Geodesics

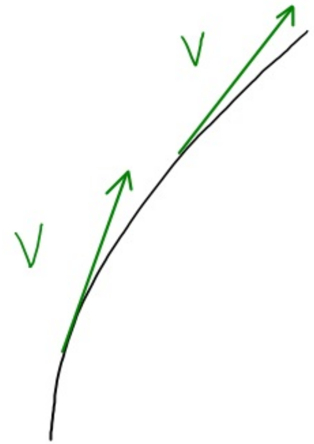
* the paths of the free (... to fall)
* straightest lines - longest proper times (locally)
* curvature makes parallel geodesics to deviate ("gravity")
* geodesics determine parallel transport and covariant derivative (*)
* Riemann Normal Coordinates:
  a convenient inertial frame

(*) torsion free

Gravitation § 10

Carroll § 3.2-3.5

* a curve is a geodesic if it parallel transports its tangent vector



$$D_V V^\mu = 0 \iff V^\nu \nabla_\nu V^\mu = 0$$

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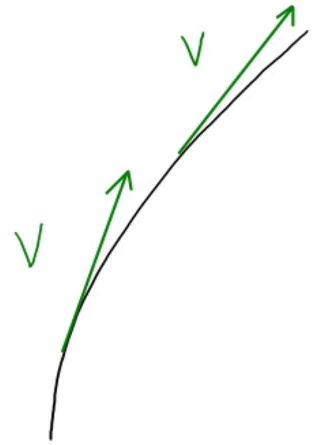

$$D_V V^\mu = 0 \iff V^\nu \nabla_\nu V^\mu = 0 \qquad (1)$$

* a weaker condition is

$$D_V V^\mu = f V^\mu, \quad f \in \mathcal{F}(M),$$

but an appropriate reparametrization it can be set to (1)
(will prove that later)

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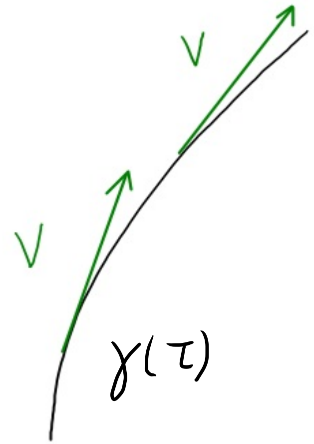

$$D_V V^\mu = 0 \Leftrightarrow V^\nu \nabla_\nu V^\mu = 0 \qquad (1)$$

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$$D_V V^\mu = f V^\mu, \quad f \in \mathcal{F}(M),$$

but an appropriate reparametrization it can be set to (1)

* For $D_V V^\mu = \frac{D}{d\tau} V^\mu = 0$, the parameter τ is an affine parameter

Any $\tau' = \alpha \tau + \beta$, $\alpha, \beta \in \mathbb{R}$ is also affine (freedom to choose units of length/time and origin)

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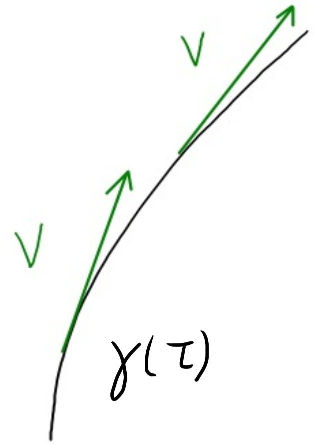

$$\Rightarrow V^\nu \partial_\nu V^\mu + \Gamma^\mu{}_{\nu\rho} V^\nu V^\rho = 0$$

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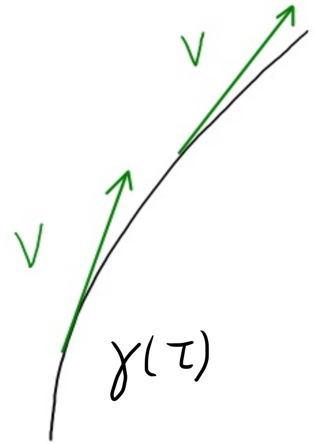

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$$\frac{dx^\nu}{d\tau} \frac{\partial}{\partial x^\nu}\left(\frac{dx^\mu}{d\tau}\right) + \Gamma^\mu{}_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0$$

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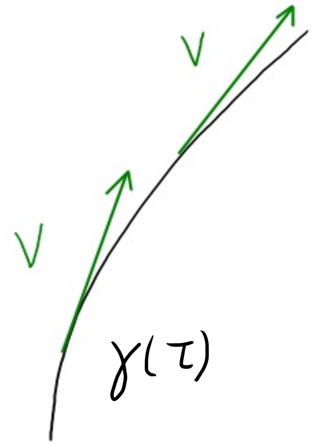

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* if $x^\mu(0)$, $\frac{dx^\mu(0)}{d\tau}$ given $\Rightarrow$ unique solution

$\Rightarrow$ P, V_P^μ determine unique geodesic through P with tangent V_P^μ

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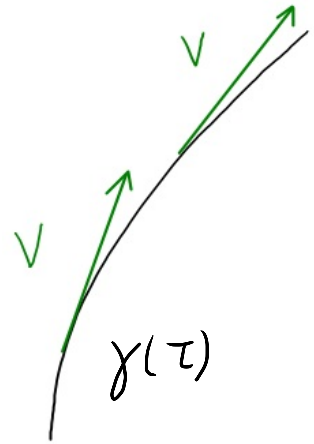

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* geodesic depends only on symmetric part of $\Gamma^\mu{}_{\nu\rho}$:
  geodesics don't depend on torsion

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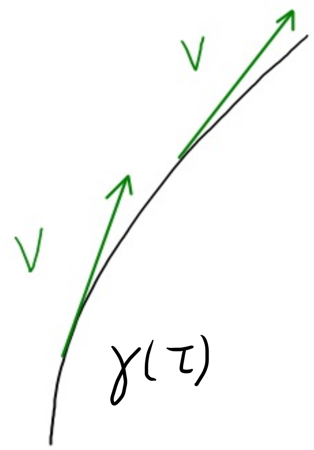

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* If $\Gamma^\mu{}_{\nu\rho} = 0$ everywhere, the solutions are straight lines

$\Rightarrow$ Flat space geodesics are straight lines

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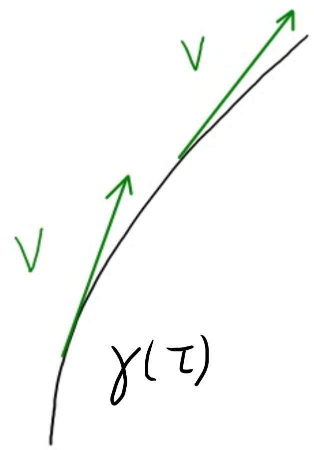

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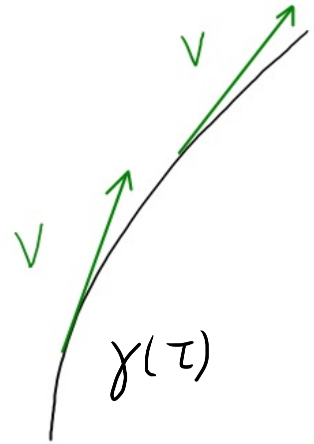
* Free particles move on geodesics

Equivalence principle: in inertial frames, free particles move on straight lines

* a curve is a geodesic if it parallel transports its tangent vector



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$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = \frac{q}{m} F^\mu{}_\nu \frac{dx^\nu}{d\tau}$$

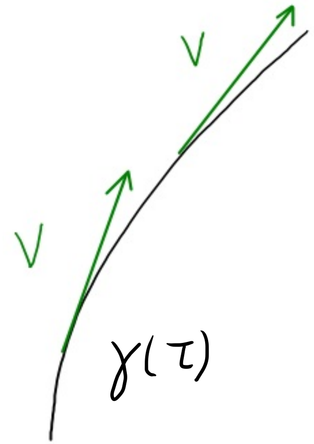
Lorentz force

* Free particles move on geodesics

Equivalence principle: in inertial frames, free particles move on straight lines

If there is a force, we add it @ RHS

* a curve is a geodesic if it parallel transports its tangent vector



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$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0$$

* character of geodesics (timelike/null/spacelike) does not change: parallel transport preserves inner product $g_{\mu\nu} V^\mu V^\nu$

* Parameter change: $s = s(\tau)$

$$\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{ds}\frac{ds}{d\tau}$$

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$$= \frac{d^2s}{d\tau^2}\frac{dx^\mu}{ds} + \left(\frac{ds}{d\tau}\right)^2\frac{d^2x^\mu}{ds^2}$$

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$$= \left(\frac{ds}{dt}\right)^2\left[\frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds}\right] + \frac{d^2s}{d\tau^2}\frac{dx^\mu}{ds}$$

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$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = -\frac{d^2s/d\tau^2}{(ds/d\tau)^2}\frac{dx^\mu}{ds}$$

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$$V^{\mu'} = \frac{dx^{\mu'}}{ds}, \quad f(s) = -\frac{s''(\tau)}{[s'(\tau)]^2}$$

* timelike paths: $U^\mu = \frac{dx^\mu}{d\tau}$ four-velocity, τ proper time

$p^\mu = m U^\mu$

$U^\nu \nabla_\nu U^\mu = 0 \iff p^\nu \nabla_\nu p^\mu = 0$

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* null paths: affine parameter λ s.t. $p^\mu = \frac{dx^\mu}{d\lambda}$

observer w/ four velocity U^μ measures

$E = -p_\mu U^\mu$ (potential energy not included)

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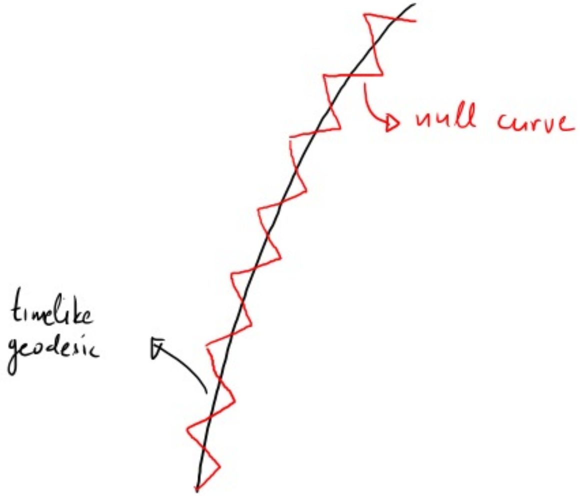

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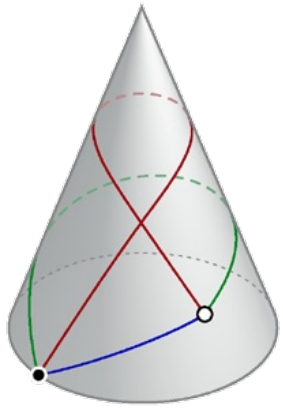

http://www.rdrop.com/~half/Creations/Puzzles/cone.geodesics/index.html

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* timelike geodesics are local maxima of propertime compared to any other curve connecting nearby events: can't be minima, always arbitrarily close to a null curve with zero length.
* These are local maxima (minima for spacelike), due to global topology two points may be connected with geodesics of different lengths

Extremization of length/proper time $\rightarrow$ geodesics

Consider a timelike curve, $ds^2 = -d\tau^2$,

$$\tau = \int d\tau \left\{ -g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right\}^{1/2}$$

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Parameterize curve s.t. $g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = g_{\mu\nu} U^\mu U^\nu = -1$, so that

$$\delta\tau = -\frac{1}{2} \int d\tau \, \delta \left(g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right)$$

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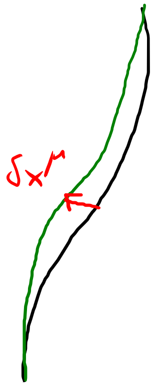

Variation: $x^\mu \to x^\mu + \delta x^\mu$ $\quad$ $g_{\mu\nu} \to g_{\mu\nu} + \partial_\lambda g_{\mu\nu} \, \delta x^\lambda$ (like Taylor series)

Extremization of length/proper time → geodesics

$$\int d\tau \, \delta\left(g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right) = \int d\tau \left\{ \delta g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + g_{\mu\nu} \, \delta\left(\frac{dx^\mu}{d\tau} \right) \frac{dx^\nu}{d\tau} + g_{\mu\nu} \frac{dx^\mu}{d\tau} \, \delta\left(\frac{dx^\nu}{d\tau} \right) \right\}$$

$$\delta\tau = -\frac{1}{2} \int d\tau \, \delta\left(g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right)$$

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$$,\quad \delta\left(\frac{dx^\mu}{d\tau}\right) = \frac{d}{d\tau}\delta x^\mu$$

$$= \int d\tau \left\{ \partial_\lambda g_{\mu\nu}\,\delta x^\lambda \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + 2\, g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d}{d\tau}\delta x^\nu \right\}$$

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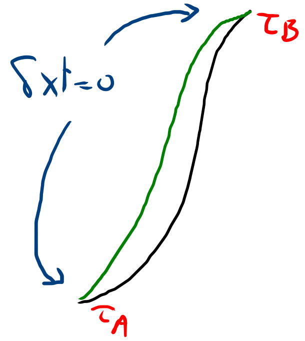

