Differentiable(*) Manifolds

Read: S. Carroll
B. Schutz
M. Nakahara

- Topological Spaces
- Differentiable Manifolds
- Charts, transition functions, atlases
- Examples: S^1, S^2, T^2, $S^1 \times \mathbb{R}$, Klein Bottle, Rotations

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(*) or differential Manifolds

Differentiable Manifolds

Topological Spaces + locally like $\mathbb{R}^n$

$\downarrow$

local coordinate systems

$\downarrow$

differentiable coordinate transformations

$\downarrow$

Differential Structure

$\swarrow$

rate of change of functions on curves $\longrightarrow$ vectors $\rightarrow$ one forms $\longrightarrow$ tensors

vectors $\downarrow$ Lie Derivative

tensors $\downarrow$ differential forms

differential forms $\rightarrow$ integration

differential forms $\searrow$ exterior derivative

Differentiable Manifolds

Topological Spaces + locally like $\mathbb{R}^n$

$\downarrow$

local coordinate systems

$\downarrow$

differentiable coordinate transformations

$\downarrow$

Differential Structure

Additional Sructure

Affine Connection $\rightarrow$ Covariant Derivative

$\rightarrow$ Parallel Tranport

$\rightarrow$ geodesics

$\rightarrow$! CURVATURE !

Metric

$\rightarrow$ "geometry"

$\rightarrow$ distance

Differentiable Manifolds

Topological Spaces + locally like $\mathbb{R}^n$

↓

local coordinate systems

↓

differentiable coordinate transformations

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Differential Structure

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Additional Sructure

↙

Affine Connection → Covariant Derivative
→ Parallel Tranport
→ geodesics
→ ! CURVATURE !

Background: Differential Structure

Dynamics: Metric

- a field
- → curvature = gravity

GRAVITY!

↑

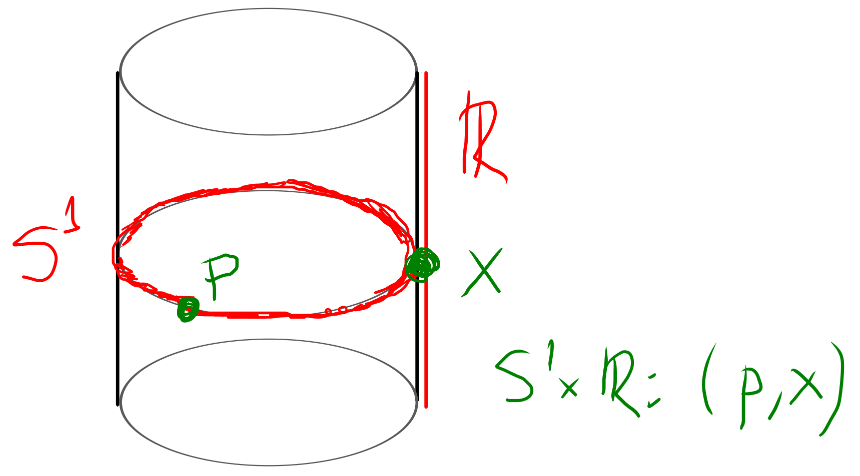
Metric

"geometry"

distance

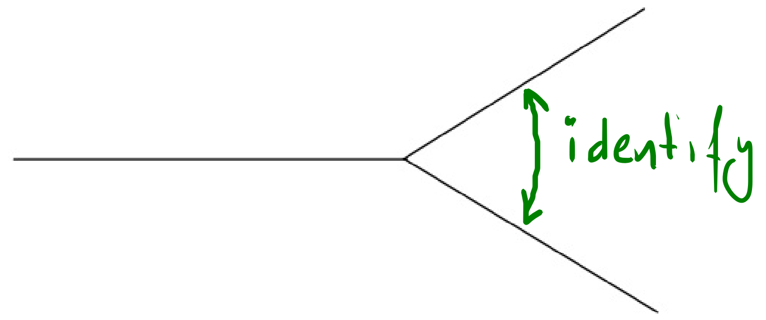
Examples of Manifolds

- $\mathbb{R}^n$: $\mathbb{R}$ (line), $\mathbb{R}^2$ (plane), $\mathbb{R}^3$ (space), ...
- S^n: S^0 (2 points), S^1 (circle), S^2 (sphere), ...
- T^n: T^2 (torus), ...
- Lie Groups: rotations, Lorentz transformations, ...
- $M = M_1 \times M_2$: $\quad \underline{p} = (\underline{p}_1, \underline{p}_2) \quad \underline{p}_1 \in M_1, \ \underline{p}_2 \in M_2$

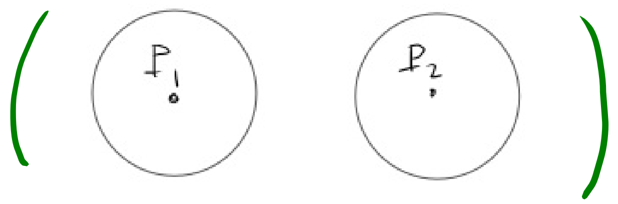


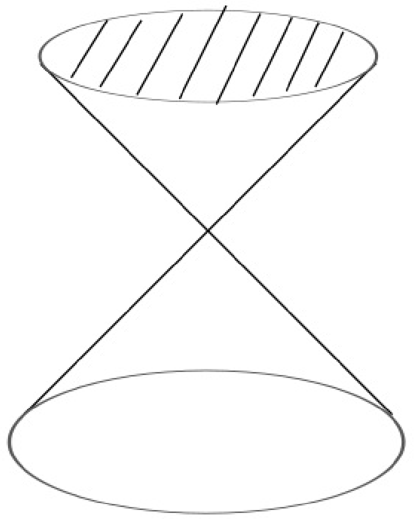
$S^1 \times \mathbb{R}$ (cylinder), $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, $T^2 = S^1 \times S^1$

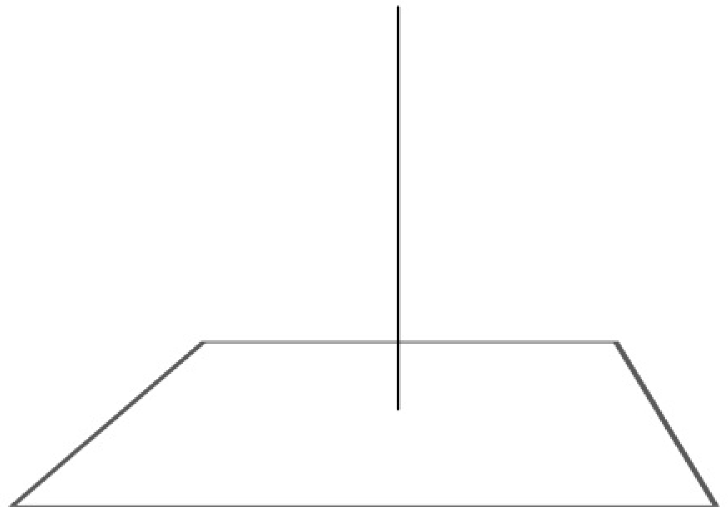
Examples of non-Manifolds



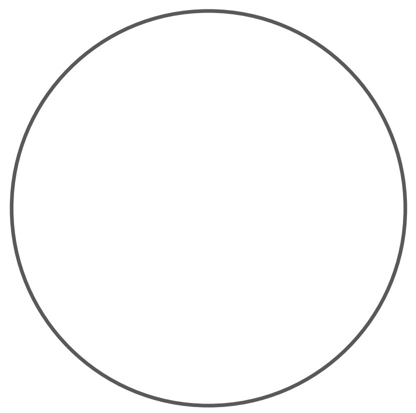
non Hausdorff





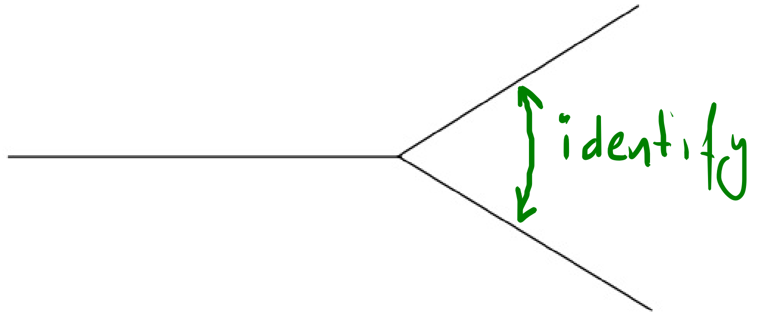


not locally everywhere $\cong \mathbb{R}^n$



Manifolds with boundary

Examples of non-Manifolds



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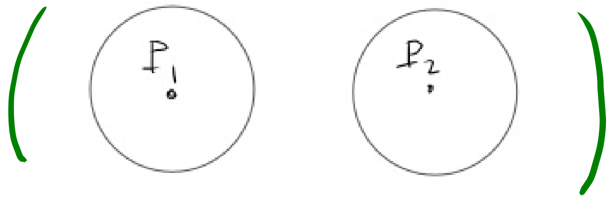
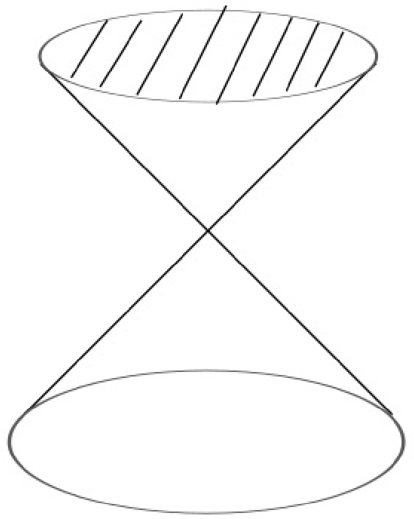
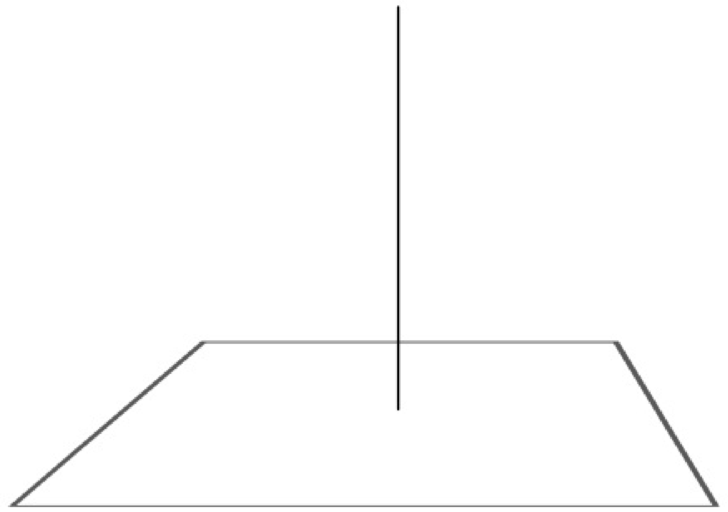
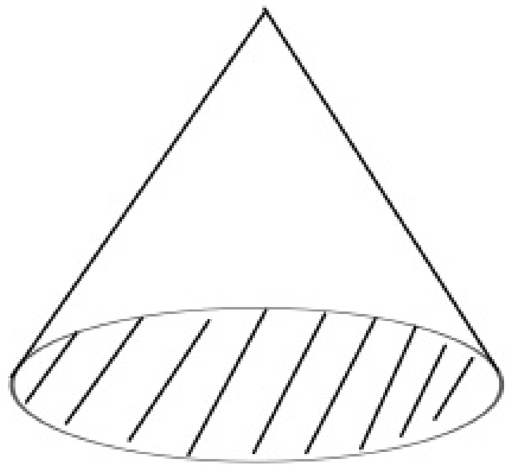




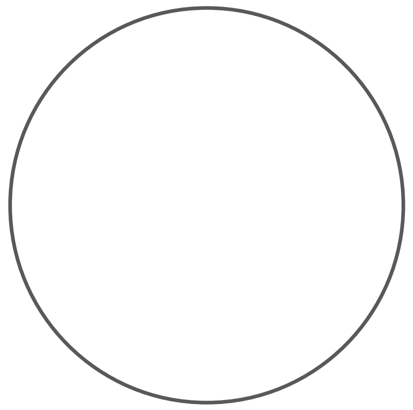


not locally everywhere $\cong \mathbb{R}^n$

the "tip" comes from choice of metric, not a manifold singularity



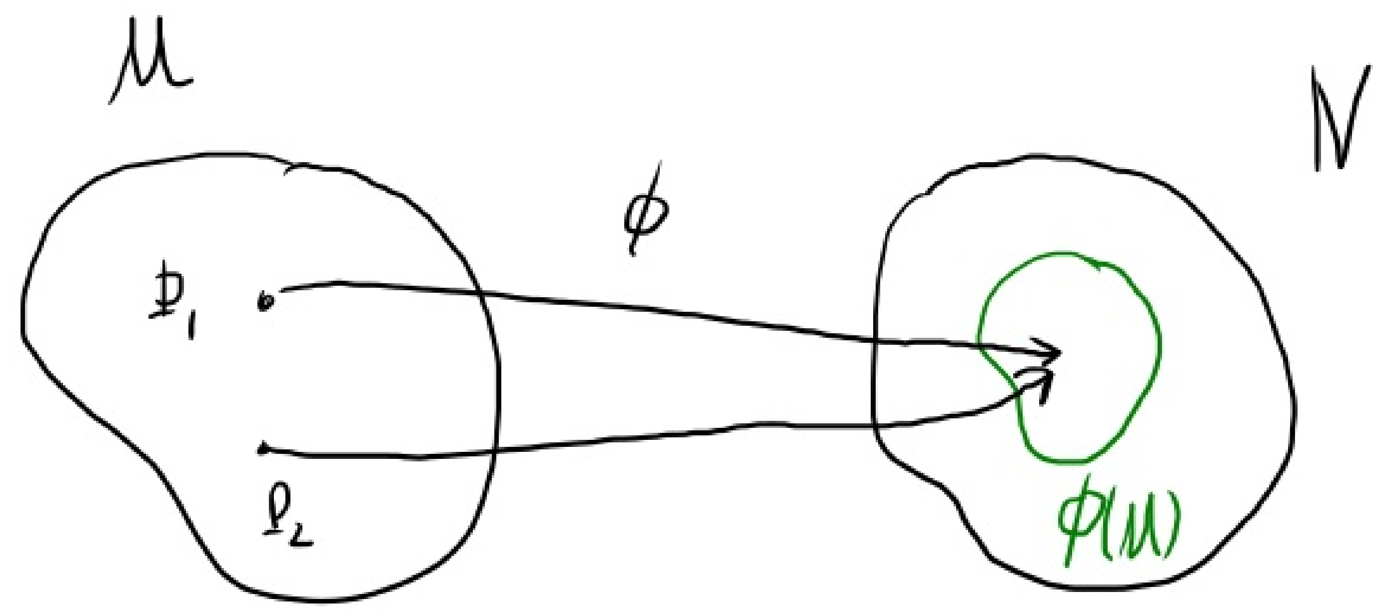
this IS a Manifold! $\cong \mathbb{R}^2$



Manifolds with boundary

Maps

$$\phi : M \to N$$



- M : domain
- N : codomain
- $\phi(M)$: range or image of M

- onto: $\phi(M) = N$ (surjective)
- 1-1 : $p_1 \neq p_2 \Rightarrow \phi(p_1) \neq \phi(p_2)$ (injective)
- invertible: 1-1 and onto (bijective)

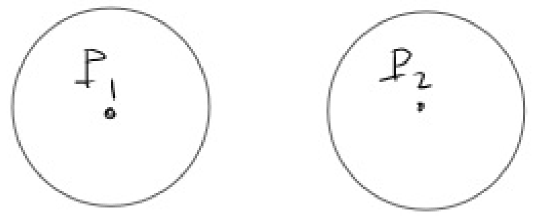
  $\phi^{-1}: N \to M$ a map

Topology

- $\mathcal{M}$ a topological space: covered by open sets $\{U_\alpha\}$ s.t. $\left.\begin{matrix} U_\alpha \cap U_\beta \\ \bigcup_\alpha U_\alpha \end{matrix}\right\}$ open

  e.g. open ball $B(r) \subset \mathbb{R}^n$, $B(r) = \{ x \mid x \in \mathbb{R}^n, |x - x_0| < r \}$

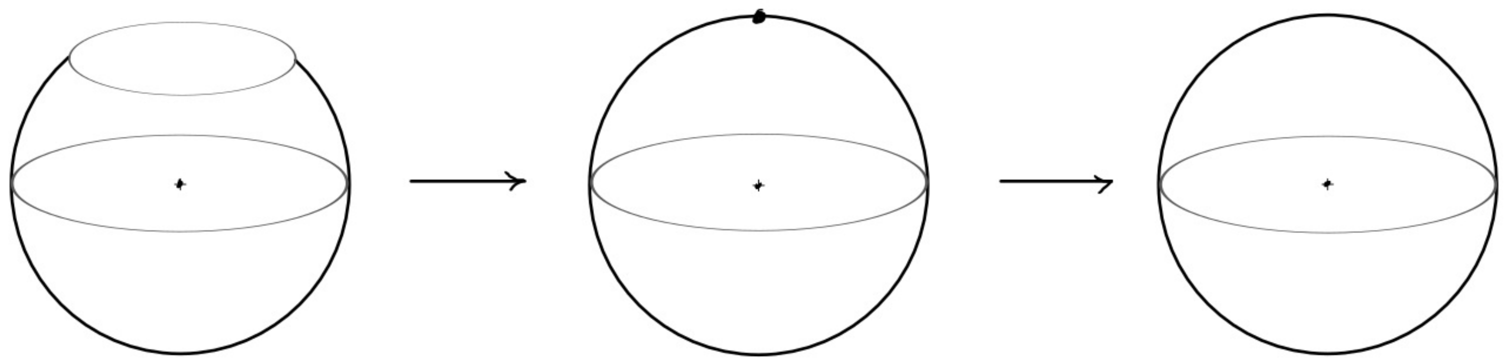
- $\mathcal{M}$ is Hausdorff: $\forall\, p_1 \neq p_2$ $\exists$ open U_1, U_2 s.t. $U_1 \cap U_2 = \emptyset$



- $W \subseteq \mathcal{M}$ is a neighborhood of $p \in W$, iff $\exists$ open $U \subseteq W$ with $p \in U$
- W is closed iff $\mathcal{M} \setminus W$ open
- $\overline{W}$ is the closure of W if the smallest closed superset of W
- W° is the interior of W if the largest open subset of W
- ∂W is the boundary of W if $\partial W = \overline{W} - W^\circ$

Topology

- W is compact if $\forall \{U_\alpha\}$ open covering of W, $\exists$ finite $\{U_{\alpha'}\} \subset \{U_\alpha\}$ covering W
  - in $\mathbb{R}^n$: $W \subseteq \mathbb{R}^n$ compact $\Leftrightarrow$ W is closed + bounded
    e.g. $\bar{B}(r) = \{x \mid |x - x_0| \leq r\}$, S^n, T^n, ... (S^n, T^n closed + bounded in $\mathbb{R}^{n+1}$)
  - compactification



$\mathbb{R}^2 \rightarrow \mathbb{R}^2 \cup \{\infty\} = S^2$

- W connected if $\nexists\, U_1, U_2$ s.t. $U_1 \cup U_2 = W$ and $U_1 \cap U_2 = \emptyset$

  disconnected if not connected

- W simply connected if all loops contractible to a point

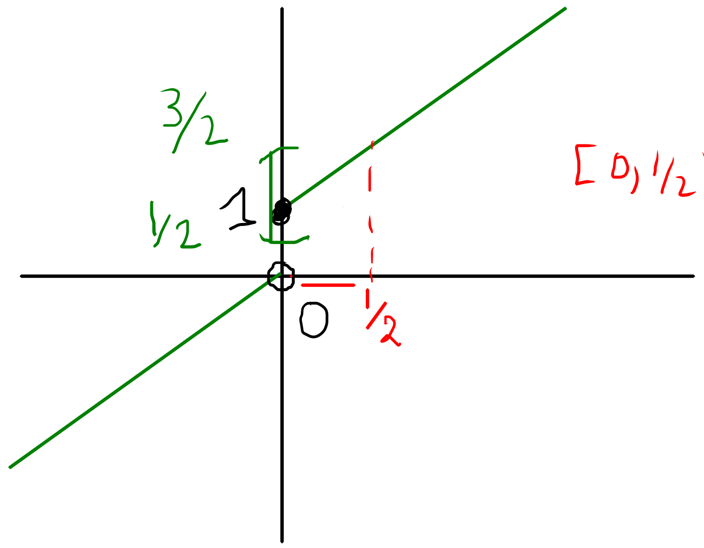
  e.g. $\mathbb{R}^2$, S^2, simply connected, $\mathbb{R}^2 - \{0\}$, T^2, S^1 not simply connected

Topology

– Continuity: $\phi: M \to N$ continuous iff $\forall$ open $V \subseteq N \Rightarrow \phi^{-1}(V) \subseteq M$ open

e.g. $f(x) = \begin{cases} x & x<0 \\ x+1 & x \geq 0 \end{cases}$

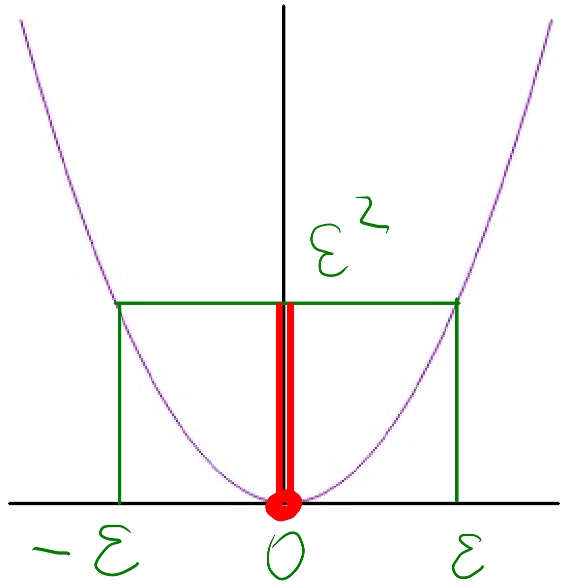
$f^{-1}\left(\left(\frac{1}{2}, \frac{3}{2}\right)\right) = \left[0, \frac{1}{2}\right)$ not open



$[0, 1/2) \to [1, 3/2) \subset (1/2, 3/2)$

continuous ϕ does not necessarily map open to open:

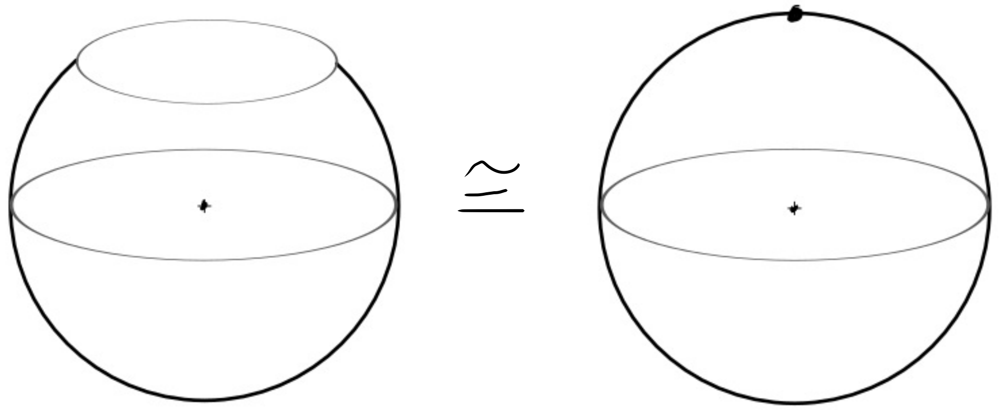
for $f(x) = x^2 \Rightarrow f((-\epsilon, \epsilon)) = [0, \epsilon^2)$ not open

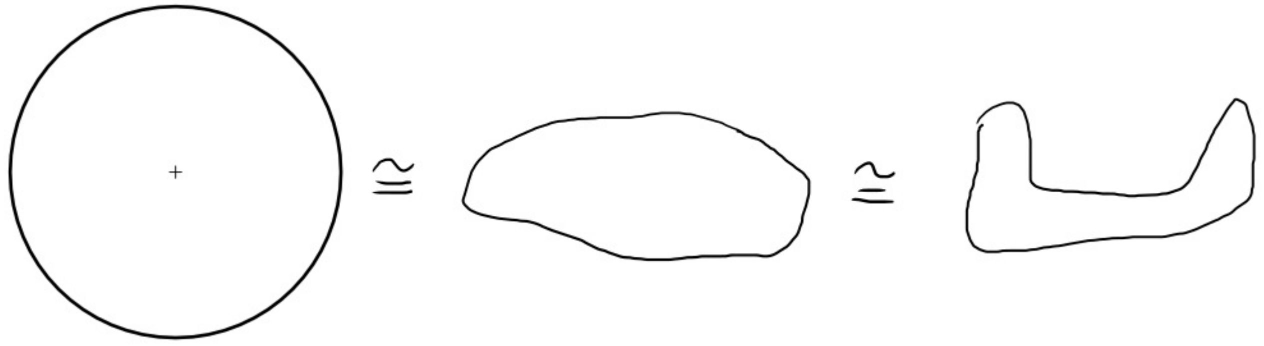


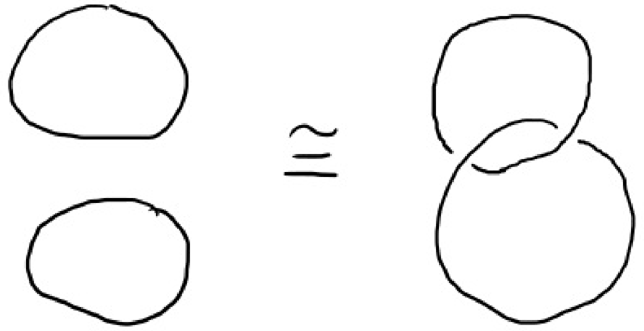
Topology

- Continuity: $\phi : M \to N$ continuous iff $\forall$ open $V \subseteq N \Rightarrow \phi^{-1}(V) \subseteq M$ open

- Homeomorphism: $\phi : M \to N$ continuous + invertible with ϕ^{-1} also continuous
    - M, N are homeomorphic
      topologically equivalent $M \cong N$







