Symmetries – Killing Vectors

Isometries

Killing Vector Fields

Conserved Quantities (during geodesic motion)

Symmetries

Carroll §3.8–3.9

Ferrari et al Ch. 8

Isometries

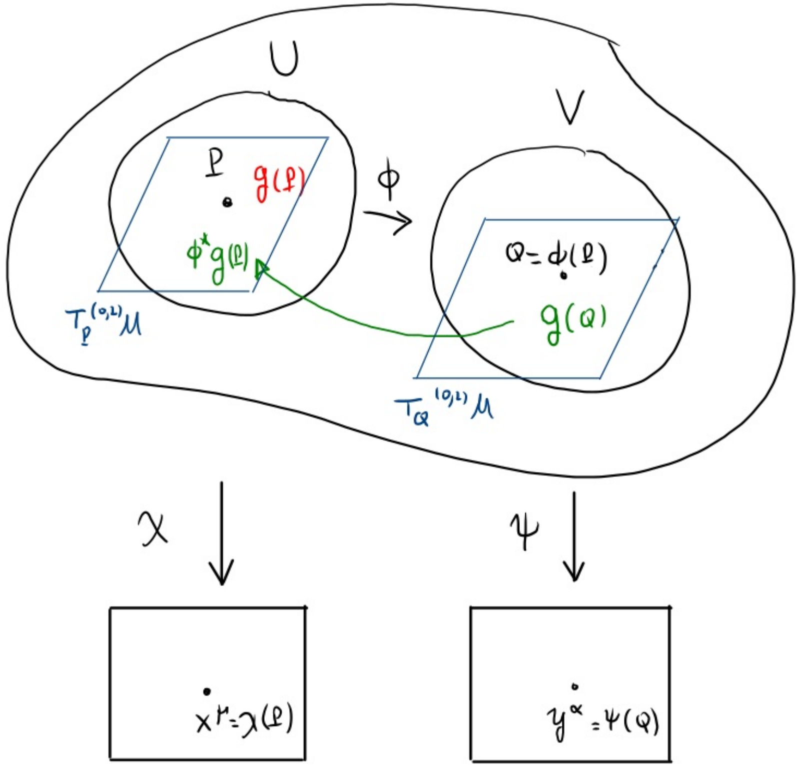
invertible, differentiable

* Let $\phi : \mathcal{M} \to \mathcal{M}$ a diffeomorphism

$$P \mapsto Q = \phi(P)$$

and charts (U, χ), (V, ψ) s.t.

$$x^\mu = \chi(P) \qquad y^\alpha = \psi(Q)$$



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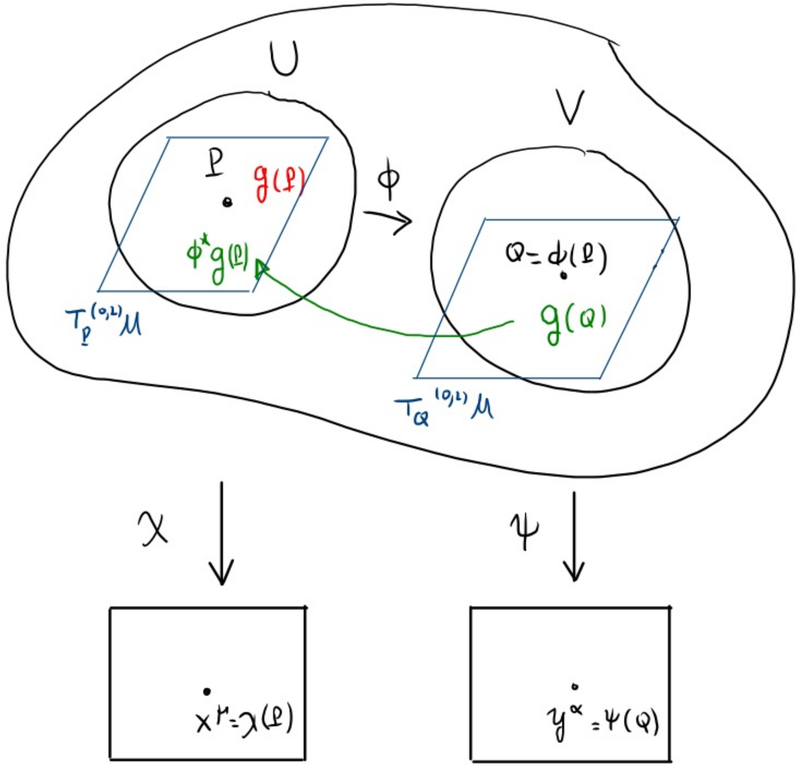
$g(x) \in T_P^{(0,2)}\mathcal{M}$ acts on $T_P\mathcal{M}$

$g(y) \in T_Q^{(0,2)}\mathcal{M}$ " " $T_Q\mathcal{M}$



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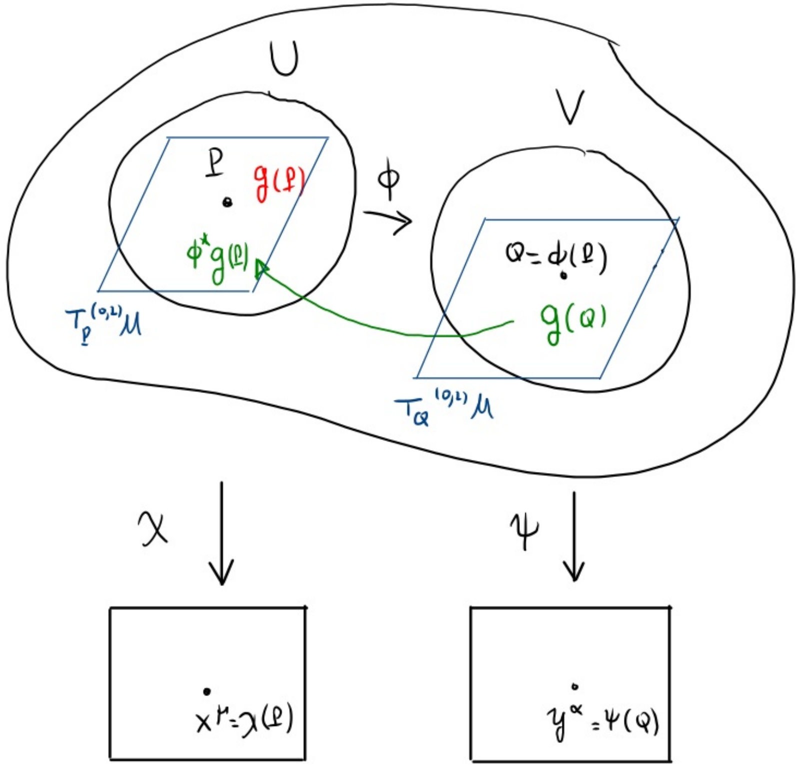

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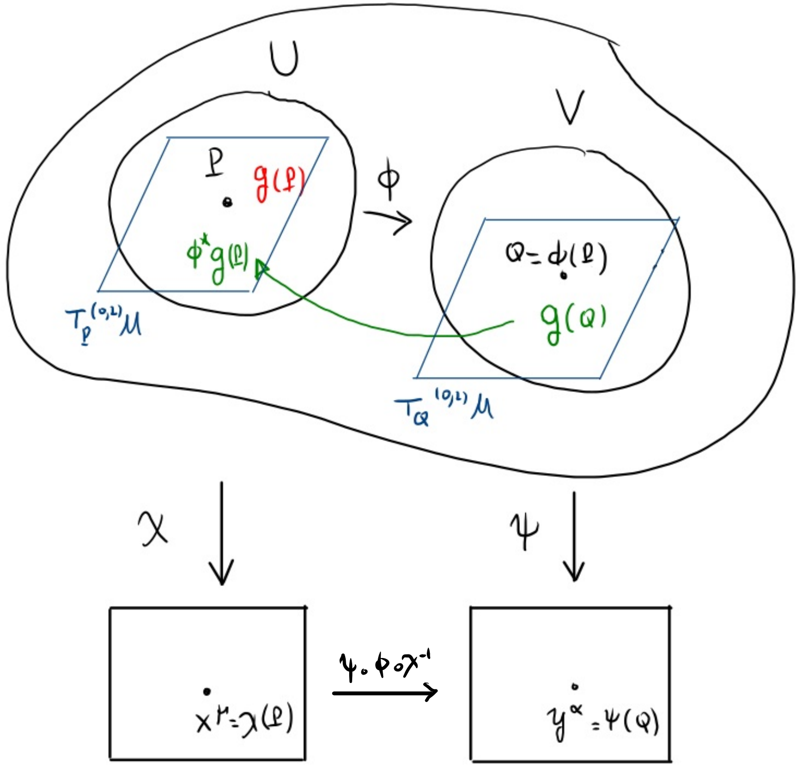

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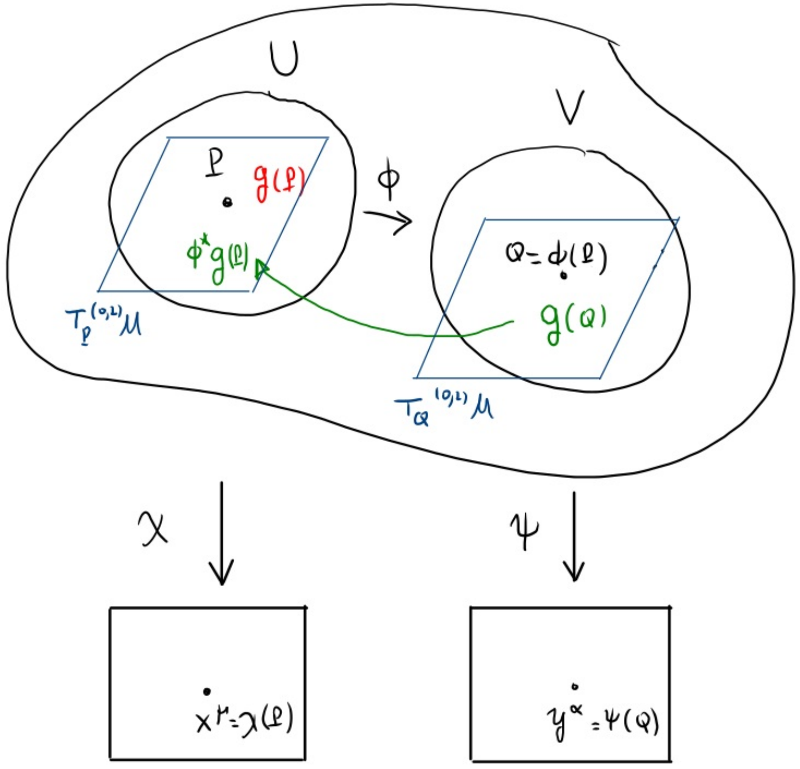
Isometries

* ϕ is an isometry if $\phi^* g(x) = g(x)$



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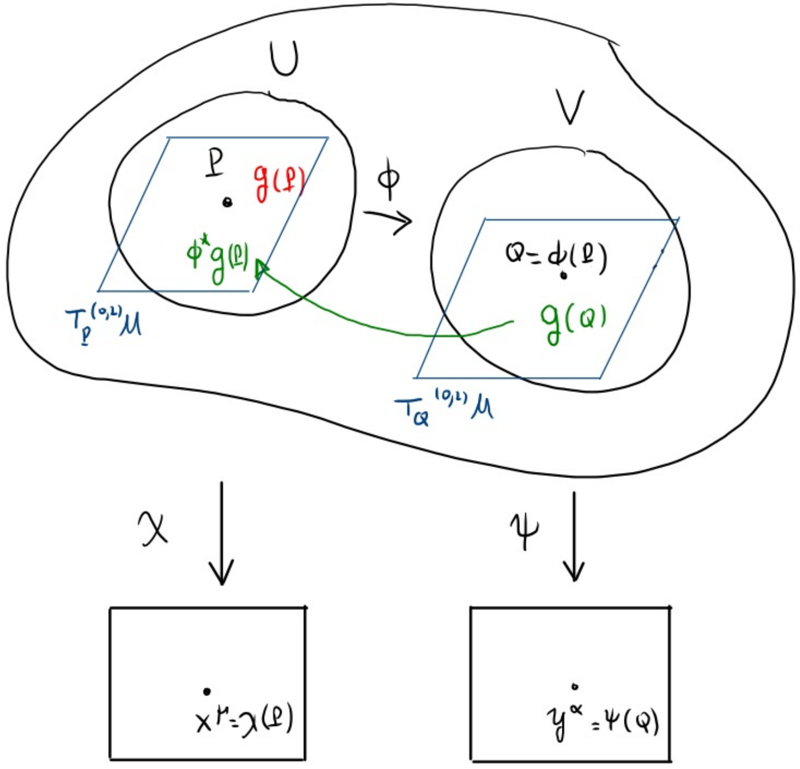