Boundary term $= 0$ because $\delta x^\nu = 0$ at initial + final point of variation

$$2g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d}{d\tau}\delta x^\nu = \frac{d}{d\tau}\left[2g_{\mu\nu}\frac{dx^\mu}{d\tau}\delta x^\nu\right] - \frac{d}{d\tau}\left[2g_{\mu\nu}\frac{dx^\mu}{d\tau}\right]\delta x^\nu$$

$$\int_{\tau_A}^{\tau_B} d\tau \frac{d}{d\tau}\left[2g_{\mu\nu}\frac{dx^\mu}{d\tau}\delta x^\nu\right] = 2g_{\mu\nu}\frac{dx^\mu}{d\tau}\delta x^\nu\Big|_{\tau_B} - 2g_{\mu\nu}\frac{dx^\mu}{d\tau}\delta x^\nu\Big|_{\tau_A}, \quad \delta x^\nu(\tau_B) = \delta x^\nu(\tau_A) = 0$$

Extremization of length/proper time → geodesics

$$\int d\tau\, \delta\left(g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right) = \int d\tau \left\{ \delta g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + g_{\mu\nu}\, \delta\left(\frac{dx^\mu}{d\tau}\right) \frac{dx^\nu}{d\tau} + g_{\mu\nu} \frac{dx^\mu}{d\tau}\, \delta\left(\frac{dx^\nu}{d\tau}\right) \right\}$$

$\lambda \leftrightarrow \nu$

$$= \int d\tau \left\{ \partial_\lambda g_{\mu\nu}\, \delta x^\lambda \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + 2\, g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d}{d\tau} \delta x^\nu \right\}$$

$$= \int d\tau \left\{ \partial_\nu g_{\mu\lambda}\, \delta x^\nu \frac{dx^\mu}{d\tau} \frac{dx^\lambda}{d\tau} - 2 \frac{d}{d\tau}\left[g_{\mu\nu} \frac{dx^\mu}{d\tau} \right] \delta x^\nu \right\}$$

Boundary term = 0 because $\delta x^\nu = 0$ at initial + final point of variation

$$2 g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d}{d\tau} \delta x^\nu = \frac{d}{d\tau}\left[2 g_{\mu\nu} \frac{dx^\mu}{d\tau} \delta x^\nu \right] - \frac{d}{d\tau}\left[2 g_{\mu\nu} \frac{dx^\mu}{d\tau} \right] \delta x^\nu$$

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Extremization of length/proper time → geodesics

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$$= \int d\tau \left\{ \partial_\nu g_{\mu\lambda}\, \delta x^\nu \frac{dx^\mu}{d\tau} \frac{dx^\lambda}{d\tau} - 2 \frac{d}{d\tau}\left[g_{\mu\nu} \frac{dx^\mu}{d\tau} \right] \delta x^\nu \right\}$$

$$= \int d\tau\, \delta x^\nu \left\{ \partial_\nu g_{\mu\lambda} \frac{dx^\mu}{d\tau} \frac{dx^\lambda}{d\tau} - 2\left[\frac{d}{d\tau} g_{\mu\nu} \right] \frac{dx^\mu}{d\tau} - 2\, g_{\mu\nu} \frac{d^2 x^\mu}{d\tau^2} \right\}$$

Extremization of length/proper time → geodesics

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$\lambda \leftrightarrow \nu$

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$$= \int d\tau \left\{ \partial_\nu g_{\mu\lambda}\, \delta x^\nu \frac{dx^\mu}{d\tau}\frac{dx^\lambda}{d\tau} - 2\frac{d}{d\tau}\left[g_{\mu\nu}\frac{dx^\mu}{d\tau}\right]\delta x^\nu \right\}$$

$$\frac{d}{d\tau}g_{\mu\nu} = \frac{dx^\lambda}{d\tau}\frac{\partial g_{\mu\nu}}{\partial x^\lambda}$$

$$= \int d\tau\, \delta x^\nu \left\{ \partial_\nu g_{\mu\lambda}\frac{dx^\mu}{d\tau}\frac{dx^\lambda}{d\tau} - 2\left[\frac{d}{d\tau}g_{\mu\nu}\right]\frac{dx^\mu}{d\tau} - 2\, g_{\mu\nu}\frac{d^2x^\mu}{d\tau^2}\right\}$$

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Extremization of length/proper time → geodesics

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symmetric in $\mu \leftrightarrow \lambda$

$$= \int d\tau\, \delta x^\nu \left\{ \partial_\nu g_{\mu\lambda}\frac{dx^\mu}{d\tau}\frac{dx^\lambda}{d\tau} - 2\,\partial_\lambda g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\lambda}{d\tau} - 2 g_{\mu\nu}\frac{d^2x^\mu}{d\tau^2}\right\}$$

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$$= -\int d\tau\, \delta x^\nu \left\{ 2 g_{\nu\mu}\frac{d^2x^\mu}{d\tau^2} + \left(\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\mu\lambda}\right)\frac{dx^\mu}{d\tau}\frac{dx^\lambda}{d\tau}\right\}$$

Extremization of length/proper time → geodesics

Extremization for arbitrary δx^ν implies that the integrant must be zero $\forall x$

(simply choose $\delta x^\nu = \delta(x^\nu - x^\nu_0)$ for all x_0^ν on the curve)

$$2 g_{\mu\nu} \frac{d^2 x^\mu}{d\tau^2} + \left(\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\mu\lambda} \right) \frac{dx^\mu}{d\tau} \frac{dx^\lambda}{d\tau} = 0$$

---

symmetric in $\mu \leftrightarrow \lambda$

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$$g^{\sigma\nu} g_{\mu\nu} \frac{d^2 x^\mu}{d\tau^2} + \frac{1}{2} g^{\sigma\nu} \left(\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\mu\lambda}\right) \frac{dx^\mu}{d\tau} \frac{dx^\lambda}{d\tau} = 0$$

---

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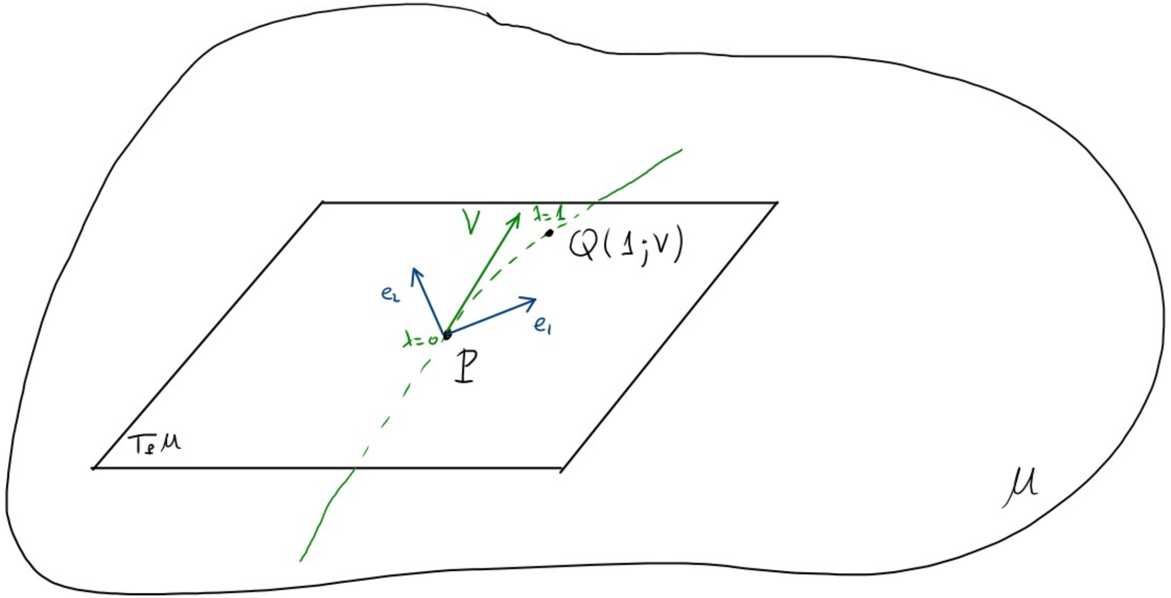
$$\underbrace{g^{\sigma\mu} g_{\mu\nu}}_{\delta^\sigma{}_\nu} \frac{d^2 x^\mu}{d\tau^2} + \frac{1}{2} g^{\sigma\mu} \left(\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\mu\lambda} \right) \frac{dx^\mu}{d\tau} \frac{dx^\lambda}{d\tau} = 0$$

$$\frac{d^2 x^\sigma}{d\tau^2} + \Gamma^\sigma{}_{\mu\lambda} \frac{dx^\mu}{d\tau} \frac{dx^\lambda}{d\tau} = 0$$

Riemann Normal Coordinates

* $\Gamma^{\mu}{}_{\nu\rho} = 0$ everywhere only on flat manifolds
* Construct local inertial frame using geodesics:



Riemann Normal Coordinates

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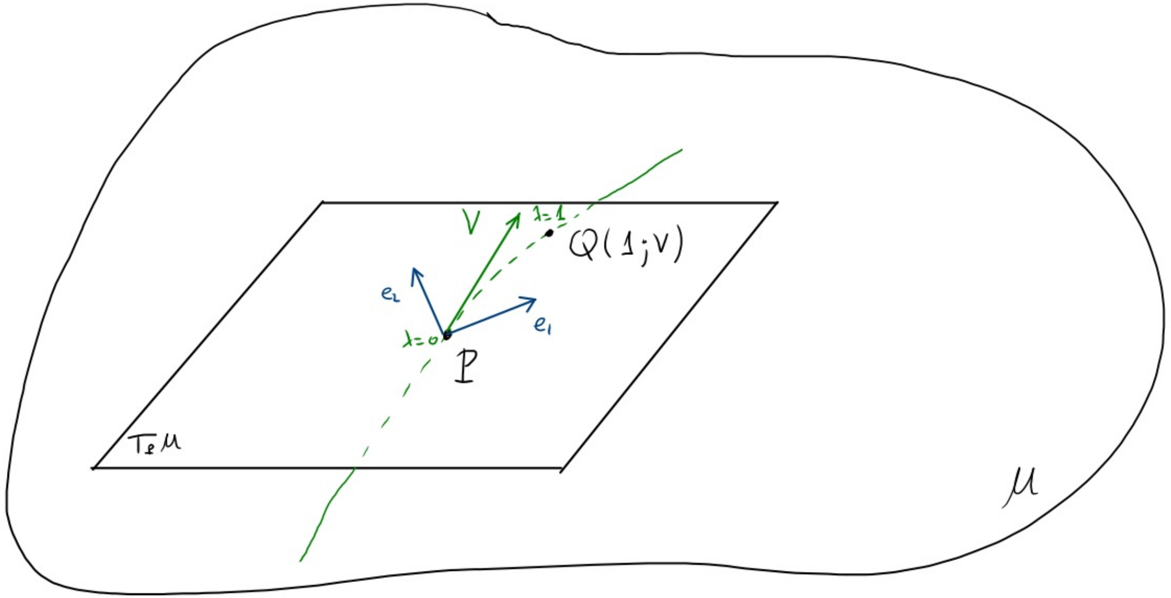
* Construct local inertial frame using geodesics:

 - Pick an event P and all geodesics through P



Riemann Normal Coordinates

* $\Gamma^{\mu}{}_{\nu\rho} = 0$ everywhere only on flat manifolds
* Construct local inertial frame using geodesics:
  - Pick an event P and all geodesics through P
  - Each vector V at P determines a geodesic with affine parameter λ.