    - punctures, cuts, ... lead to topologically inequivalent spaces

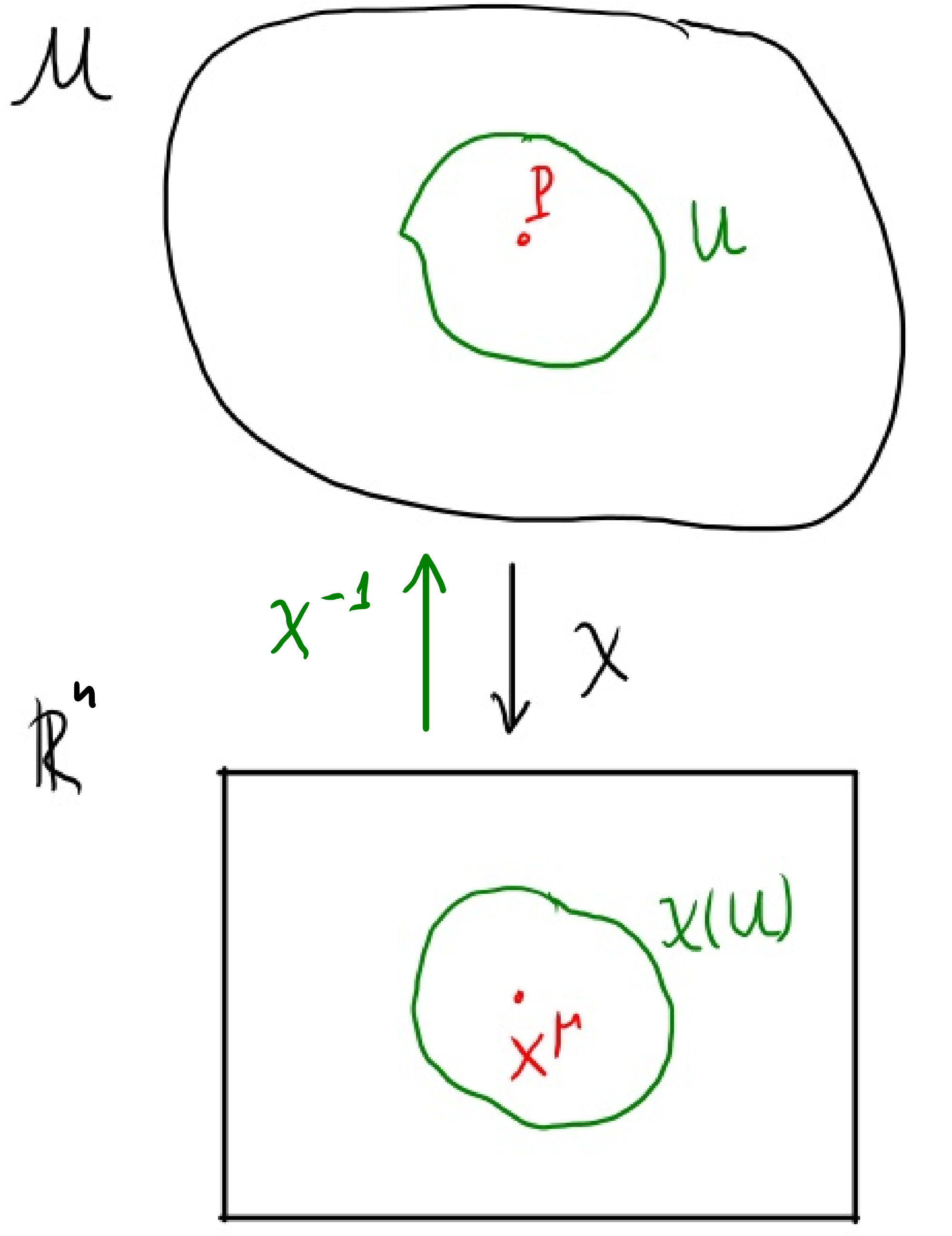
Differentiable Manifolds

- M with a maximal atlas is a differentiable manifold of $\dim M = n$:
  - M is a topological space (+ Hausdorff)
  - M is locally like $\mathbb{R}^n$, i.e. each point $P \in M$ is in a chart (U, χ)

    $\dim M = n$

    - U open neighborhood of P
    - $\chi : \; U \to \chi(U) \subseteq \mathbb{R}^n$ a homeomorphism

      $P \mapsto x^\mu(P) = \chi(P)$



→ $x^\mu(P)$ coordinates of P

→ homeo: χ is 1-1, onto, continuous

χ^{-1} " " "

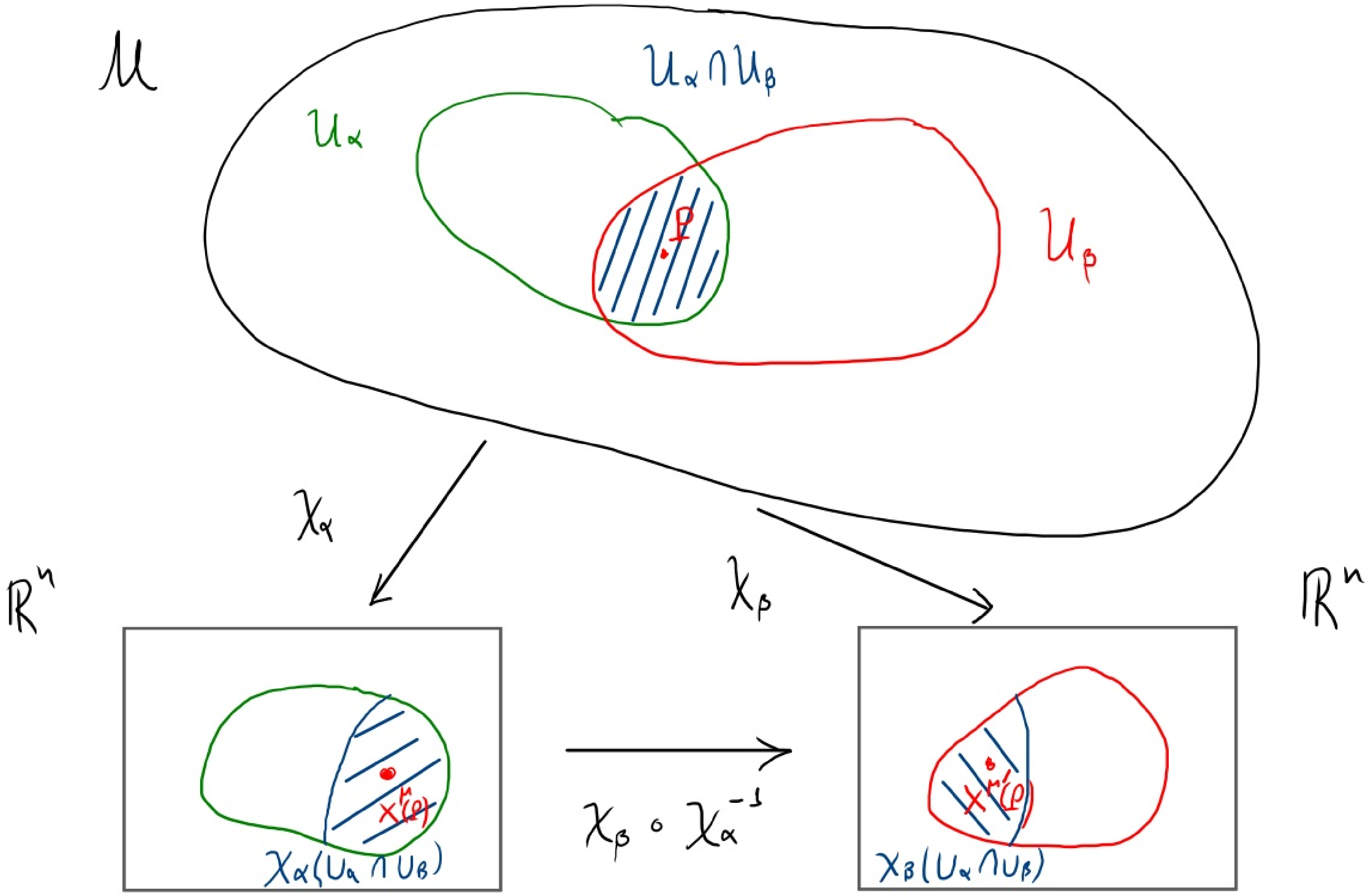
→ $U \cong \chi(U) \subseteq \mathbb{R}^n$

Differentiable Manifolds

- M with a maximal atlas is a differentiable manifold of $\dim M = n$:
  - M is a topological space
  - M is locally like $\mathbb{R}^n$, i.e. each point $P \in M$ is in a chart (U, x)

    $\dim M = n$
  - Coordinate transformations are differentiable

$\chi_\beta \circ \chi_\alpha^{-1}$: transition function



$$\chi_\beta \circ \chi_\alpha^{-1} : \chi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \chi_\beta(U_\alpha \cap U_\beta)$$

$$x^\mu \mapsto x^{\mu'}(x^\mu)$$

coordinate tranformation

Differentiable Manifolds

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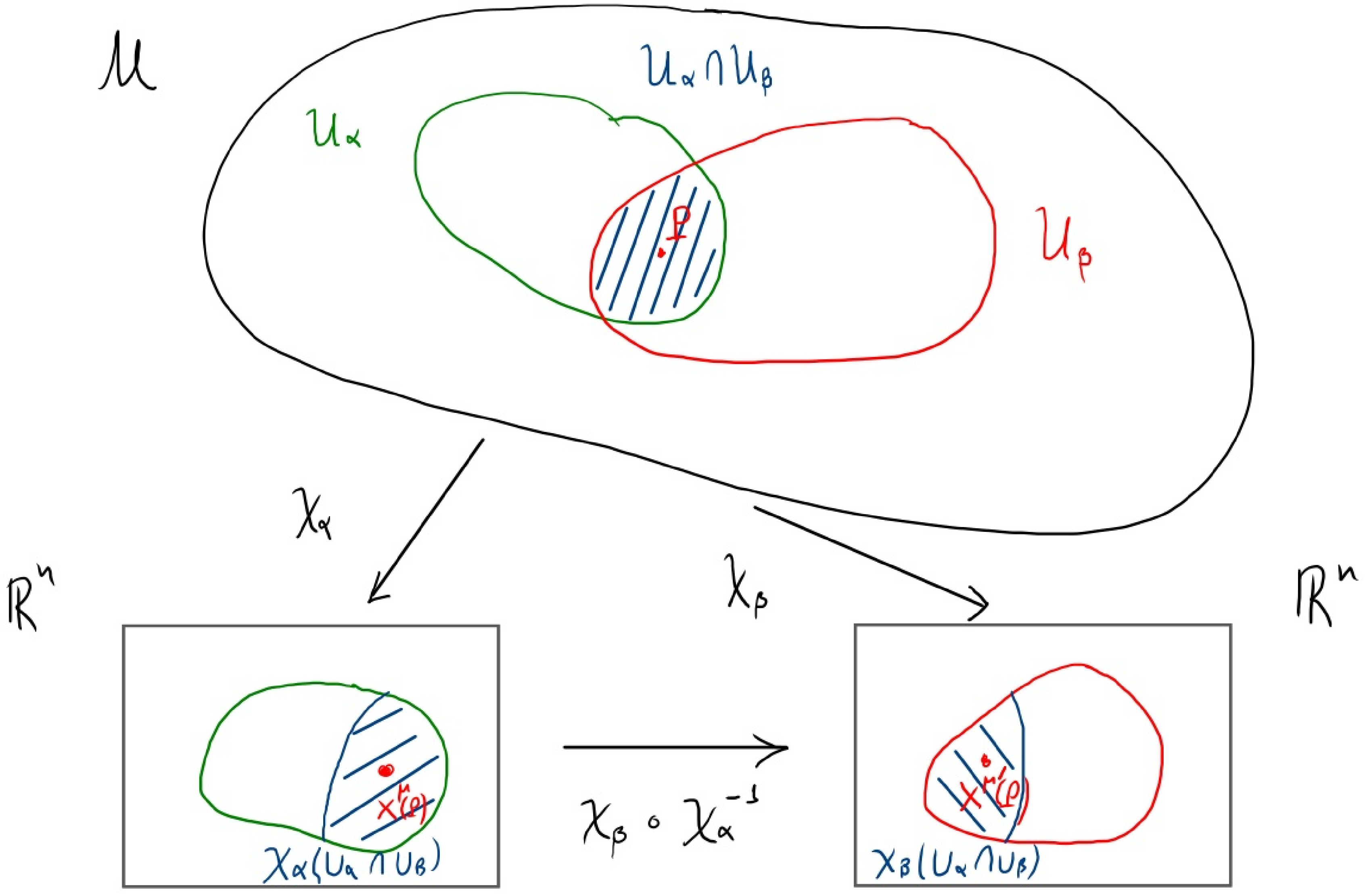


$\chi_\beta \circ \chi_\alpha^{-1}$: transition function

$$x^{\mu'} = x^{\mu'}(x^\mu) = \chi_\beta \circ \chi_\alpha^{-1}(x^\mu)$$

differentiability: $\dfrac{\partial x^{\mu'}}{\partial x^\mu}$ continuous C^1

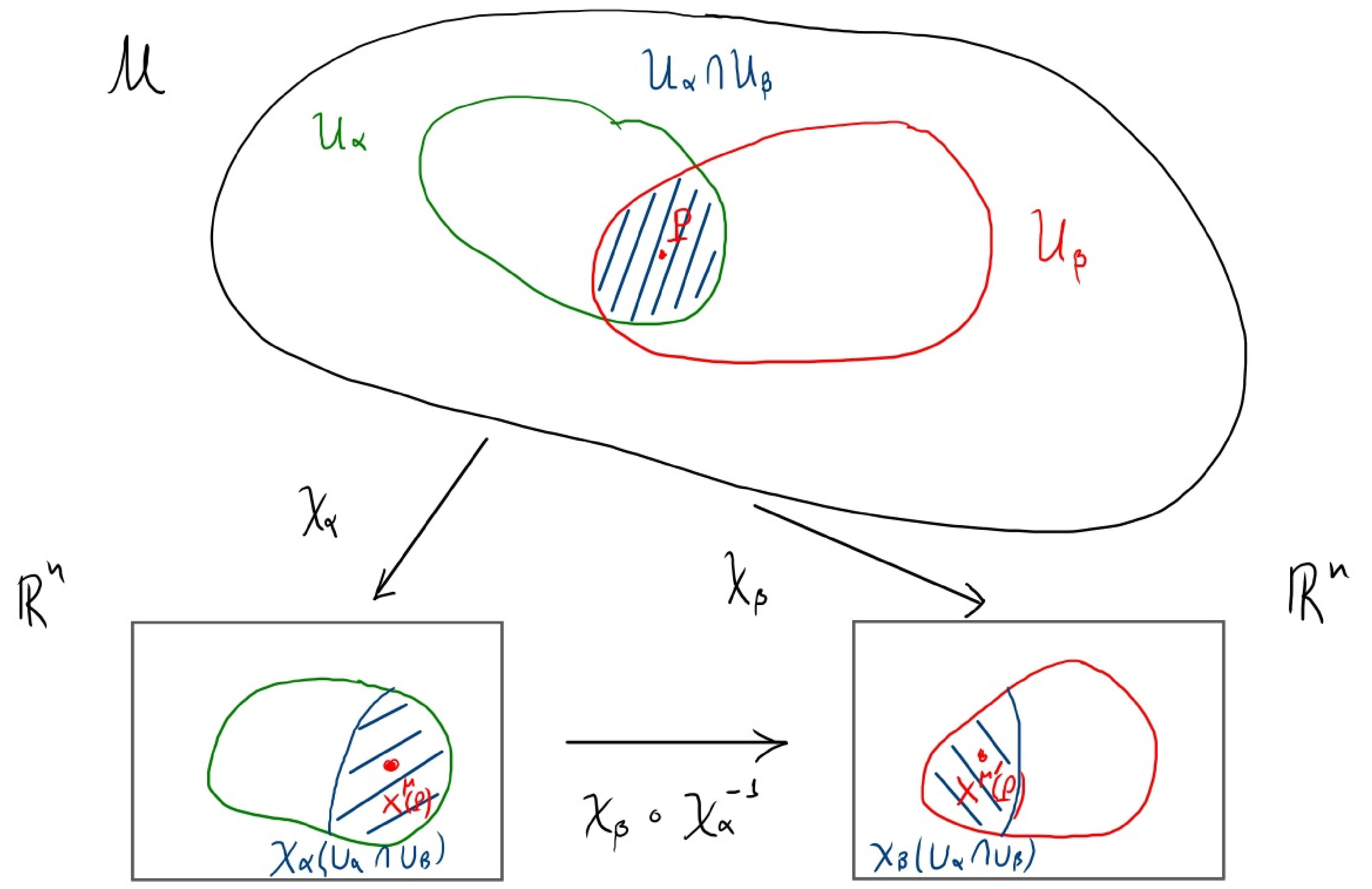
$\dfrac{\partial^p x^{\mu'}}{\partial x^{\mu_1} \cdots \partial x^{\mu_p}}$ " C^p

analytic C^∞

Differentiable Manifolds

- M with a maximal atlas is a differentiable manifold of $\dim M = n$:
  - M is a topological space
  - M is locally like $\mathbb{R}^n$, i.e. each point $P \in M$ is in a chart (U, x)

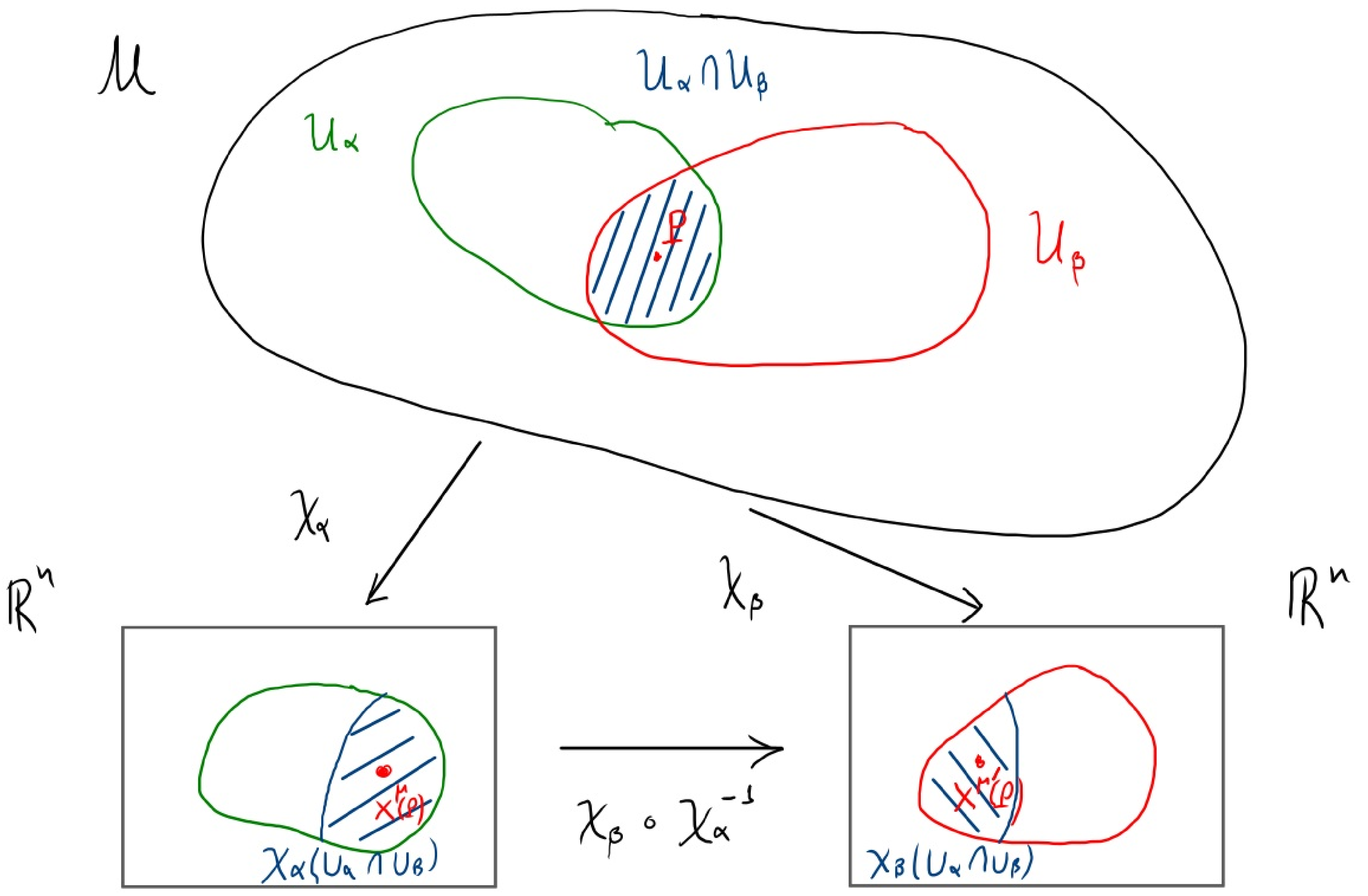
    $\dim M = n$
  - Coordinate transformations are differentiable
  - $\{(U_\alpha, x_\alpha)\}$, U_α open covering of M is an atlas of M
  - Maximal atlas: contains all compatible charts



Differentiable Manifolds

Remarks:

* We usually define manifolds using embeddings (e.g. surfaces)
  Manifolds need not embeddings to exist (e.g. spacetime)
  Manifolds can be embedded in many different ways
* Embeddings are useful: Any n-dim manifold is embeddable in $\mathbb{R}^{2n}$ (Whitney's embedding thm)
* Manifolds may need more than one chart to be covered
  e.g circle S^1: $0 < \varphi < 2\pi$ leaves one point out



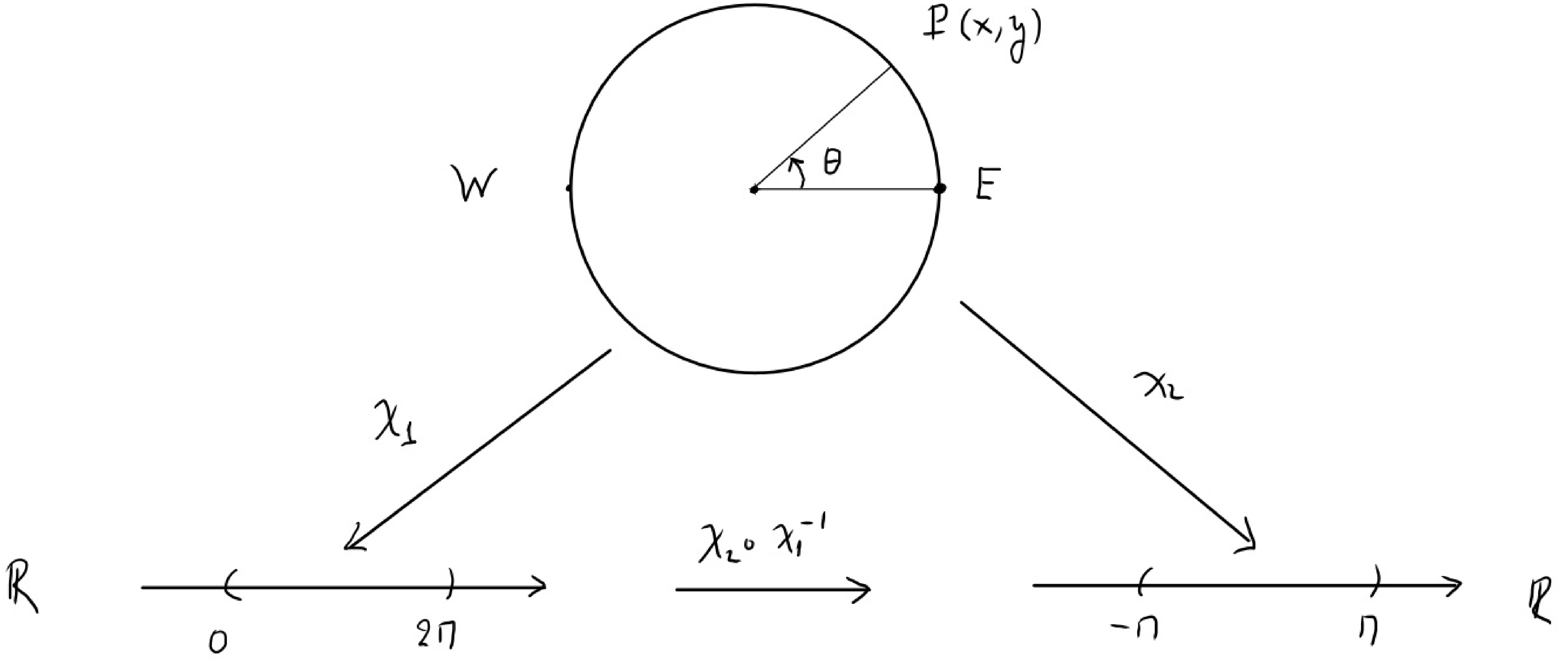
Examples: S^1

(U_1, χ_1):

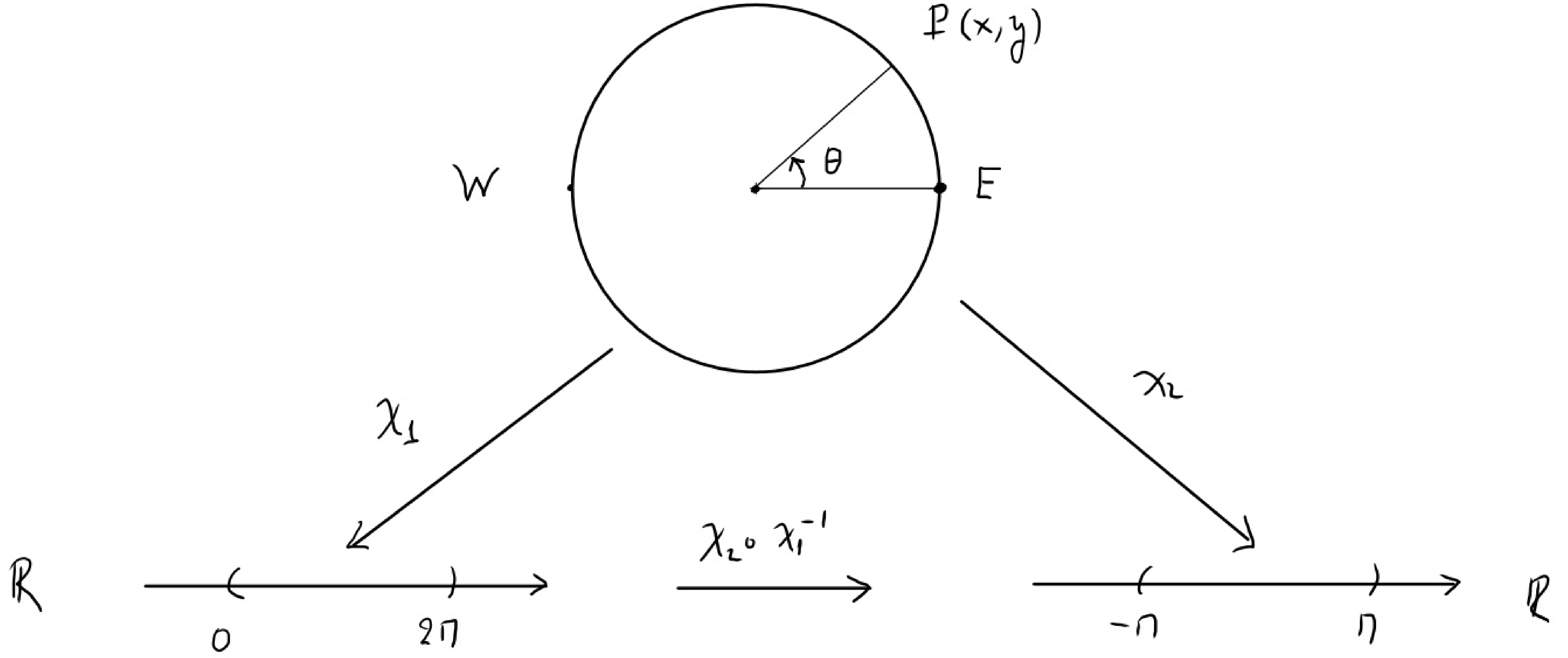
$U_1 = S^1 \setminus \{E\}$

$\chi_1(P) = \theta \qquad 0 < \theta < 2\pi$

$\chi_1 : (x, y) \mapsto \theta$



Examples: S^1



$(\mathcal{U}_1, \chi_1)$:

