① Let $\phi: M \to N$ a diffeomorphism, and $S \in T^{(2,0)}N$, $T \in T^{(0,2)}N$. Calculate the components of $\phi^*(S \otimes T)$ from the components of $S, T$ in a coordinate basis

② Let $V, U, W$ be vector fields. Use the identity $\mathcal{L}_V W = [V, W]$ to show that:

$$\mathcal{L}_{\alpha V + \beta W} U = \alpha \mathcal{L}_V U + \beta \mathcal{L}_W U \qquad \alpha, \beta \in \mathbb{R}$$

$$\mathcal{L}_{[V,W]} U = [\mathcal{L}_V, \mathcal{L}_W] U \equiv \mathcal{L}_V(\mathcal{L}_W U) - \mathcal{L}_W(\mathcal{L}_V U)$$

③ From the definition $[V, W](f) = V(W(f)) - W(V(f))$, show that

$$[V, W]^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu = (\mathcal{L}_V W)^\mu$$

(4) Show that $\mathcal{L}_V W = -\mathcal{L}_W V$

(5) Use the definition $\mathcal{L}_V T(0) = \lim_{t \to 0} \frac{1}{t}\left[ \phi_t^* T(0) - T(0) \right]$ to prove that

($\alpha$) $\mathcal{L}_V (fW) = \mathcal{L}_V f \, W + f \mathcal{L}_V W$ , $f$ a function

($\beta$) $\mathcal{L}_V(\omega(W)) = \mathcal{L}_V \omega\,(W) + \omega(\mathcal{L}_V W)$ , $\omega$ a 1-form field

($\gamma$) use $\mathcal{L}_V W = [V, W]$ to prove ($\alpha$)

(6) Show that $\mathcal{L}_{fV} W = f \mathcal{L}_V W - \mathcal{L}_W f \cdot W$ $f$ a function

(7) Consider the sphere $x^2+y^2+z^2=1$ and the coordinate systems $(\theta,\varphi)$, $(u,v)$. Let $V = u\,\partial_v - v\,\partial_u$ , $W = \sin\theta\,\partial_\varphi$ , $\omega = d\theta + \sin\theta\, d\varphi$ , $\sigma = \frac{dv}{1-v}$ . Compute $\mathcal{L}_V W$, $\mathcal{L}_V \omega$, $\mathcal{L}_W \sigma$