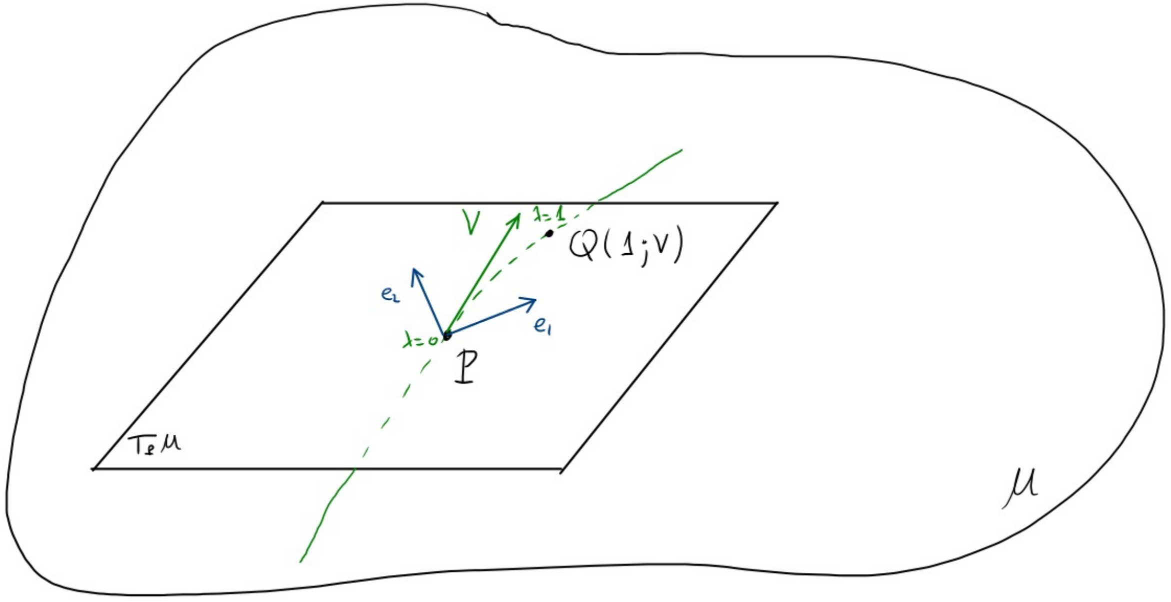
    Consider the point

$$Q = Q(\lambda; V)$$

which geodesic

where on geodesic

on such a geodesic



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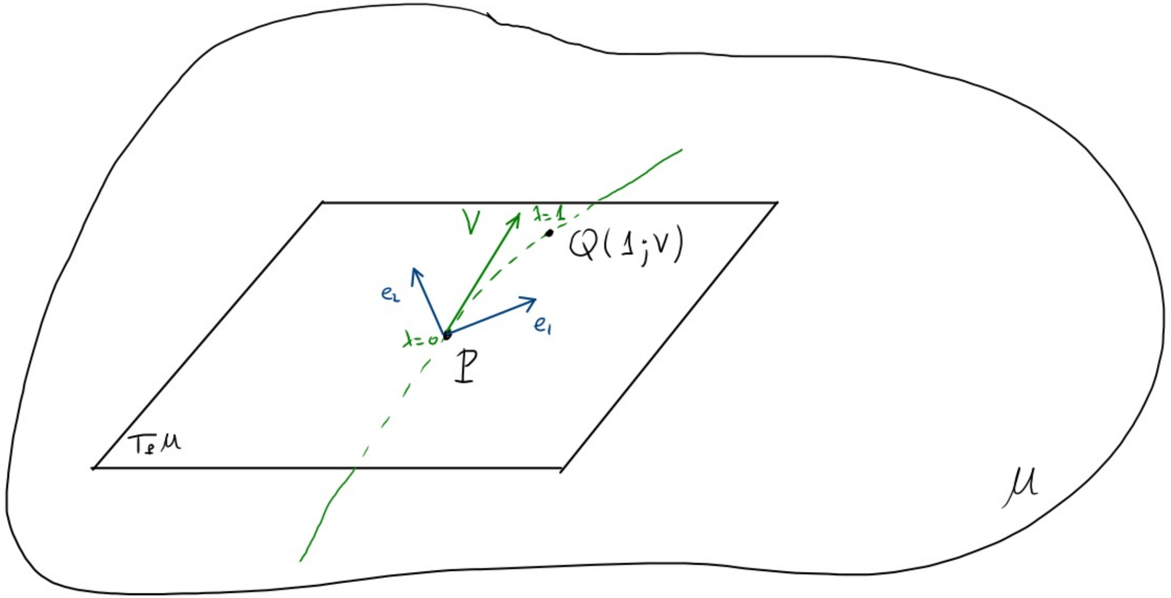
  Notice that $Q = Q(\lambda; V) = Q(\tfrac{1}{2}\lambda; 2V)$,
  so we fix $\lambda = 1$, vary V



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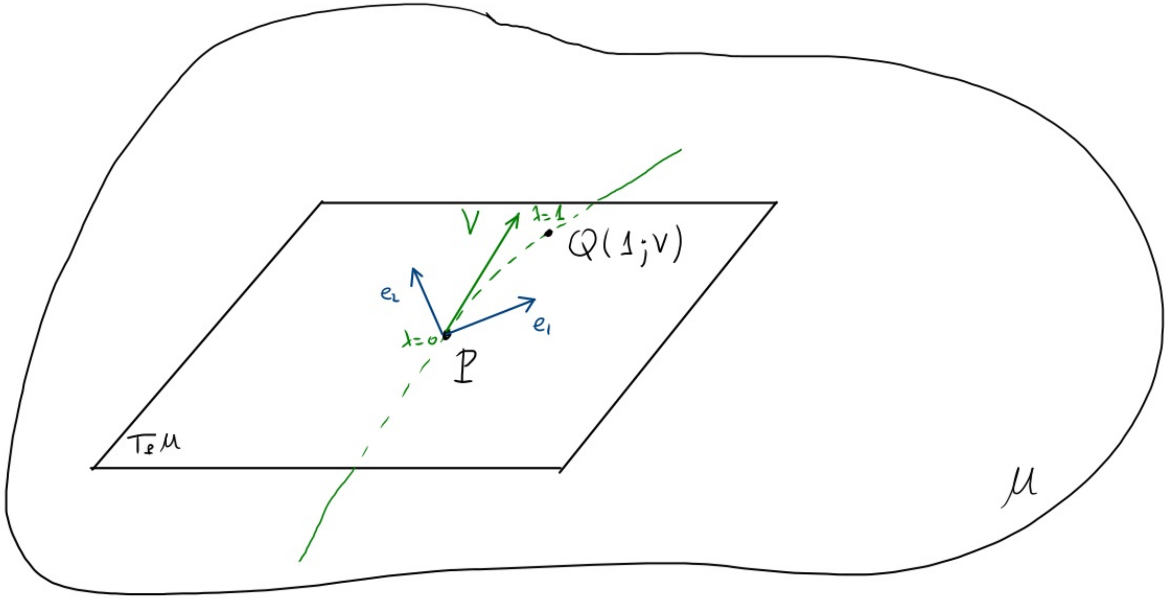

* In a neighborhood of P where geodesics don't intersect, we have 1-1 map of some vectors + points

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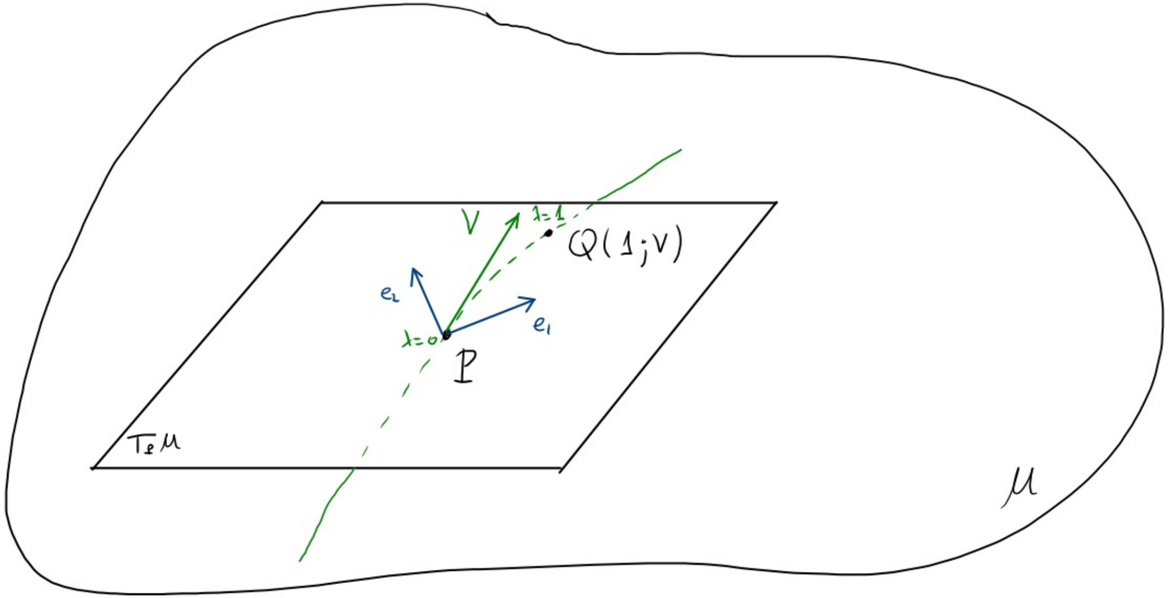


* In a neighborhood of P where geodesics don't intersect, we have 1-1 map of some vectors + points

  $\Rightarrow$ Choose orthonormal basis e_μ at T_PM

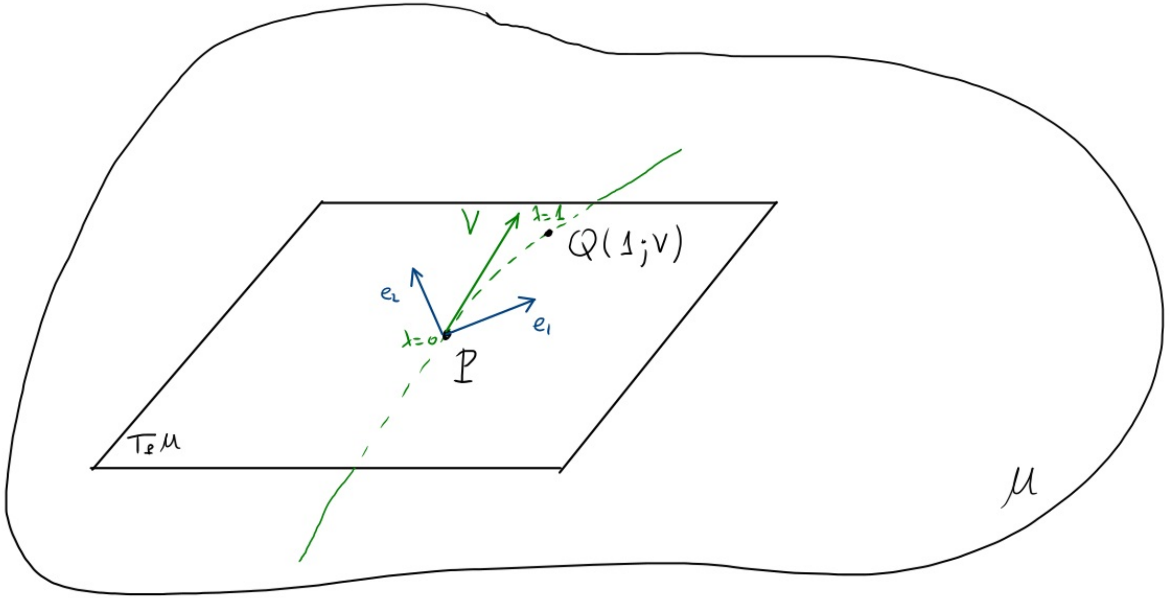
  $$g(e_\mu, e_\nu) = g_{\alpha\beta}\, e^\alpha_\mu e^\beta_\nu = \eta_{\mu\nu}\,, \text{ so that}$$

  $V = x^\mu e_\mu$ and define coordinates of $Q(1;V)$ to be $\{x^\mu\}$

Riemann Normal Coordinates

* Prove that: (problem 11.9, Gravitation)

- $e_\mu = \partial_\mu |_P$
- $\Gamma^\mu{}_{\nu\rho}(P) = 0$
- $\partial_\sigma \Gamma^\mu{}_{\nu\rho}(P) = -\frac{1}{3}\left(R^\mu{}_{\nu\rho\sigma} + R^\mu{}_{\rho\nu\sigma}\right)$
- $g_{\mu\nu}(P) = \eta_{\mu\nu}$
- $\partial_\sigma g_{\mu\nu}(P) = 0$
- $\partial_\rho \partial_\sigma g_{\mu\nu}(P) = -\frac{1}{3}\left(R_{\mu\sigma\nu\rho} + R_{\mu\rho\nu\sigma}\right)$
- $R_{\mu\nu\rho\sigma}(P) = \partial_\rho \partial_\nu g_{\mu\sigma}(P) - \partial_\sigma \partial_\nu g_{\mu\rho}(P)$



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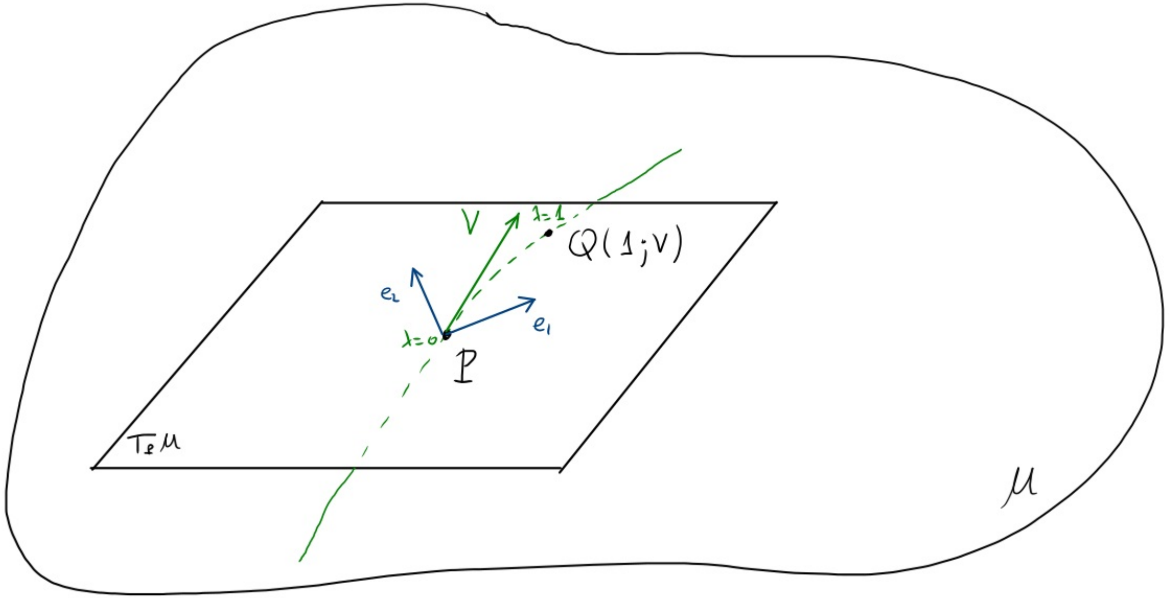