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$\chi_1 : (x, y) \mapsto \theta$

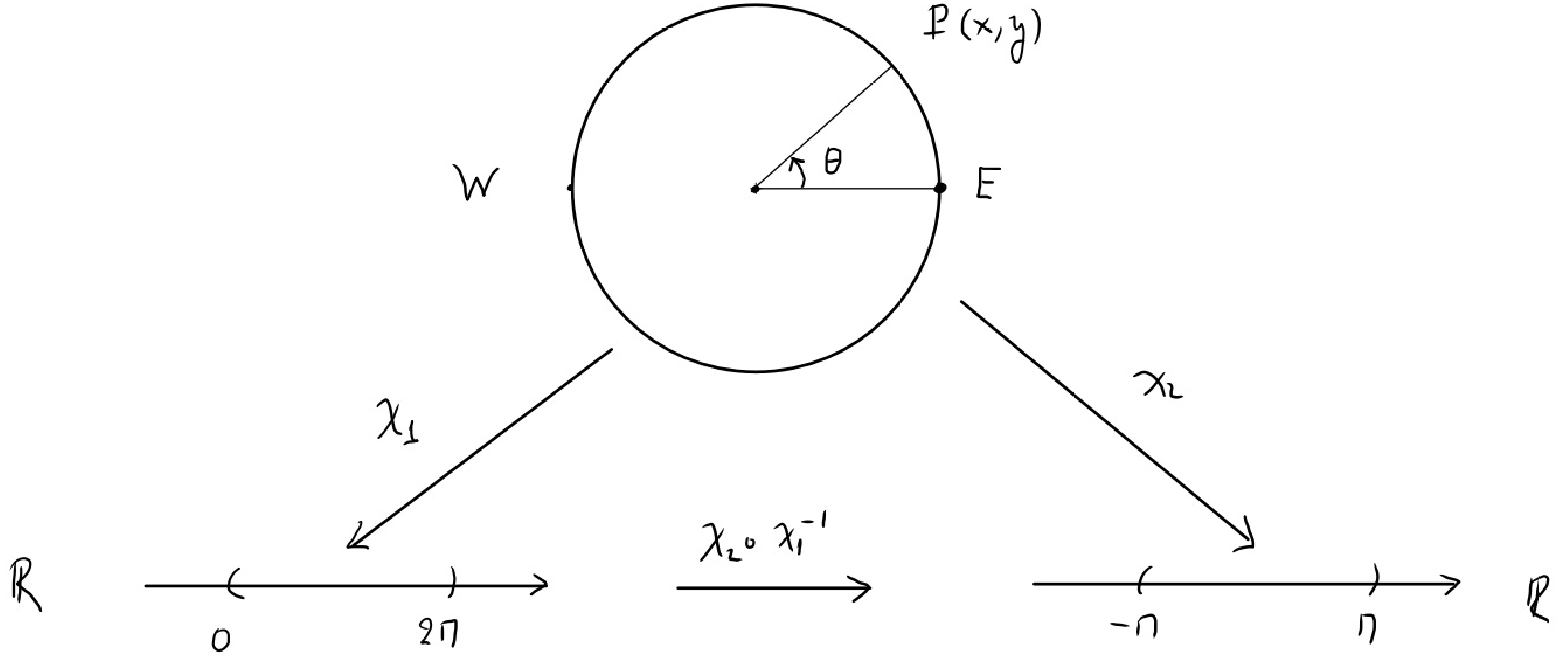
$(\mathcal{U}_2, \chi_2)$:

$\mathcal{U}_2 = S^1 \setminus \{W\}$

$\chi_2(P) = \theta \qquad -\pi < \theta < \pi$

$\chi_2 : (x, y) \mapsto \theta$

Examples: S^1



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$(\mathcal{U}^2, \chi_2)$:

$\mathcal{U}_2 = S^1 \setminus \{W\}$

$\chi_2(P) = \theta \qquad -\pi < \theta < \pi$

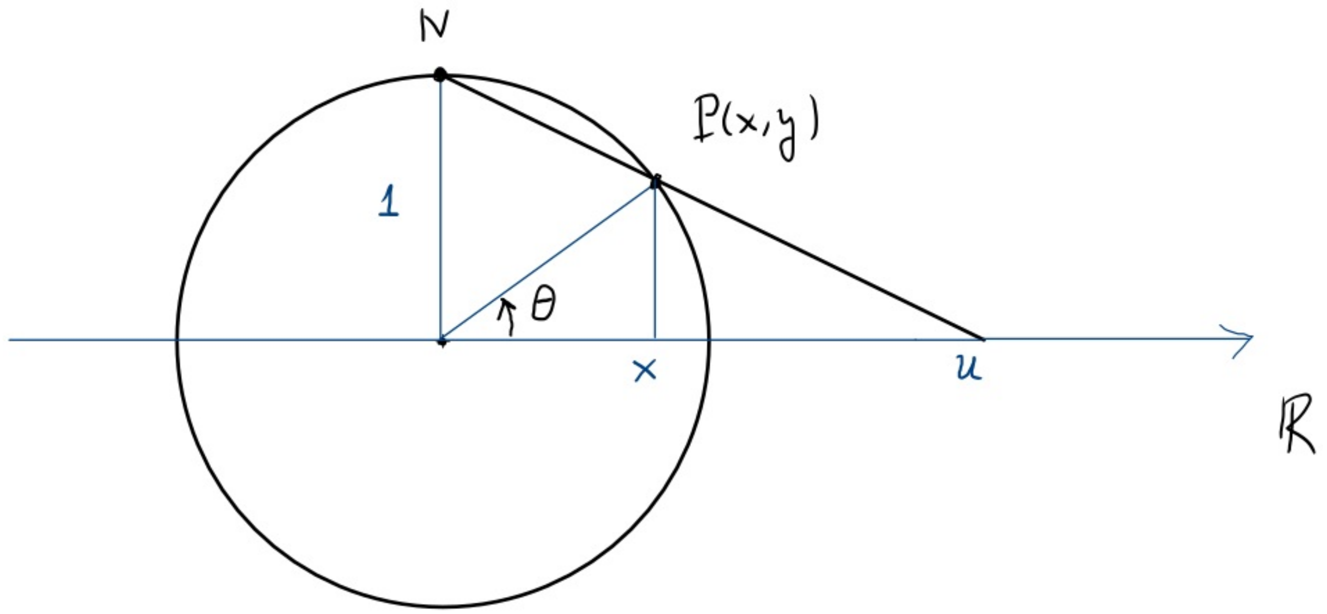
$\chi_2 : (x, y) \mapsto \theta$

$$\chi_2 \circ \chi_1^{-1}(\theta) = \begin{cases} \theta & 0 < \theta < \pi \\ \theta - 2\pi & \pi < \theta < 2\pi \end{cases}$$

differentiable

$S^1 = \mathcal{U}^1 \cup \mathcal{U}^2$ so $\{(\mathcal{U}^1, \chi_1), (\mathcal{U}^2, \chi_2)\}$ an atlas of S^1

There are more atlases



$(\mathcal{U}^3, \chi_3)$: $\mathcal{U}^3 = S^1 \setminus \{N\}$

$$\chi_3 : (x,y) \mapsto u = \frac{x}{1-y} \qquad -\infty < u < +\infty$$

from similarity of triangles:

$$\frac{u}{1} = \frac{u-x}{y} \Rightarrow u = \frac{x}{1-y}$$

Also, since $\left.\begin{array}{l} x = 1 \cdot \cos\theta \\ y = 1 \cdot \sin\theta \end{array}\right\} \Rightarrow u = \dfrac{\cos\theta}{1-\sin\theta}$

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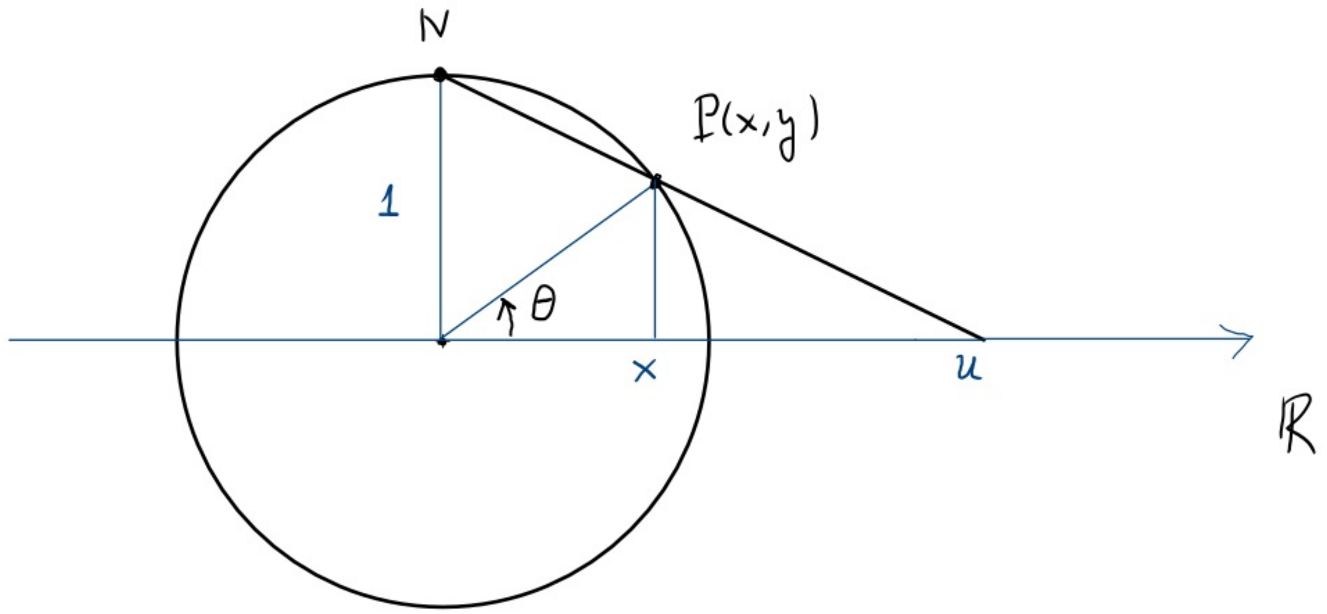

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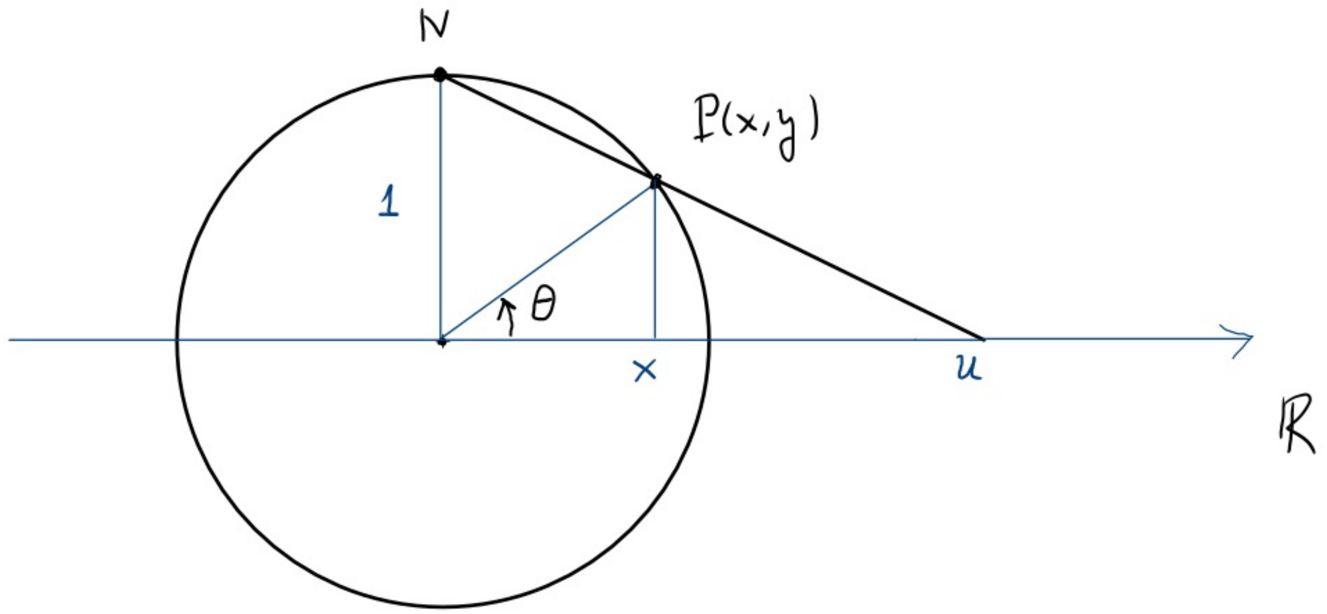
Note that we just proved that:

a. $S_1 \setminus \{N\} \cong \mathbb{R}$

b. a line segment $\cong \mathbb{R}$

There are more atlases



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$$u = \chi_3 \circ \chi_1^{-1} = \frac{\cos\theta}{1-\sin\theta}$$ differentiable

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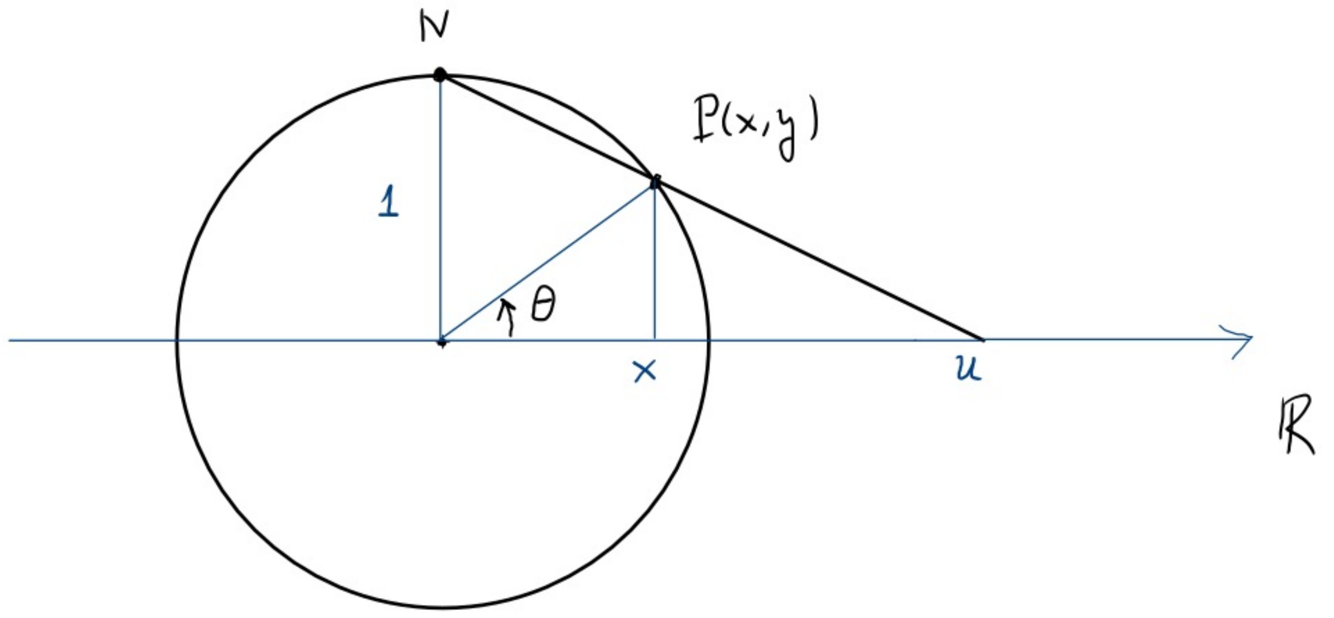

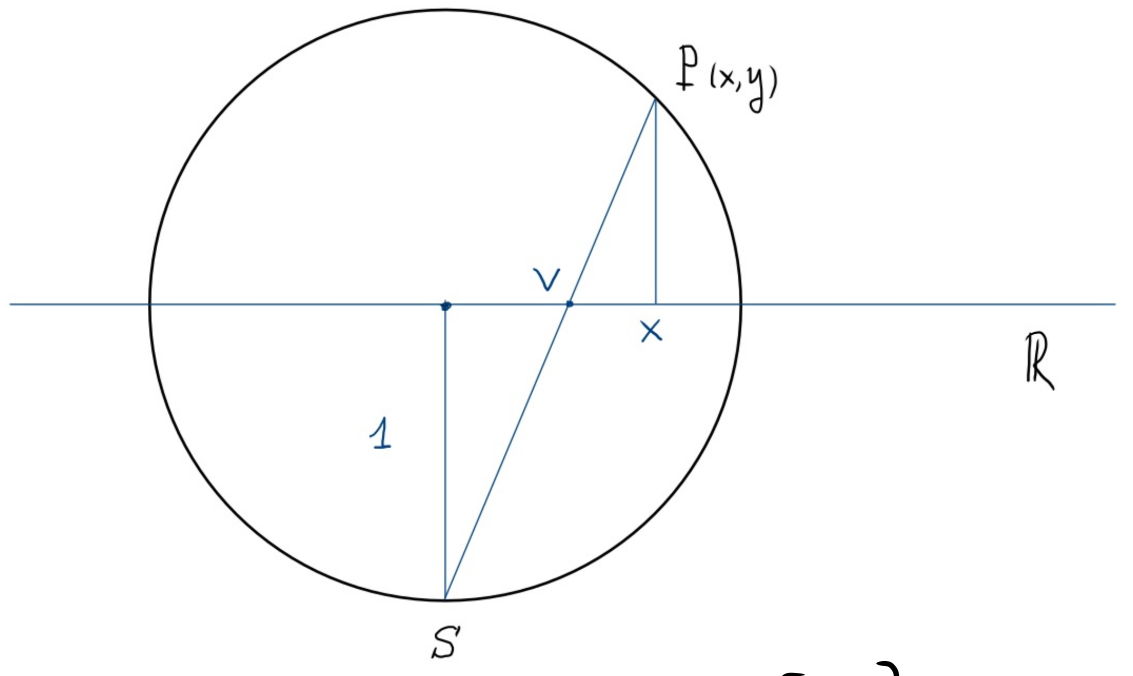

$(U^3, \chi_3): U^3 = S^1 \setminus \{N\}$

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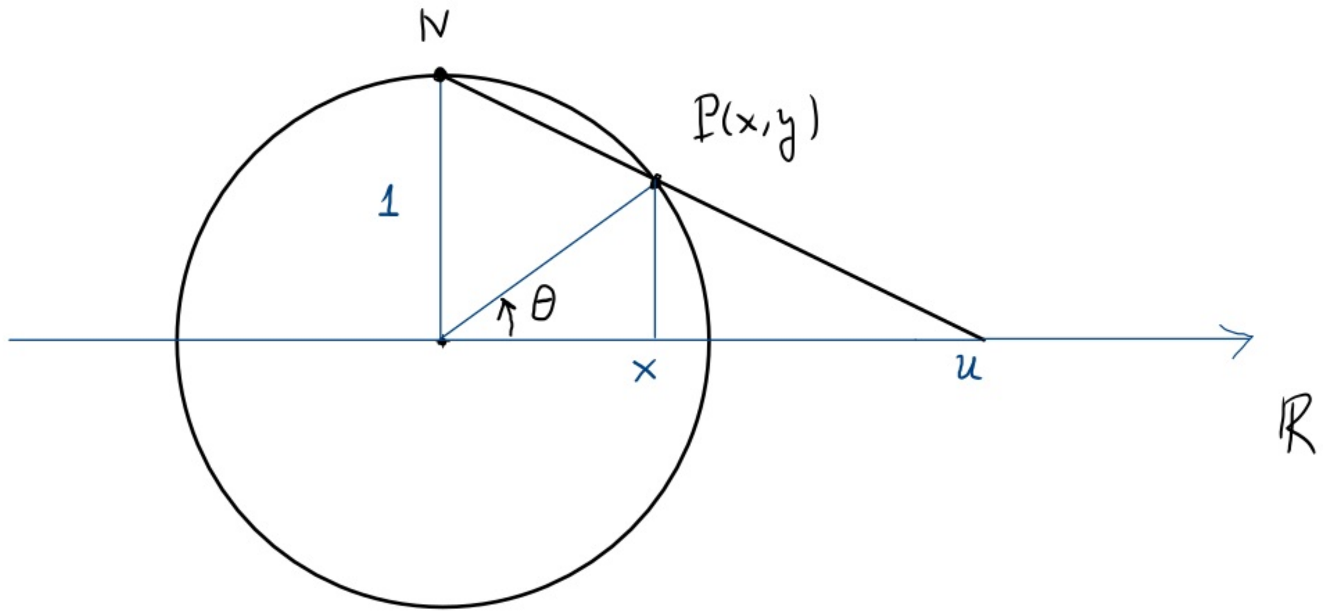
$u = \chi_3 \circ \chi_1^{-1} = \dfrac{\cos\theta}{1-\sin\theta}$ differentiable

$(U^4, \chi_4): U^4 = S^1 \setminus \{S\}$

$\chi_4 : (x,y) \mapsto v = \dfrac{x}{1+y} \qquad -\infty < v < +\infty$

- similarity of triangles gives: $\dfrac{v}{1} = \dfrac{x-v}{y} \Rightarrow v = \dfrac{x}{1+y}$
- for $x = \cos\theta$, $y = \sin\theta$ $\qquad v = \dfrac{\cos\theta}{1+\sin\theta}$

There are more atlases



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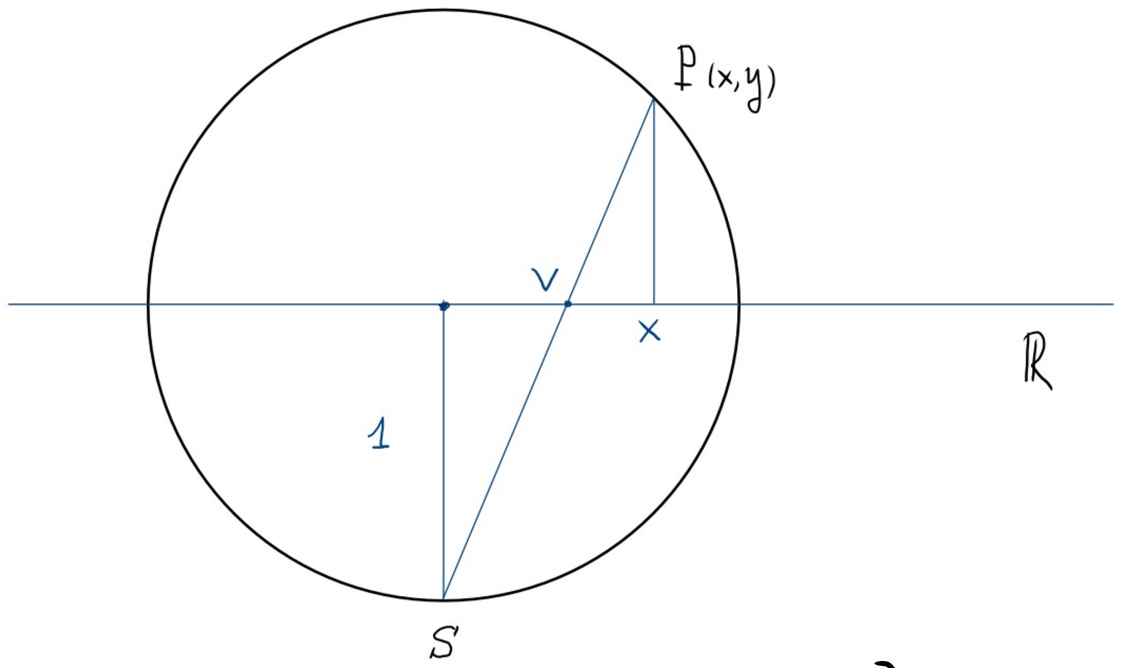
$u = \chi_3 \circ \chi_1^{-1} = \dfrac{\cos\theta}{1-\sin\theta}$ differentiable



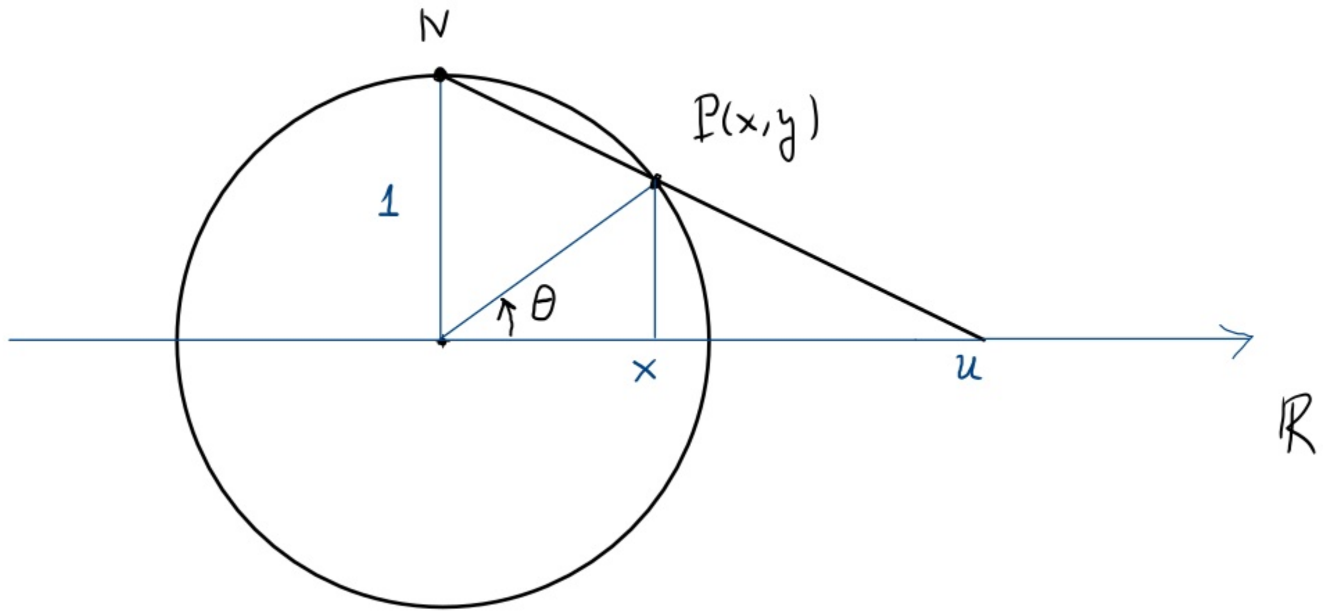
$(\mathcal{U}^4, \chi_4)$: $\mathcal{U}^4 = S^1 \setminus \{S\}$

