquad y = uv\sin\phi \qquad z = \frac{1}{2}(u^2 - v^2)$$

$$ds^2 = dx^2 + dy^2 + dz^2$$

- Compute $g_{\mu\nu}$ in the (u, v, ϕ) coordinate system
- if $V^\mu = v\,\partial_u - u\,\partial_v$ compute the components of V_μ and $V_\mu V^\mu$
- if $U^\mu = \sin\phi\,\partial_u - \cos\phi\,\partial_v$ compute $V_\mu U^\mu$

Carroll § 2.7 : Misner space

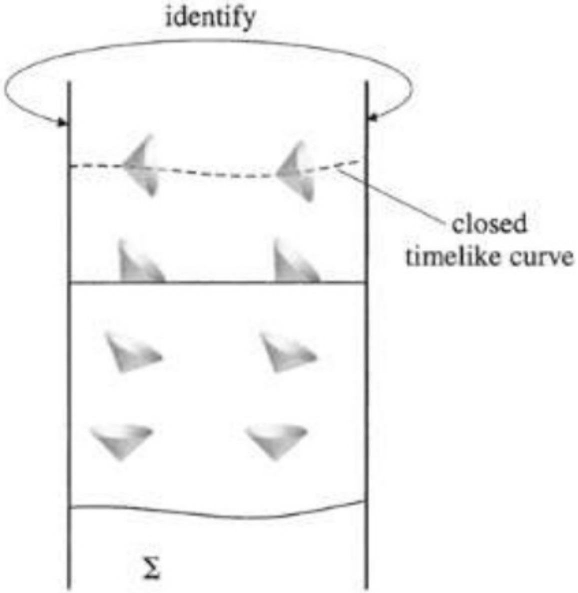
Consider the metric

$$ds^2 = -\cos\lambda\, dt^2 - 2\sin\lambda\, dt\, dx + \cos\lambda\, dx^2$$

on the cylinder $S^1 \times \mathbb{R}$ with

$$0 < x < 1 \qquad -\infty < t < +\infty \qquad t = \cot\lambda \qquad 0 < \lambda < \pi$$

where we identify the lines $x = 0$ and $x = 1$



(Carroll Fig 2.25)

(i) Compute the differential equation for the null curves $t(x)$

(ii) Compute null vector fields (U^μ, V^μ) in the direction of the forward light cone such that for $t \to -\infty$ $V^\mu = (V^0, V^1) \to (1,1)$ $U^\mu = (U^0, U^1) \to (1,-1)$

(iii) Draw the (U, V) vector fields for $t \ll -1$, $t=0$, $t \gg 1$ at $x = \frac{1}{2}$, and show qualitatively how the light cone tilts as t increases from $t \to -\infty$ to $t \to +\infty$

At each light cone, draw the vectors ∂_t and ∂_x, and specify their kind (timelike or space like)

(iv) At each light cone above, draw the direction of increasing time

(v) Make qualitative drawings of the paths of time forward null lines that start at an event in the far past at $x = 1/2$, one in the direction of V^μ and the other in the direction of U^μ

(vi) make the necessary corrections in Fig 2.25 of Carroll