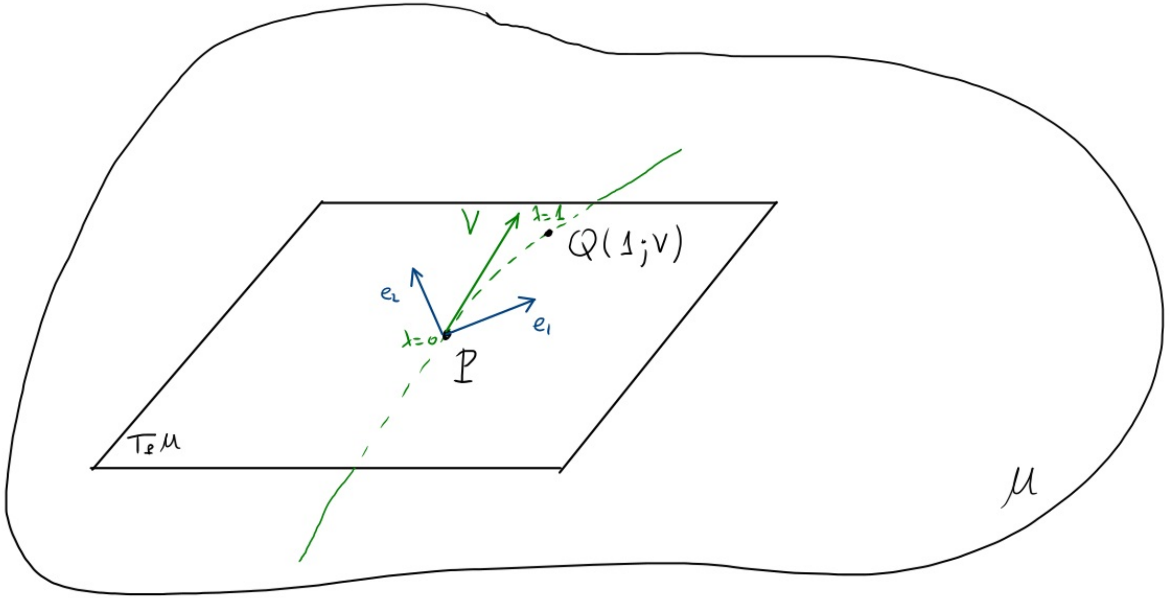
* any other normal coordinate system same to 2nd order

$$x^{\mu\prime}(P) = x^\mu(P) + O\left[(x^\mu)^3\right]$$

Riemann Normal Coordinates

* Prove that: (problem 11.9, Gravitation)

- $e_\mu = \partial_\mu |_P$
- $\Gamma^\mu{}_{\nu\rho}(P) = 0$
- $\partial_\sigma \Gamma^\mu{}_{\nu\rho}(P) = -\frac{1}{3}\left(R^\mu{}_{\nu\rho\sigma} + R^\mu{}_{\rho\nu\sigma}\right)$ (1)
- $g_{\mu\nu}(P) = \eta_{\mu\nu}$
- $\partial_\sigma g_{\mu\nu}(P) = 0$
- $\partial_\rho \partial_\sigma g_{\mu\nu}(P) = -\frac{1}{3}\left(R_{\mu\sigma\nu\rho} + R_{\mu\rho\nu\sigma}\right)$
- $R_{\mu\nu\rho\sigma}(P) = \partial_\rho \partial_\nu g_{\mu\sigma}(P) - \partial_\sigma \partial_\nu g_{\mu\rho}(P)$ (2)



* any other normal coordinate system same to 2nd order

$$x^{\mu\prime}(P) = x^\mu(P) + O[(x^\mu)^3]$$

* if they are same to 3rd order

$$x^{\mu\prime}(P) = x^\mu(P) + O[(x^\mu)^4], \text{ then}$$

eqs. (1) + (2) are preserved

Geodesics $\Rightarrow$ Parallel transport

so we will have established

$$\text{Geodesics} \Leftrightarrow \underset{\text{Affine Connection}}{\text{Parallel Transport}} \Leftrightarrow \underset{\text{(torsion free)}}{\text{Cov. Derivative}}$$

Geodesics $\Rightarrow$ Parallel transport

so we will have established

Geodesics $\Leftrightarrow$ Parallel Transport $\Leftrightarrow$ Cov. Derivative
Affine Connection (torsion free)

* Assume that we know all geodesics on $\mathcal{M}$

$\Rightarrow$ construct parallel transport using Shild's ladder

(see Gravitation §10)

Geodesics ⇒ Parallel transport

so we will have established

Geodesics ⇔ Parallel Transport ⇔ Cov. Derivative
Affine Connection (torsion free)

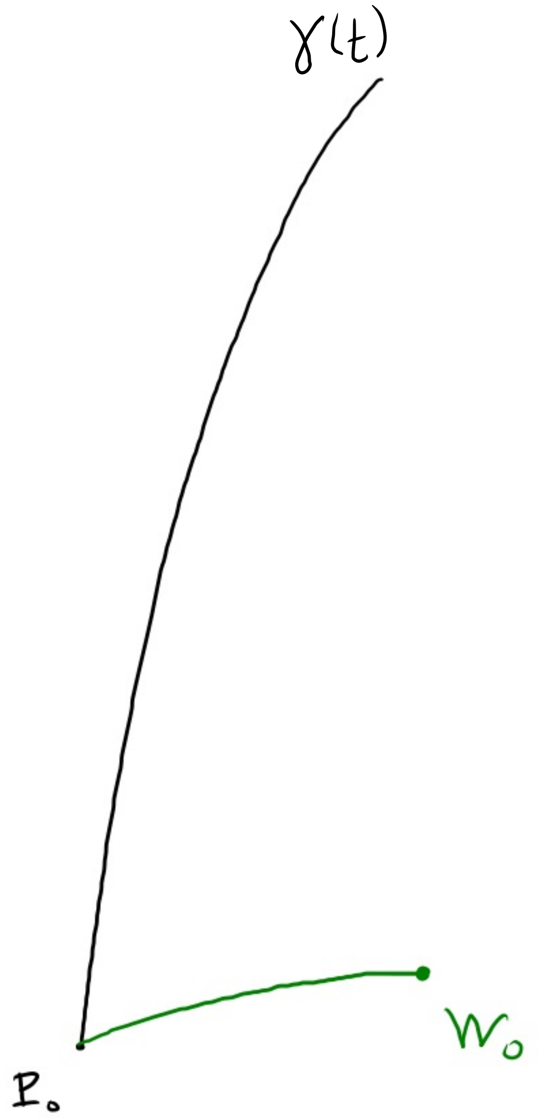
* Assume that we know all geodesics on $\mathcal{M}$

⇒ construct parallel transport using Shild's ladder
(see Gravitation §10)

not necessarily a geodesic!

* Consider a curve $\gamma(t)$ and a point P_0 on it

Let $W(P_0)$ be a vector at P_0 and the geodesic passing through P_0 in the direction of $W(P_0)$



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Affine Connection (torsion free)

* Assume that we know all geodesics on $\mathcal{M}$

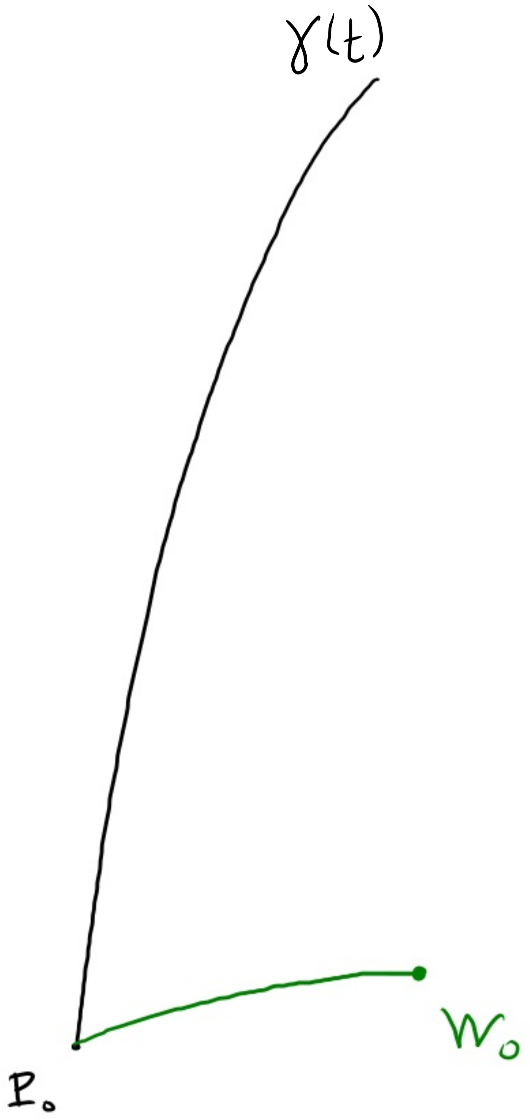
$\Rightarrow$ construct parallel transport using Shild's ladder
(see Gravitation §10)

* Consider a curve $\gamma(t)$ and a point P_0 on it

Let $W(P_0)$ be a vector at P_0 and the geodesic passing through P_0 in the direction of $W(P_0)$
Let W_0 be a point on the geodesic, affine parameter distance ϵ from P_0. For small ϵ, we may write

$$\overline{P_0 W_0} \approx \epsilon W(P_0)$$

("equivalence principle")



* take a point P_1 close to P_0

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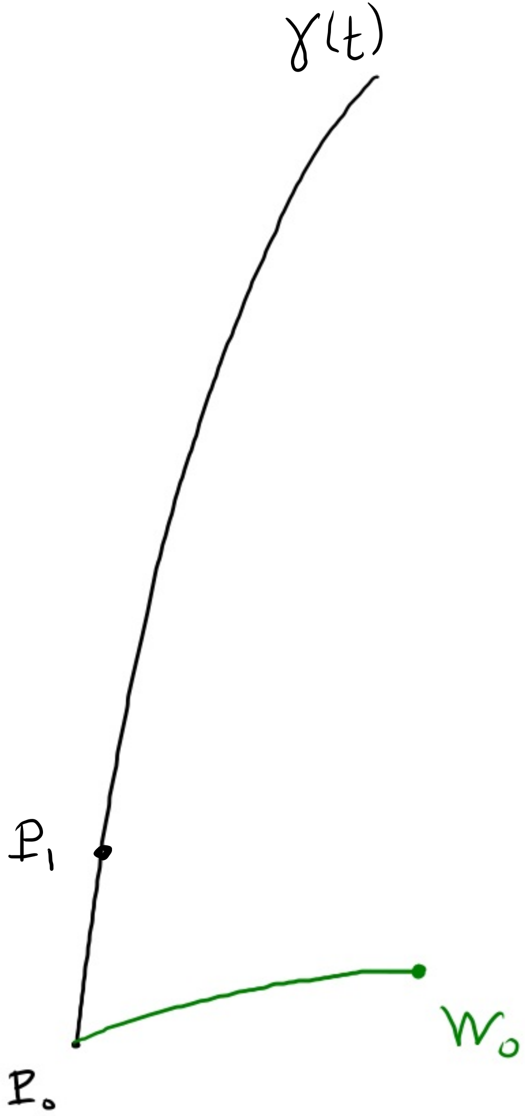
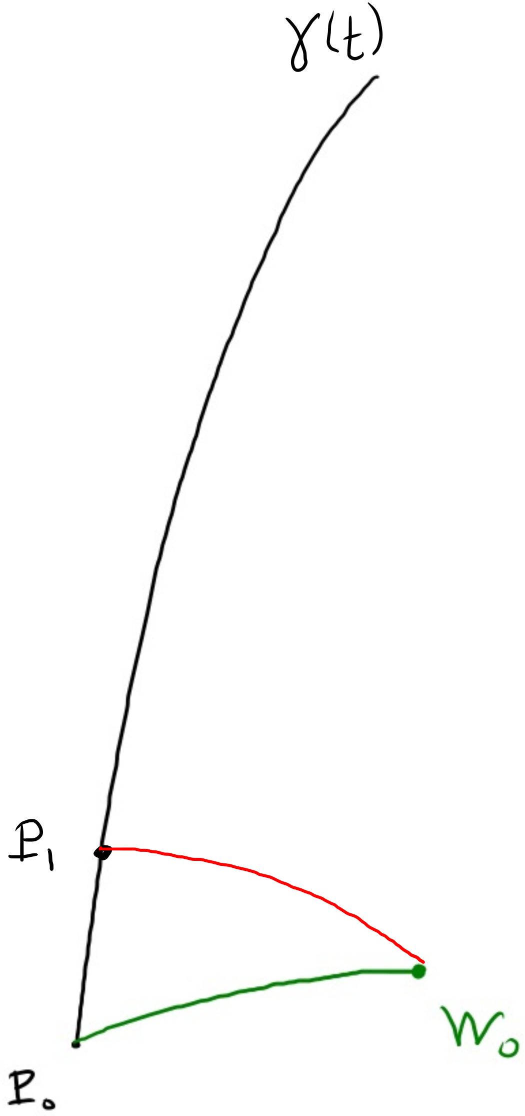


* take a point P_1 close to P_0
* draw the geodesic $P_1 W_0$



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* take a point P_1 close to P_0
* draw the geodesic P_1W_0
* compute N_1, midway between P_1, W_0

$$\lambda_{N_1} = \frac{\lambda_{P_1} + \lambda_{W_0}}{2}$$

---

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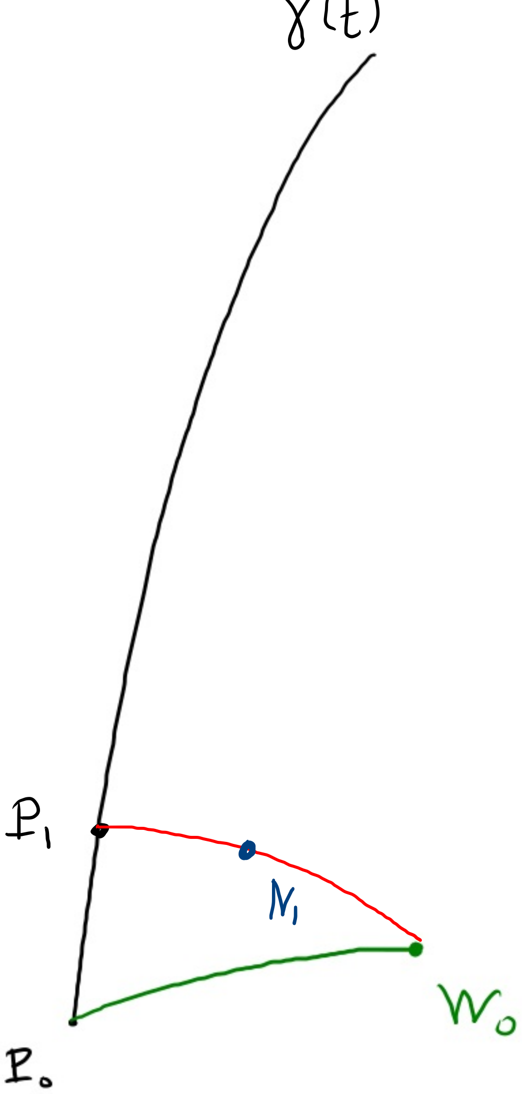

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* compute N_1, midway between P_1, W_0

$$\lambda_{N_1} = \frac{\lambda_{P_1} + \lambda_{W_0}}{2}$$

* compute geodesic $P_0 N_1$, extend it to W_1 s.t.