$\chi_4 : (x, y) \mapsto v = \dfrac{x}{1+y}$ $\quad -\infty < v < +\infty$

$v = \chi_4 \circ \chi_1^{-1} = \dfrac{\cos\theta}{1+\sin\theta}$

differentiable

There are more atlases



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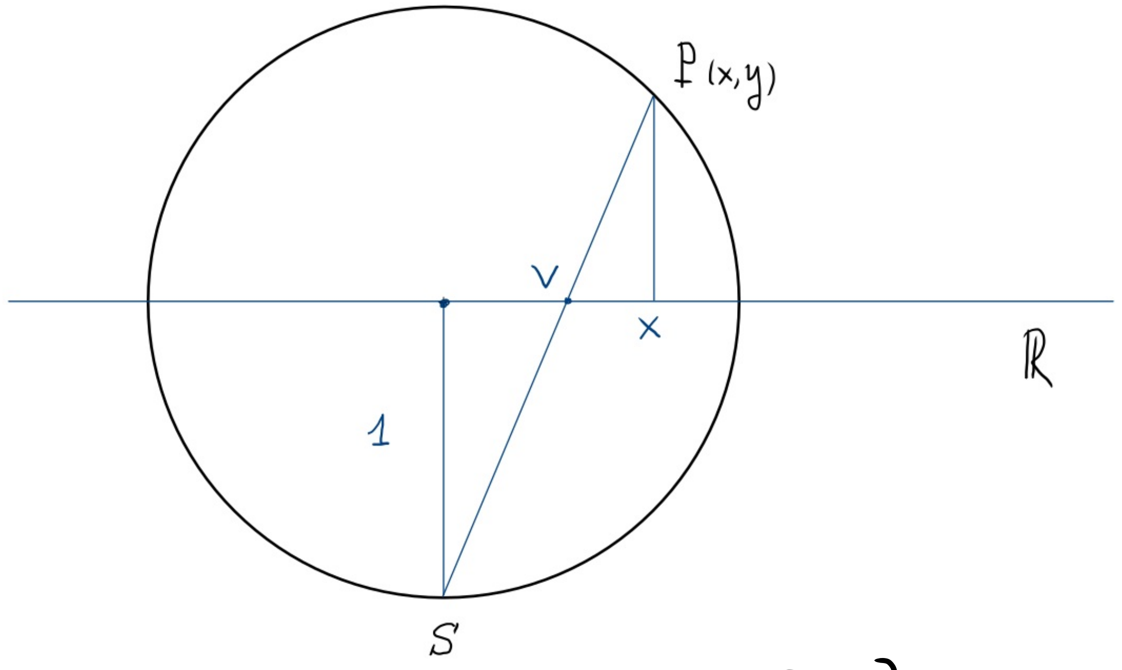


$(U^4, \chi_4): U^4 = S^1 \setminus \{S\}$

$\chi_4 : (x,y) \mapsto v = \dfrac{x}{1+y} \qquad -\infty < v < +\infty$

$v = \chi_4 \circ \chi_1^{-1} = \dfrac{\cos\theta}{1+\sin\theta}$

differentiable

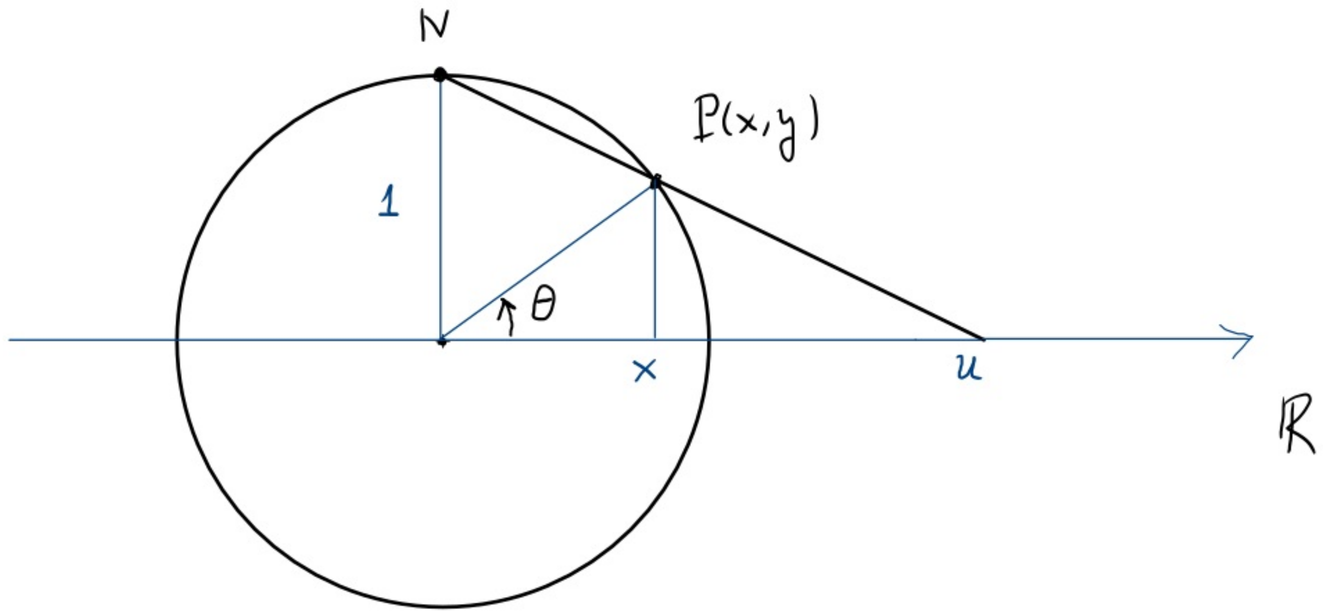
So

$v = \chi_4 \circ \chi_3^{-1}(u) = \dfrac{1}{u}$

differentiable

Note that: $\left. \begin{array}{l} u \cdot v = \dfrac{x}{1-y} \cdot \dfrac{x}{1+y} = \dfrac{x^2}{1-y^2} \\ x^2 + y^2 = 1 \end{array} \right\} \Rightarrow u \cdot v = \dfrac{x^2}{x^2} = 1 \Rightarrow v = \dfrac{1}{u}$

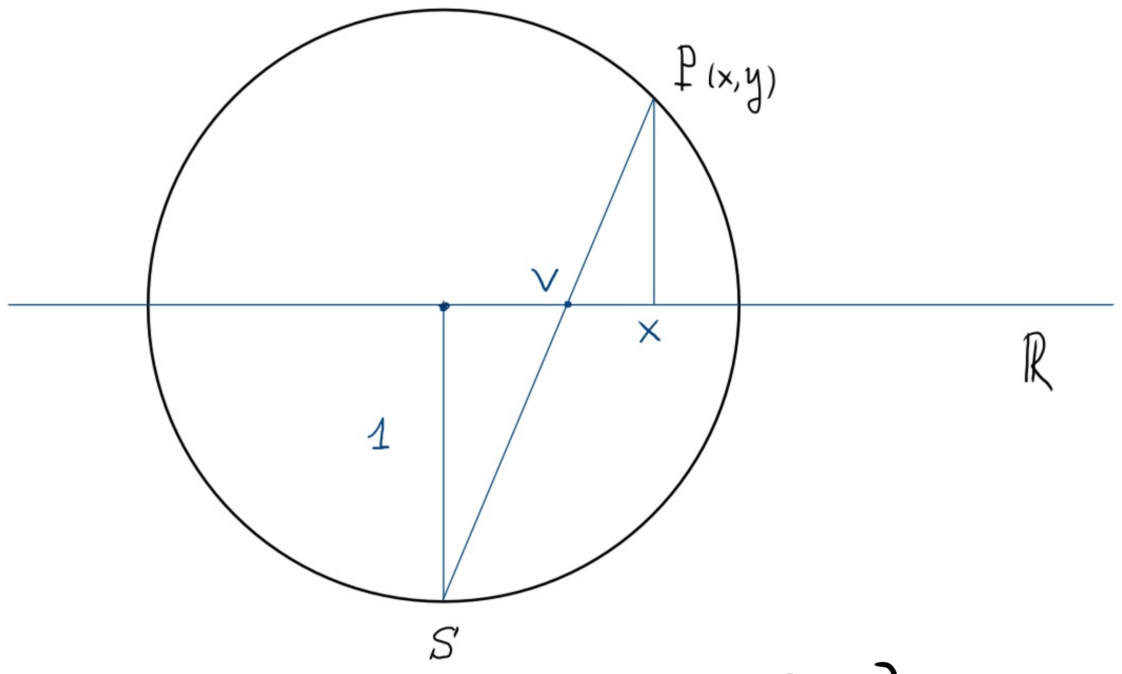
There are more atlases



(U^3, χ_3) : $U^3 = S^1 \setminus \{N\}$

$\chi_3 : (x,y) \mapsto u = \dfrac{x}{1-y}$ $\quad -\infty < u < +\infty$

$u = \chi_3 \circ \chi_1^{-1} = \dfrac{\cos\theta}{1-\sin\theta}$ differentiable



(U^4, χ_4) : $U^4 = S^1 \setminus \{S\}$

$\chi_4 : (x,y) \mapsto v = \dfrac{x}{1+y}$ $\quad -\infty < v < +\infty$

$v = \chi_4 \circ \chi_1^{-1} = \dfrac{\cos\theta}{1+\sin\theta}$

differentiable

So

$v = \chi_4 \circ \chi_3^{-1}(u) = \dfrac{1}{u}$

differentiable

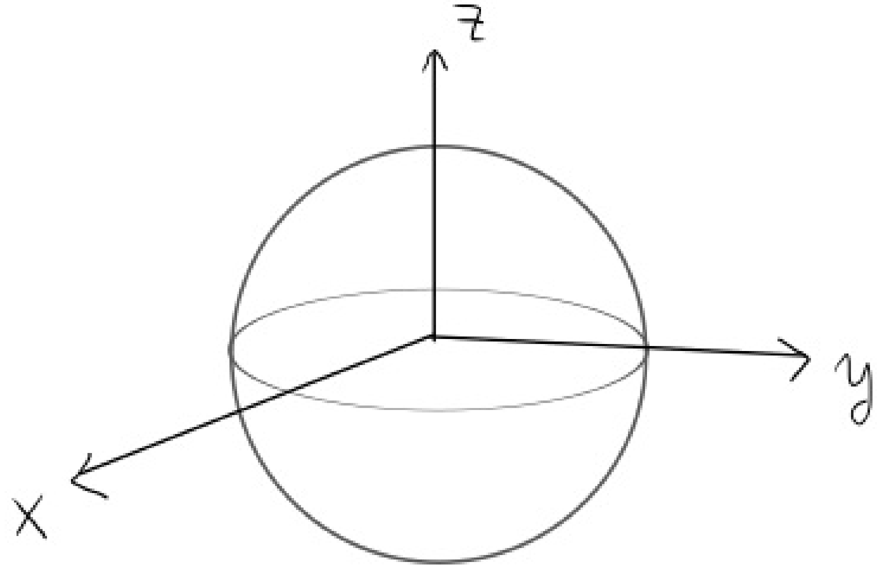
and

$\{(U^3, \chi_3), (U^4, \chi_4)\}$ an atlas of S^1

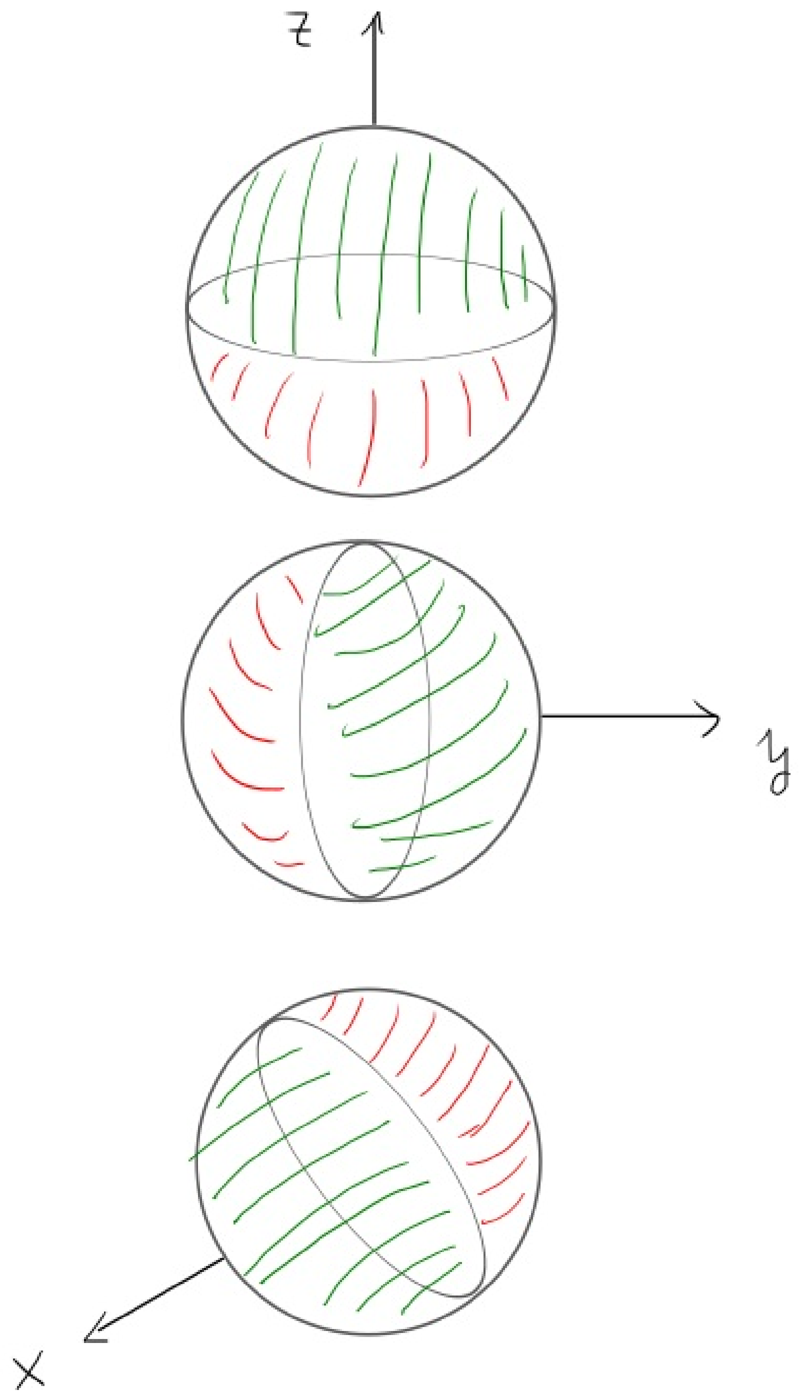
Note : We cannot cover S^1 using only one chart!

S^2 : the sphere

$P(x,y,z)$: $x^2+y^2+z^2=1$



Define 6 charts - hemispheres:



$U_{z+} = \{(x,y,z) \in S^2 \mid z>0\}$ $\quad \chi_{z+} : (x,y,+\sqrt{1-x^2-y^2}) \mapsto (x,y)$

$U_{z-} = \{(x,y,z) \in S^2 \mid z<0\}$ $\quad \chi_{z-} : (x,y,-\sqrt{1-x^2-y^2}) \mapsto (x,y)$

projection on $\angle xy$ plane

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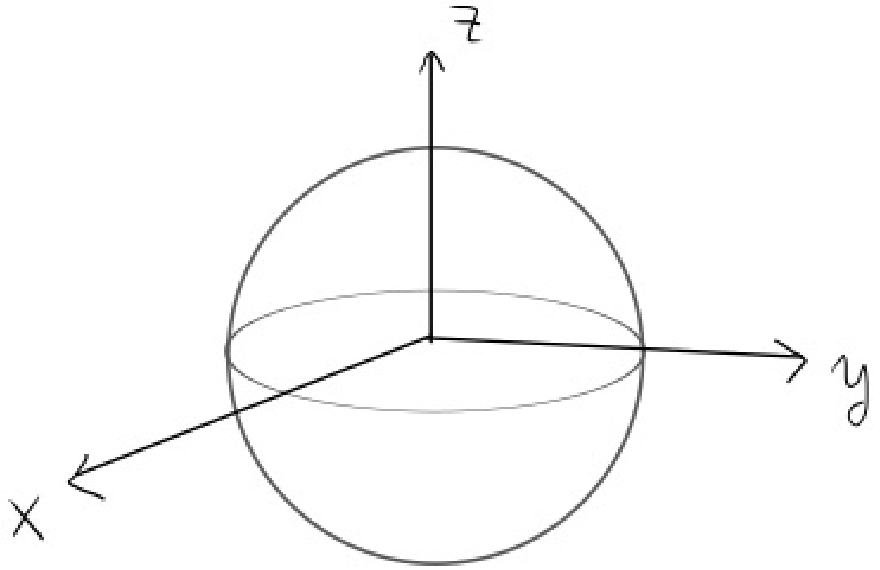
$P(x,y,z)$: $x^2+y^2+z^2=1$



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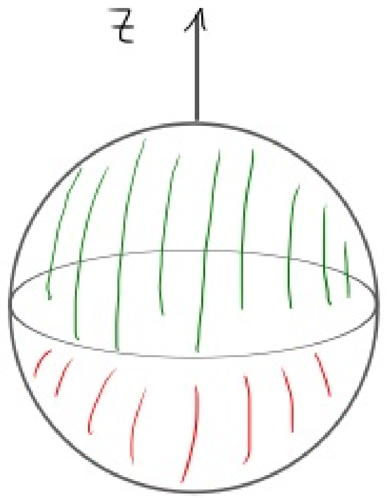

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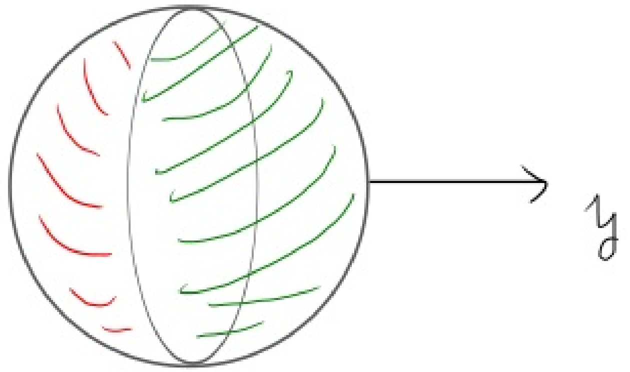

$U_{y+} = \{(x,y,z) \in S^2 \mid y>0\}$ $\quad \chi_{y+}: (x,+\sqrt{1-x^2-z^2},z) \mapsto (x,z)$

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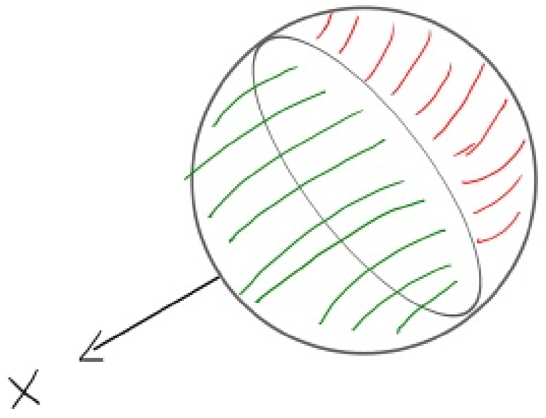
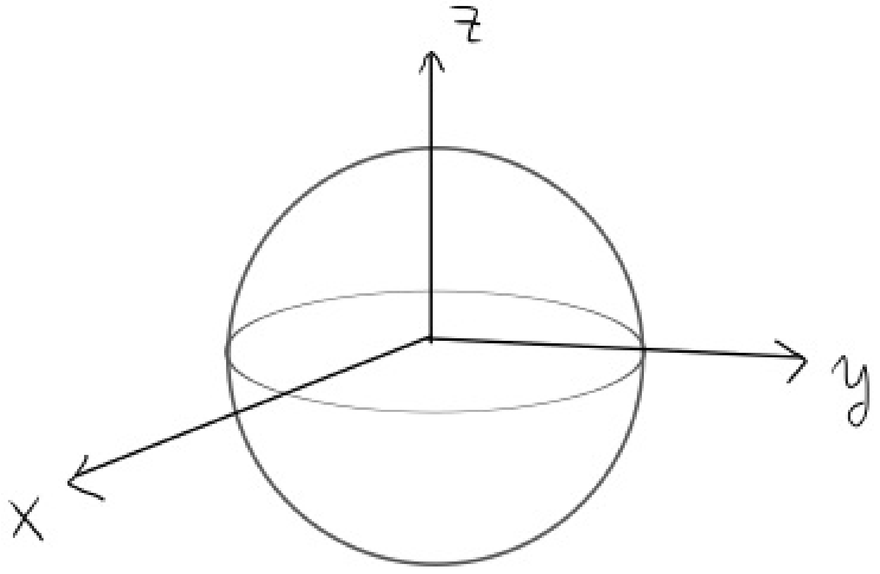
projection on $\angle xz$ plane



S^2 : the sphere

$P(x,y,z): \; x^2+y^2+z^2=1$



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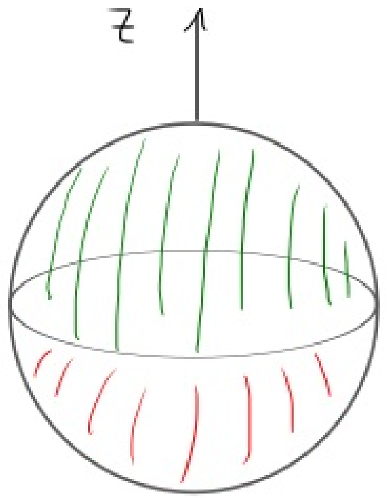

$\mathcal{U}_{z+} = \{(x,y,z) \in S^2 \mid z>0\} \qquad \chi_{z+}: (x, y, +\sqrt{1-x^2-y^2}) \mapsto (x,y)$

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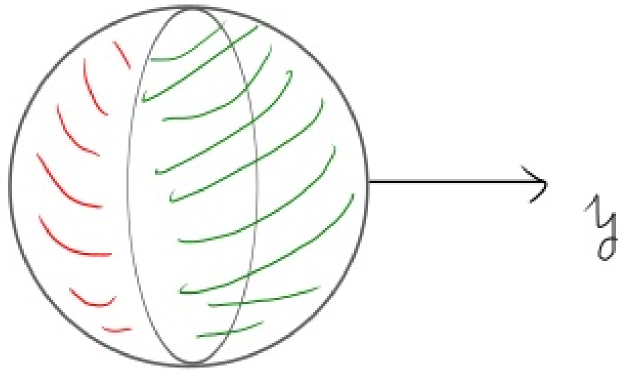

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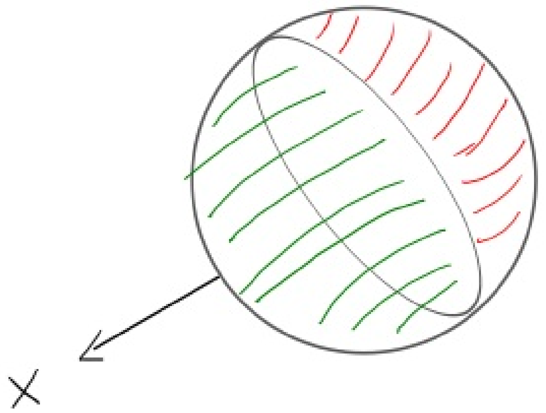

$\mathcal{U}_{x+} = \{(x,y,z) \in S^2 \mid x>0\} \qquad \chi_{x+}: (+\sqrt{1-y^2-z^2}, y, z) \mapsto (y,z)$

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S^2 : the sphere

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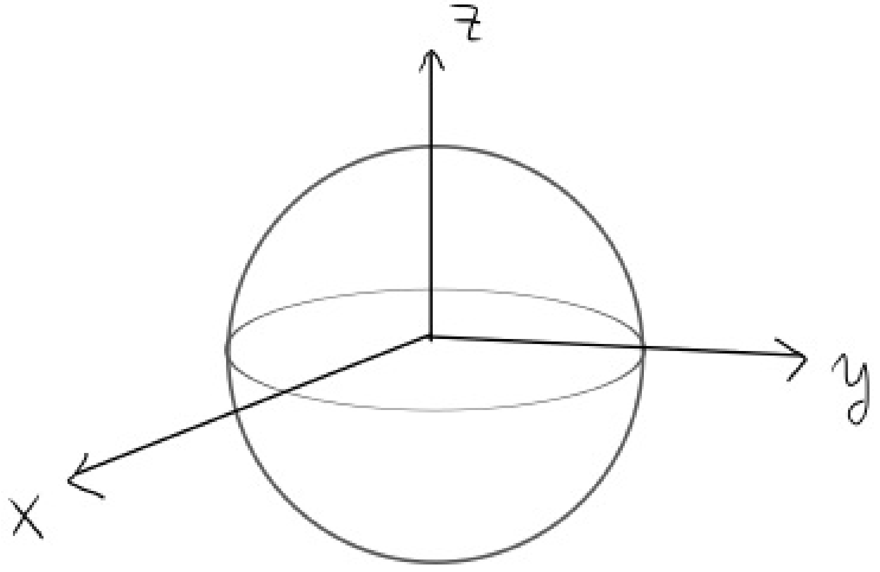

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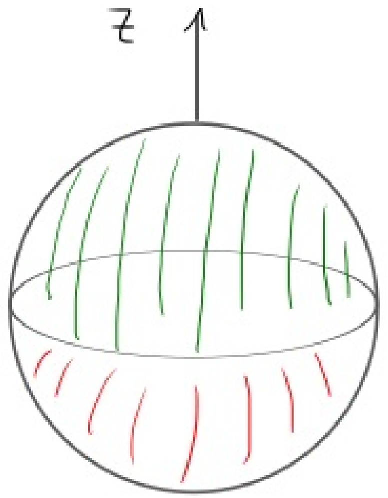

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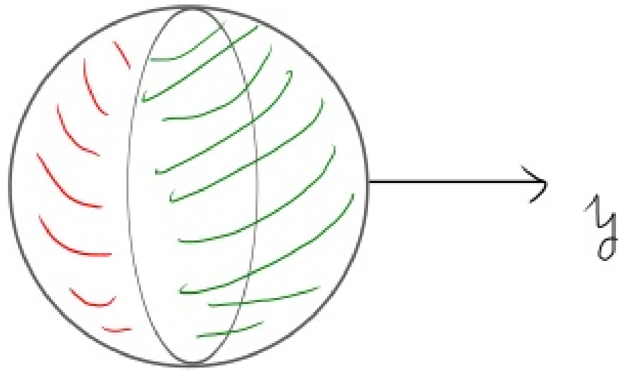