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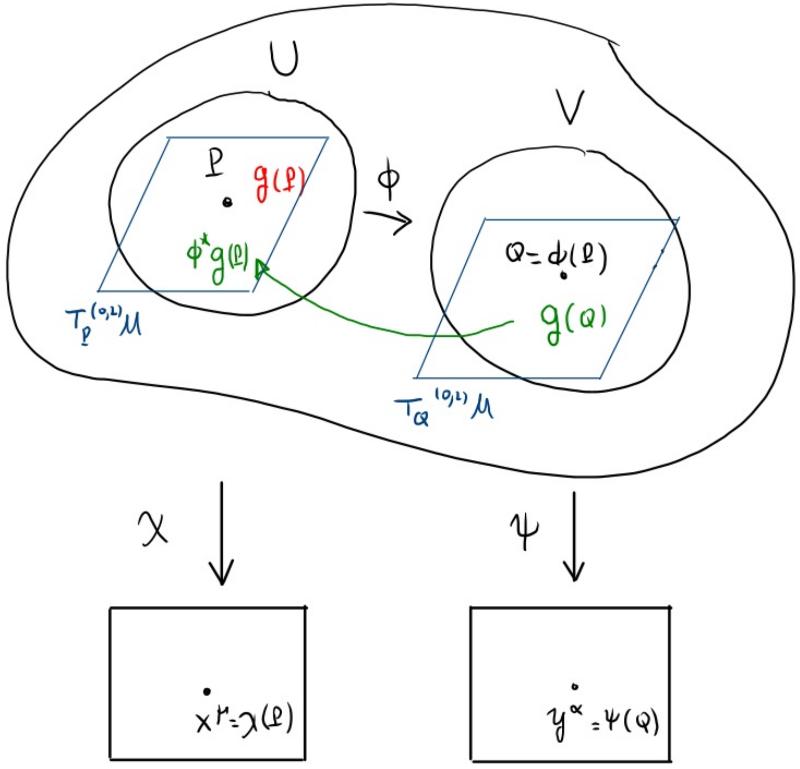

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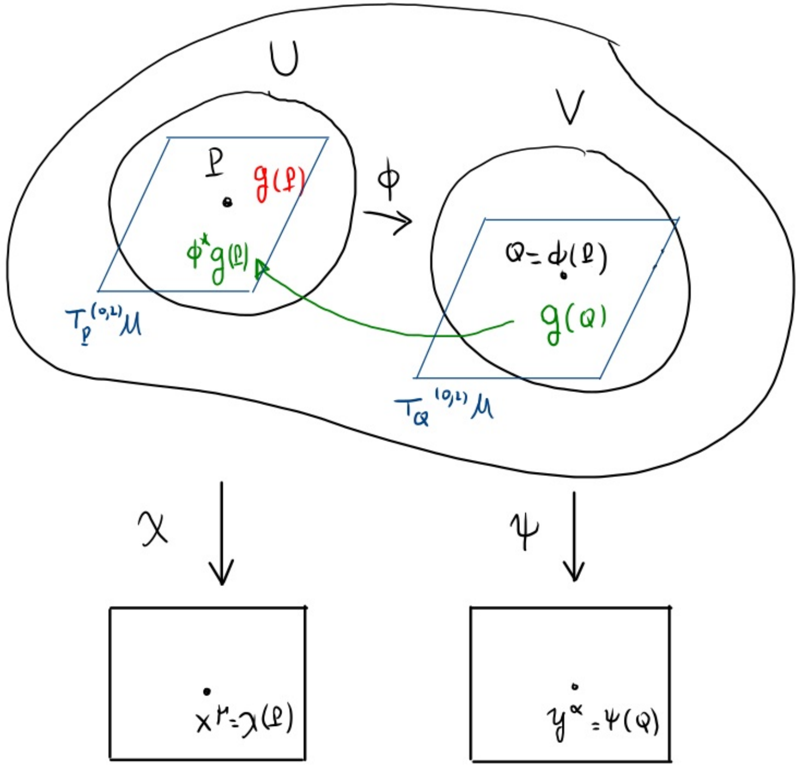
- non trivial equations, hard to solve
- study isometries via infinitesimal, one parameter, family of diffeos, generated by vector fields



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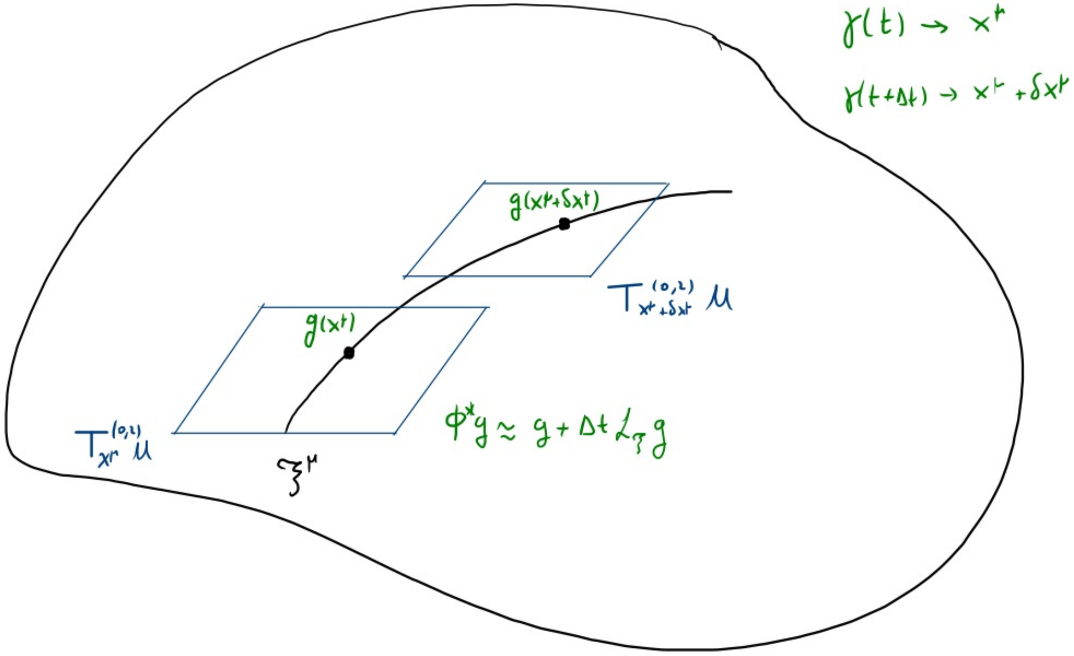

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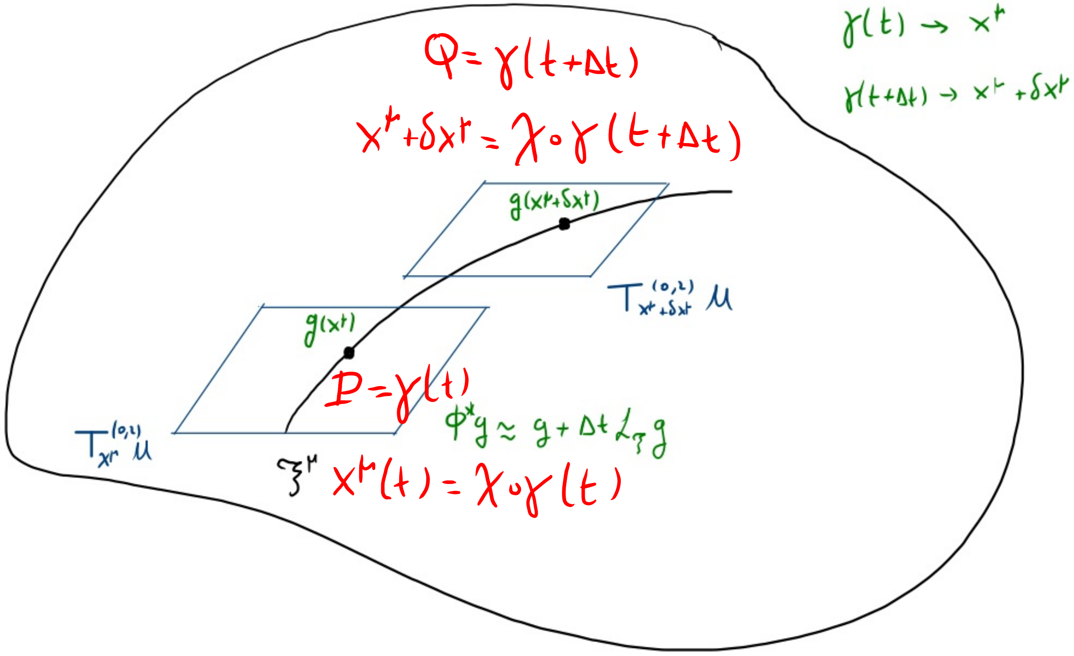

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The curve goes through $P = \gamma(t)$ and $Q = \gamma(t+\Delta t)$, $Q = \phi_{\Delta t}(P) \equiv \phi(P)$

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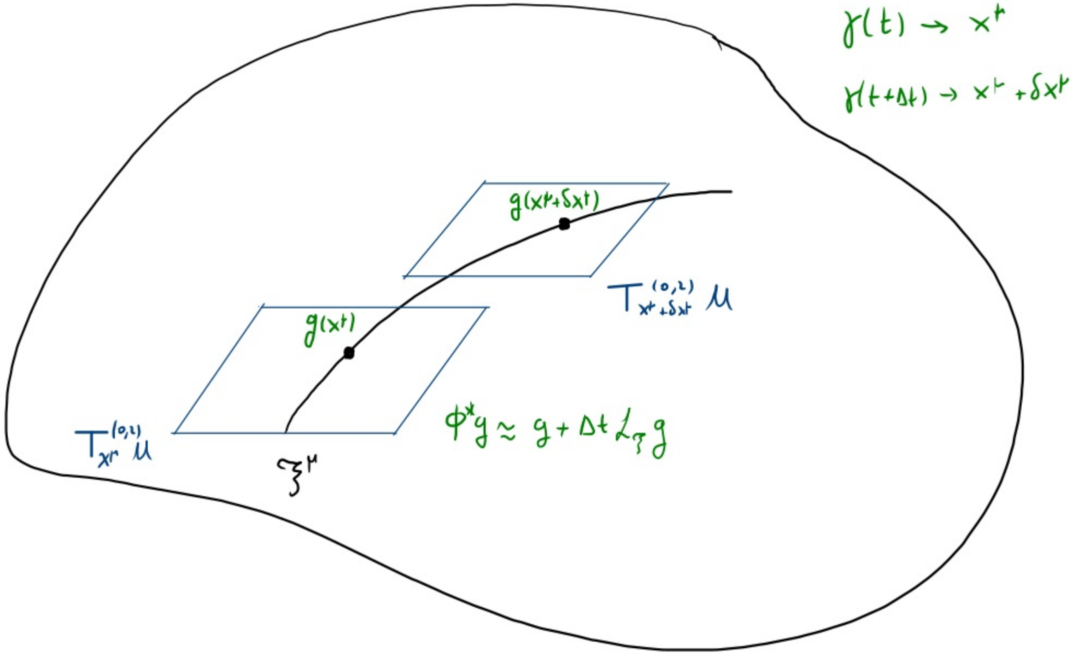

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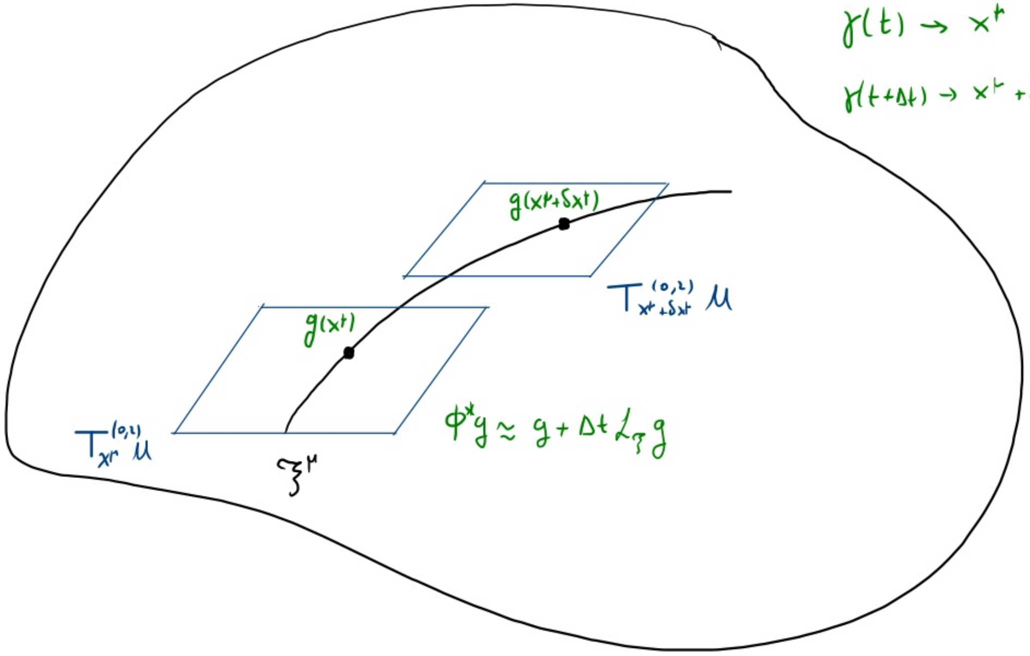
Take $\Delta t = \epsilon$, ignore $O(\epsilon^2)$ $\quad \xi^\alpha = \dfrac{dx^\alpha}{dt}$