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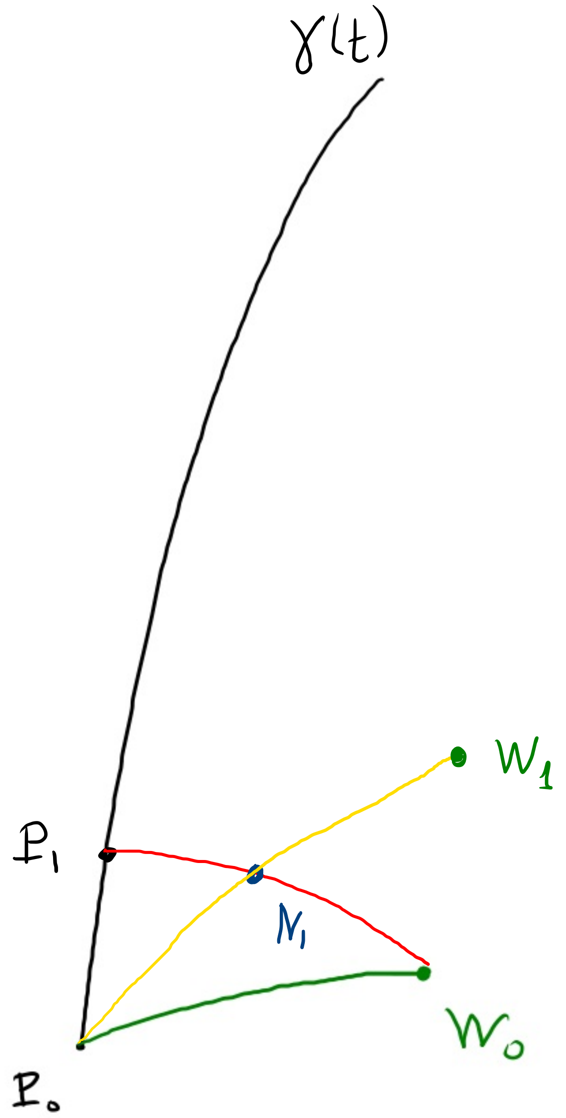

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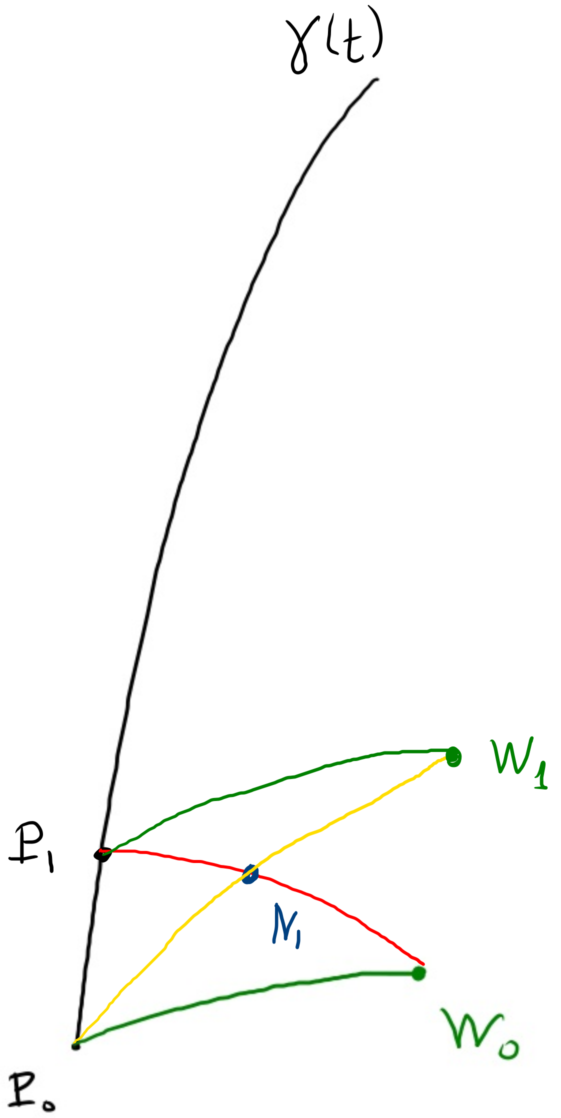

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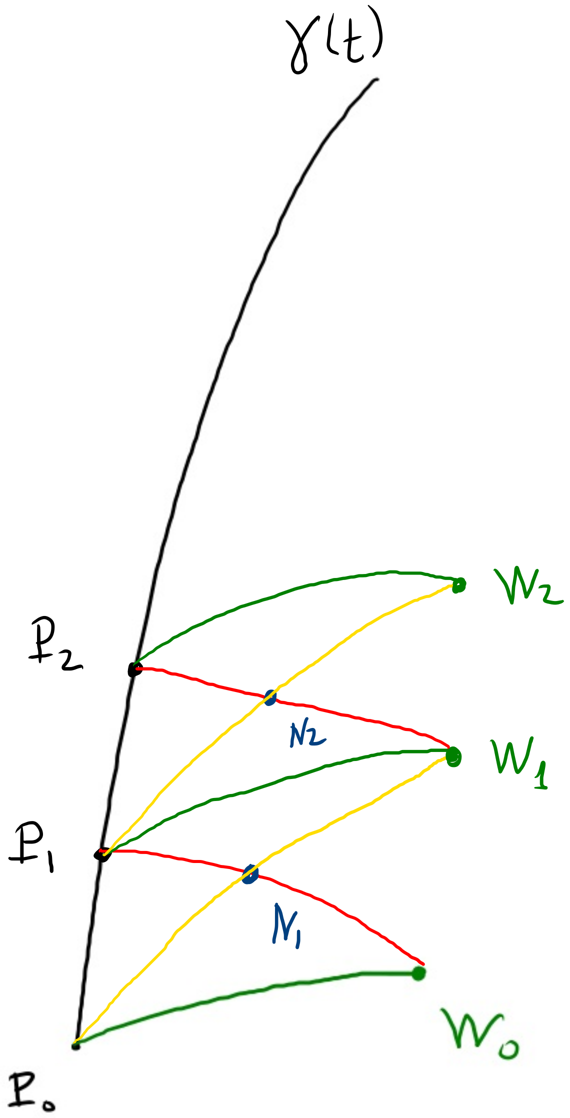
* Repeat:
  - choose P_2
  - draw geodesic $P_2 W_1$
  - draw N_2 at $\lambda_{N_2} = (\lambda_{P_2} + \lambda_{W_1})/2$
  - draw geodesic $P_1 N_2$, extend it to W_2 s.t.

    $$\lambda_{N_2} = (\lambda_{P_1} + \lambda_{W_2})/2$$

  - $P_2 W_2 \parallel P_1 W_1 \parallel P_0 W_0$



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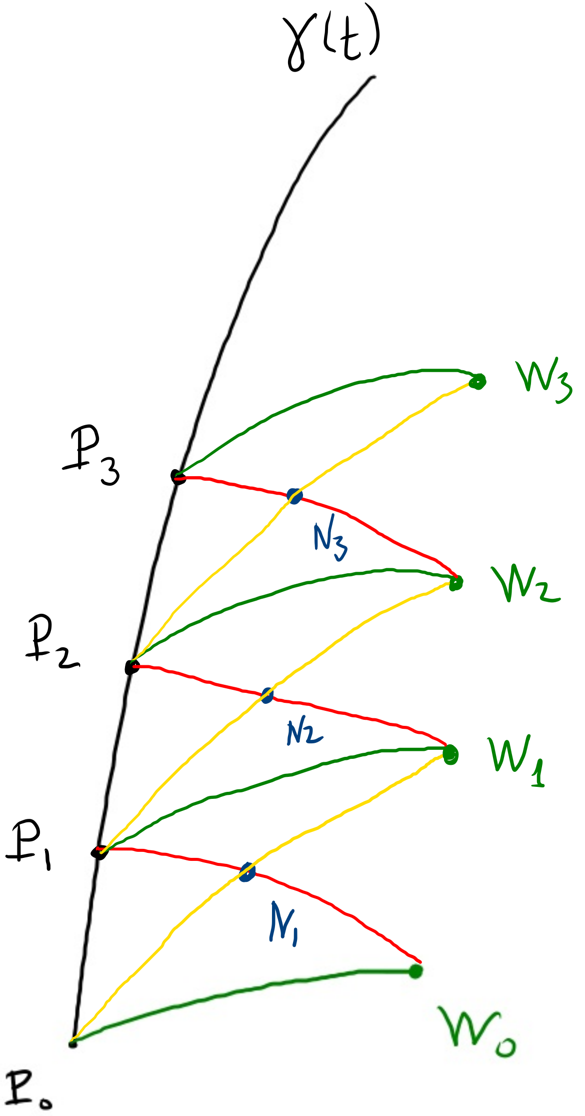
  - $P_2 W_2 \parallel P_1 W_1 \parallel P_0 W_0 \parallel P_3 W_3$



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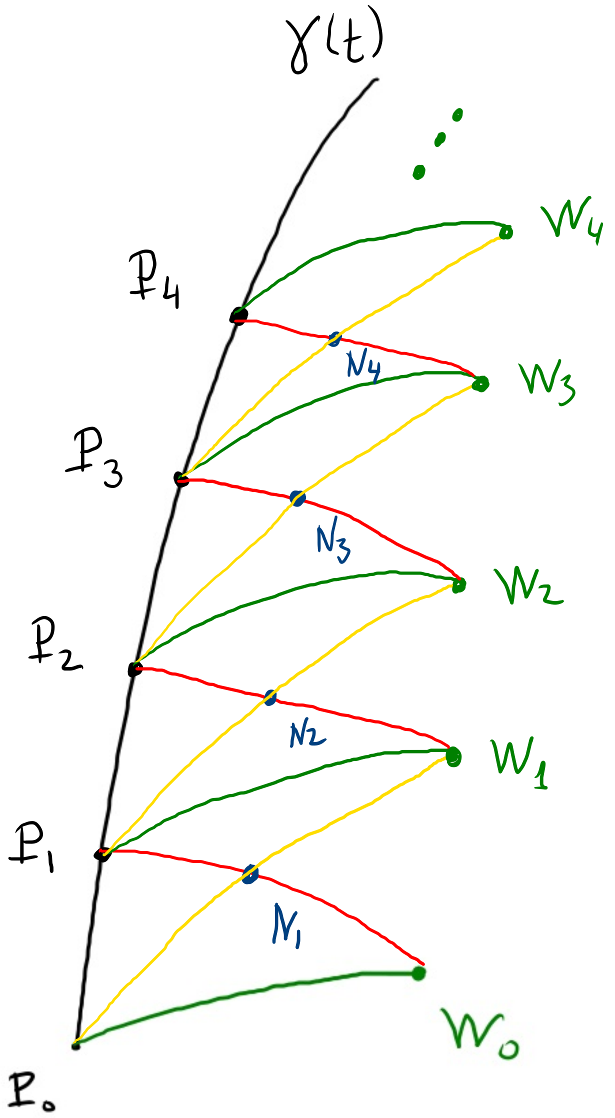
    $\lambda_{N_2} = (\lambda_{P_1} + \lambda_{W_2})/2$
  - $P_2 W_2 \parallel P_1 W_1 \parallel P_0 W_0 \parallel P_3 W_3 \parallel P_4 W_4 \parallel \dots$



* Works for flat space ...