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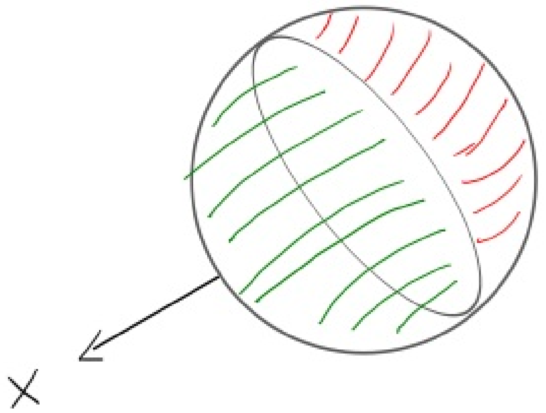

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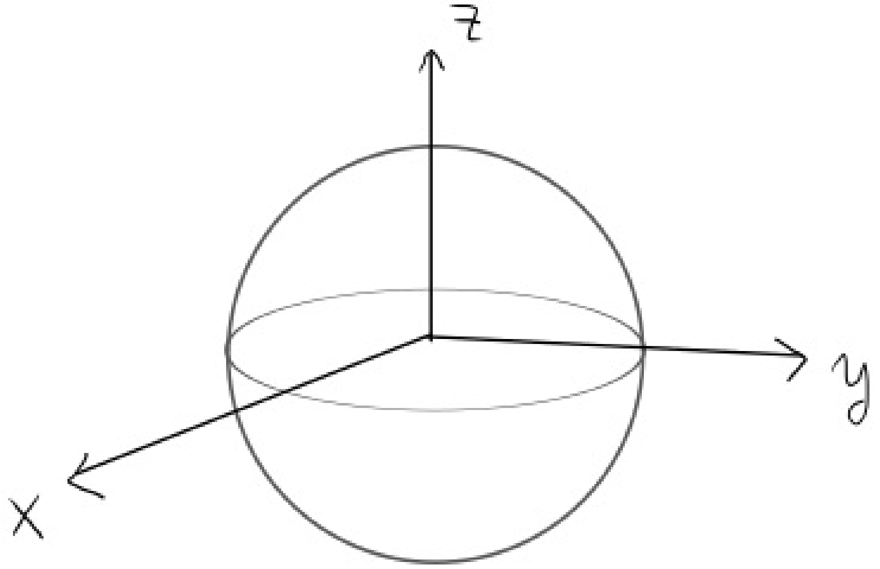
$U_{x-} = \{(x,y,z) \in S^2 \mid x<0\}$ $\quad \chi_{x-}: (-\sqrt{1-y^2-z^2},y,z) \mapsto (y,z)$

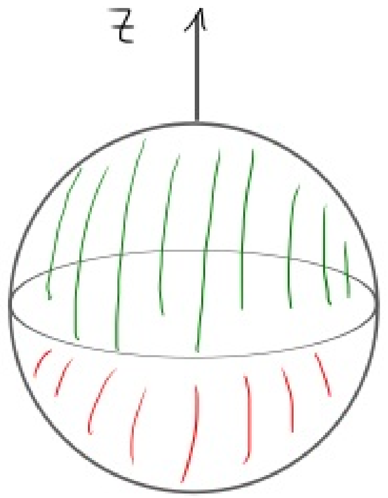
A transition function: $\chi_{y-} \circ \chi_{x+}^{-1} : (y,z) \mapsto (x,z)$ with $\begin{array}{l} x=\sqrt{1-y^2-z^2} \\ z=z \end{array}$ is differentiable

S^2: the sphere

$P(x,y,z):\ x^2+y^2+z^2=1$



Define 6 charts - hemispheres:



$U_{z+} = \{(x,y,z)\in S^2 \mid z>0\}$ $\quad \chi_{z+}: (x,y,+\sqrt{1-x^2-y^2}) \mapsto (x,y)$

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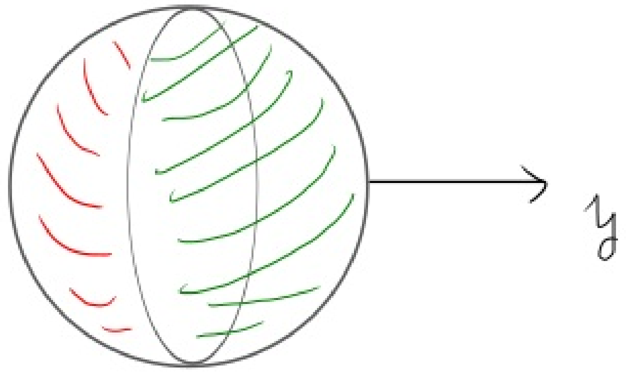

$U_{y+} = \{(x,y,z)\in S^2 \mid y>0\}$ $\quad \chi_{y+}: (x,+\sqrt{1-x^2-z^2},z) \mapsto (x,z)$

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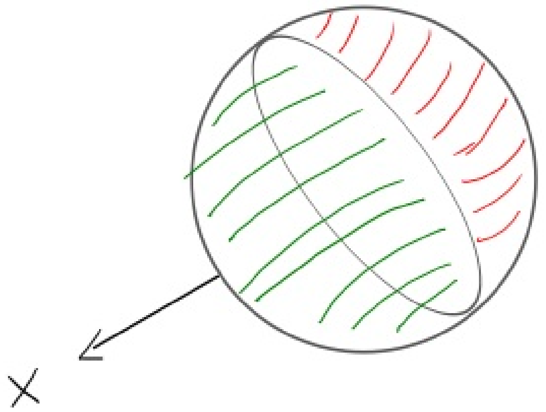

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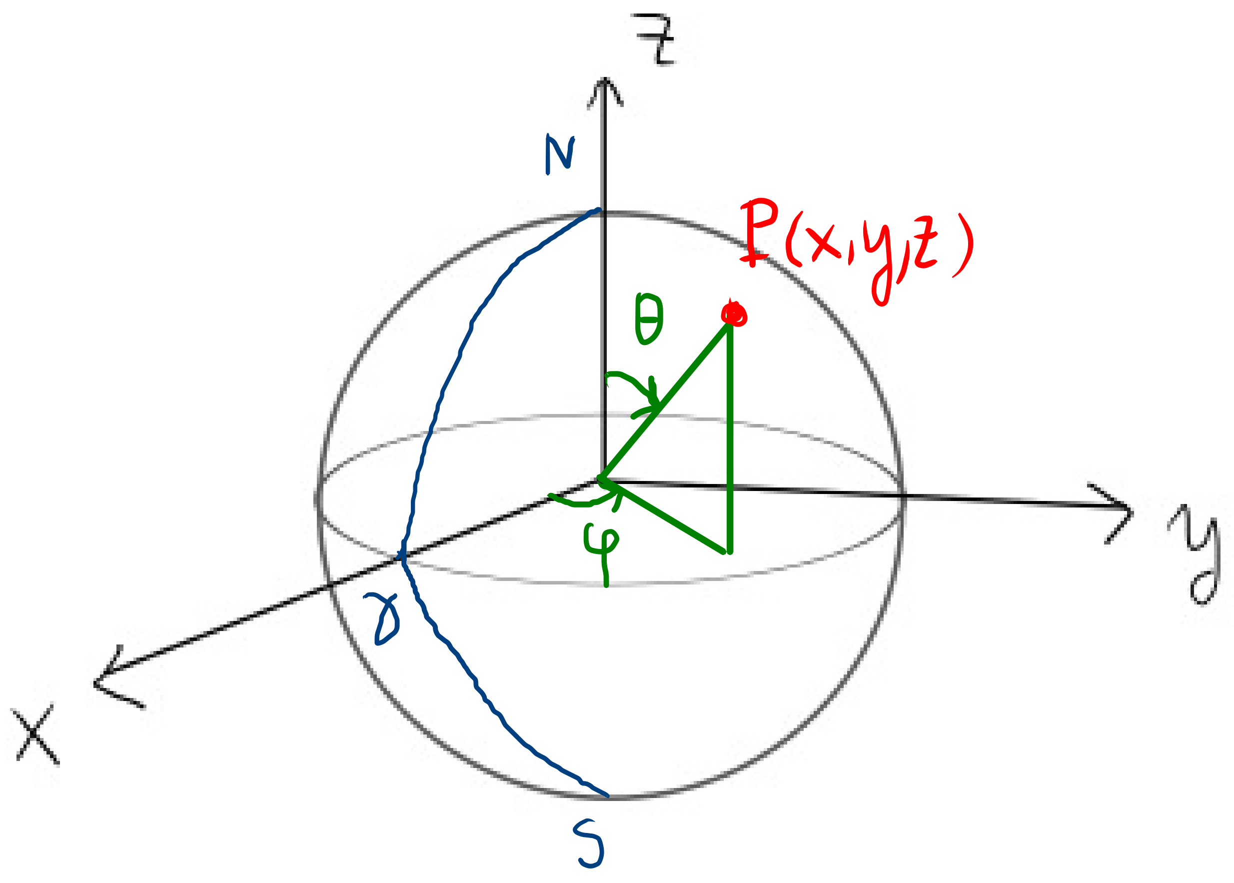
We need all 6 charts to cover S^2

A transition function: $\chi_{y-}\circ\chi_{x+}^{-1}: (y,z)\mapsto(x,z)$ with $x=\sqrt{1-y^2-z^2}$, $z=z$ is differentiable



(U_θ, χ_θ): $\quad U_\theta = S^2 - \{\gamma\}$

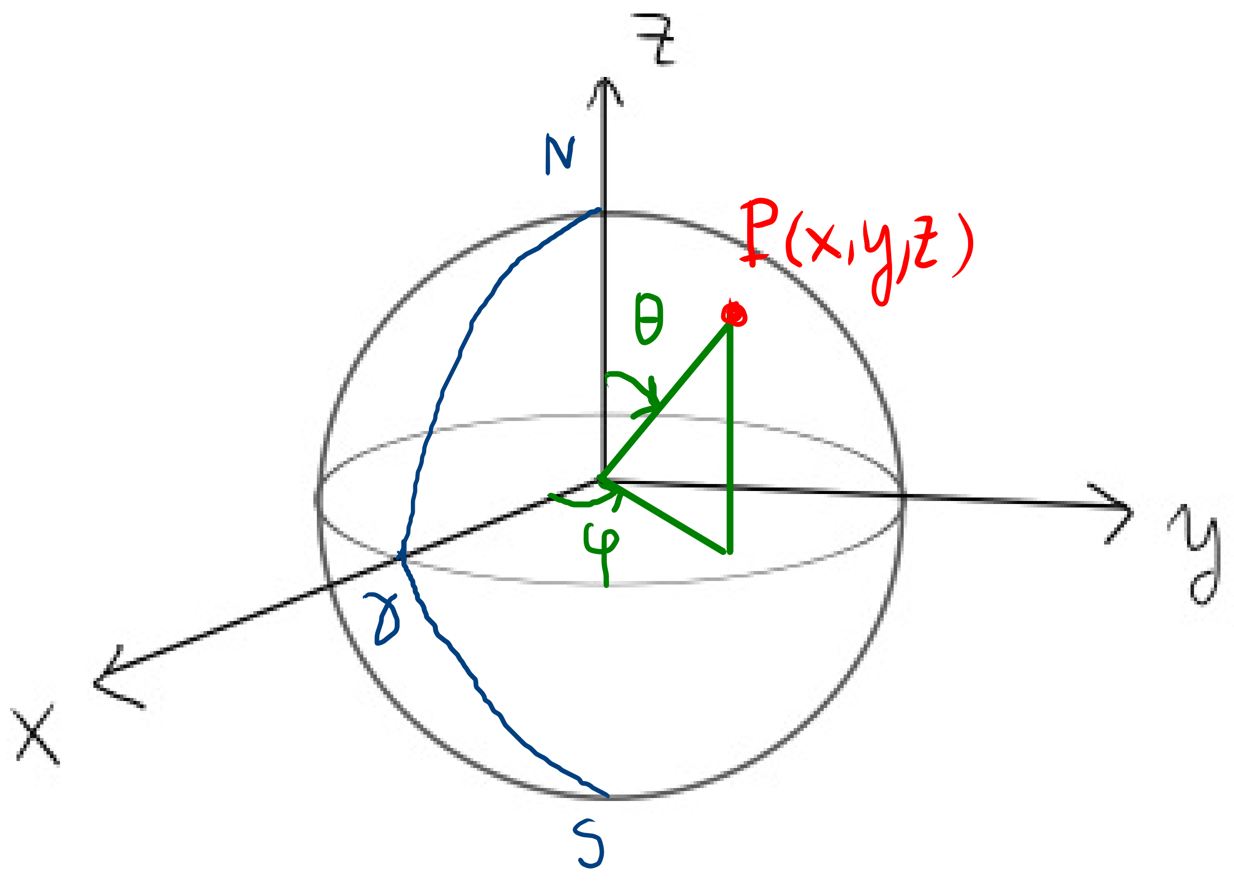
$\chi_\theta : (x, y, z) \mapsto (\theta, \varphi)$ $\quad 0 < \theta < \pi$

$0 < \varphi < 2\pi$

$x = \sin\theta \cos\phi$

$y = \sin\theta \sin\phi$

$z = \cos\theta$



(U_θ, χ_θ): $\quad U_\theta = S^2 - \{\gamma\}$

$\chi_\theta : (x,y,z) \mapsto (\theta,\varphi) \qquad 0<\theta<\pi$
$\qquad\qquad\qquad\qquad\qquad\qquad\quad 0<\varphi<2\pi$

$x = \sin\theta\cos\phi$

$y = \sin\theta\sin\phi$

$z = \cos\theta$

Examples of transition functions:

$$\chi_{y+} \circ \chi_\theta^{-1} : \quad (\theta,\varphi) \mapsto (x,z)$$

$x = \sin\theta\cos\varphi$

$z = \cos\theta$

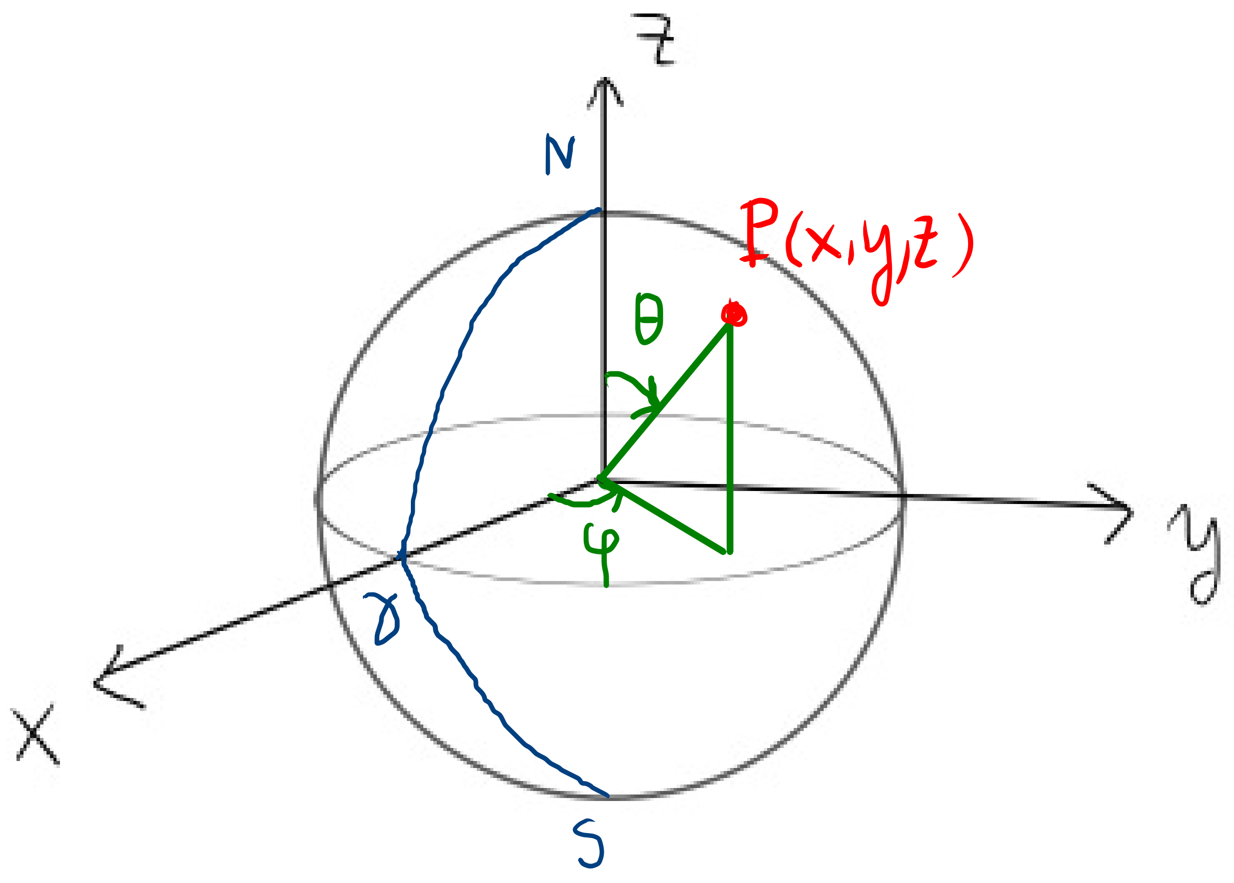
for $\quad 0<\theta<\pi$

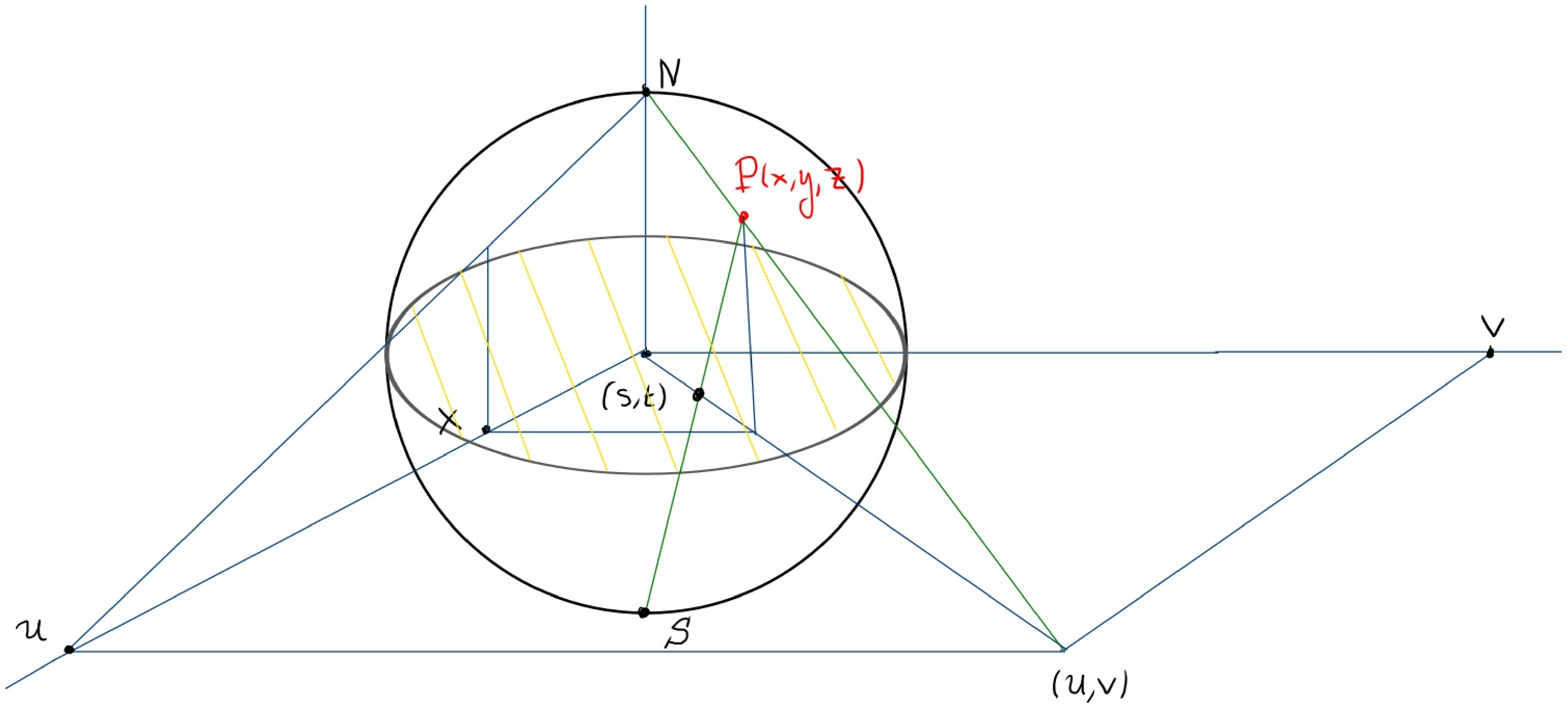
$\qquad 0<\varphi<\pi$

$$\chi_\theta \circ \chi_{y+}^{-1} : \quad (x,z) \mapsto (\theta,\varphi)$$

$0 < \theta = \tan^{-1}\sqrt{\frac{1}{z^2}-1} < \pi$

$0 < \varphi = \tan^{-1}\frac{\sqrt{1-x^2-z^2}}{x} < \pi$





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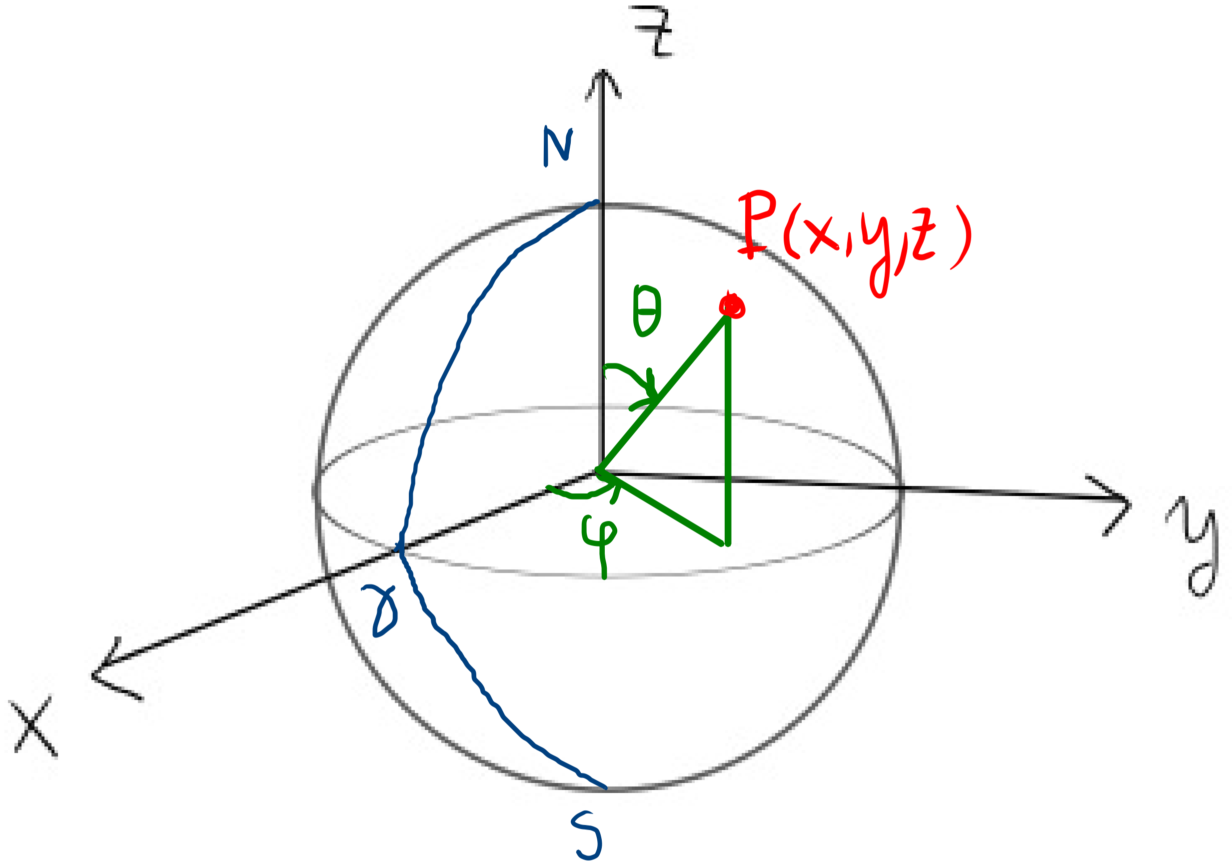
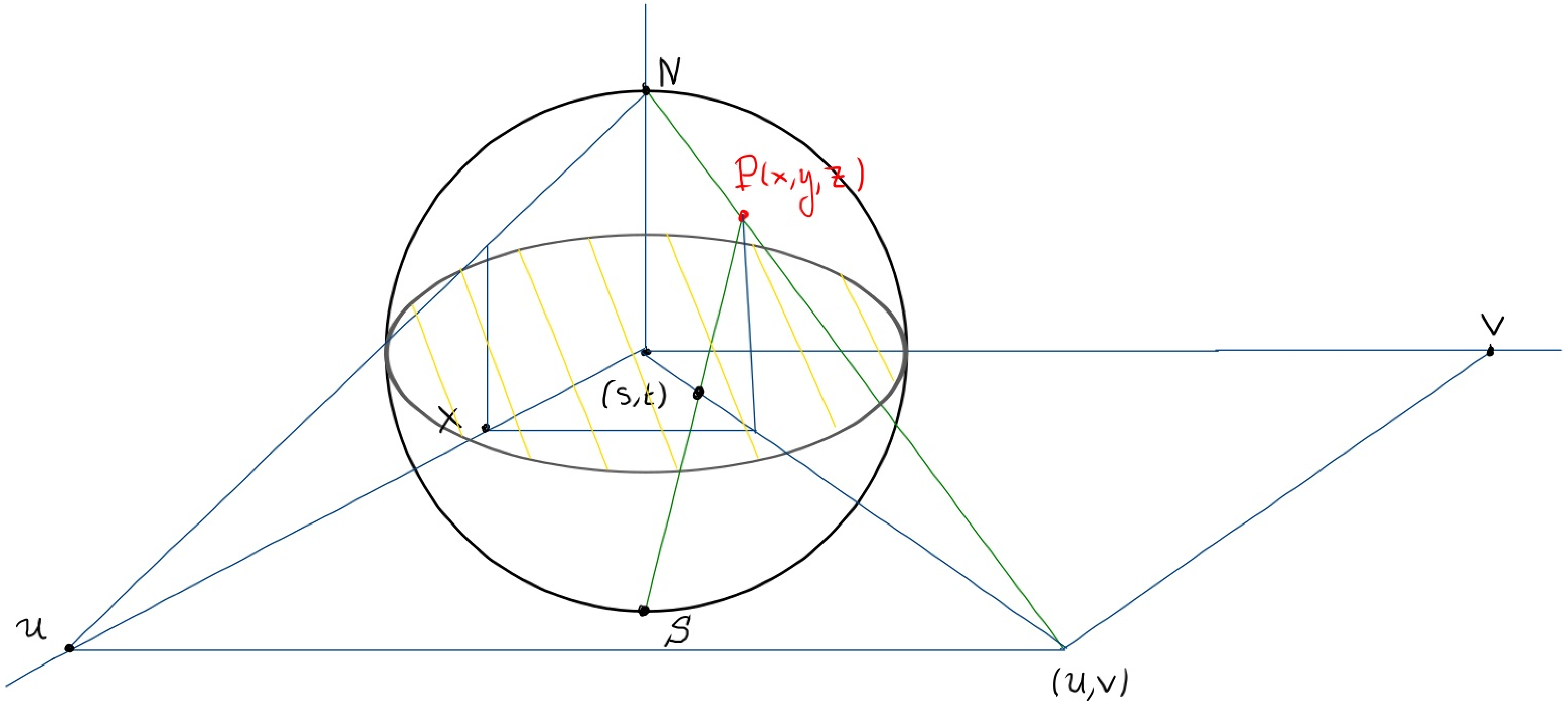
$z = \cos\theta$