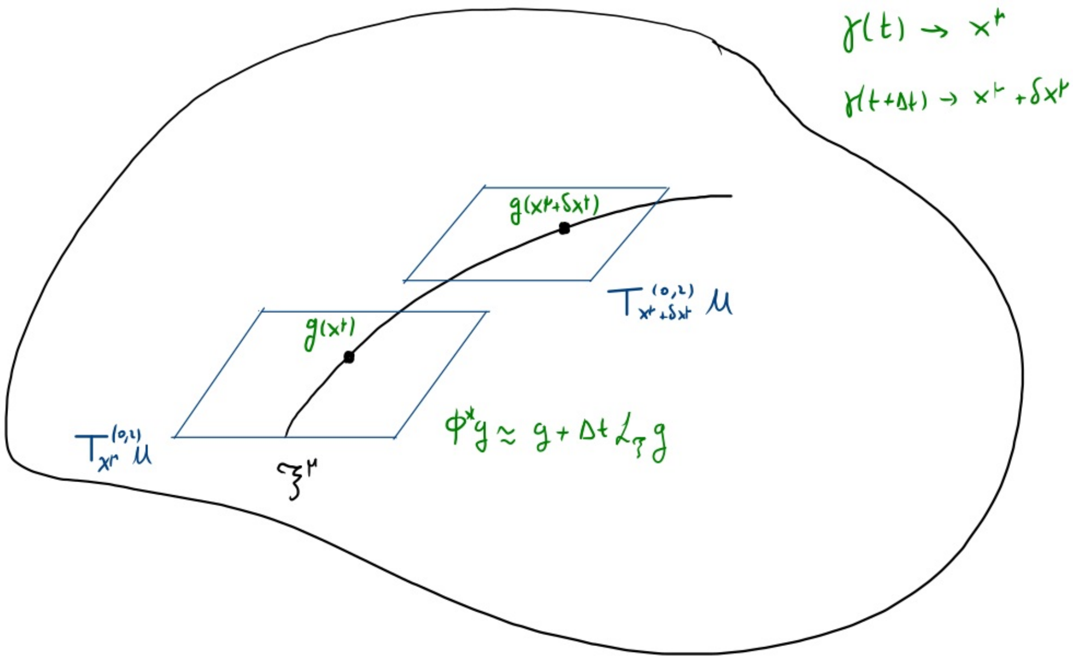
$$y^\alpha \equiv x^\alpha + \delta x^\alpha = x^\alpha + \epsilon \xi^\alpha \Rightarrow \frac{\partial y^\alpha}{\partial x^\mu} = \delta^\alpha{}_\mu + \epsilon \partial_\mu \xi^\alpha$$

$\gamma(t) \to x^\mu$

$\gamma(t+\Delta t) \to x^\mu + \delta x^\mu$



Isometries



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$$g_{\alpha\beta}(y) = g_{\alpha\beta}(x+\delta x) = g_{\alpha\beta}(x) + \epsilon \frac{d}{dt} g_{\alpha\beta}(x) + O(\epsilon^2) \approx g_{\alpha\beta}(x) + \epsilon \frac{dx^\lambda}{dt} \frac{\partial g_{\alpha\beta}(x)}{\partial x^\lambda} = g_{\alpha\beta}(x) + \epsilon \, \xi^\lambda \partial_\lambda g_{\alpha\beta}(x)$$

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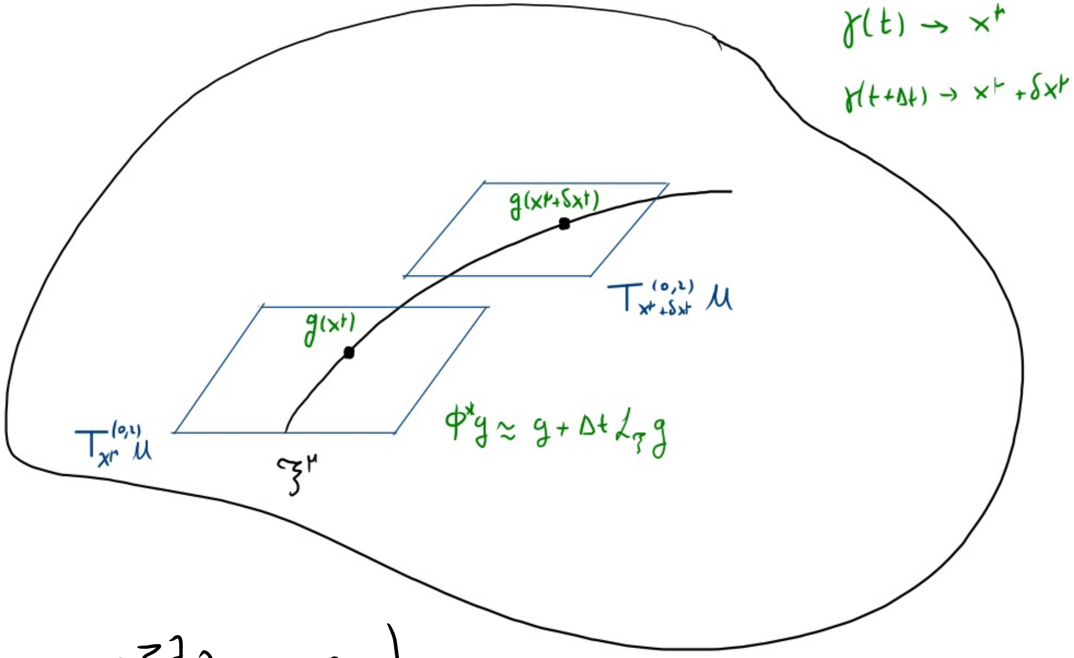

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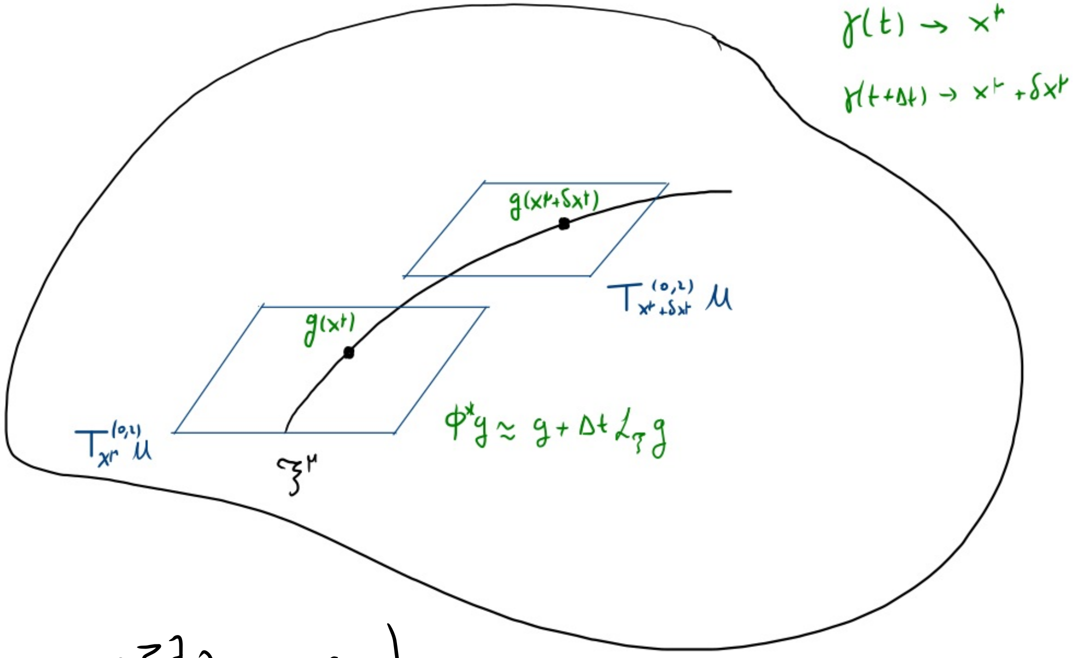
$$\xi^\lambda \partial_\lambda g_{\alpha\nu}$$

$$= \left(\delta^\alpha{}_\mu + \epsilon \partial_\mu \xi^\alpha\right)\left(g_{\alpha\nu} + \epsilon \xi^\lambda \partial_\lambda g_{\alpha\nu} + \epsilon \partial_\nu \xi^\rho g_{\alpha\rho} + \mathcal{O}(\epsilon^2) \right)$$



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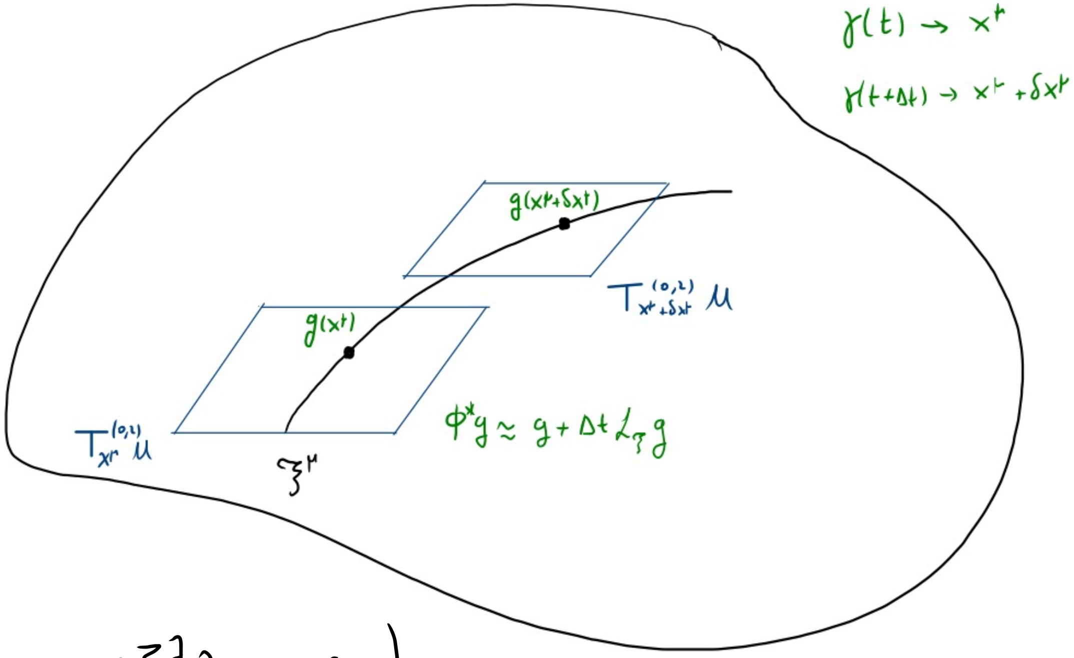

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$\partial_\nu \xi^\beta g_{\mu\beta}$ $\xi^\lambda \partial_\lambda g_{\mu\nu}$ $\partial_\mu \xi^\alpha g_{\alpha\nu}$

$$= \underbrace{\left(\delta^\alpha{}_\mu + \epsilon\, \partial_\mu \xi^\alpha\right)\left(g_{\alpha\nu}\right.}_{g_{\mu\nu}} \left. + \epsilon\, \xi^\lambda \partial_\lambda g_{\alpha\nu} + \epsilon\, \partial_\nu \xi^\beta g_{\alpha\beta} + \cancel{O(\epsilon^2)}\right)$$