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* Repeat:
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  - draw geodesic $P_2 W_1$
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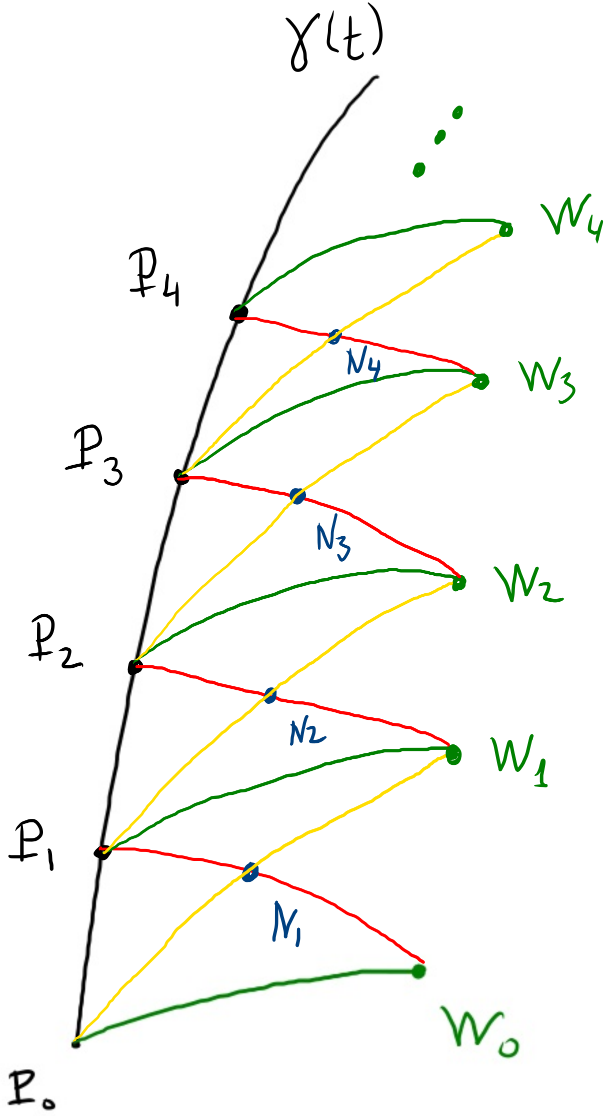
  - draw geodesic $P_1 N_2$, extend it to W_2 s.t. $\lambda_{N_2} = (\lambda_{P_1} + \lambda_{W_2})/2$
  - $P_2 W_2 \,//\, P_1 W_1 \,//\, P_0 W_0 \,//\, P_3 W_3 \,//\, P_4 W_4 \,//\, \ldots$



* Works for flat space ...

* Works for curved space over infinitesimal distances
  (equivalence principle, inertial frames)

---

* Repeat:
  - choose P_2
  - draw geodesic P_2W_1
  - draw N_2 at $\lambda_{N_2} = (\lambda_{P_2} + \lambda_{W_1})/2$
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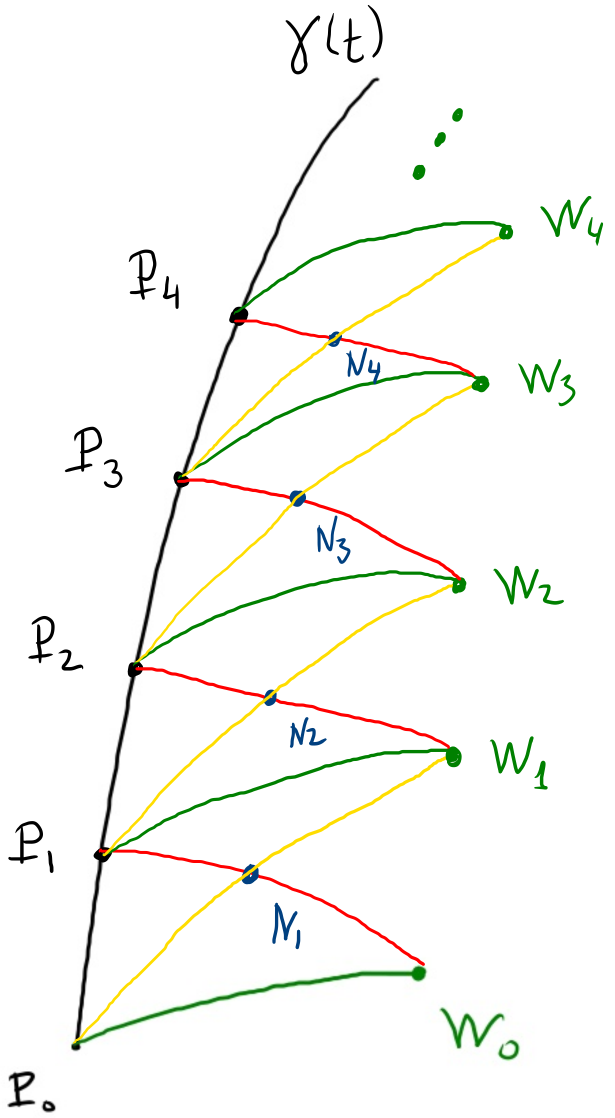
  - $P_2W_2 \,//\, P_1W_1 \,//\, P_0W_0 \,//\, P_3W_3 \,//\, P_4W_4 \,//\, \cdots$



* Works for flat space ...
* Works for curved space over infinitesimal distances
  (equivalence principle, inertial frames)
* Defines parallel transport over finite distance by integrating the procedure

$$W(t_0) \to \tau_{t_0 t}\, W(t_0) \in T_{\gamma(t)} M$$

$$W(t) \to \tau_{t t_0}\, W(t) \in T_{\gamma(t_0)} M$$

---

* Repeat:
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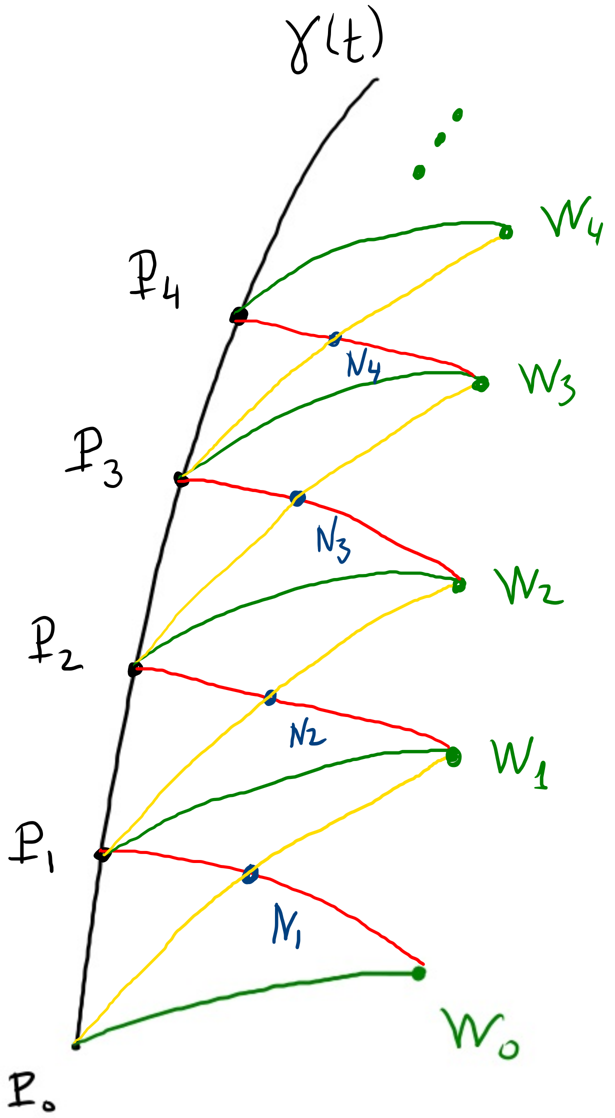
  - $P_2 W_2 \parallel P_1 W_1 \parallel P_0 W_0 \parallel P_3 W_3 \parallel P_4 W_4 \parallel \dots$



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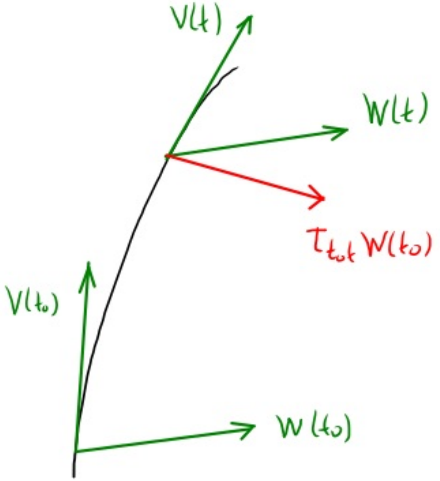

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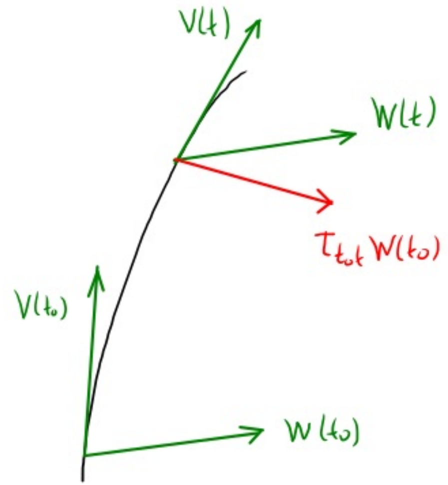
* Defines parallel transport over finite distance by integrating the procedure

$$W(t_0) \to \tau_{t_0 t} W(t_0) \quad \in T_{\gamma(t)} M$$

$$W(t) \to \tau_{t t_0} W(t) \quad \in T_{\gamma(t_0)} M$$



such that

$$\tau_{t_0 t} f(t_0) W(t_0) = f(t_0) \tau_{t_0 t} W(t_0)$$

$$\tau_{t_0 t} \left(W(t_0) + U(t_0) \right) = \tau_{t_0 t} W(t_0) + \tau_{t_0 t} U(t_0)$$

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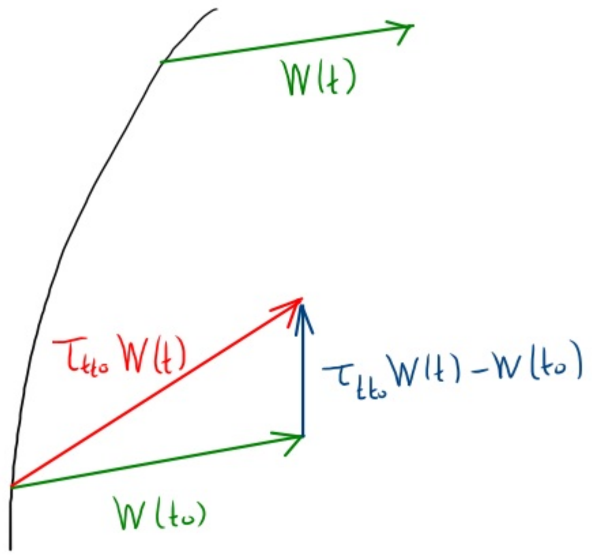
* defines covariant derivative: $D_v W(t_0) = \lim_{t \to t_0} \dfrac{\tau_{t t_0} W(t) - W(t_0)}{t - t_0}$



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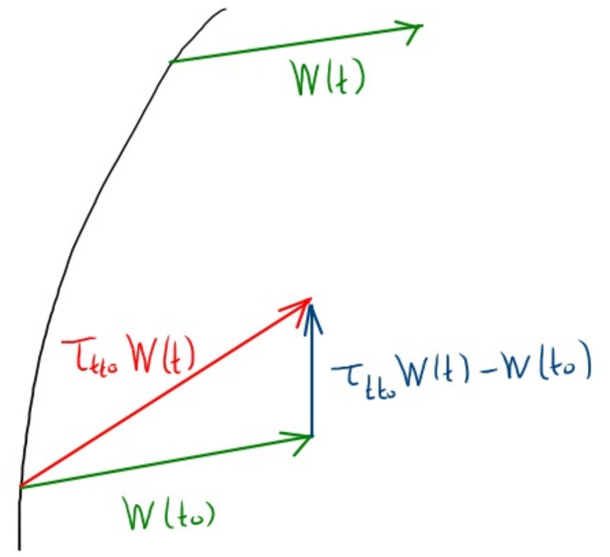
s.t. $D_V(fW) = f D_V W + V(f)\, W \quad (1)$

$D_V(W+U) = D_V W + D_V U \quad (2)$

$D_{fV} W = f D_V W \quad (3)$

$D_V W - D_W V = [V, W] \quad (4)$

$D_{V+U} W = D_V W + D_U W \quad (5)$



* (1), (2), (3) easy to prove (exercise ...)

* (2) + (4) $\Rightarrow$ (5) (see previous slides ...)