(U_N, χ_N): $U_N = S^2 \setminus \{N\}$

$\chi_N : (x,y,z) \mapsto (u,v)$ $\quad -\infty<u<+\infty$, $-\infty<v<+\infty$

$u = \dfrac{x}{1-z} \qquad v = \dfrac{y}{1-z}$





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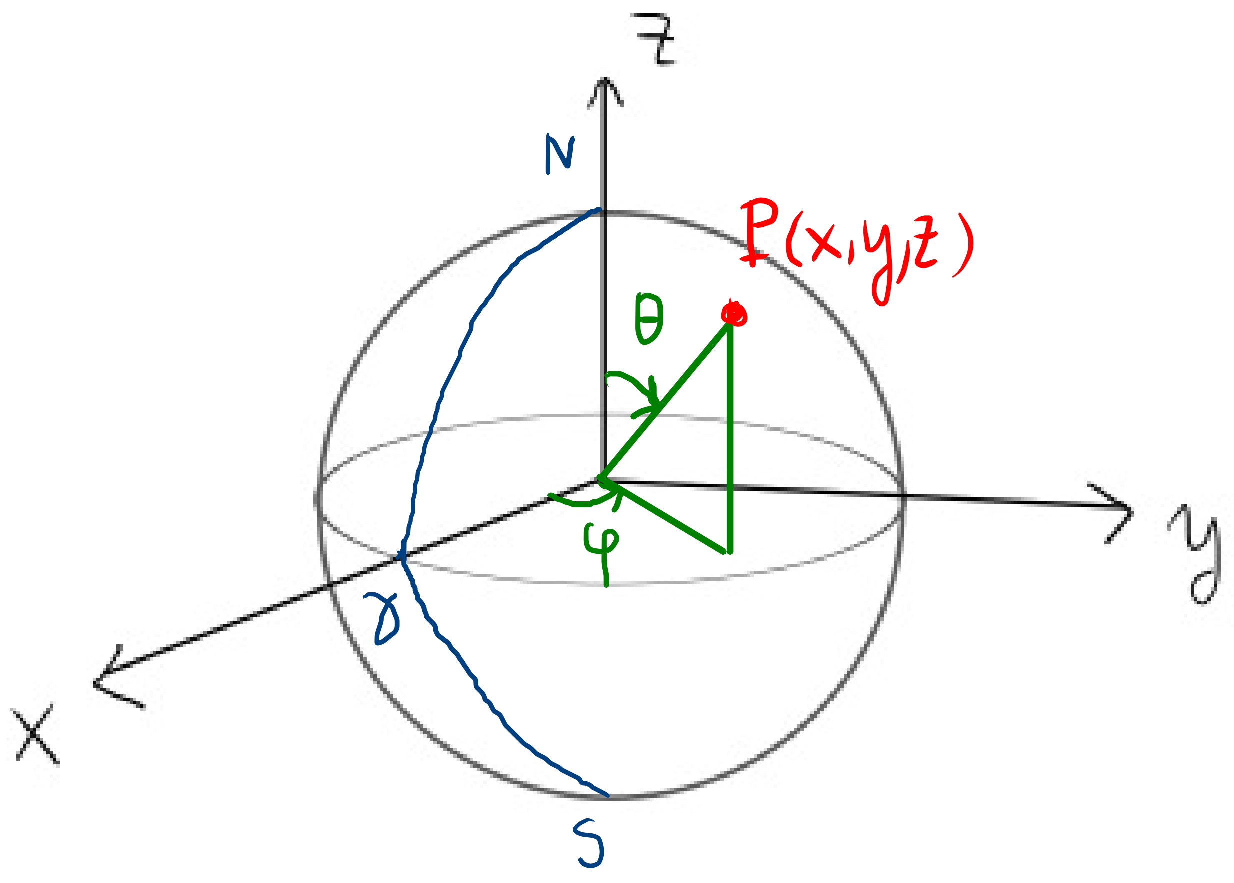
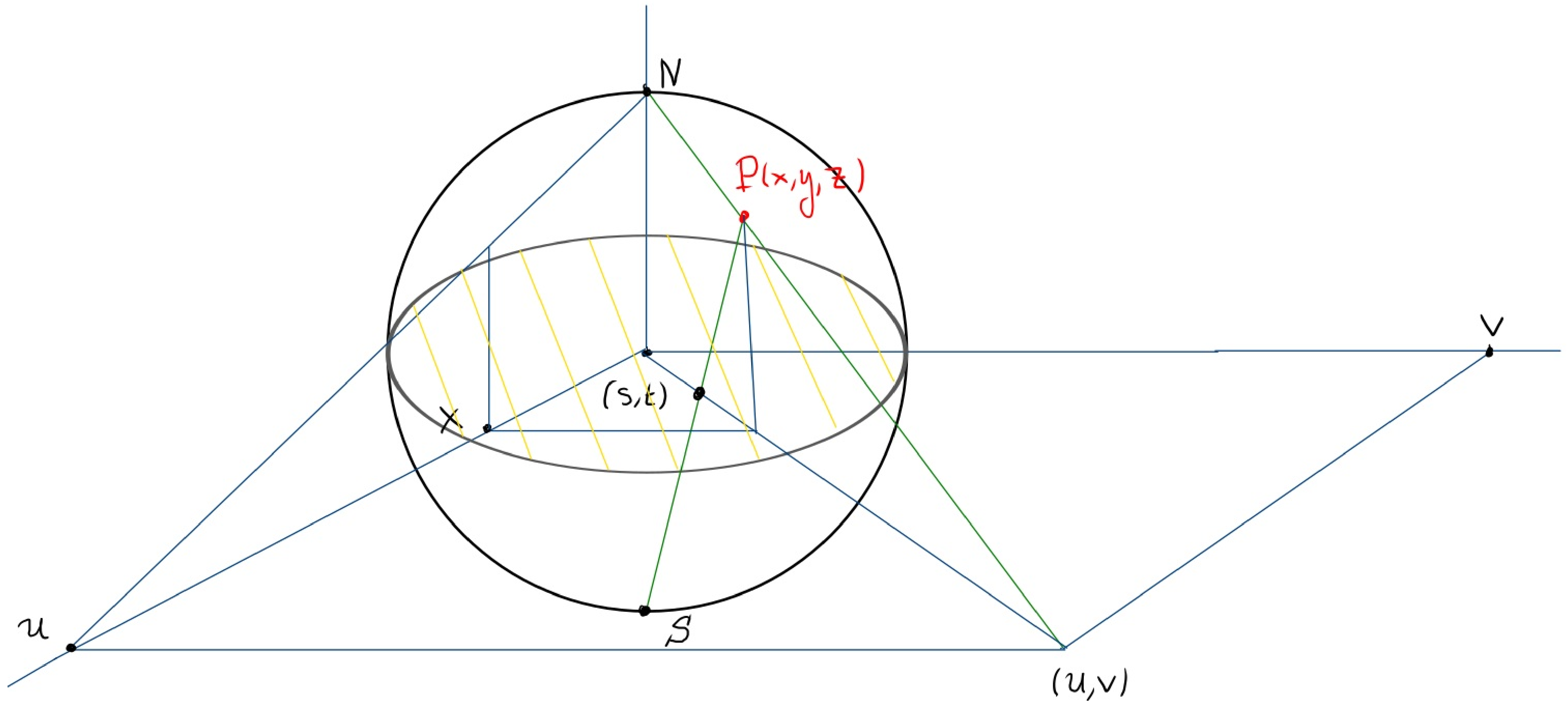
$\chi_N \circ \chi_\theta^{-1} : (\theta,\varphi) \mapsto (u,v)$

$$u = \frac{\sin\theta\cos\varphi}{1-\cos\theta} \qquad v = \frac{\sin\theta\sin\varphi}{1-\cos\theta}$$





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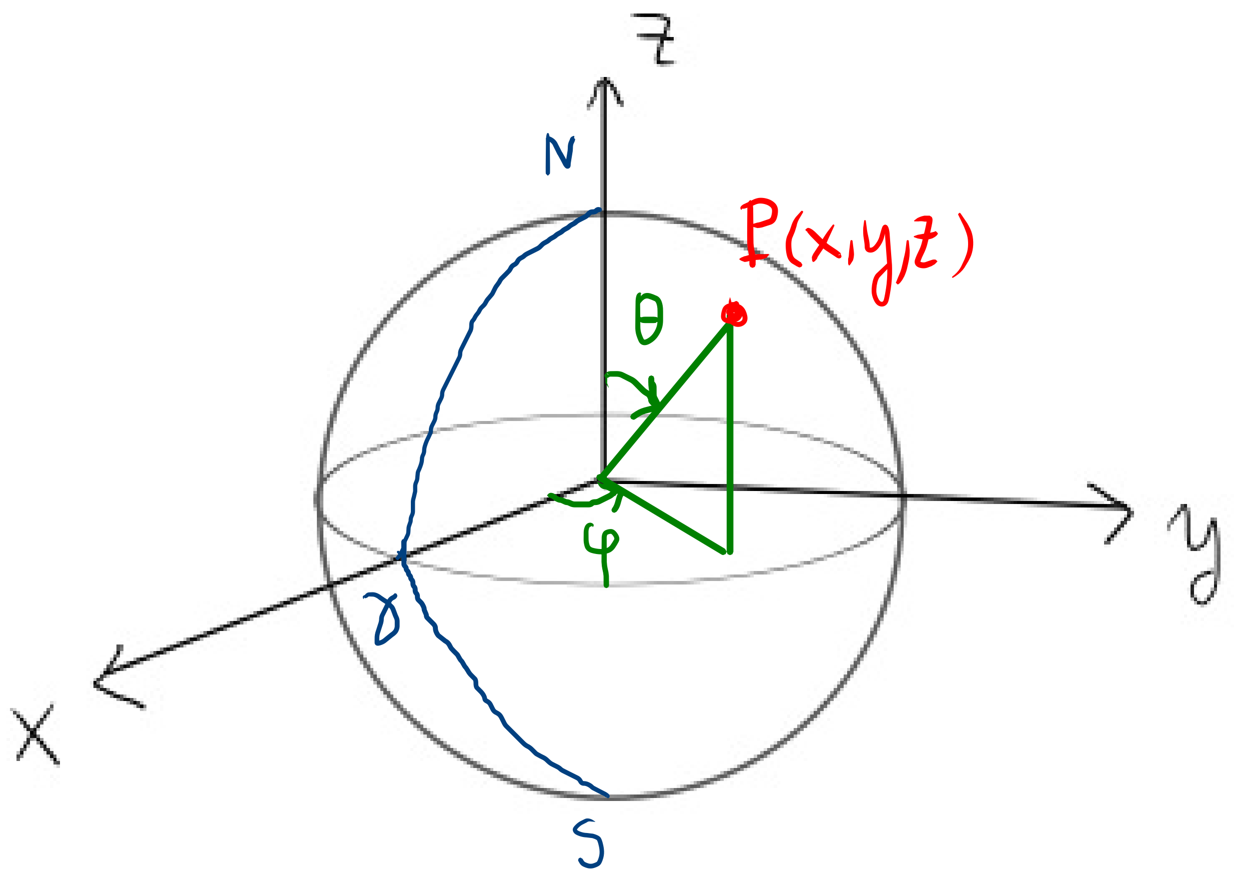
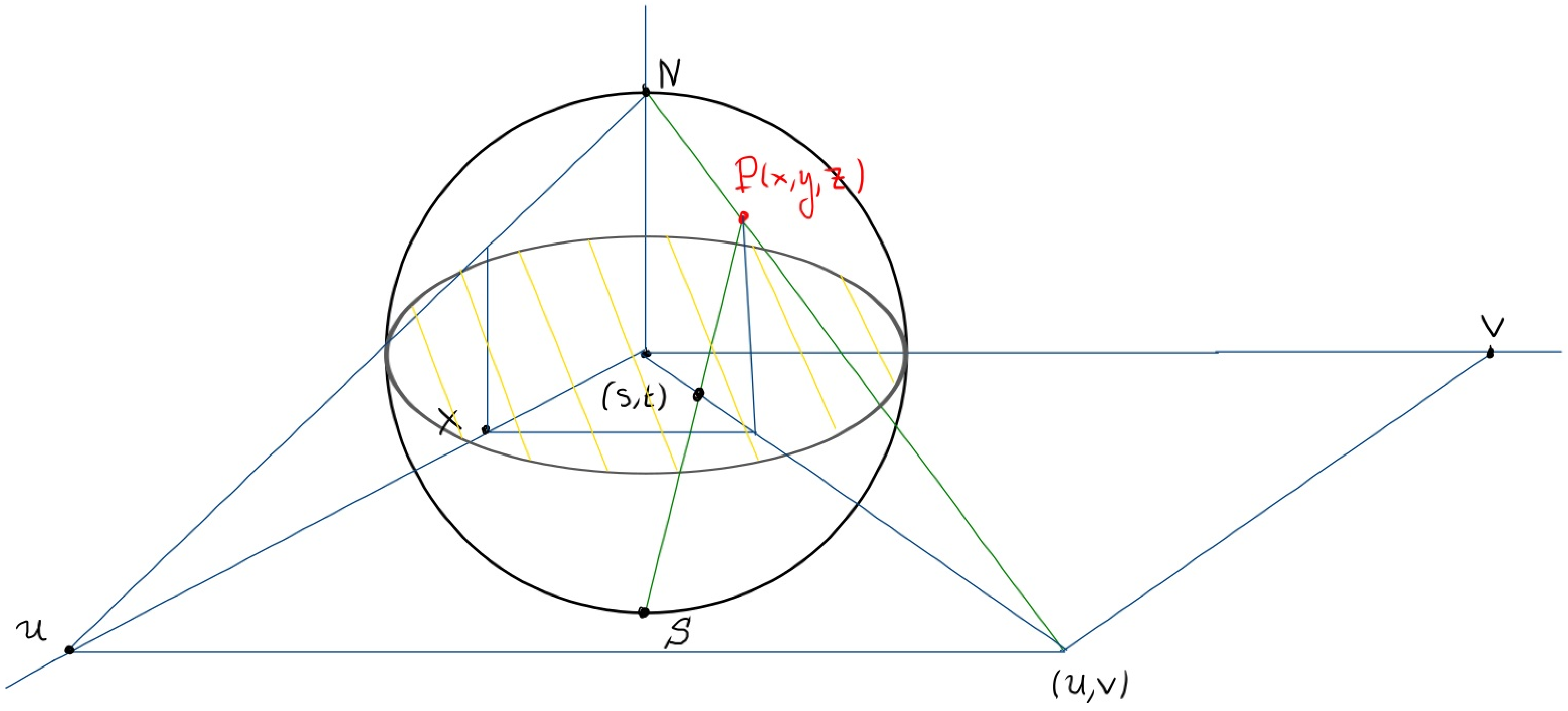
(U_S, χ_S): $U_S = S^2 - \{S\}$

$\chi_S : (x,y,z) \mapsto (s,t)$ $\quad -\infty<s<+\infty$, $-\infty<t<+\infty$

$$s = \frac{x}{1+z} \qquad t = \frac{y}{1+z}$$





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$$\chi_S \circ \chi_\theta^{-1} : (\theta,\varphi) \mapsto (s,t)$$

$$s = \frac{\sin\theta \cos\varphi}{1+\cos\theta} \qquad t = \frac{\sin\theta \sin\varphi}{1+\cos\theta}$$

* $\{(\mathcal{U}_N, \chi_N), (\mathcal{U}_S, \chi_S)\}$ an atlas on S^2

Minimal: Smaller than hemispheres, but cannot find an atlas with one chart

$(\mathcal{U}_\theta, \chi_\theta)$: $\mathcal{U}_\theta = S^2 - \{\gamma\}$	$(\mathcal{U}_N, \chi_N)$: $\mathcal{U}_N = S^2 \setminus \{N\}$	$(\mathcal{U}_S, \chi_S)$: $\mathcal{U}_S = S^2 - \{S\}$
$\chi_\theta : (x, y, z) \mapsto (\theta, \varphi)$ $\quad 0 < \theta < \pi$, $0 < \varphi < 2\pi$	$\chi_N : (x, y, z) \mapsto (u, v)$ $\quad -\infty < u < +\infty$, $-\infty < v < +\infty$	$\chi_S : (x, y, z) \mapsto (s, t)$ $\quad -\infty < s < +\infty$, $-\infty < t < +\infty$
$x = \sin\theta \cos\phi$	$u = \dfrac{x}{1-z}$ $\quad v = \dfrac{y}{1-z}$	$s = \dfrac{x}{1+z}$ $\quad t = \dfrac{y}{1+z}$
$y = \sin\theta \sin\phi$	$\chi_N \circ \chi_\theta^{-1} : (\theta, \varphi) \mapsto (u, v)$	$\chi_S \circ \chi_\theta^{-1} : (\theta, \phi) \mapsto (s, t)$
$z = \cos\theta$	$u = \dfrac{\sin\theta \cos\phi}{1 - \cos\theta}$ $\quad v = \dfrac{\sin\theta \sin\phi}{1 - \cos\theta}$	$s = \dfrac{\sin\theta \cos\phi}{1 + \cos\theta}$ $\quad t = \dfrac{\sin\theta \sin\phi}{1 + \cos\theta}$

* $\{(U_N,\chi_N),(U_S,\chi_S)\}$ an atlas on S^2
Minimal: Smaller than hemispheres, but cannot find an atlas with one chart

Notice that $$us+vt = \frac{x^2}{1-z^2}+\frac{y^2}{1-z^2} = \frac{x^2+y^2}{x^2+y^2} = 1$$ $$ut = vs = \frac{xy}{1-z^2}$$ $\Rightarrow$ $$\left.\begin{array}{l} us+vt=1 \\ vs-ut=0 \end{array}\right\} \Rightarrow \quad s=\frac{u}{u^2+v^2}, \quad t=\frac{v}{u^2+v^2}$$

(U_θ,χ_θ): $U_\theta = S^2-\{\gamma\}$

$\chi_\theta : (x,y,z)\mapsto(\theta,\varphi)$ $\quad 0<\theta<\pi$, $0<\varphi<2\pi$

$x=\sin\theta\cos\phi$

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* $\{(\mathcal{U}_N, \chi_N), (\mathcal{U}_S, \chi_S)\}$ an atlas on S^2
  Minimal: Smaller than hemispheres, but cannot find an atlas with one chart

* $\chi_S \circ \chi_N^{-1} : (u,v) \mapsto (s,t)$ $\qquad s = \dfrac{u}{u^2+v^2} \qquad t = \dfrac{v}{u^2+v^2}$

  differentiable

$(\mathcal{U}_\theta, \chi_\theta)$: $\quad \mathcal{U}_\theta = S^2 - \{\gamma\}$

$\chi_\theta : (x,y,z) \mapsto (\theta,\varphi)$ $\qquad 0<\theta<\pi$, $\; 0<\varphi<2\pi$

$x = \sin\theta\cos\phi$

$y = \sin\theta\sin\phi$

$z = \cos\theta$

$(\mathcal{U}_N, \chi_N)$: $\quad \mathcal{U}_N = S^2 \setminus \{N\}$

$\chi_N : (x,y,z) \mapsto (u,v)$ $\qquad -\infty<u<+\infty$, $\; -\infty<v<+\infty$

$u = \dfrac{x}{1-z} \qquad v = \dfrac{y}{1-z}$

$\chi_N \circ \chi_\theta^{-1} : (\theta,\varphi) \mapsto (u,v)$

$u = \dfrac{\sin\theta\cos\phi}{1-\cos\theta} \qquad v = \dfrac{\sin\theta\sin\phi}{1-\cos\theta}$

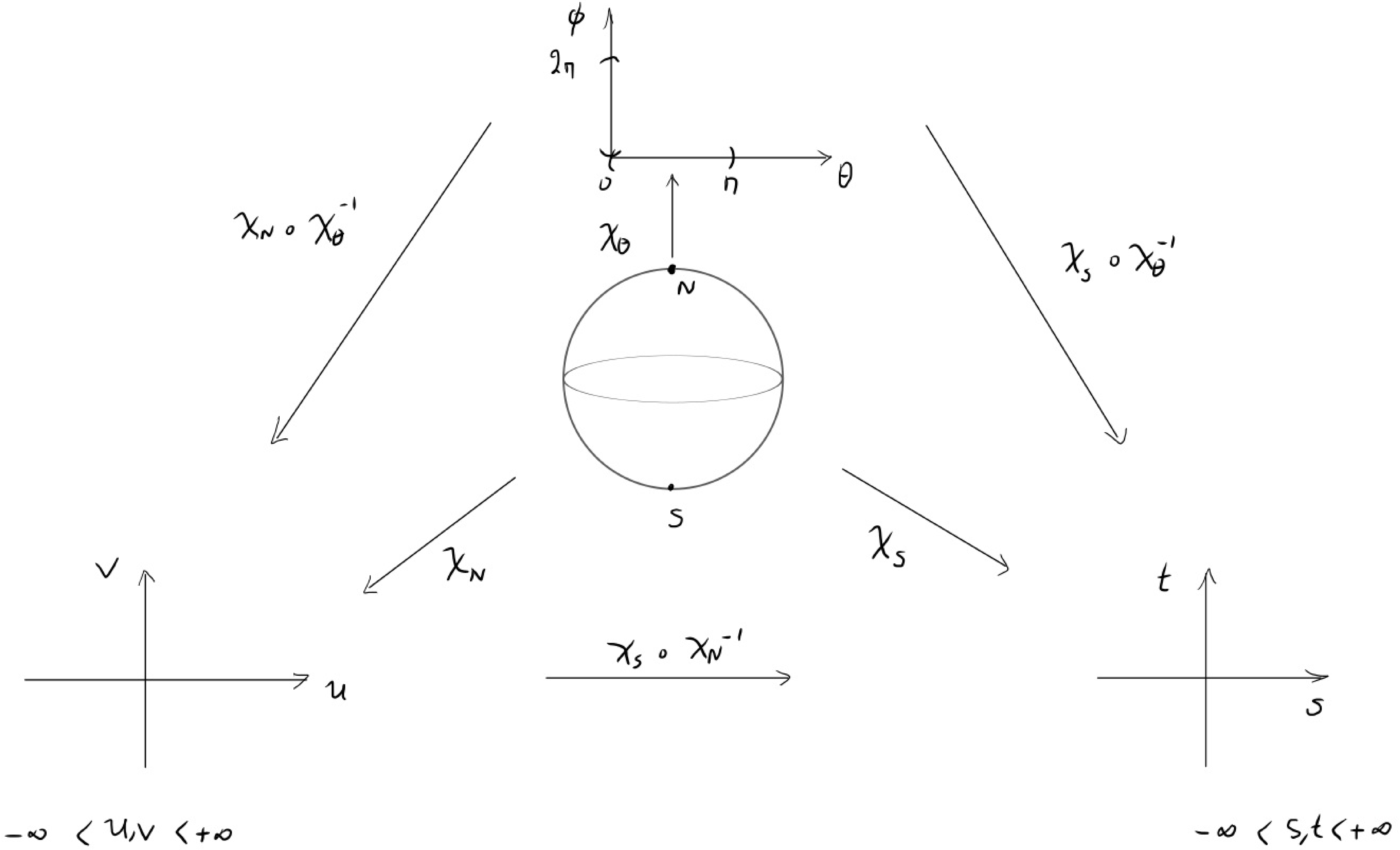
$(\mathcal{U}_S, \chi_S)$: $\quad \mathcal{U}_S = S^2 - \{S\}$

$\chi_S : (x,y,z) \mapsto (s,t)$ $\qquad -\infty<s<+\infty$, $\; -\infty<t<+\infty$

$s = \dfrac{x}{1+z} \qquad t = \dfrac{y}{1+z}$

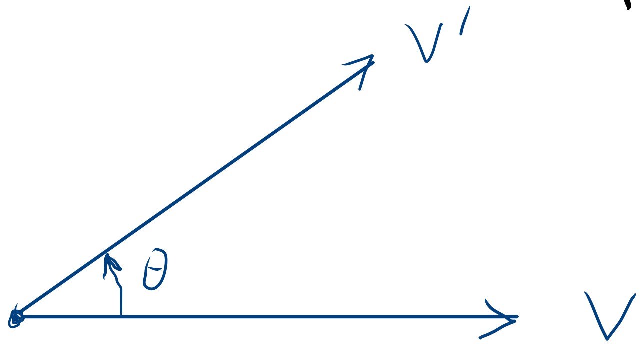
$\chi_S \circ \chi_\theta^{-1} : (\theta,\varphi) \mapsto (s,t)$

$s = \dfrac{\sin\theta\cos\phi}{1+\cos\theta} \qquad t = \dfrac{\sin\theta\sin\phi}{1+\cos\theta}$



ϕ
2π
0
π
θ
χ_θ
N
S
$\chi_N \circ \chi_\theta^{-1}$
$\chi_S \circ \chi_\theta^{-1}$
χ_N
χ_S
v
u
$\chi_S \circ \chi_N^{-1}$
t
s
$-\infty < u, v < +\infty$
$-\infty < s, t < +\infty$