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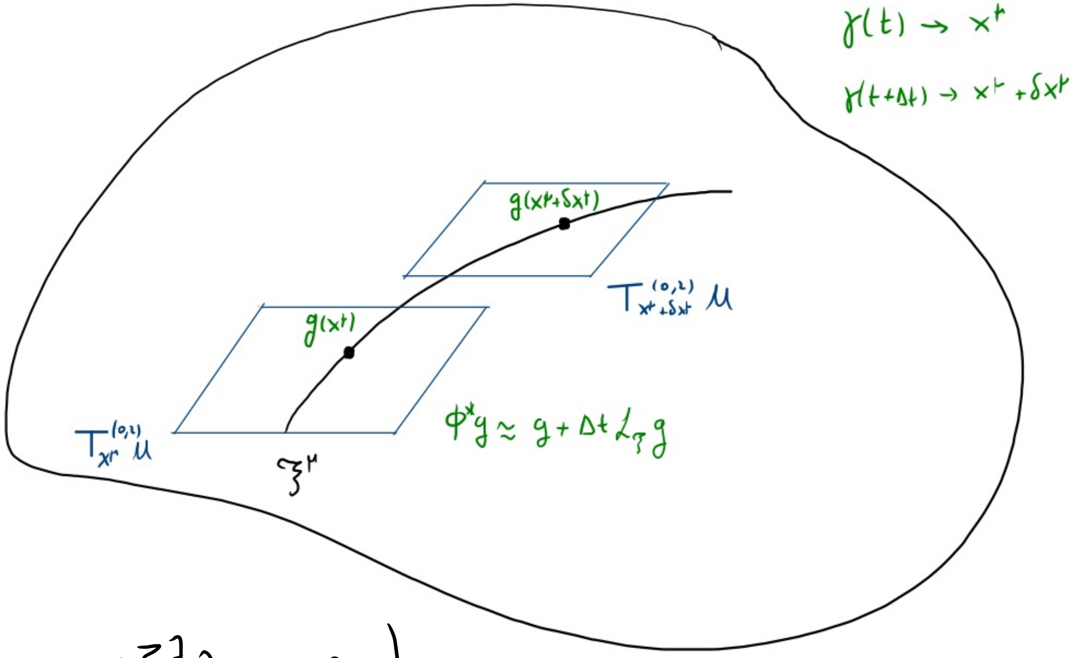

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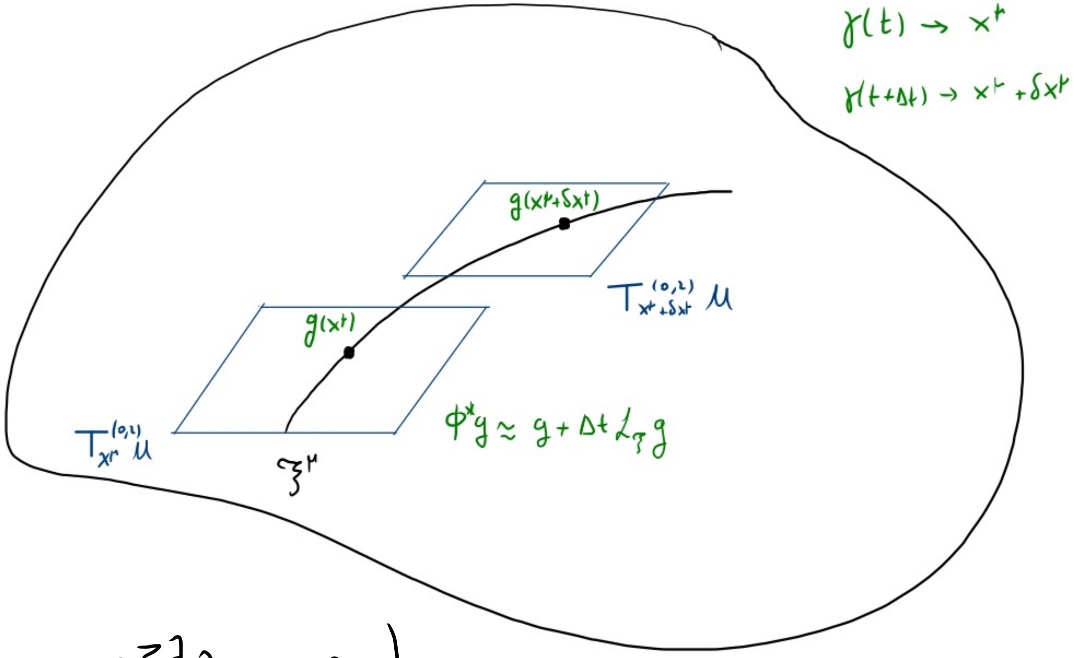

$$\Rightarrow \cancel{g_{\mu\nu}}(x) \approx (\delta^\alpha{}_\mu + \epsilon \partial_\mu \xi^\alpha)(\delta^\beta{}_\nu + \epsilon \partial_\nu \xi^\beta)\left(g_{\alpha\beta}(x) + \epsilon \xi^\lambda \partial_\lambda g_{\alpha\beta}(x)\right)$$

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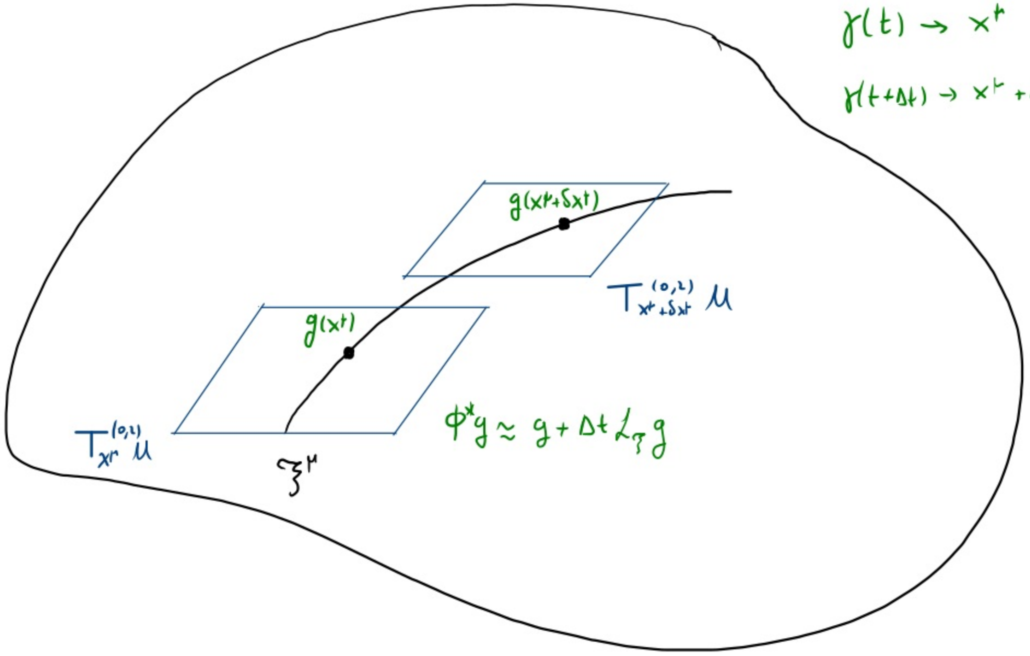
$= 0$!

Isometries



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- No surprise:

$$\mathcal{L}_\xi g(t) = \lim_{\epsilon\to 0} \frac{\phi^*_{t+\epsilon} g(t) - g(t)}{\epsilon} \Rightarrow \phi^* g = g + \epsilon \mathcal{L}_\xi g + \mathcal{O}(\epsilon^2)$$

- A local condition: differential equations to be solved
- May or may not have solutions. E.g. on compact manifolds with $R<0$, no solutions!

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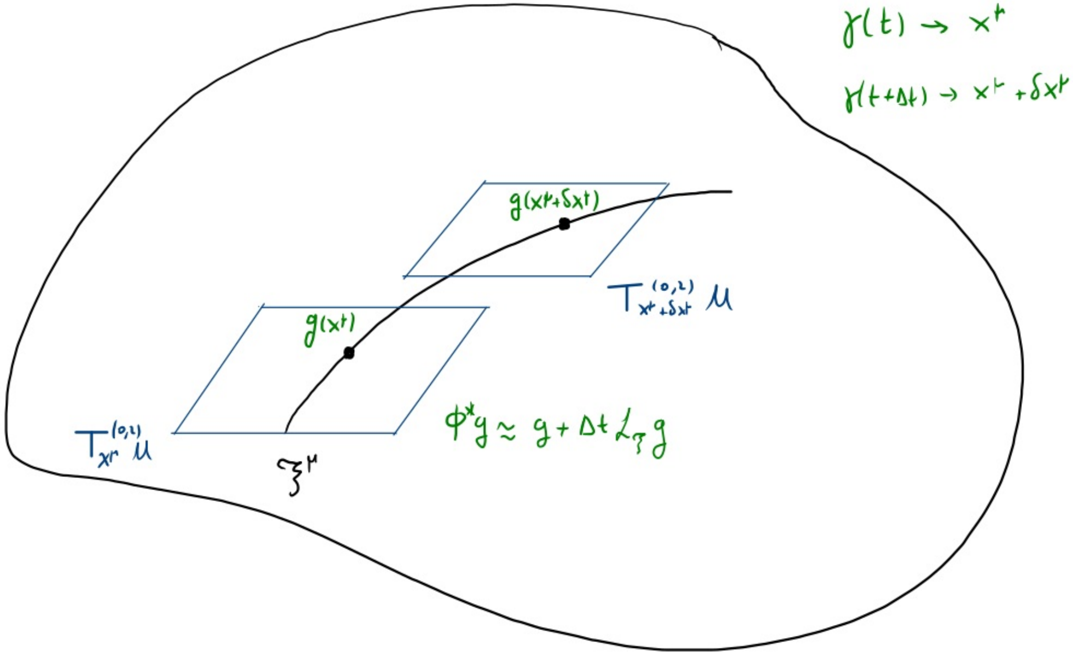

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- A local condition: differential equations to be solved

- May or may not have solutions. E.g. on compact manifolds with $R<0$, no solutions!
- May have local solutions, but not global

  e.g. in a coordinate system

Show that $\mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$ ∇: Christoffel connection ($\nabla g = 0$, $T = 0$)

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Show that $\mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$ $\qquad \nabla$: Christoffel connection $(\nabla g = 0, T = 0)$

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Show that $\mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$ $\qquad$ ∇ : Christoffel connection $(\nabla g = 0,\ T = 0)$

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$\mu \leftrightarrow \nu$

$$\nabla_\nu \xi_\mu = g_{\mu\lambda}\partial_\nu \xi^\lambda + \frac{1}{2}\left(\partial_\nu g_{\rho\mu} + \partial_\rho g_{\nu\mu} - \partial_\mu g_{\nu\rho}\right)\xi^\rho$$

Show that $\mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$ $\qquad$ ∇: Christoffel connection $(\nabla g = 0, T = 0)$

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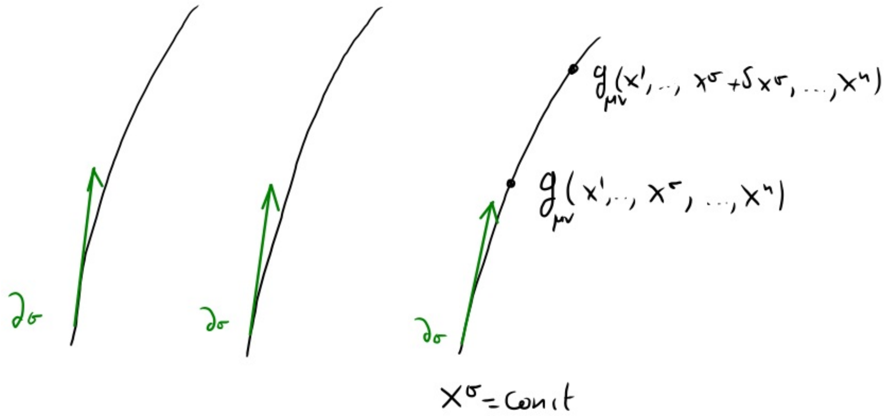
$$\Rightarrow \mathcal{L}_{[\xi,\chi]} g = [\mathcal{L}_\xi, \mathcal{L}_\chi] g = (\mathcal{L}_\xi \mathcal{L}_\chi g - \mathcal{L}_\chi \mathcal{L}_\xi) g = \mathcal{L}_\xi(\overset{0}{\cancel{\mathcal{L}_\chi g}}) - \mathcal{L}_\chi(\overset{0}{\cancel{\mathcal{L}_\xi g}}) = 0$$

Killing Vector Fields & Coordinate Independence of $g_{\mu\nu}$

If $\exists$ coordinate system, s.t. $g_{\mu\nu}(x)$ is independent of x^σ $\Rightarrow$ $\partial_\sigma g_{\mu\nu} = 0$