* so we prove (4):

Consider two vectors $V(t_0)$, $W(t_0)$ and their tangent curves. Consider infinitesimal $\epsilon V(t_0)$, $\epsilon W(t_0)$

s.t. $D_V(fW) = f D_V W + V(f) W$ (1)

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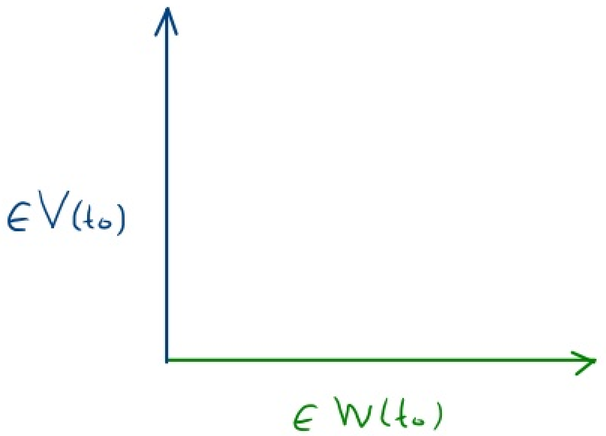

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  Consider two vectors $V(t_0), W(t_0)$ and their tangent curves. Consider infinitesimal $\epsilon V(t_0)$, $\epsilon W(t_0)$

• Use Shild's ladder to parallel transport $\epsilon W(t_0)$ along $V(t_0)$

s.t. $D_V(fW) = f D_V W + V(f)\, W$ (1)

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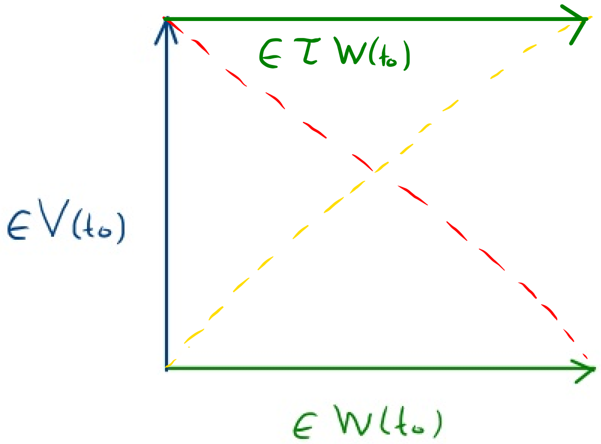

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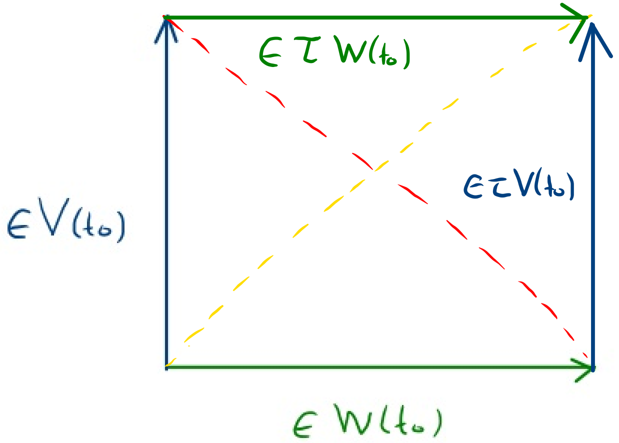

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Consider two vectors $V(t_0), W(t_0)$ and their tangent curves. Consider infinitesimal $\epsilon V(t_0)$, $\epsilon W(t_0)$

- Use Shild's ladder to parallel transport $\epsilon W(t_0)$ along $V(t_0)$
- Use Shild's ladder to parallel transport $\epsilon V(t_0)$ along $W(t_0)$
- Consider values of vector fields at $t_0+\epsilon$

  $\epsilon W(t_0+\epsilon)$, $\epsilon V(t_0+\epsilon)$ along respective curves

s.t. $D_V(fW) = f D_V W + V(f)\, W$ (1)

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$D_{fV} W = f D_V W$ (3)

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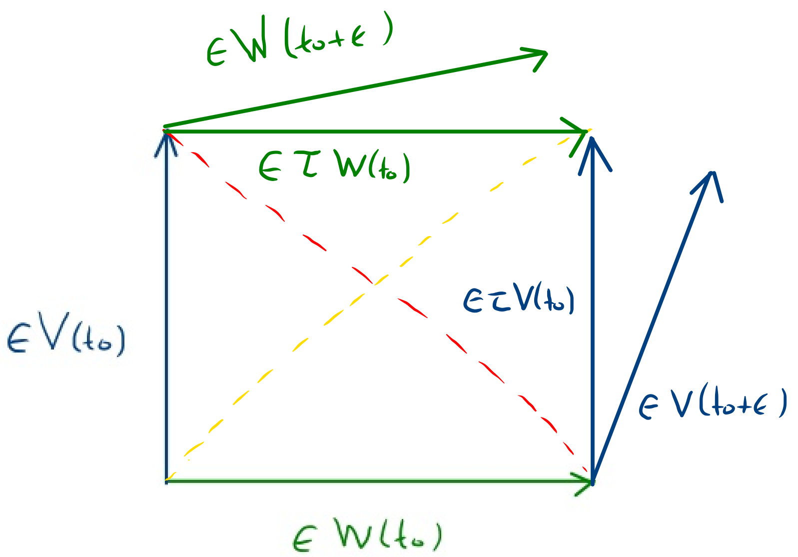

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Consider two vectors $V(t_0), W(t_0)$ and their tangent curves. Consider infinitesimal $\epsilon V(t_0)$, $\epsilon W(t_0)$

- Use Shild's ladder to parallel transport $\epsilon W(t_0)$ along $V(t_0)$
- Use Shild's ladder to parallel transport $\epsilon V(t_0)$ along $W(t_0)$
- Consider values of vector fields at $t_0+\epsilon$
  $\epsilon W(t_0+\epsilon)$, $\epsilon V(t_0+\epsilon)$ along respective curves
- Consider the differences

$$\underbrace{\epsilon W(t_0+\epsilon) - \epsilon \tau W(t_0)}_{} = \epsilon^2 D_V W \qquad \underbrace{\epsilon V(t_0+\epsilon) - \epsilon \tau V(t_0)}_{} = \epsilon^2 D_W V$$

notice the correct signs!
(prove it!)

s.t. $D_V(fW) = f D_V W + V(f)\, W$ (1)

$D_V(W+U) = D_V W + D_V U$ (2)

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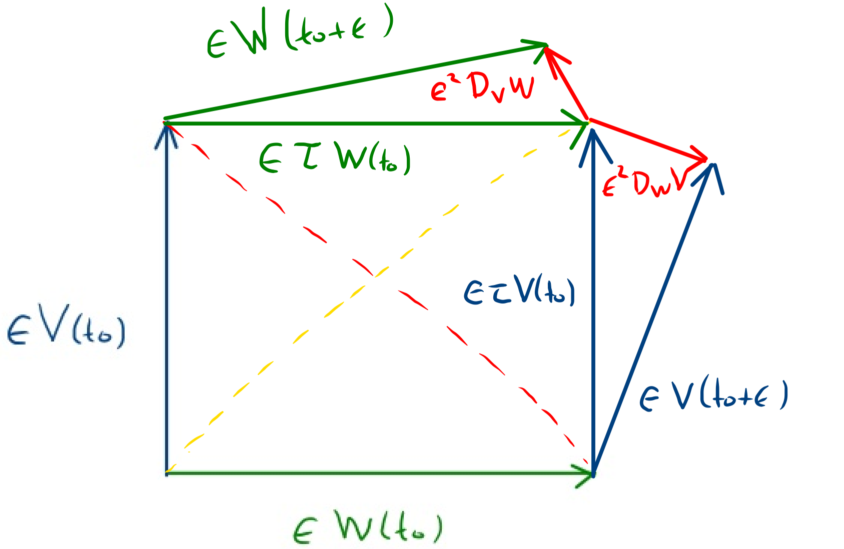

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  - Use Shild's ladder to parallel transport $\epsilon V(t_0)$ along $W(t_0)$
  - Consider values of vector fields at $t_0+\epsilon$

    $\epsilon W(t_0+\epsilon)$, $\epsilon V(t_0+\epsilon)$ along respective curves
  - Consider the differences

    $\epsilon W(t_0+\epsilon) - \epsilon \tau W(t_0) = \epsilon^2 D_V W \qquad \epsilon V(t_0+\epsilon) - \epsilon \tau V(t_0) = \epsilon^2 D_W V$
  - The difference $\epsilon^2 D_V W - \epsilon^2 D_W V$ closes the rectangle formed by V, W $= \epsilon^2 [V,W]$!

s.t.

$$D_V(fW) = f D_V W + V(f)\, W \qquad (1)$$

$$D_V(W+U) = D_V W + D_V U \qquad (2)$$

$$D_{fV} W = f D_V W \qquad (3)$$

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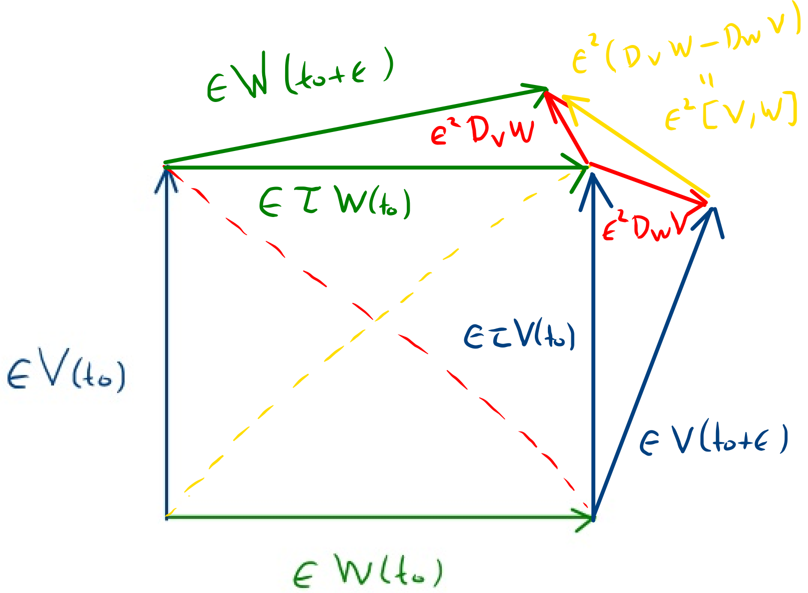

Example: Expanding Universe (Carroll §3.5)

$$ds^2 = -dt^2 + a(t)^2 (dx^2 + dy^2 + dz^2)$$

flat space cosmology

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$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)$$

flat space cosmology

$$= -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j$$

Calculate $\Gamma^\lambda{}_{\mu\nu}$:

$$I = \frac{1}{2}\int d\tau \left[-\left(\frac{dt}{d\tau}\right)^2 + a^2(t)\,\delta_{ij}\,\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}\right]$$

$$x^\mu \to x^\mu + \delta x^\mu \quad , \quad \delta I = 0$$

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flat space cosmology

$$= -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j$$

Calculate $\Gamma^\lambda{}_{\mu\nu}$:

$$I = \frac{1}{2} \int d\tau \left[-\left(\frac{dt}{d\tau}\right)^2 + a^2(t) \delta_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} \right]$$

$$x^\mu \to x^\mu + \delta x^\mu \quad , \quad \delta I = 0$$

(a) $t \to t + \delta t$ $\qquad$ $\delta a(t) = \frac{da}{dt} \delta t \equiv \dot{a}(t) \delta t$

Example: Expanding Universe (Carroll §3.5)

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)$$ flat space cosmology

$$= -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j$$

Calculate $\Gamma^\lambda{}_{\mu\nu}$:

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$x^\mu \to x^\mu + \delta x^\mu$, $\delta I = 0$

(a) $t \to t + \delta t$ $\qquad \delta a(t) = \frac{da}{dt}\delta t \equiv \dot{a}(t)\,\delta t$

$$\delta I = \frac{1}{2}\int d\tau \left[-2\frac{dt}{d\tau}\frac{d\delta\tau}{d\tau} + \delta a^2\,\delta_{ij}\,\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} \right]$$

$$= \frac{1}{2}\int d\tau \left[+2\frac{d^2 t}{d\tau^2}\,\delta t + 2 a \dot{a}\,\delta t\, \delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} \right]$$

(a) $t \to t + \delta t \qquad \delta a(t) = \frac{da}{dt}\delta t \equiv \dot{a}(t)\,\delta t$

boundary term, vanishes since $\delta t(\tau_A) = \delta t(\tau_B) = 0$

$$\delta I = \frac{1}{2}\int d\tau \left[-2\frac{dt}{d\tau}\frac{d\delta\tau}{d\tau} + \delta a^2\, \delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} \right] =$$

$$-\frac{dt}{d\tau}\frac{d\delta t}{d\tau} = -\frac{d}{d\tau}\left[\frac{dt}{d\tau}\delta t\right] + \frac{d}{d\tau}\frac{dt}{d\tau}\,\delta t$$

$$= \frac{1}{2} \int d\tau \left[+2 \frac{d^2 t}{d\tau^2} \delta t + 2 a \dot{a} \delta t \, \delta_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} \right]$$

$$= \int d\tau \left[\frac{d^2 t}{d\tau^2} + a \dot{a} \, \delta_{ij} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} \right] \delta t$$

(a) $t \to t + \delta t \qquad \delta a(t) = \frac{da}{dt} \delta t \equiv \dot{a}(t) \delta t$

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$$= \frac{1}{2}\int d\tau \left[+2\frac{d^2t}{d\tau^2}\,\delta t + 2a\dot{a}\,\delta t\,\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}\right]$$

$$= \int d\tau \left[\frac{d^2t}{d\tau^2} + a\dot{a}\,\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}\right]\delta t$$

$$\delta I = 0 \;\; \forall\, \delta t \;\Rightarrow\; \frac{d^2t}{d\tau^2} + a\dot{a}\,\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = 0$$

(a) $t \to t + \delta t$ $\qquad \delta a(t) = \frac{da}{dt}\delta t \equiv \dot{a}(t)\,\delta t$