Rotations on the plane:



$$|V| = |V'|$$

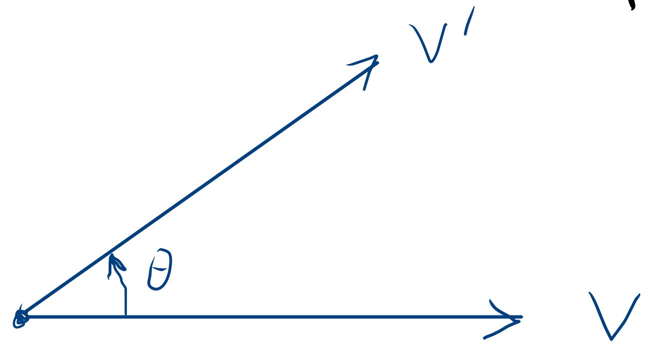
$$\begin{pmatrix} V^{x'} \\ V^{y'} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} V^x \\ V^y \end{pmatrix}$$

$$V' = R(\theta) \cdot V$$

$$(U, \chi): \quad U = \{ R(\theta) \mid 0 < \theta < 2\pi \}$$

$$\chi : R(\theta) \mapsto \theta$$

Rotations on the plane:



$$|V| = |V'|$$

$$\begin{pmatrix} V^{x'} \\ V^{y'} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} V^x \\ V^y \end{pmatrix}$$

$$V' = R(\theta) \cdot V$$

$$(U, \chi): \quad U = \{ R(\theta) \mid 0 < \theta < 2\pi \}$$

$$\chi : R(\theta) \mapsto \theta$$

Rotations in space:

$$R_x(\theta_x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{pmatrix}, \quad R_y(\theta_y) = \begin{pmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{pmatrix}$$

$$R_z(\theta_z) = \begin{pmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R(\theta_x, \theta_y, \theta_z) = R_z(\theta_z) \cdot R_y(\theta_y) \cdot R_x(\theta_x)$$

$$\begin{pmatrix} V^{x'} \\ V^{y'} \\ V^{z'} \end{pmatrix} = R(\theta_x, \theta_y, \theta_z) \cdot \begin{pmatrix} V^x \\ V^y \\ V^z \end{pmatrix}$$

Cylinder : $S^1 \times \mathbb{R}$

(U, χ): $U = S^1 \times \mathbb{R} \setminus \{x = R \text{ line}\}$

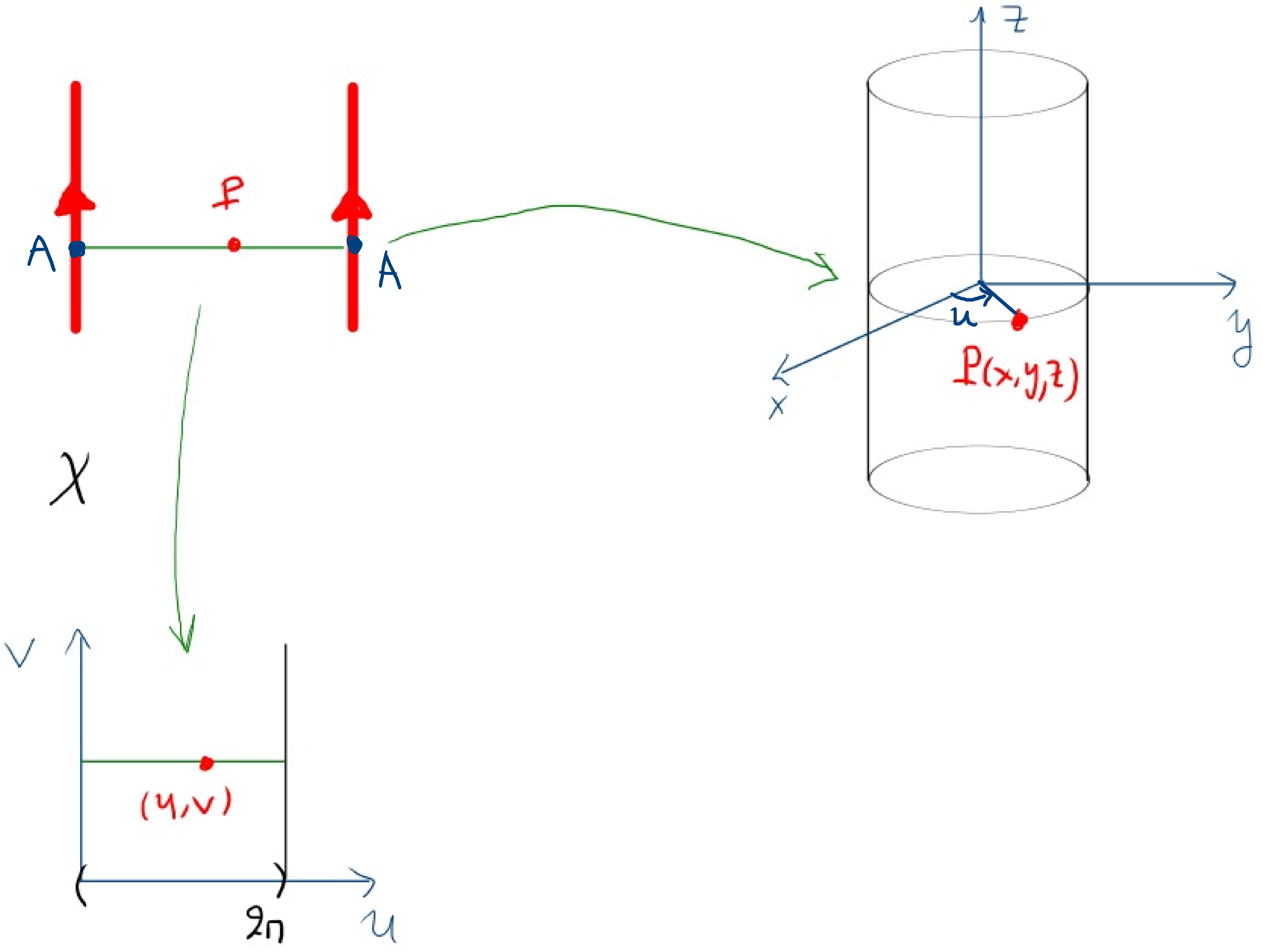
$\chi : (x, y, z) \mapsto (u, v)$ $\quad 0 < u < 2\pi$