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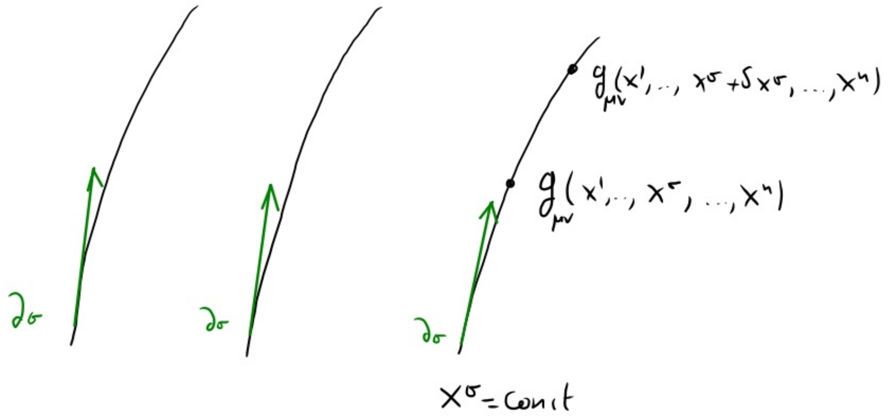

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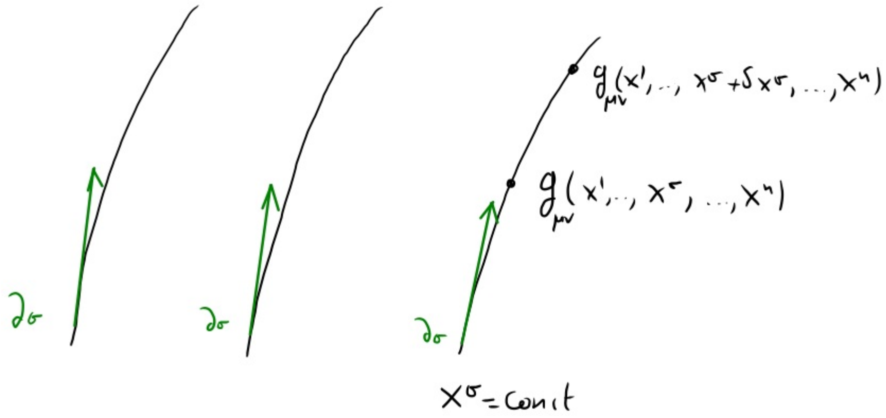

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$$\mathcal{L}_\xi g_{\mu\nu} = \xi^\lambda \partial_\lambda g_{\mu\nu} + g_{\mu\lambda} \partial_\nu \xi^\lambda + g_{\lambda\nu} \partial_\mu \xi^\lambda$$

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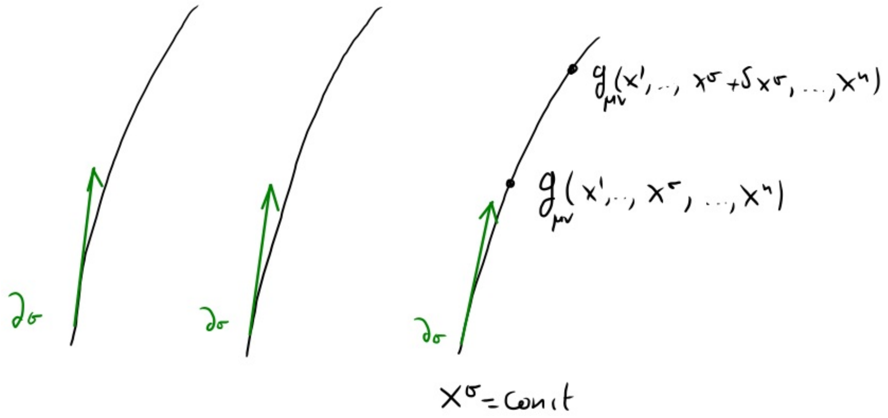

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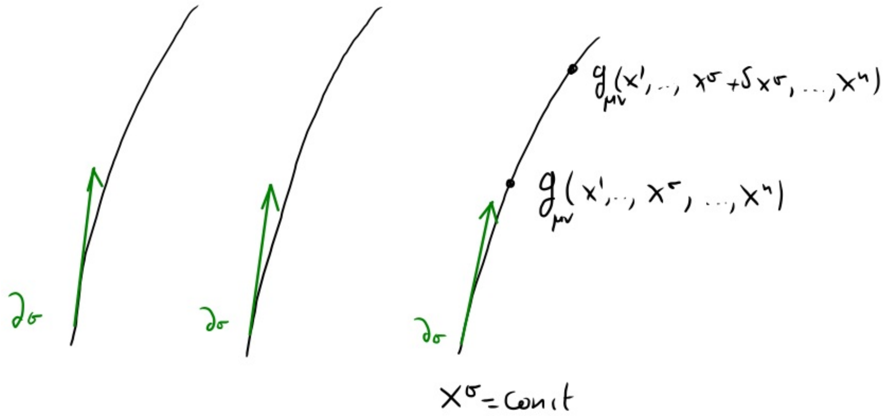

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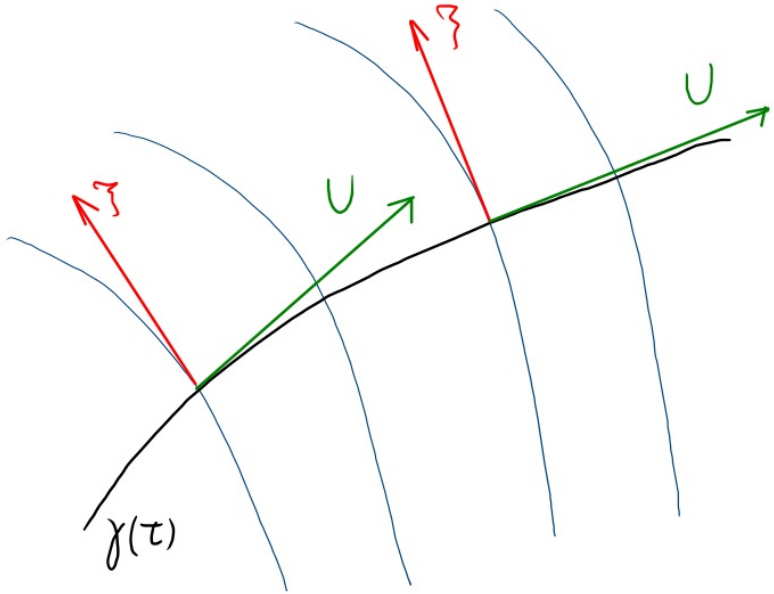
$$= \partial_\sigma g_{\mu\nu}$$

$$= 0$$

Conserved Quantities

- Let ξ be a Killing vector field
- " $\gamma(\tau)$ " " geodesic with $\begin{cases} \text{affine parameter} & \tau \\ \text{tangent vector} & U \end{cases}$



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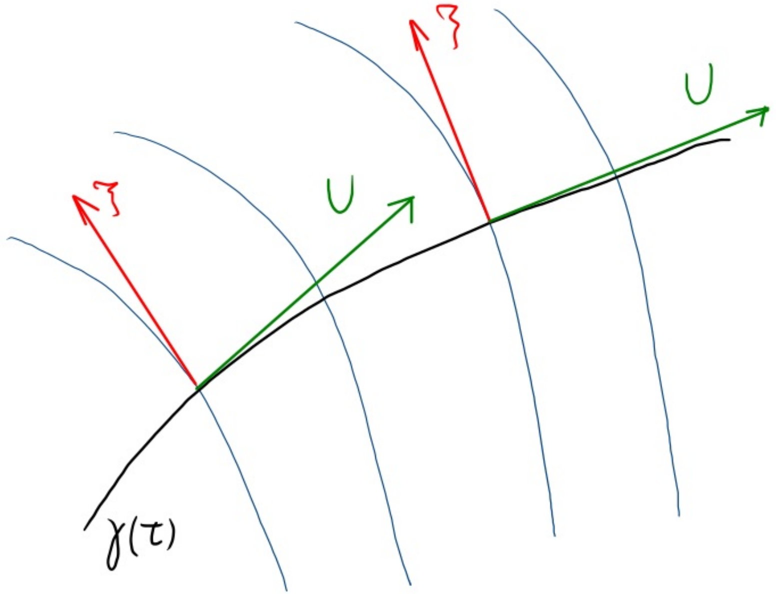

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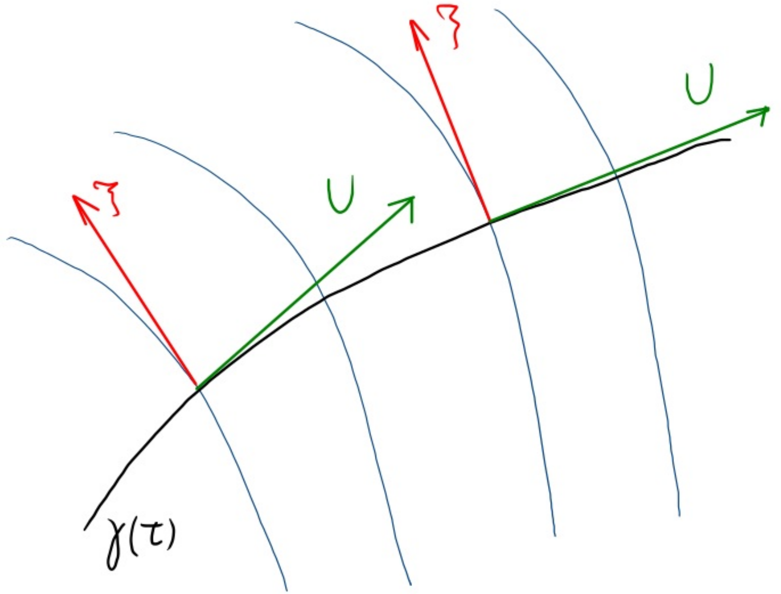

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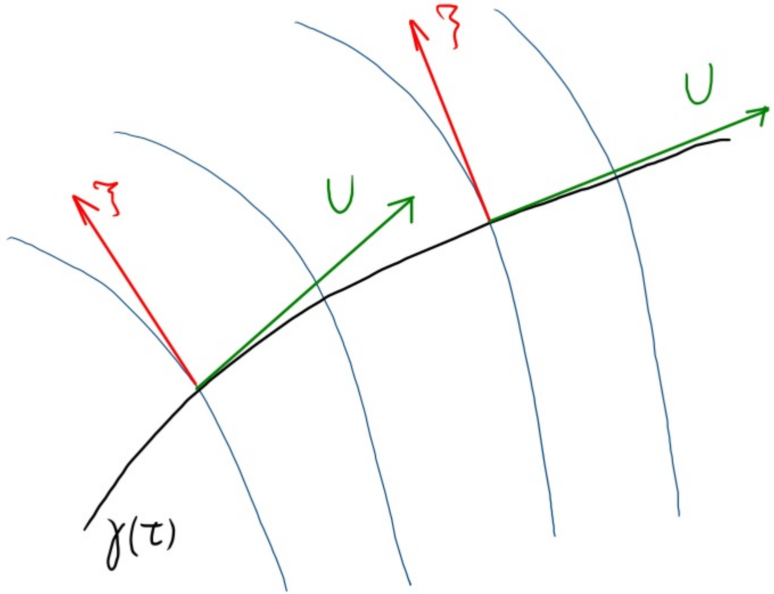
$$\Rightarrow \quad \xi \cdot U = \xi_\mu U^\mu = g_{\mu\nu} \xi^\mu U^\nu$$ conserved along the geodesic $\gamma(\tau)$

Conserved Quantities



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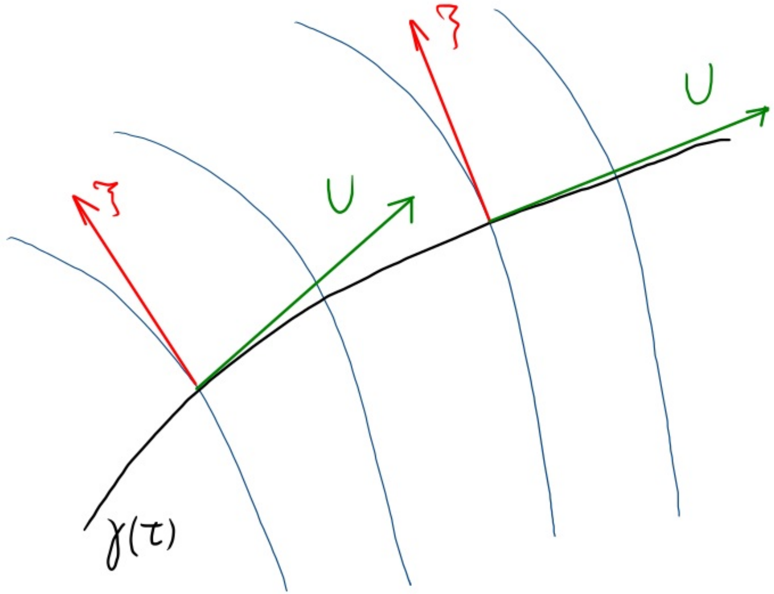
$$\Rightarrow \quad D_U(\xi \cdot U) = U^\nu \nabla_\nu (\xi_\mu U^\mu) = 0$$