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$$\delta I = 0 \;\forall\, \delta t \Rightarrow \quad \frac{d^2t}{d\tau^2} + a\dot{a}\,\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = 0$$

compare with $\quad \frac{d^2t}{d\tau^2} + \Gamma^0{}_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = 0 \Rightarrow \quad \Gamma^0{}_{00} = \Gamma^0{}_{i0} = \Gamma^0{}_{0i} = 0$

$$\Gamma^0{}_{ij} = a\dot{a}\,\delta_{ij}$$

(a) $t \to t + \delta t \qquad \delta a(t) = \frac{da}{dt}\delta t \equiv \dot{a}(t)\,\delta t$

$$\delta I = \frac{1}{2}\int d\tau \left[-2\frac{dt}{d\tau}\frac{d\delta\tau}{d\tau} + \delta a^2\,\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} \right] =$$

$$x^i \to x^i + \delta x^i$$

$$I = \frac{1}{2}\int d\tau \left[-\left(\frac{dt}{d\tau}\right)^2 + a^2(t)\,\delta_{ij}\,\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} \right] \longrightarrow I + \delta I$$

$$x^i \to x^i + \delta x^i$$

$$I = \frac{1}{2}\int d\tau \left[-\left(\frac{dt}{d\tau}\right)^2 + a^2(t)\,\delta_{ij}\,\frac{dx^i}{d\tau}\,\frac{dx^j}{d\tau} \right] \longrightarrow I + \delta I \qquad \delta\,\frac{dx^i}{d\tau} = \frac{d}{d\tau}(\delta x^i)$$

$$\delta I = \frac{1}{2}\int d\tau \left[0 + a^2(t)\,\delta_{ij}\,\frac{d\delta x^i}{d\tau}\,\frac{dx^j}{d\tau} + a^2(t)\,\delta_{ij}\,\frac{dx^i}{d\tau}\,\frac{d\delta x^j}{d\tau} \right]$$

$$x^i \to x^i + \delta x^i$$

$$I = \frac{1}{2}\int d\tau \left[-\left(\frac{dt}{d\tau}\right)^2 + a^2(t)\,\delta_{ij}\,\frac{dx^i}{d\tau}\,\frac{dx^j}{d\tau}\right] \longrightarrow I + \delta I$$

symm $i \leftrightarrow j$

$$\delta I = \frac{1}{2}\int d\tau \left[0 + a^2(t)\,\delta_{ij}\,\frac{d\delta x^i}{d\tau}\,\frac{dx^j}{d\tau} + a^2(t)\,\delta_{ij}\,\frac{dx^i}{d\tau}\,\frac{d\delta x^j}{d\tau}\right]$$

$$= \frac{1}{2}\int d\tau\, 2\, a^2(t)\,\delta_{ij}\,\frac{dx^i}{d\tau}\,\frac{d}{d\tau}\delta x^j$$

$$x^i \to x^i + \delta x^i$$

$$I = \frac{1}{2}\int d\tau \left[-\left(\frac{dt}{d\tau}\right)^2 + a^2(t)\,\delta_{ij}\,\frac{dx^i}{d\tau}\,\frac{dx^j}{d\tau} \right] \longrightarrow I + \delta I$$

$$\delta I = \frac{1}{2}\int d\tau \left[0 + a^2(t)\,\delta_{ij}\,\frac{d\delta x^i}{d\tau}\,\frac{dx^j}{d\tau} + a^2(t)\,\delta_{ij}\,\frac{dx^i}{d\tau}\,\frac{d\delta x^j}{d\tau} \right]$$

$$= \frac{1}{2}\int d\tau\, 2\, a^2(t)\,\delta_{ij}\,\frac{dx^i}{d\tau}\,\frac{d\delta x^j}{d\tau}$$

$$= -\int d\tau\, \frac{d}{d\tau}\left[a^2(t)\,\delta_{ij}\,\frac{dx^i}{d\tau} \right] \delta x^j$$

boundary term, vanishes for $dx^j(\tau_a) = dx^j(\tau_B) = 0$

$$a^2\,\delta_{ij}\,\frac{dx^i}{d\tau}\,\frac{d\delta x^j}{d\tau} = \frac{d}{d\tau}\left[a^2\,\delta_{ij}\,\frac{dx^i}{d\tau}\,\delta x^j\right] - \frac{d}{d\tau}\left[a^2\,\delta_{ij}\,\frac{dx^i}{d\tau}\right]\delta x^j$$

$$x^i \to x^i + \delta x^i$$

$$I = \frac{1}{2}\int d\tau \left[-\left(\frac{dt}{d\tau}\right)^2 + a^2(t)\,\delta_{ij}\,\frac{dx^i}{d\tau}\,\frac{dx^j}{d\tau} \right] \longrightarrow I + \delta I$$

$$\delta I = \frac{1}{2}\int d\tau \left[0 + a^2(t)\,\delta_{ij}\,\frac{d\delta x^i}{d\tau}\,\frac{dx^j}{d\tau} + a^2(t)\,\delta_{ij}\,\frac{dx^i}{d\tau}\,\frac{d\delta x^j}{d\tau} \right]$$

$$= \frac{1}{2}\int d\tau\, 2\, a^2(t)\,\delta_{ij}\,\frac{dx^i}{d\tau}\,\frac{d}{d\tau}\delta x^j$$

$$= -\int d\tau\, \frac{d}{d\tau}\left[a^2(t)\,\delta_{ij}\,\frac{dx^i}{d\tau} \right] \delta x^j$$

$$\delta I = 0 \;\; \forall \delta x^j \;\Rightarrow\; \frac{d}{d\tau}\left[a^2(t)\,\frac{dx^i}{d\tau} \right] = 0 \;\Rightarrow\; 2a\,\frac{da}{d\tau}\,\frac{dx^i}{d\tau} + a^2\,\frac{d^2x^i}{d\tau^2} = 0$$

$$\Rightarrow \quad \frac{d^2x^i}{d\tau^2} + 2\,\frac{a\dot{a}}{a^2}\,\frac{dt}{d\tau}\,\frac{dx^i}{d\tau} = 0$$

$$\delta I = 0 \quad \forall\, \delta x^i \quad \Rightarrow \quad \frac{d}{d\tau}\left[a^2(t)\,\frac{dx^i}{d\tau}\right] = 0 \quad \Rightarrow \quad 2a\,\frac{da}{d\tau}\,\frac{dx^i}{d\tau} + a^2\,\frac{d^2x^i}{d\tau^2} = 0$$

$$\frac{da}{d\tau} = \frac{da}{dt}\,\frac{dt}{d\tau} = \dot{a}\,\frac{dt}{d\tau}$$

$$\Rightarrow \quad \frac{d^2x^i}{d\tau^2} + 2\,\frac{a\dot{a}}{a^2}\,\frac{dt}{d\tau}\,\frac{dx^i}{d\tau} = 0$$

$$\Rightarrow \quad \frac{d^2x^i}{d\tau^2} + \frac{\dot{a}}{a}\,\frac{dt}{d\tau}\,\frac{dx^i}{d\tau} + \frac{\dot{a}}{a}\,\frac{dx^i}{d\tau}\,\frac{dt}{d\tau} = 0$$

$$\delta I = 0 \quad \forall\, \delta x^i \quad \Rightarrow \quad \frac{d}{d\tau}\left[a^2(t)\,\frac{dx^i}{d\tau}\right] = 0 \quad \Rightarrow \quad 2\,a\,\frac{da}{d\tau}\,\frac{dx^i}{d\tau} + a^2\,\frac{d^2x^i}{d\tau^2} = 0$$

$$\frac{da}{d\tau} = \frac{da}{dt}\,\frac{dt}{d\tau} = \dot{a}\,\frac{dt}{d\tau}$$

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$$\Rightarrow \frac{d^2x^i}{d\tau^2} + \frac{\dot{a}}{a}\frac{dt}{d\tau}\frac{dx^i}{d\tau} + \frac{\dot{a}}{a}\frac{dx^i}{d\tau}\frac{dt}{d\tau} = 0$$

Compare with:

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \Leftrightarrow \frac{d^2x^i}{d\tau^2} + \Gamma^i{}_{00}\frac{dt}{d\tau}\frac{dt}{d\tau} + \Gamma^i{}_{0j}\frac{dt}{d\tau}\frac{dx^j}{d\tau} + \Gamma^i{}_{j0}\frac{dx^j}{d\tau}\frac{dt}{d\tau} + \Gamma^i{}_{jk}\frac{dx^j}{d\tau}\frac{dx^k}{d\tau} = 0$$

$$\delta I = 0 \ \forall \delta x^j \Rightarrow \frac{d}{d\tau}\left[a^2(t)\frac{dx^i}{d\tau}\right] = 0 \Rightarrow 2a\frac{da}{d\tau}\frac{dx^i}{d\tau} + a^2\frac{d^2x^i}{d\tau^2} = 0$$

$$\frac{da}{d\tau} = \frac{da}{dt}\frac{dt}{d\tau} = \dot{a}\frac{dt}{d\tau}$$

$$\Rightarrow \frac{d^2x^i}{d\tau^2} + 2\frac{a\dot{a}}{a^2}\frac{dt}{d\tau}\frac{dx^i}{d\tau} = 0$$

$$\Rightarrow \frac{d^2x^i}{d\tau^2} + \frac{\dot{a}}{a}\frac{dt}{d\tau}\frac{dx^i}{d\tau} + \frac{\dot{a}}{a}\frac{dx^i}{d\tau}\frac{dt}{d\tau} = 0$$

Compare with:

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \iff \frac{d^2x^i}{d\tau^2} + \Gamma^i_{00}\frac{dt}{d\tau}\frac{dt}{d\tau} + \Gamma^i_{0j}\frac{dt}{d\tau}\frac{dx^j}{d\tau} + \Gamma^i_{j0}\frac{dx^j}{d\tau}\frac{dt}{d\tau} + \Gamma^i_{jk}\frac{dx^j}{d\tau}\frac{dx^k}{d\tau} = 0$$

$$\rightsquigarrow \Gamma^i_{00} = \Gamma^i_{jk} = 0 \quad , \quad \Gamma^i_{0j} = \Gamma^i_{j0} = \frac{\dot{a}}{a}\delta^i_j$$

$$\delta I = 0 \;\; \forall \delta x^j \Rightarrow \frac{d}{d\tau}\left[a^2(t)\frac{dx^i}{d\tau}\right] = 0 \Rightarrow 2a\frac{da}{d\tau}\frac{dx^i}{d\tau} + a^2\frac{d^2x^i}{d\tau^2} = 0$$

$$\frac{da}{d\tau} = \frac{da}{dt}\frac{dt}{d\tau} = \dot{a}\frac{dt}{d\tau}$$

Null Geodesics:

consider photons moving on +x axis: $x^\mu(\lambda) = (t(\lambda), x(\lambda), 0, 0)$

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$$\frac{d^2t}{d\lambda^2} + a\dot{a}\,\delta_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = 0 \Rightarrow \frac{d^2t}{d\lambda^2} + a\dot{a}\left(\frac{dx}{d\lambda}\right)^2 = 0$$

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$$ds^2 = 0 \Rightarrow -dt^2 + a^2(t)\,dx^2 = 0 \Rightarrow \frac{dx}{d\lambda} = \frac{1}{a}\frac{dt}{d\lambda} \qquad (+x \text{ direction}) \qquad (1)$$

$$\frac{d^2t}{d\lambda^2} + a\dot{a}\,\delta_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = 0 \Rightarrow \frac{d^2t}{d\lambda^2} + a\dot{a}\left(\frac{dx}{d\lambda}\right)^2 = 0 \qquad (2)$$

$$(1),(2) \Rightarrow \frac{d^2t}{d\lambda^2} + a\dot{a}\frac{1}{a^2}\left(\frac{dt}{d\lambda}\right)^2 = 0 \Rightarrow \frac{d^2t}{d\lambda^2} + \frac{\dot{a}}{a}\left(\frac{dt}{d\lambda}\right)^2 = 0$$

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solution: $\frac{dt}{d\lambda} = \frac{\omega_0}{a(t)}$

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solution: $$\frac{dt}{d\lambda} = \frac{\omega_0}{a(t)} \tag{3}$$

* $a(t)$ is a known function (solution of Einstein's Eqs), therefore (3) gives $t = t(\lambda)$ and (1) $x = x(\lambda)$

Null Geodesics:

consider photons moving on +x axis: $x^\mu(\lambda) = (t(\lambda), x(\lambda), 0, 0)$

$$ds^2 = 0 \Rightarrow -dt^2 + a^2(t)\,dx^2 = 0 \Rightarrow \frac{dx}{d\lambda} = \frac{1}{a}\frac{dt}{d\lambda} \quad (+x \text{ direction}) \qquad (1)$$

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* $a(t)$ is a known function (solution of Einstein's Eqs), therefore (3) gives $t = t(\lambda)$ and (1) $x = x(\lambda)$
* consider comoving observers $U^\mu = (1,0,0,0)$. Then $E = -p_\mu U^\mu = -g_{00}\frac{dx^0}{d\lambda}U^0 = +\frac{\omega_0}{a}$

remember: affine parameter is defined by $p^\mu = \frac{dx^\mu}{d\lambda}$

Null Geodesics:

consider photons moving on +x axis: $x^\mu(\lambda) = (t(\lambda), x(\lambda), 0, 0)$

$$ds^2 = 0 \Rightarrow -dt^2 + a^2(t)\,dx^2 = 0 \Rightarrow \frac{dx}{d\lambda} = \frac{1}{a}\frac{dt}{d\lambda} \qquad (+x \text{ direction}) \qquad (1)$$

$$\frac{d^2t}{d\lambda^2} + a\dot{a}\,\delta_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = 0 \Rightarrow \frac{d^2t}{d\lambda^2} + a\dot{a}\left(\frac{dx}{d\lambda}\right)^2 = 0 \qquad (2)$$

$$(1),(2) \Rightarrow \frac{d^2t}{d\lambda^2} + a\dot{a}\frac{1}{a^2}\left(\frac{dt}{d\lambda}\right)^2 = 0 \Rightarrow \frac{d^2t}{d\lambda^2} + \frac{\dot{a}}{a}\left(\frac{dt}{d\lambda}\right)^2 = 0$$

solution: $$\frac{dt}{d\lambda} = \frac{\omega_0}{a(t)} \qquad (3)$$

* $a(t)$ is a known function (solution of Einstein's Eqs), therefore (3) gives $t = t(\lambda)$ and (1) $x = x(\lambda)$
* consider comoving observers $U^\mu = (1,0,0,0)$. Then $E = -p_\mu U^\mu = -g_{00}\frac{dx^0}{d\lambda}U^0 = +\frac{\omega_0}{a}$
* ω_0 is freq. of photon when $a(t)=1$. Cosmological redshift:

$$\frac{E_2}{E_1} = \frac{\hbar\omega_2}{\hbar\omega_1} = \frac{\omega_0/a_2}{\omega_0/a_1} = \frac{a_1}{a_2}$$

(note: energy reduces, photon not "stretched". A quantum mechanical "miracle")