$\quad -\infty < v < +\infty$

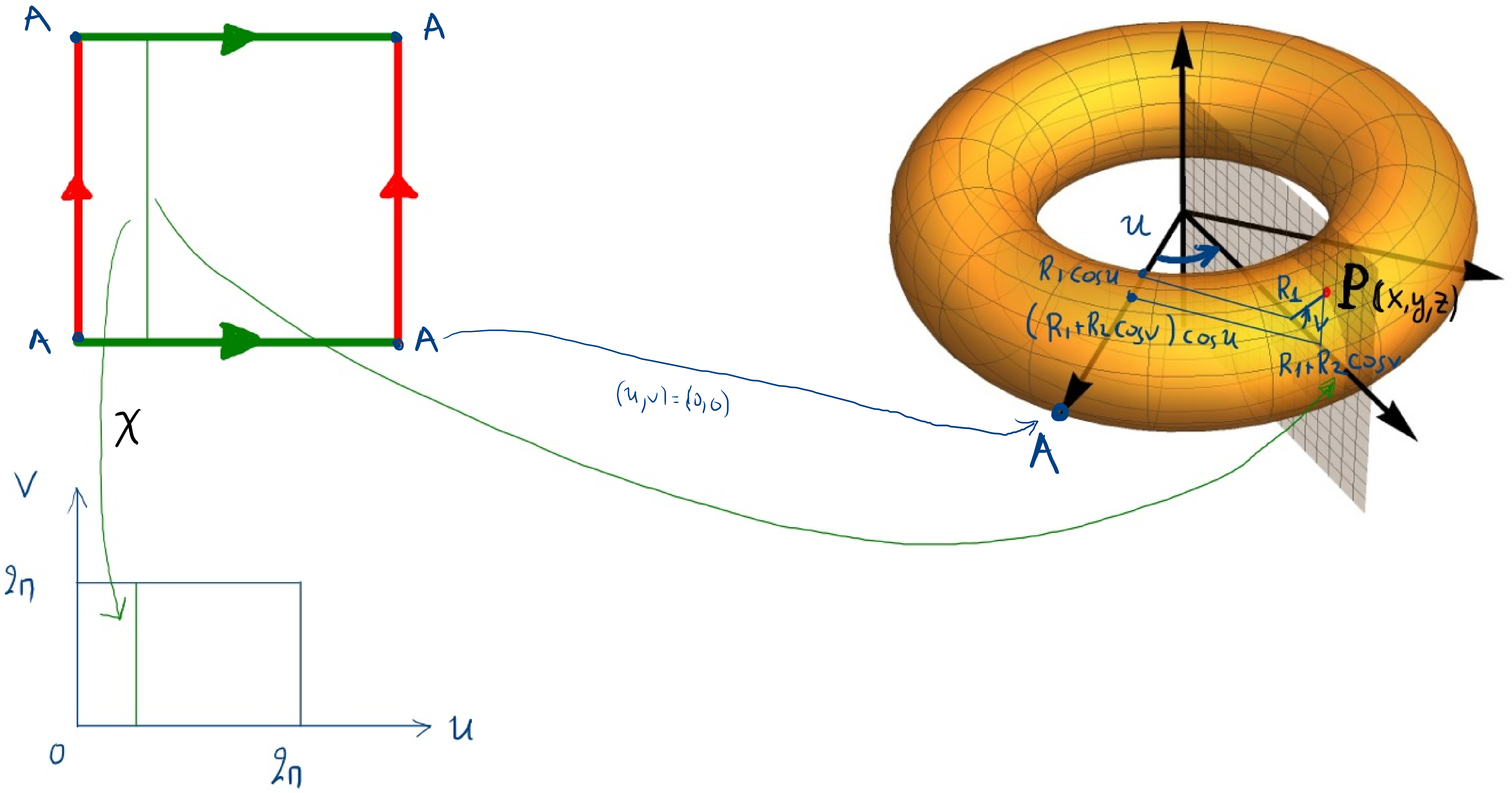
where $x = R\cos u$

$y = R\sin u$

$z = v$



wikipedia.org/wiki/Torus

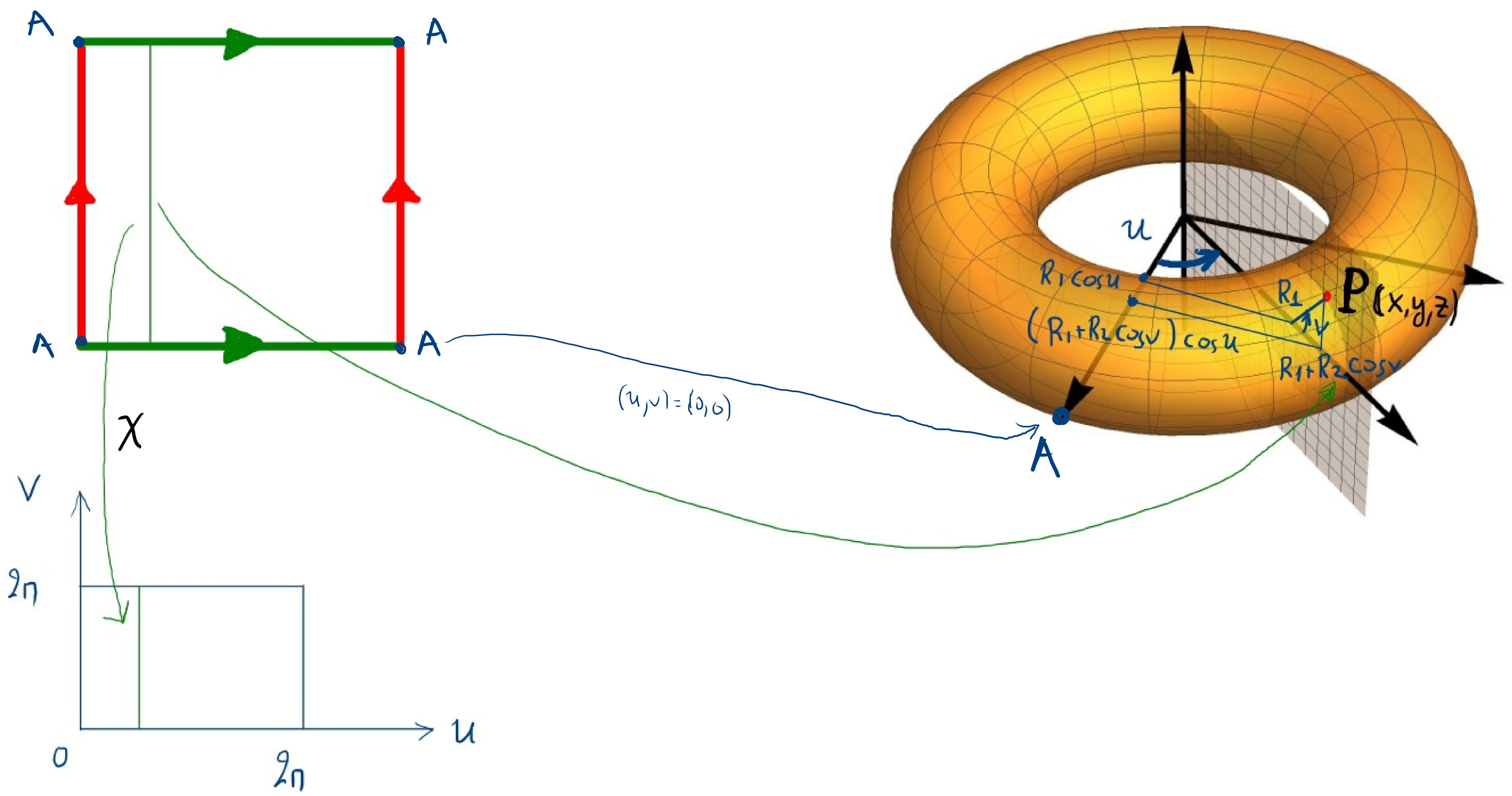


Torus $T^2 = S^1 \times S^1$

$$x = (R_1 + R_2 \cos v)\cos u$$

$$y = (R_1 + R_2 \cos v)\sin u$$

$$z = R_2 \sin v$$



wikipedia.org/wiki/Torus

Torus $T^2 = S^1 \times S^1$

(U,χ): $\quad U = T^2 \setminus (C_1 \cup C_2)$

$C_1 = \{$the "red" circle$\}$ $\quad C_2 = \{$the "green" circle$\}$

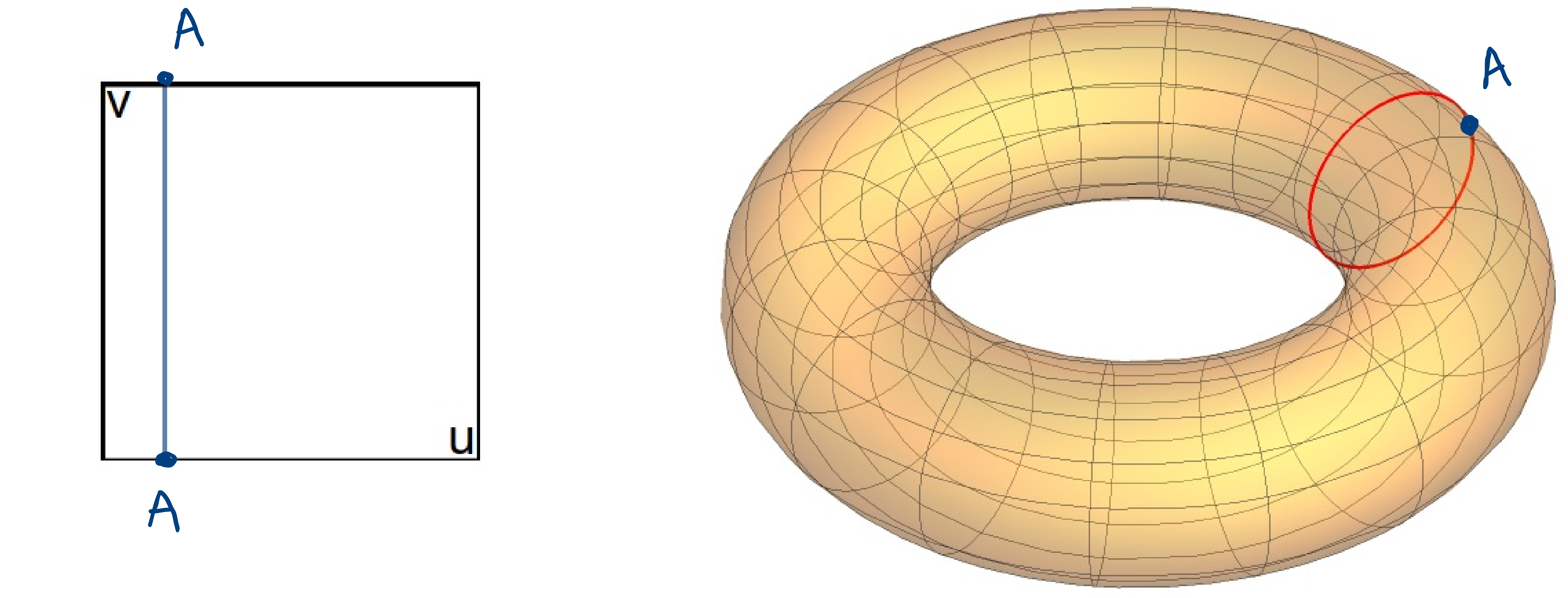
$\chi : (x,y,z) \mapsto (u,v)$ where

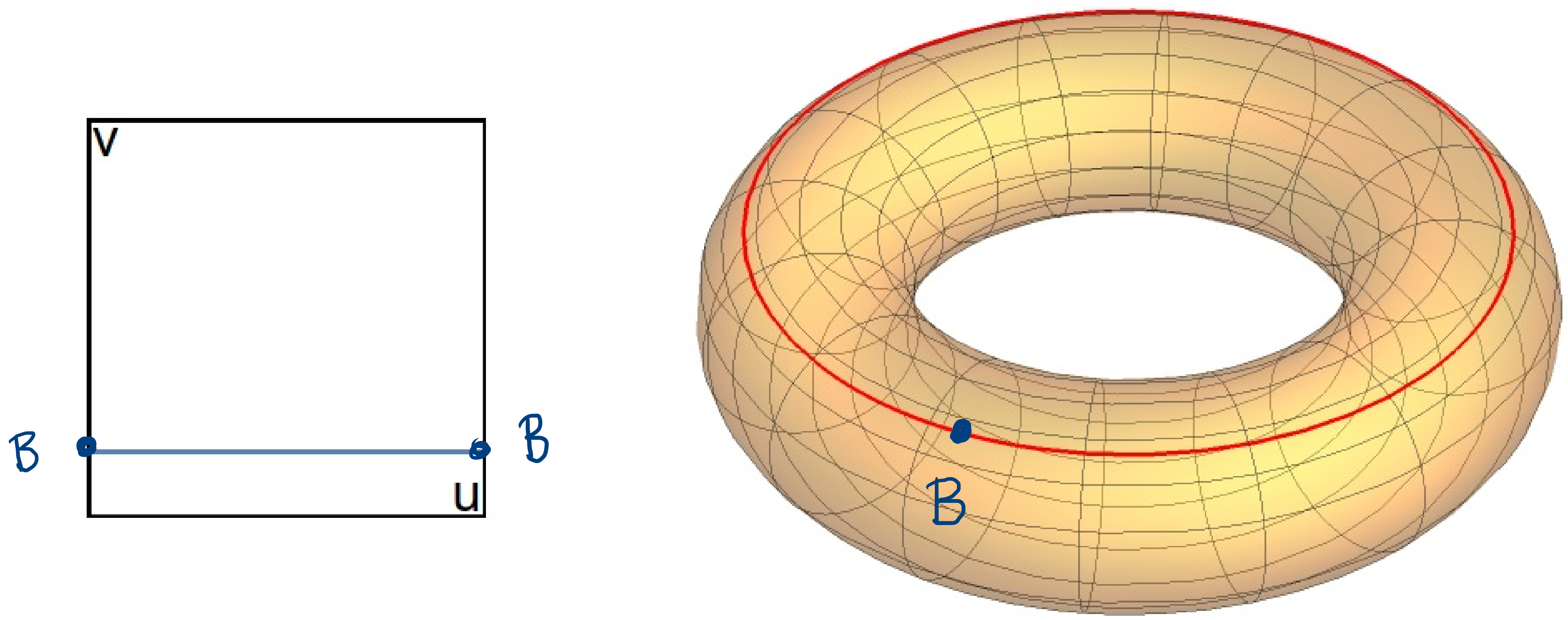
$0<u<2\pi \qquad 0<v<2\pi$

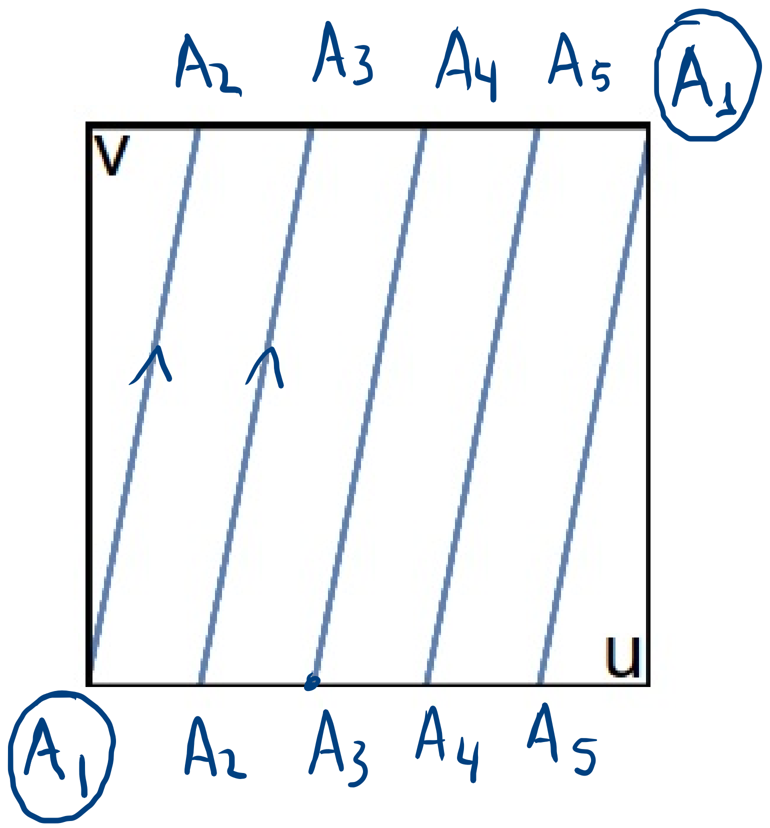
$$x = (R_1 + R_2 \cos v)\cos u$$

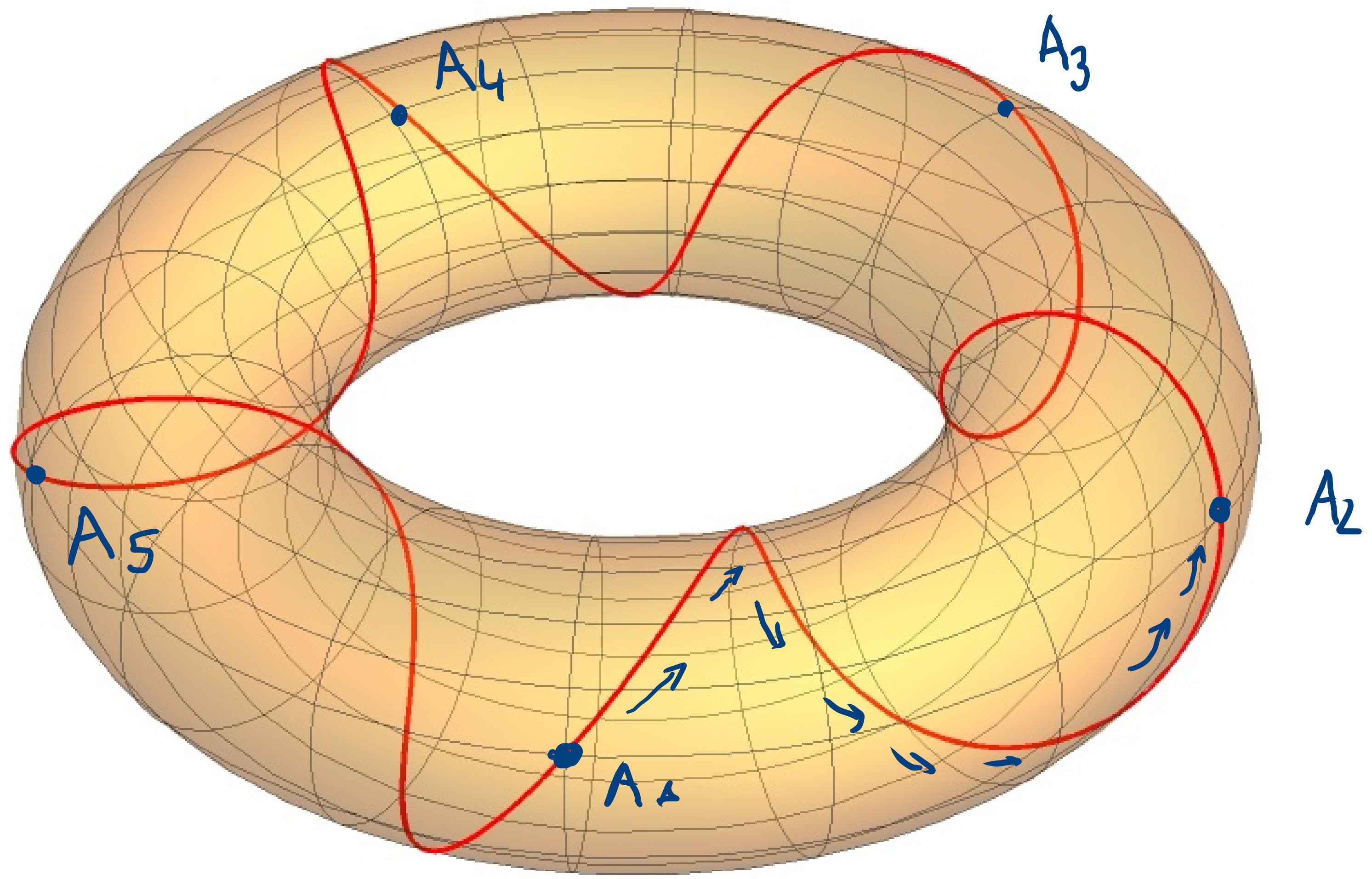
$$y = (R_1 + R_2 \cos v)\sin u$$

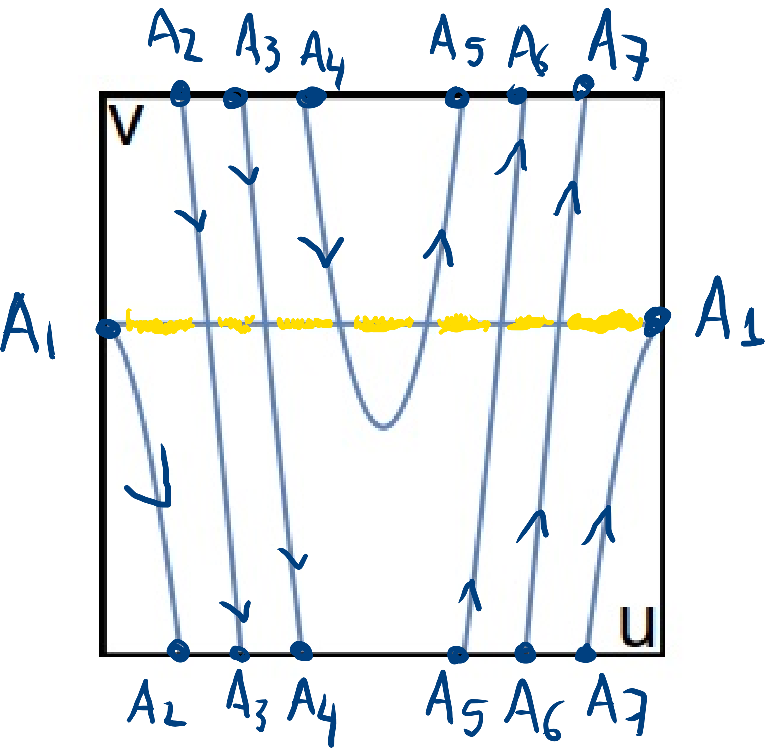
$$z = R_2 \sin v$$

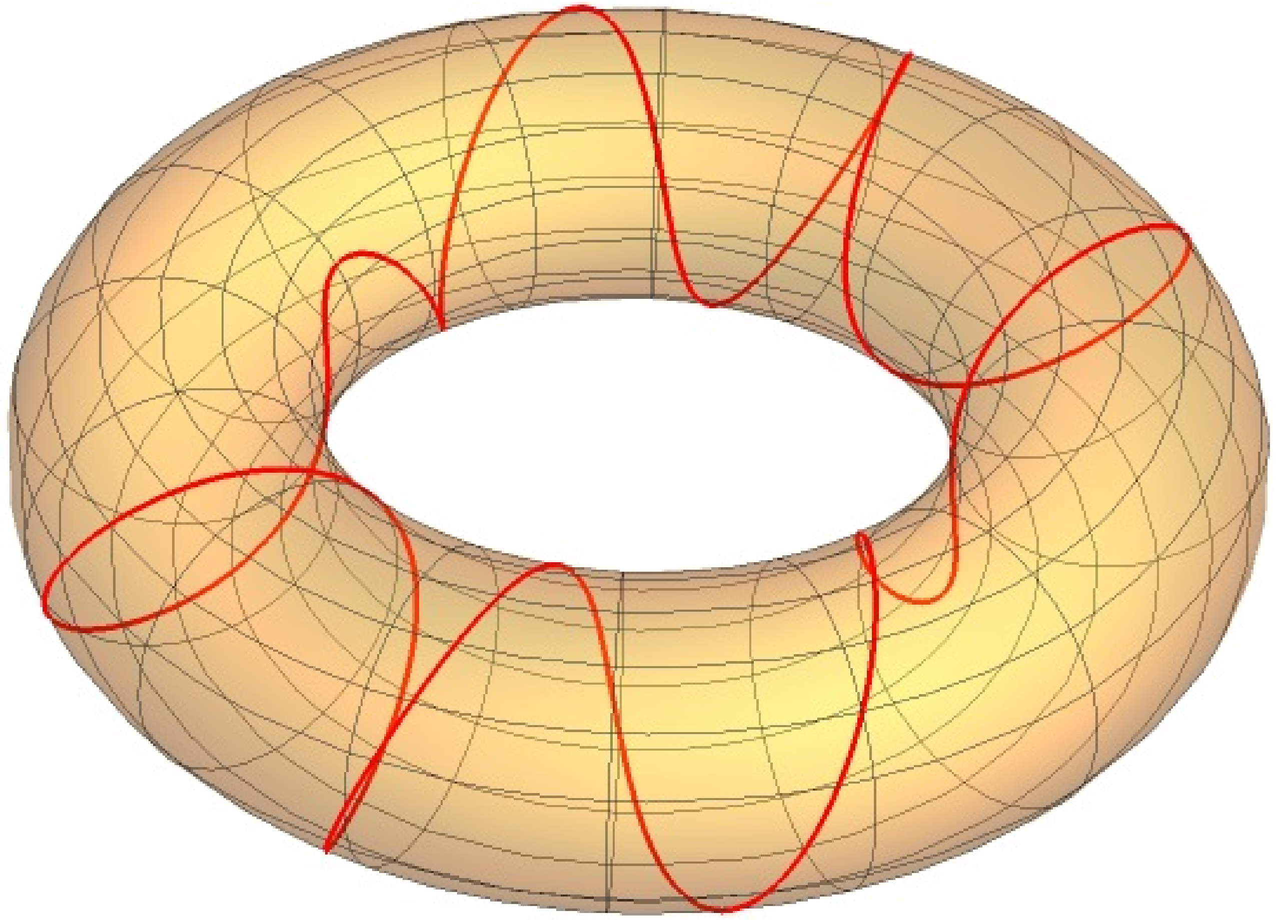

A
v
u
A
A

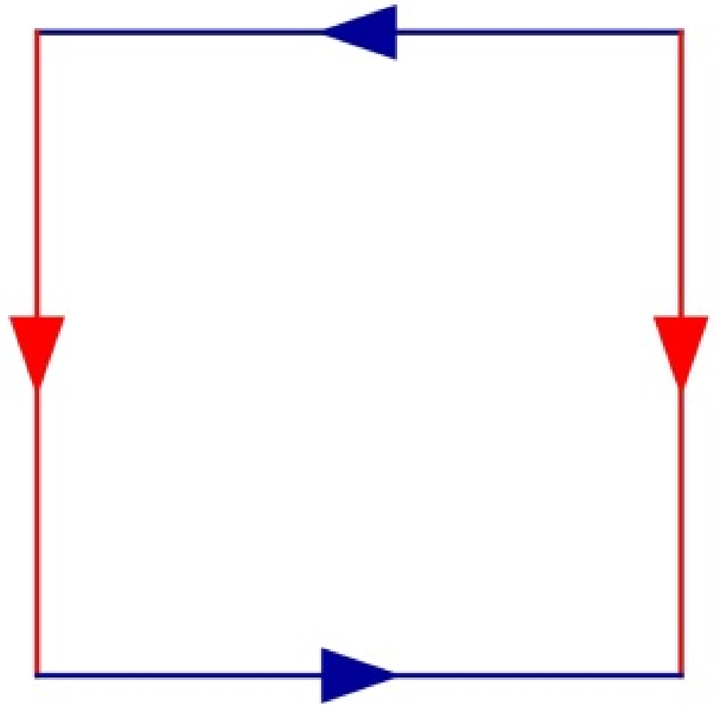

v
B
B
u
B


A_2 A_3 A_4 A_5 A_1
v
u
A_1 A_2 A_3 A_4 A_5

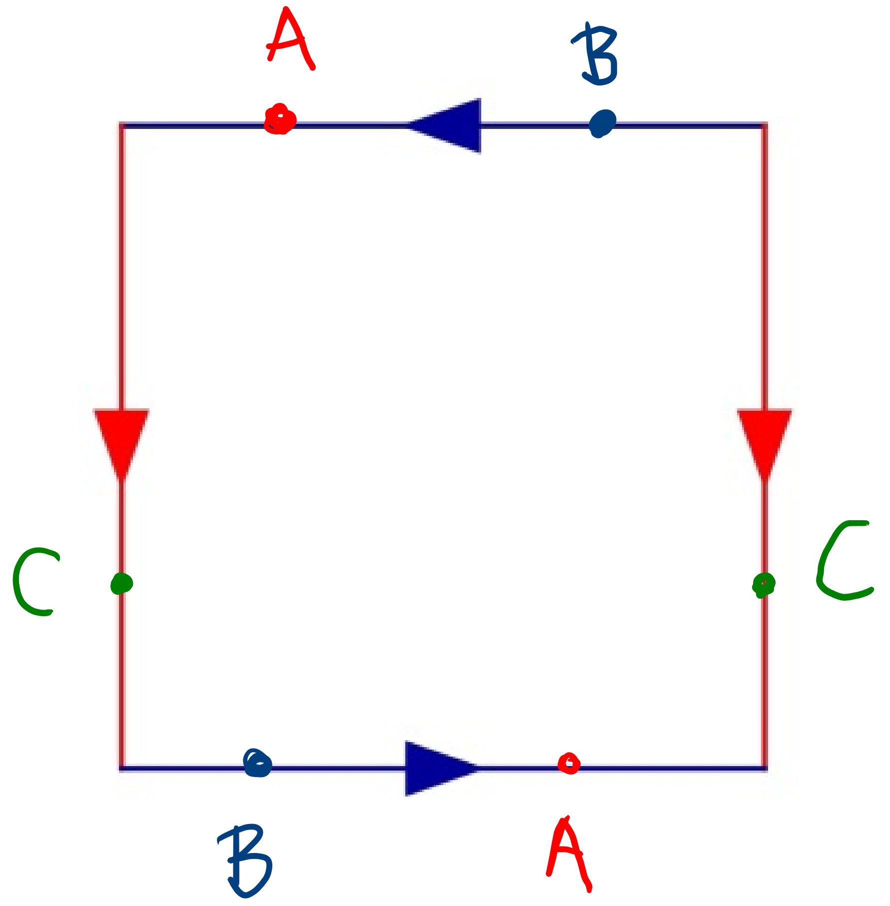

A_4
A_3
A_5
A_2
A_1

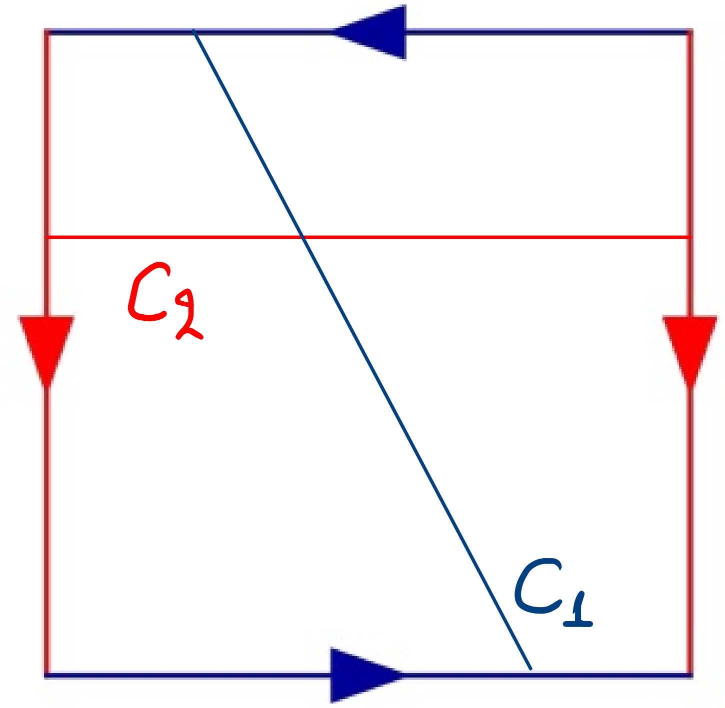

A_2 A_3 A_4 A_5 A_6 A_7
v
A_1
A_1
u
A_2 A_3 A_4 A_5 A_6 A_7



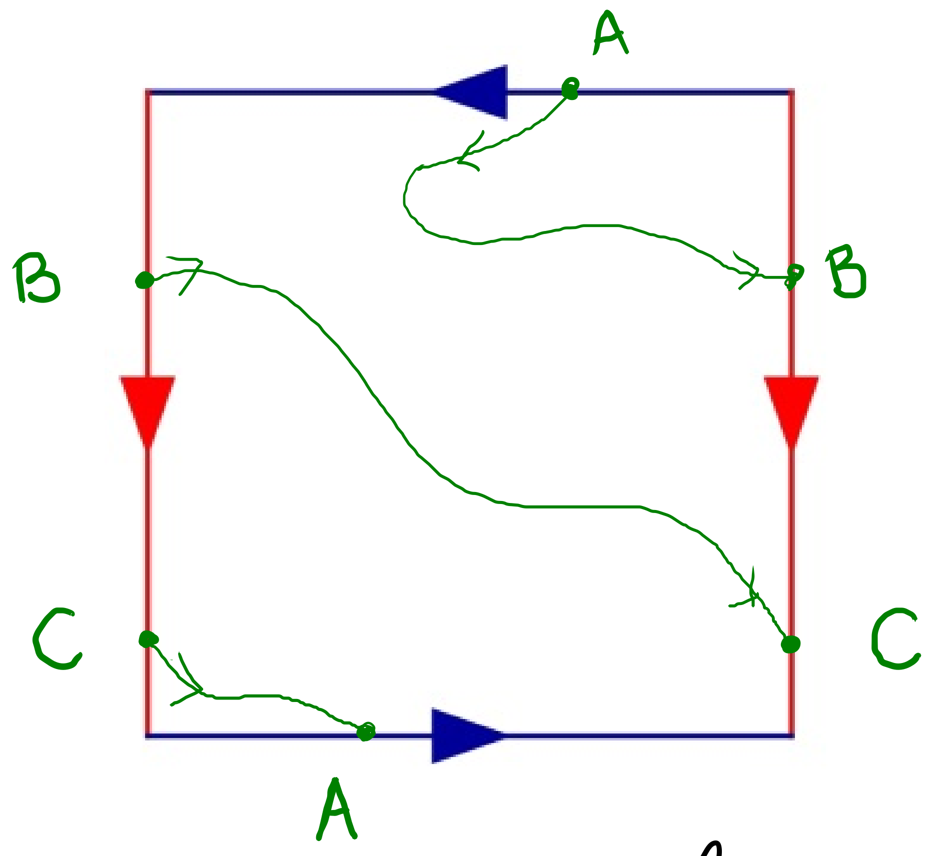


<u>Klein Bottle</u>: Non-orientable, not embeddable in $\mathbb{R}^3$ (ok in $\mathbb{R}^4$)



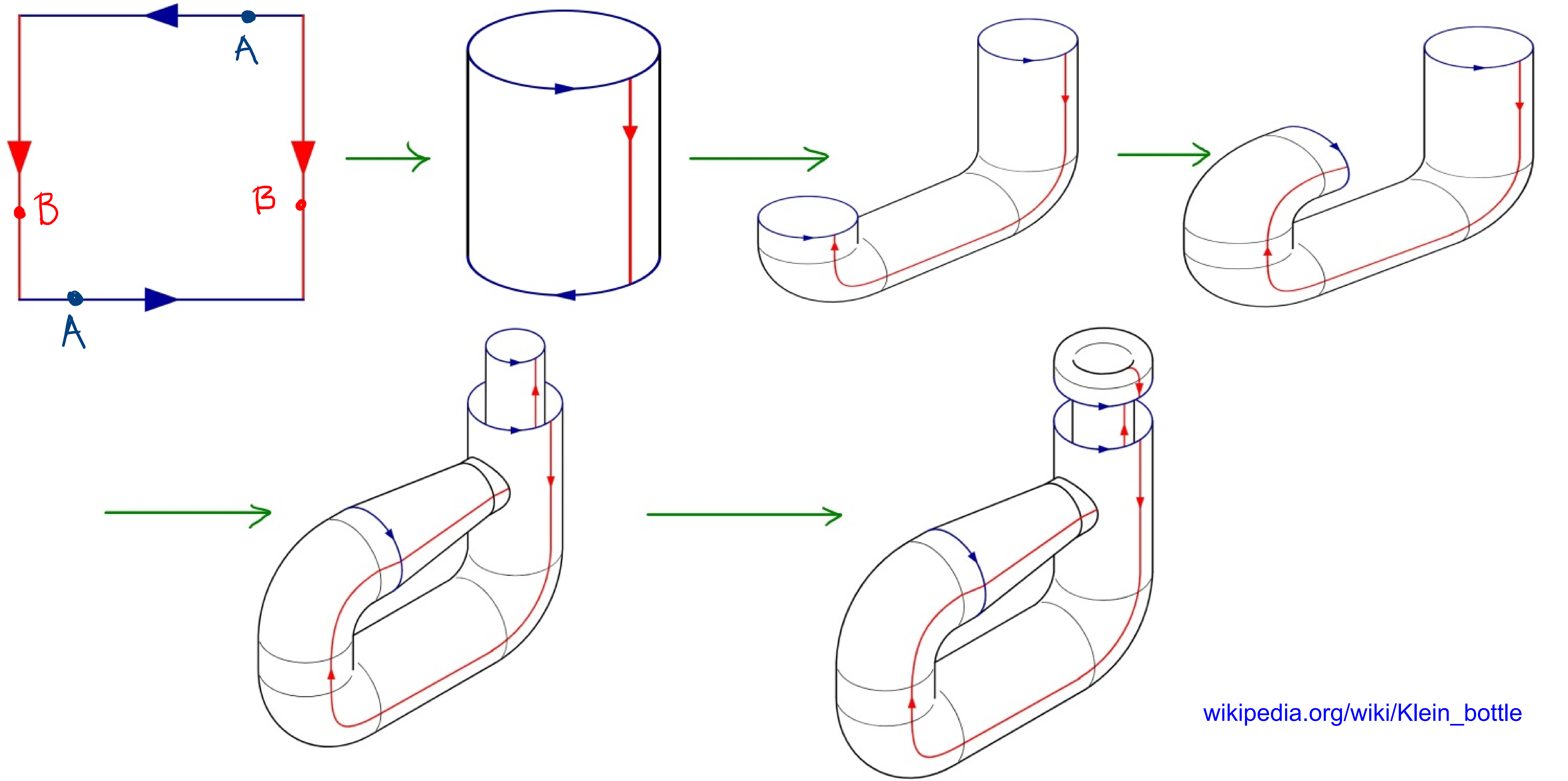


C_1 and C_2 are circles



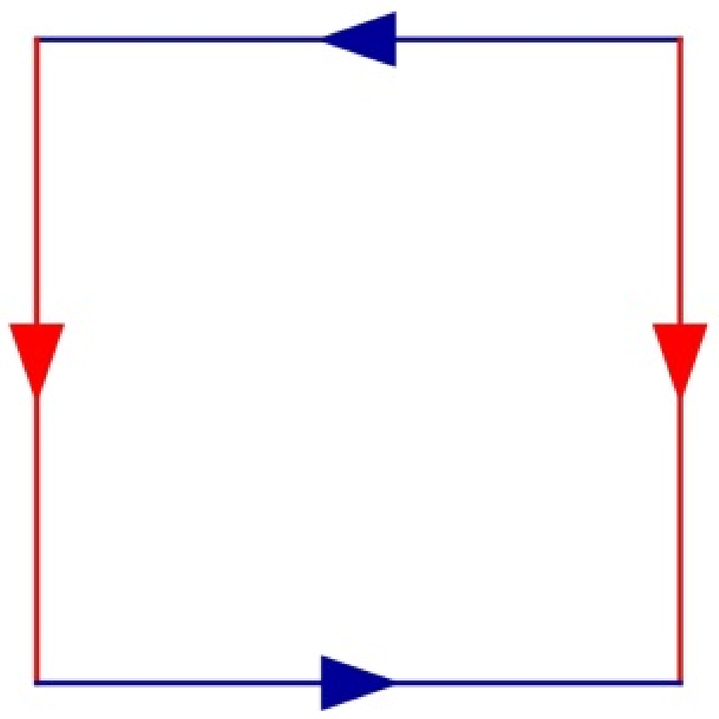
- a closed loop
- notice the direction of velocities at A, C

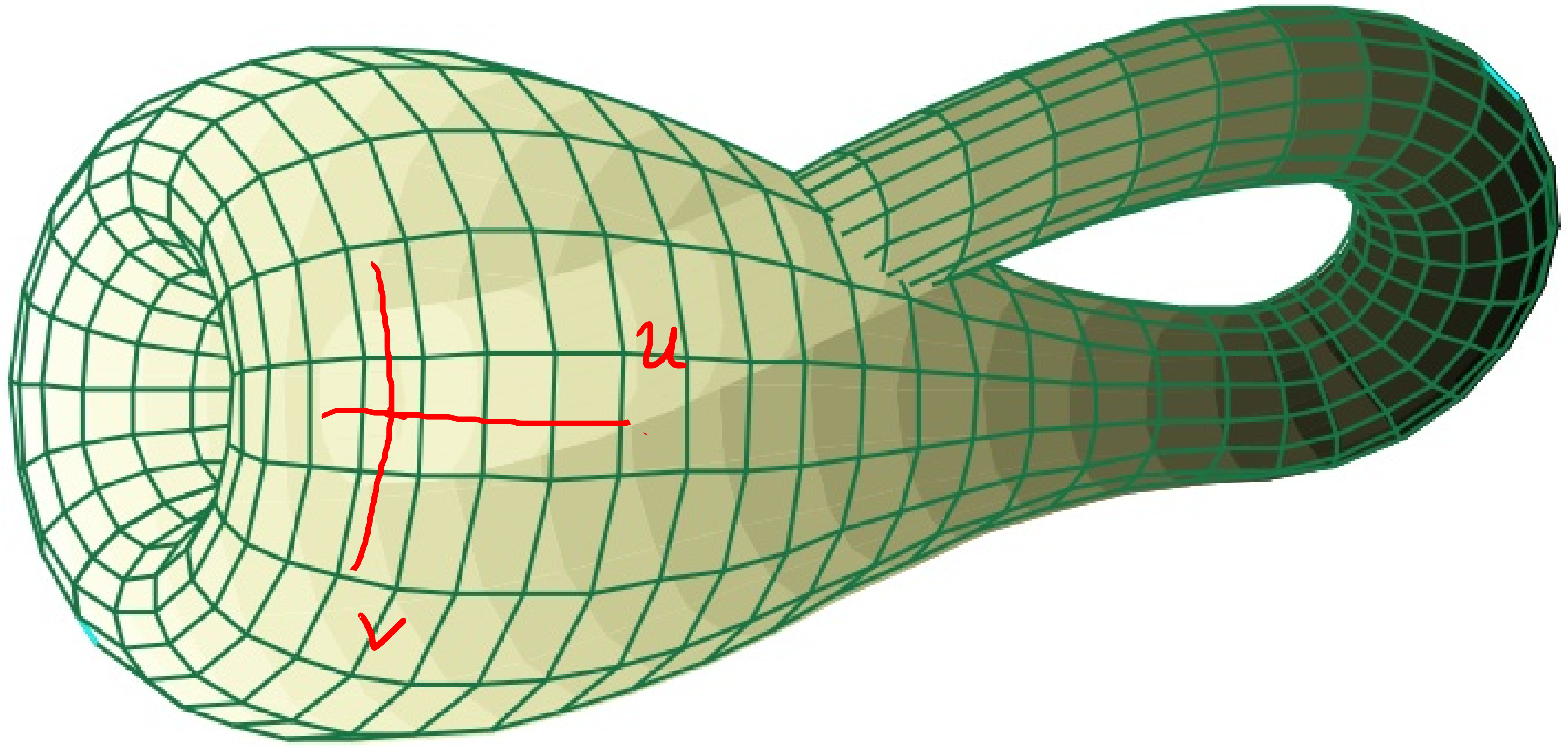
<u>**Klein Bottle**</u>: Non-orientable, not embeddable in $\mathbb{R}^3$ (ok in $\mathbb{R}^4$)



Klein Bottle: Non-orientable, not embeddable in $\mathbb{R}^3$ (ok in $\mathbb{R}^4$)

wikipedia.org/wiki/Klein_bottle





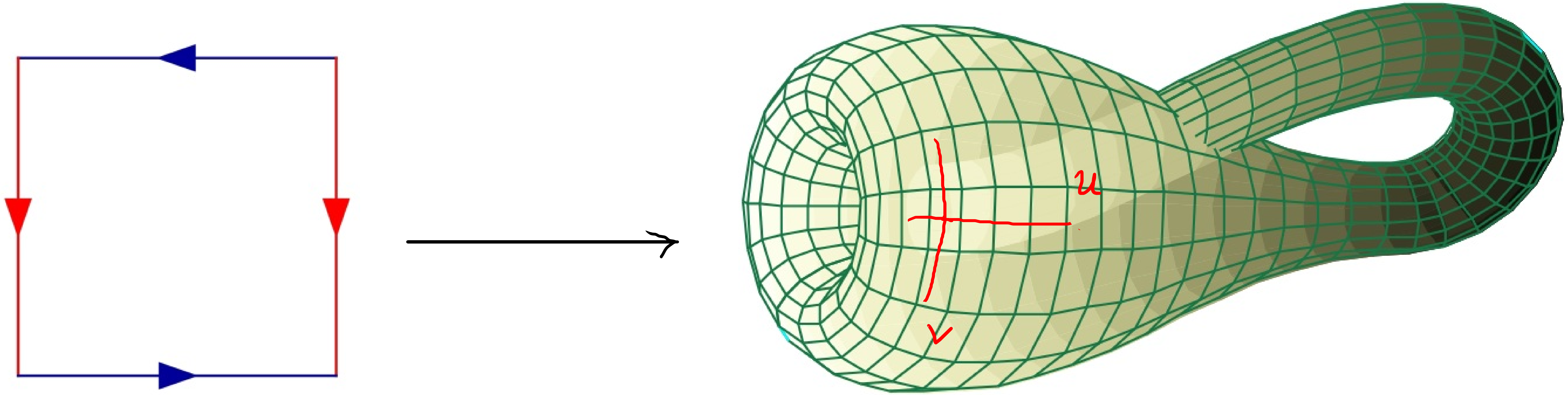
$$x(u,v) = -\frac{2}{15}\cos u\,\big(3\cos v - 30\sin u + 90\cos^4 u\sin u - 60\cos^6 u\sin u + 5\cos u\cos v\sin u\big)$$

$$y(u,v) = -\frac{1}{15}\sin u\,\big(3\cos v - 3\cos^2 u\cos v - 48\cos^4 u\cos v + 48\cos^6 u\cos v - 60\sin u + 5\cos u\cos v\sin u - 5\cos^3 u\cos v\sin u - 80\cos^5 u\cos v\sin u + 80\cos^7 u\cos v\sin u\big)$$

$$z(u,v) = \frac{2}{15}\left(3 + 5\cos u\sin u\right)\sin v$$

$0 < u < \pi \qquad 0 < v < 2\pi$

wikipedia.org/wiki/Klein_bottle



$$x(u,v) = -\frac{2}{15}\cos u\left(3\cos v - 30\sin u + 90\cos^4 u\sin u - 60\cos^6 u\sin u + 5\cos u\cos v\sin u\right)$$

$$y(u,v) = -\frac{1}{15}\sin u\left(3\cos v - 3\cos^2 u\cos v - 48\cos^4 u\cos v + 48\cos^6 u\cos v - 60\sin u + 5\cos u\cos v\sin u - 5\cos^3 u\cos v\sin u - 80\cos^5 u\cos v\sin u + 80\cos^7 u\cos v\sin u\right)$$

$$z(u,v) = \frac{2}{15}\left(3 + 5\cos u\sin u\right)\sin v$$

$0 < u < \pi \qquad 0 < v < 2\pi$

4-D non-intersecting [edit] Embedding in $\mathbb{R}^4$

A non-intersecting 4-D parametrization can be modeled after that of the flat torus:

$$x = R\left(\cos\frac{\theta}{2}\cos v - \sin\frac{\theta}{2}\sin 2v\right)$$
$$y = R\left(\sin\frac{\theta}{2}\cos v + \cos\frac{\theta}{2}\sin 2v\right)$$
$$z = P\cos\theta\left(1 + \epsilon\sin v\right)$$
$$w = P\sin\theta\left(1 + \epsilon\sin v\right)$$