Conserved Quantities

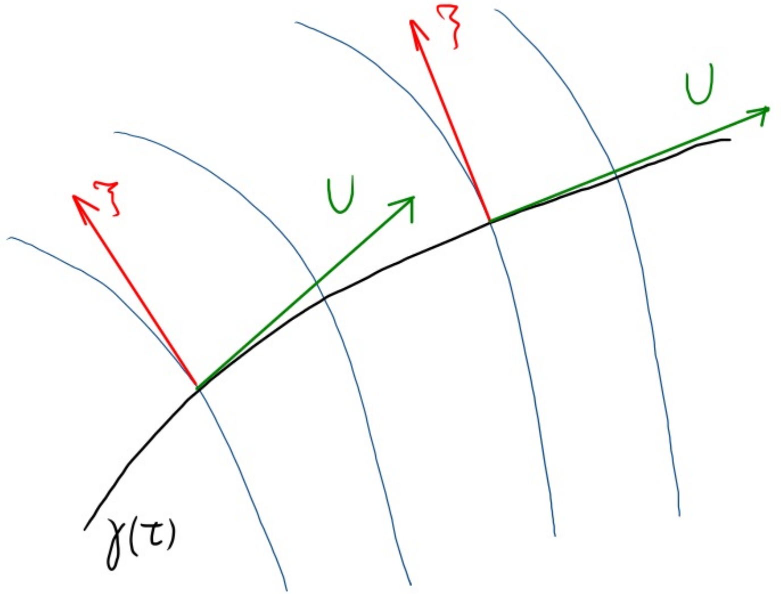


Indeed:

$$U^\nu \nabla_\nu (\xi_\mu U^\mu) = U^\nu \left[\nabla_\nu \xi_\mu \, U^\mu + \xi_\mu \nabla_\nu U^\mu \right]$$

$$\Rightarrow \quad D_U (\xi \cdot U) = U^\nu \nabla_\nu (\xi_\mu U^\mu) = 0$$

Conserved Quantities



Indeed:

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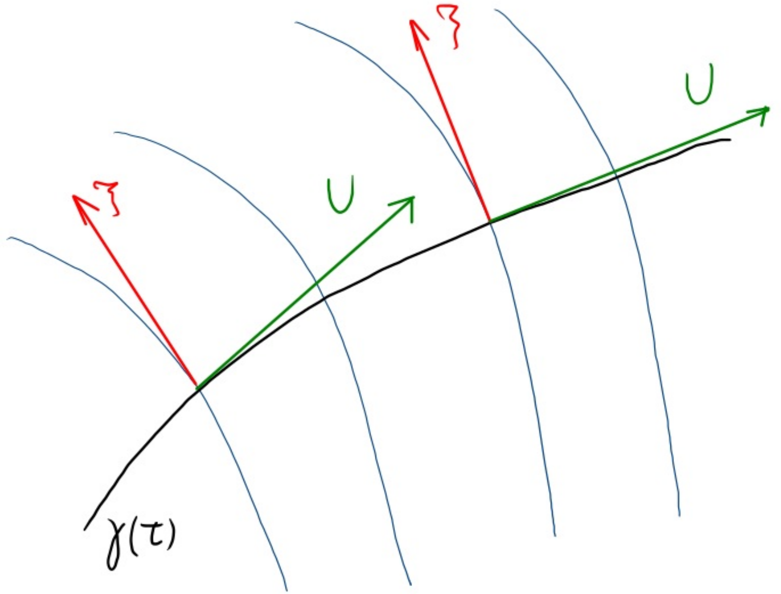
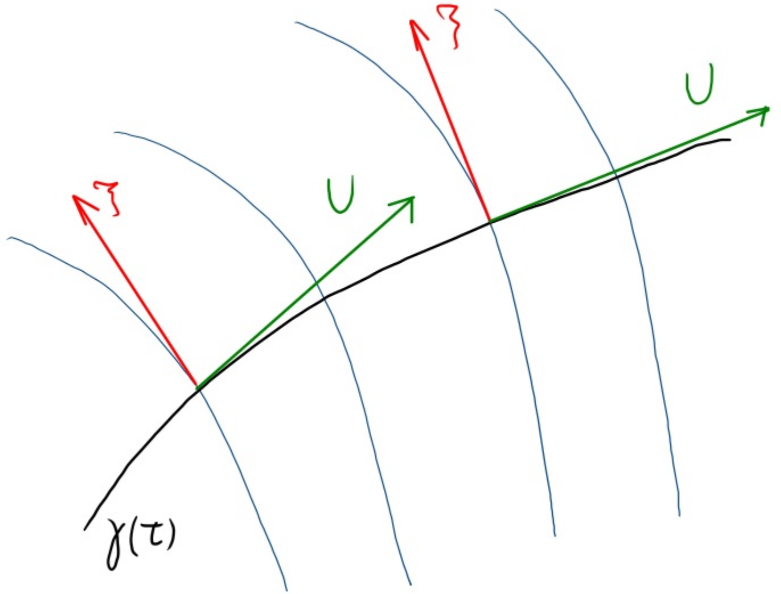


But

$\gamma(\tau)$ geodesic $\Rightarrow$ $U^\nu \nabla_\nu U^\mu = 0$

Conserved Quantities



Indeed:

$$U^\nu \nabla_\nu (\xi_\mu U^\mu) = U^\nu [\nabla_\nu \xi_\mu \, U^\mu + \xi_\mu \nabla_\nu U^\mu]$$

$$= \cancel{U^\nu U^\mu \nabla_\nu \xi_\mu} + \xi_\mu \, \cancel{U^\nu \nabla_\nu U^\mu}$$

$$= 0$$

$\gamma(\tau)$ geodesic $\Rightarrow$ $U^\nu \nabla_\nu U^\mu = 0$

$$U^\nu U^\mu \nabla_\nu \xi_\mu = U^\nu U^\mu \frac{1}{2} \underbrace{(\nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu)}_{\text{Killing Eq.} \to 0} = U^\nu U^\mu \cdot 0 = 0$$

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$\hookrightarrow$ because $T^{\mu\nu} = T^{\nu\mu}$

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* ξ timelike $\rightsquigarrow$ conserved energy

  spacelike $\rightsquigarrow$ " momentum (generalized, e.g. angular momentum)

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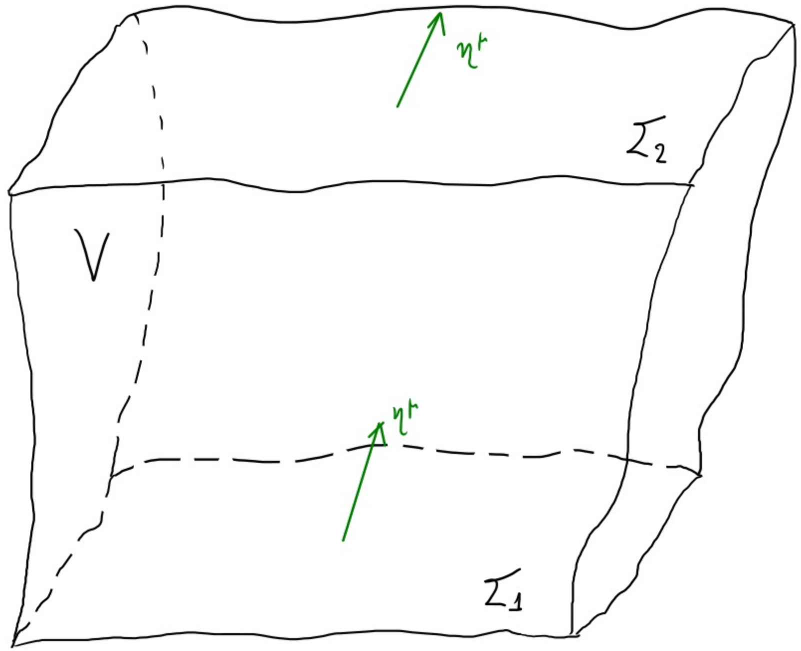
E.g. ξ timelike: Consider 4d volume V, and Σ_1, Σ_2 spacelike, extend to spatial infinity



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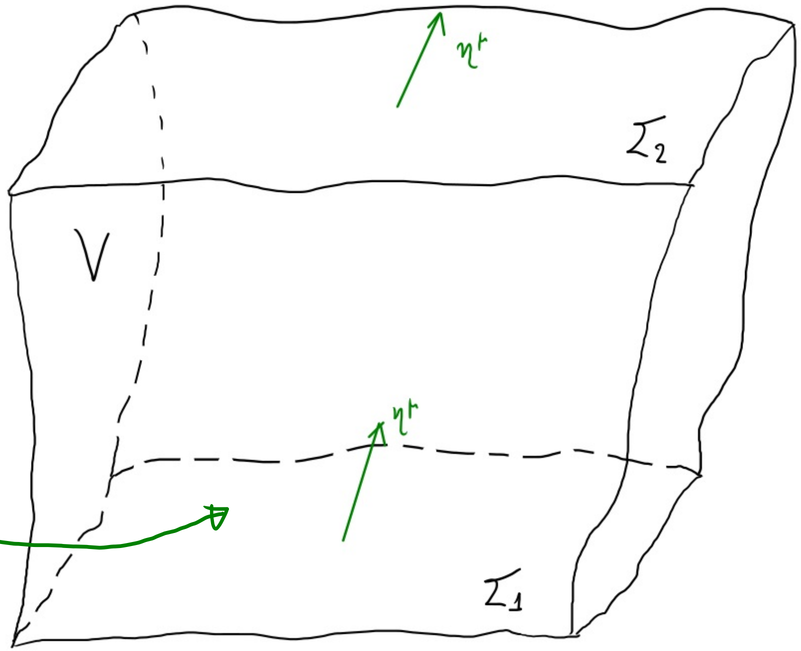

$$\int_V \nabla_\mu J^\mu \sqrt{|g|}\, d^4x = \int_{\Sigma_2} J_\mu n^\mu \sqrt{\gamma}\, d^3x - \int_{\Sigma_1} J_\mu n^\mu \sqrt{\gamma}\, d^3x$$

$$+ \int_{\substack{\text{spatial}\\ \text{infinity}}} J_\mu n^\mu$$

$\hookrightarrow$ because we chose n^μ to point "inward" in V

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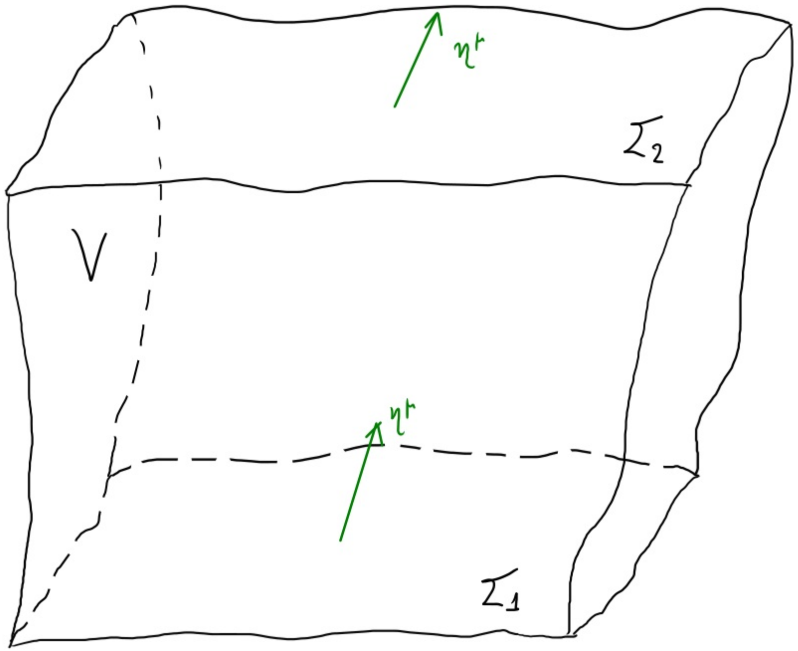

$$\int_V \nabla_\mu J^\mu \sqrt{|g|}\, d^4x = \int_{\Sigma_2} J_\mu n^\mu \sqrt{\gamma}\, d^3x - \int_{\Sigma_1} J_\mu n^\mu \sqrt{\gamma}\, d^3x + \cancel{\int_{\substack{\text{spatial}\\ \text{infinity}}} J_\mu n^\mu}$$

assume $J_\mu \to 0$ fast enough

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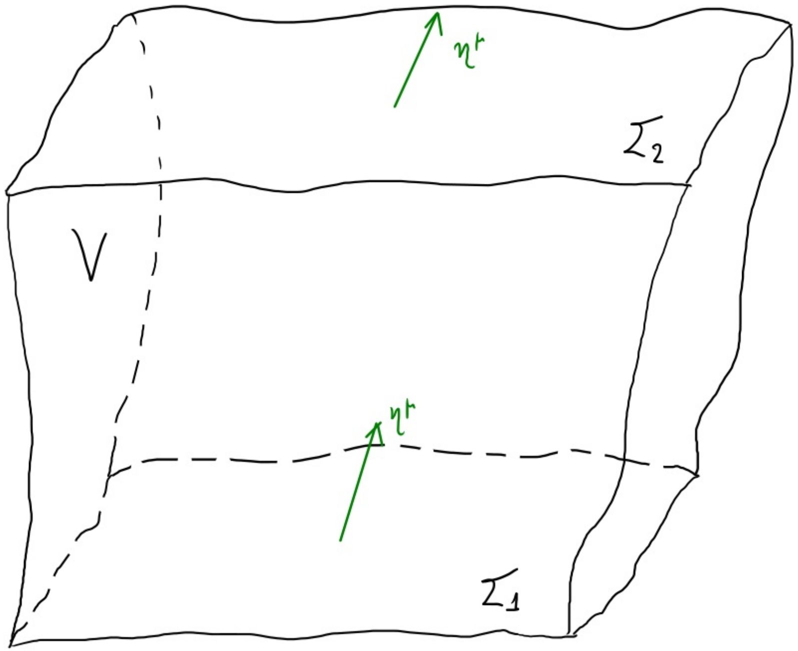

$$0 = \int_V \nabla_\mu \cancel{J}^\mu \sqrt{|g|}\, d^4x = \int_{\Sigma_2} J_\mu n^\mu \sqrt{\gamma}\, d^3x - \int_{\Sigma_1} J_\mu n^\mu \sqrt{\gamma}\, d^3x \Rightarrow$$

$$\int_{\Sigma_2} J_\mu n^\mu \sqrt{\gamma}\, d^3x = \int_{\Sigma_1} J_\mu n^\mu \sqrt{\gamma}\, d^3x = \text{const.}$$

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If we choose $\xi^\mu = n^\mu = (\partial_0)^\mu$, then

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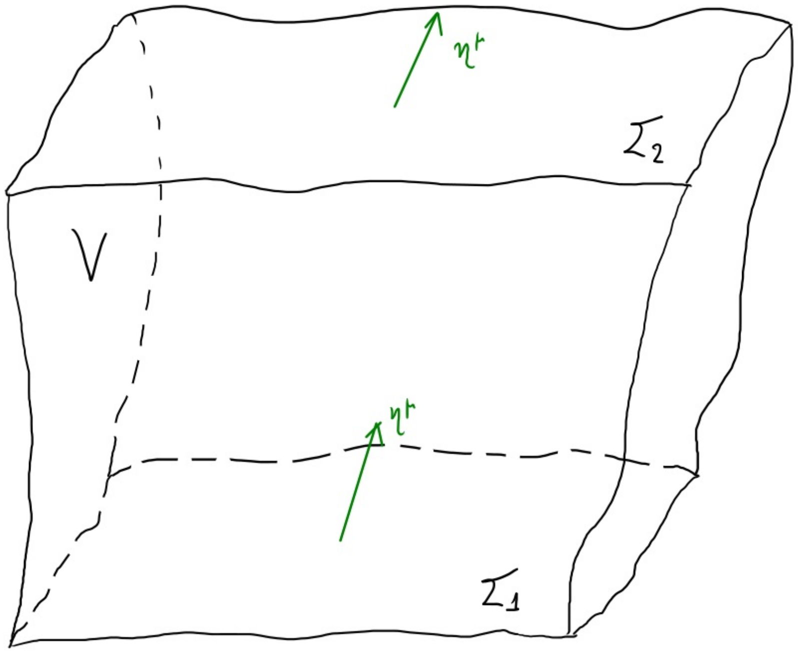
$$J_0 = \xi^\mu T_{\mu 0} = T_{00} = \text{energy density}$$



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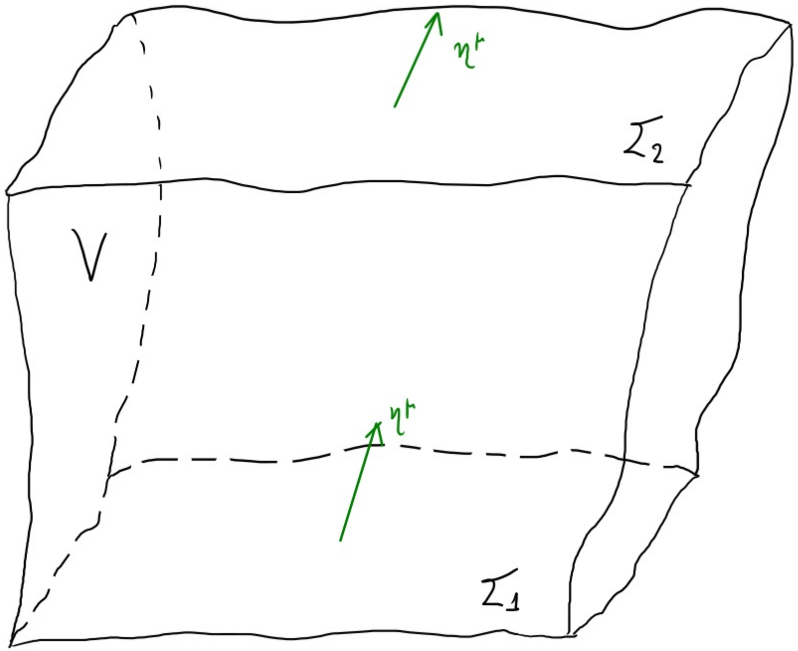
ξ timelike $\rightarrow$ time symmetry $\rightarrow$ energy conservation

↳ "static spacetimes"



Examples: S^2

$$ds^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$$

$$g_{\theta\theta} = 1 \qquad g_{\varphi\varphi} = \sin^2\theta$$

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(θ, θ)

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$$\xi^\lambda \partial_\lambda g_{\mu\nu} + g_{\lambda\nu}\partial_\mu \xi^\lambda + g_{\mu\lambda}\partial_\nu \xi^\lambda = 0$$

(θ,θ)

$$\xi^\lambda \partial_\lambda \cancel{g_{\theta\theta}} + g_{\lambda\theta}\partial_\theta \xi^\lambda + g_{\theta\lambda}\partial_\theta \xi^\lambda = 0$$

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Examples: S^2

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$$\xi^\theta \partial_\theta (\sin^2\theta) + \sin^2\theta \, \partial_\varphi \xi^\varphi + \sin^2\theta \, \partial_\varphi \xi^\varphi = 0$$

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$$\xi^\theta \, 2 \sin\theta \cos\theta + 2 \sin\theta \sin\theta \, \partial_\varphi \xi^\varphi = 0$$

Examples: S^2

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Solutions:

$$\xi^\theta = c_3 \sin(\varphi + c_1)$$

$$\xi^\varphi = c_3 \cos(\varphi + c_1) \cot\theta + c_2$$

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3 linearly independent solutions

(c_1, c_2, c_3):

$(0, 1, 0) \rightarrow \xi^{(1)} = (0, 1)$

$(0, 0, 1) \rightarrow \xi^{(2)} = (\sin\varphi, \cos\varphi \cot\theta)$

$(\frac{\pi}{2}, 0, 1) \rightarrow \xi^{(3)} = (\cos\varphi, -\sin\varphi \cot\theta)$

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Note: * S^2 is 2-dim

has 3 linearly independent $\xi^{(i)}$ $i = 1, 2, 3$

$$\lambda_1 \xi^{(1)} + \lambda_2 \xi^{(2)} + \lambda_3 \xi^{(3)} = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$$

$\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ not functions

linear independence of basis:

$$f^1 \partial_1 + f^2 \partial_2 = 0 \Rightarrow f^1 = f^2 = 0$$

at each point, so f^1, f^2 become functions if considered at all points

Examples: S^2

$\partial_\theta \xi^\theta = 0$

$\partial_\varphi \xi^\theta + \sin^2\theta\, \partial_\theta \xi^\varphi = 0$

$\partial_\varphi \xi^\varphi + \cot\theta\, \xi^\theta = 0$

Solutions:

$\xi^\theta = c_3 \sin(\varphi + c_1)$

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* $\xi^{(1)}, \xi^{(2)}, \xi^{(3)}$ generate $SO(3)$

rotations on sphere

Examples: S^2

$\partial_\theta \xi^\theta = 0$

$\partial_\varphi \xi^\theta + \sin^2\theta \, \partial_\theta \xi^\varphi = 0$

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Solutions:

$\xi^\theta = c_3 \sin(\varphi + c_1)$

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rotations on sphere

Consider geodesic along equator: $U = \partial_\varphi = (0, 1)$

$U_\mu \xi^\mu = g_{\mu\nu} U^\mu \xi^\nu = g_{\varphi\varphi} U^\varphi \xi^\varphi = \sin^2\theta \, \xi^\varphi$

For $\xi^{(1)}$: $\sin^2\theta$ is conserved

$\xi^{(2)}$: $\sin^2\theta \cos\varphi \cot\theta$ is conserved

$\xi^{(3)}$: ...

Examples: $\mathbb{R}^2$

$$ds^2 = dx^2 + dy^2$$

$$\xi^\lambda \cancel{\partial_\lambda g_{\mu\nu}} + g_{\lambda\nu}\, \partial_\mu \xi^\lambda + g_{\mu\lambda}\, \partial_\nu \xi^\lambda = 0$$

(x,x):

$$g_{\lambda x}\, \partial_x \xi^\lambda + g_{x\lambda}\, \partial_x \xi^\lambda = 0 \quad \Rightarrow$$

$$2 g_{xx}\, \partial_x \xi^x = 0 \quad \Rightarrow \quad \partial_x \xi^x = 0$$

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(x,y):

$$g_{\lambda y}\,\partial_x \xi^\lambda + g_{x\lambda}\,\partial_y \xi^\lambda = 0 \quad \Rightarrow$$

$$g_{yy}\,\partial_x \xi^y + g_{xx}\,\partial_y \xi^x = 0 \quad \Rightarrow \quad \partial_x \xi^y = -\partial_y \xi^x$$

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$$g_{\lambda y}\,\partial_y \xi^\lambda + g_{y\lambda}\,\partial_y \xi^\lambda = 0 \Rightarrow g_{yy}\,\partial_y \xi^y + g_{yy}\,\partial_y \xi^y = 0$$

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(x,x):

$$g_{\lambda x}\,\partial_x \xi^\lambda + g_{x\lambda}\,\partial_x \xi^\lambda = 0 \Rightarrow \qquad (1)$$

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$$g_{yy}\,\partial_x \xi^y + g_{xx}\,\partial_y \xi^x = 0 \Rightarrow \partial_x \xi^y = -\partial_y \xi^x$$

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$$\Rightarrow \partial_y \xi^y = 0 \qquad (3)$$

$(1),(3) \Rightarrow \quad \xi^x = f(y) \qquad \xi^y = g(x)$

$(2) \Rightarrow \quad f'(y) = -g'(x) = c_1$

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(c_1, c_2, c_3)

$(0, 1, 0)$ $\quad \xi^{(1)} = (1, 0)$

$(0, 0, 1)$ $\quad \xi^{(2)} = (0, 1)$

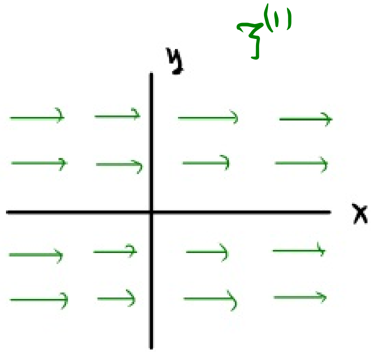
$(-1, 0, 0)$ $\quad \xi^{(3)} = (-y, x) = (-\rho\sin\varphi, \rho\cos\varphi) = \rho e_\varphi = \partial_\varphi$

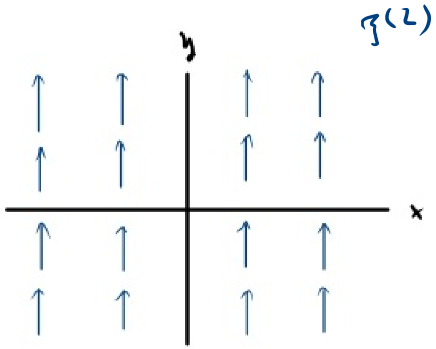
Examples: $\mathbb{R}^2$

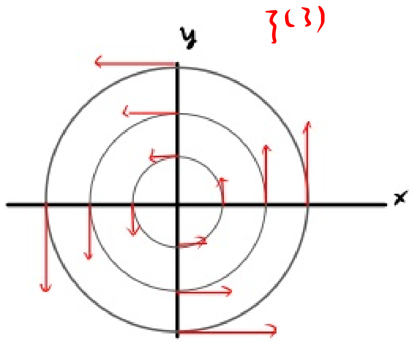
$\xi^{(1)}$: x-translations

$\xi^{(2)}$: y-translations

$\xi^{(3)}$: rotation around $(0,0)$







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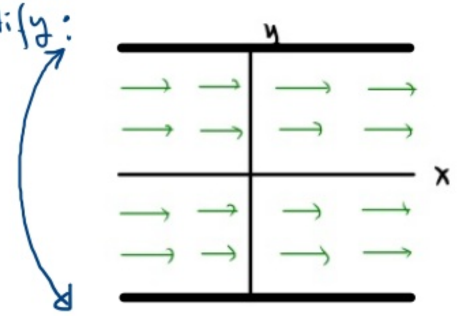
Examples: $S^1 \times \mathbb{R}$ (cylinder)

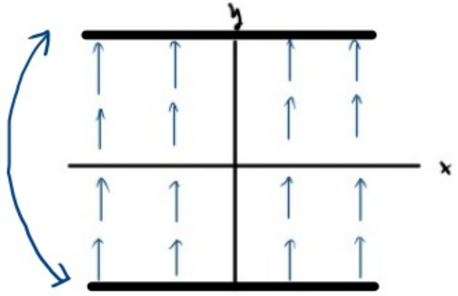
$\xi^{(1)}$: x - translations

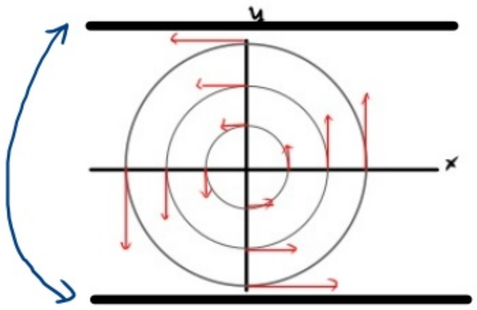
$\xi^{(2)}$: y - translations

$\xi^{(3)}$: rotation around $(0,0)$

identify:







* The Killing equations don't change: same solutions in (x,y) coordinate system

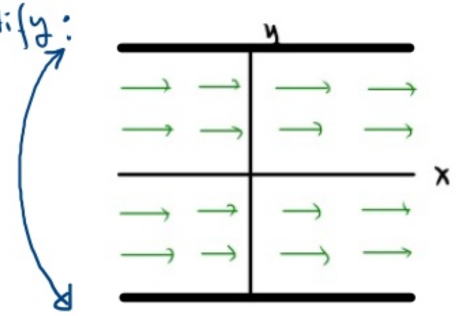
Examples: $S^1 \times \mathbb{R}$ (cylinder)

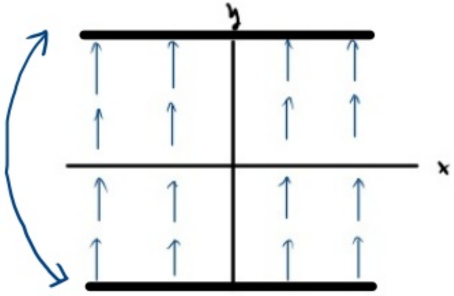
$\xi^{(1)}$: x - translations

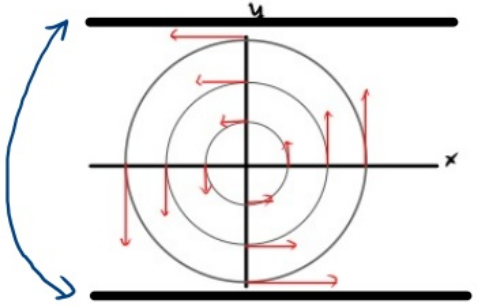
$\xi^{(2)}$: y - translations

$\xi^{(3)}$: rotations around (0,0)

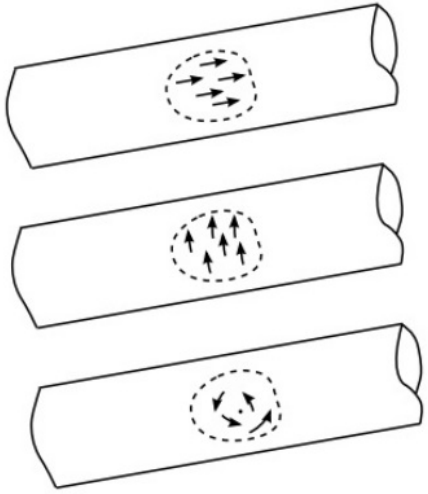
identify:







* The Killing equations don't change: same solutions in (x,y) coordinate system
* (x,y) coordinate system does not cover whole cylinder



- $\xi^{(1)}$ globally defined
- $\xi^{(2)}$ " "
- $\xi^{(3)}$ only locally defined

(notice that $\lim\limits_{y \to \pi^-} \xi^{(3)} \neq \lim\limits_{y \to -\pi^+} \xi^{(3)}$)

General Relativity, B. Crowell
https://phys.libretexts.org/Bookshelves/Relativity/General_Relativity_(Crowell)/

Example: The Schwarzschild metric

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$$

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$g_{\mu\nu}$ t-independent $\rightsquigarrow$ $g_{\mu\nu}$ static
$\Big\} \to$ energy conservation

R^μ: generates φ-translations $\rightsquigarrow$ rotations
$g_{\mu\nu}$ rotationally symmetric
$\Big\} \to$ angular momentum conservation

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For massive particle $p^\mu = mU^\mu \rightsquigarrow$ energy/ang momentum per unit mass $(\lambda = \tau)$

For massless: choose λ s.t. $p^\mu = \frac{dx^\mu}{d\lambda}$

<u>Exercise:</u> Show that for a k.v.f. ξ $\nabla_\mu \nabla_\nu \xi_\lambda = R^\rho{}_{\mu\lambda\nu} \xi_\rho$

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$$\nabla_\mu \nabla_\nu \xi_\lambda - \nabla_\nu \nabla_\mu \xi_\lambda = - R^\rho{}_{\lambda\mu\nu}\, \xi_\rho$$

$$\nabla_\lambda \nabla_\mu \xi_\nu - \nabla_\mu \nabla_\lambda \xi_\nu = - R^\rho{}_{\nu\lambda\mu}\, \xi_\rho$$

$$\nabla_\nu \nabla_\lambda \xi_\mu - \nabla_\lambda \nabla_\nu \xi_\mu = - R^\rho{}_{\mu\nu\lambda}\, \xi_\rho$$

$$\begin{tikzcd} & \mu \arrow[dr] & \\ \lambda \arrow[ur] & & \nu \arrow[ll] \end{tikzcd}$$

Exercise:

Show that for a k.v.f. ξ $\nabla_\mu \nabla_\nu \xi_\lambda = R^\rho{}_{\mu\lambda\nu}\, \xi_\rho$

$$(1)\quad \nabla_\mu \nabla_\nu \xi_\lambda - \nabla_\nu \nabla_\mu \xi_\lambda = - R^\rho{}_{\lambda\mu\nu}\, \xi_\rho$$

$$(2)\quad \nabla_\lambda \nabla_\mu \xi_\nu - \nabla_\mu \nabla_\lambda \xi_\nu = - R^\rho{}_{\nu\lambda\mu}\, \xi_\rho$$

$$(3)\quad \nabla_\nu \nabla_\lambda \xi_\mu - \nabla_\lambda \nabla_\nu \xi_\mu = - R^\rho{}_{\mu\nu\lambda}\, \xi_\rho$$

$(1) + (2) - (3) \Rightarrow$

$$\begin{aligned}
&\left(\nabla_\mu \nabla_\nu \xi_\lambda - \nabla_\mu \nabla_\lambda \xi_\nu\right) && -\left(R^\rho{}_{\lambda\mu\nu} + R^\rho{}_{\nu\lambda\mu}\right)\xi_\rho \\
&-\left(\nabla_\nu \nabla_\mu \xi_\lambda + \nabla_\nu \nabla_\lambda \xi_\mu\right) = && \\
&+\left(\nabla_\lambda \nabla_\mu \xi_\nu + \nabla_\lambda \nabla_\nu \xi_\mu\right) && + R^\rho{}_{\mu\nu\lambda}\, \xi_\rho
\end{aligned}$$

Exercise: Show that for a k.v.f. ξ $\quad \nabla_\mu \nabla_\nu \xi_\lambda = R^\rho{}_{\mu\lambda\nu}\, \xi_\rho$

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$$(3) \quad \nabla_\nu \nabla_\lambda \xi_\mu - \nabla_\lambda \nabla_\nu \xi_\mu = - R^\rho{}_{\mu\nu\lambda}\, \xi_\rho$$

$$R^\rho{}_{\lambda\mu\nu} + R^\rho{}_{\nu\lambda\mu} + R^\rho{}_{\mu\nu\lambda} = 0 \Rightarrow R^\rho{}_{\lambda\mu\nu} + R^\rho{}_{\nu\lambda\mu} = - R^\rho{}_{\mu\nu\lambda}$$

$(1) + (2) - (3) \Rightarrow$

$$\begin{aligned}
&(\nabla_\mu \nabla_\nu \xi_\lambda - \nabla_\mu \nabla_\lambda \xi_\nu) \\
&- (\nabla_\nu \cancel{\nabla_\mu \xi_\lambda} + \nabla_\nu \cancel{\nabla_\lambda \xi_\mu}) \\
&+ (\nabla_\lambda \cancel{\nabla_\mu \xi_\nu} + \nabla_\lambda \cancel{\nabla_\nu \xi_\mu})
\end{aligned}
=
\begin{aligned}
&- (R^\rho{}_{\lambda\mu\nu} + R^\rho{}_{\nu\lambda\mu})\, \xi_\rho \\
&+ R^\rho{}_{\mu\nu\lambda}\, \xi_\rho
\end{aligned}$$

$\hookrightarrow$ use $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$

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$(1) + (2) - (3) \Rightarrow$ $\quad -\nabla_\nu \xi_\lambda$ (Killing Eqs)

$$\left(\nabla_\mu \nabla_\nu \xi_\lambda - \nabla_\mu \nabla_\lambda \xi_\nu\right) \qquad\qquad -\left(R^\rho{}_{\lambda\mu\nu} + R^\rho{}_{\nu\lambda\mu}\right)\xi_\rho$$

$$-\left(\nabla_\nu \cancel{\nabla_\mu \xi_\lambda} + \nabla_\nu \cancel{\nabla_\lambda \xi_\mu}\right) =$$

$$+\left(\nabla_\lambda \cancel{\nabla_\mu \xi_\nu} + \nabla_\lambda \cancel{\nabla_\nu \xi_\mu}\right) \qquad\qquad + R^\rho{}_{\mu\nu\lambda} \qquad \xi_\rho$$

$$\Rightarrow \nabla_\mu \nabla_\nu \xi_\lambda + \nabla_\mu \nabla_\nu \xi_\lambda = -\left(-R^\rho{}_{\mu\nu\lambda}\right)\xi_\rho + R^\rho{}_{\mu\nu\lambda}\, \xi_\rho$$

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$$(3)\quad \nabla_\nu \nabla_\lambda \xi_\mu - \nabla_\lambda \nabla_\nu \xi_\mu = -R^\rho{}_{\mu\nu\lambda}\xi_\rho$$

$R^\rho{}_{\lambda\mu\nu} + R^\rho{}_{\nu\lambda\mu} + R^\rho{}_{\mu\nu\lambda} = 0 \Rightarrow R^\rho{}_{\lambda\mu\nu} + R^\rho{}_{\nu\lambda\mu} = -R^\rho{}_{\mu\nu\lambda}$

(1) + (2) − (3) $\Rightarrow$ $\quad -\nabla_\nu\xi_\lambda$ (killing Eqs)

$$\left(\nabla_\mu \nabla_\nu \xi_\lambda - \nabla_\mu \nabla_\lambda \xi_\nu\right) - \left(\nabla_\nu \nabla_\mu \xi_\lambda + \nabla_\nu \nabla_\lambda \xi_\mu\right) + \left(\nabla_\lambda \nabla_\mu \xi_\nu + \nabla_\lambda \nabla_\nu \xi_\mu\right) = -\left(R^\rho{}_{\lambda\mu\nu} + R^\rho{}_{\nu\lambda\mu}\right)\xi_\rho + R^\rho{}_{\mu\nu\lambda}\xi_\rho$$

$$\Rightarrow \nabla_\mu \nabla_\nu \xi_\lambda + \nabla_\mu \nabla_\nu \xi_\lambda = -(-R^\rho{}_{\mu\nu\lambda})\xi_\rho + R^\rho{}_{\mu\nu\lambda}\xi_\rho$$

$$\Rightarrow \cancel{2}\,\nabla_\mu \nabla_\nu \xi_\lambda = \cancel{2}\,R^\rho{}_{\mu\nu\lambda}\xi_\rho$$