Plan:

- Prove $\mathcal{L}_V W = [V, W]$

  → review vectors + vector fields, $[V, W]$

- Prove $\mathcal{L}_V f = V(f)$

- Compute $\mathcal{L}_V \omega$

- Geometric interpretation of $\mathcal{L}_V W$ and $[V, W]$

Review of vectors + vector fields

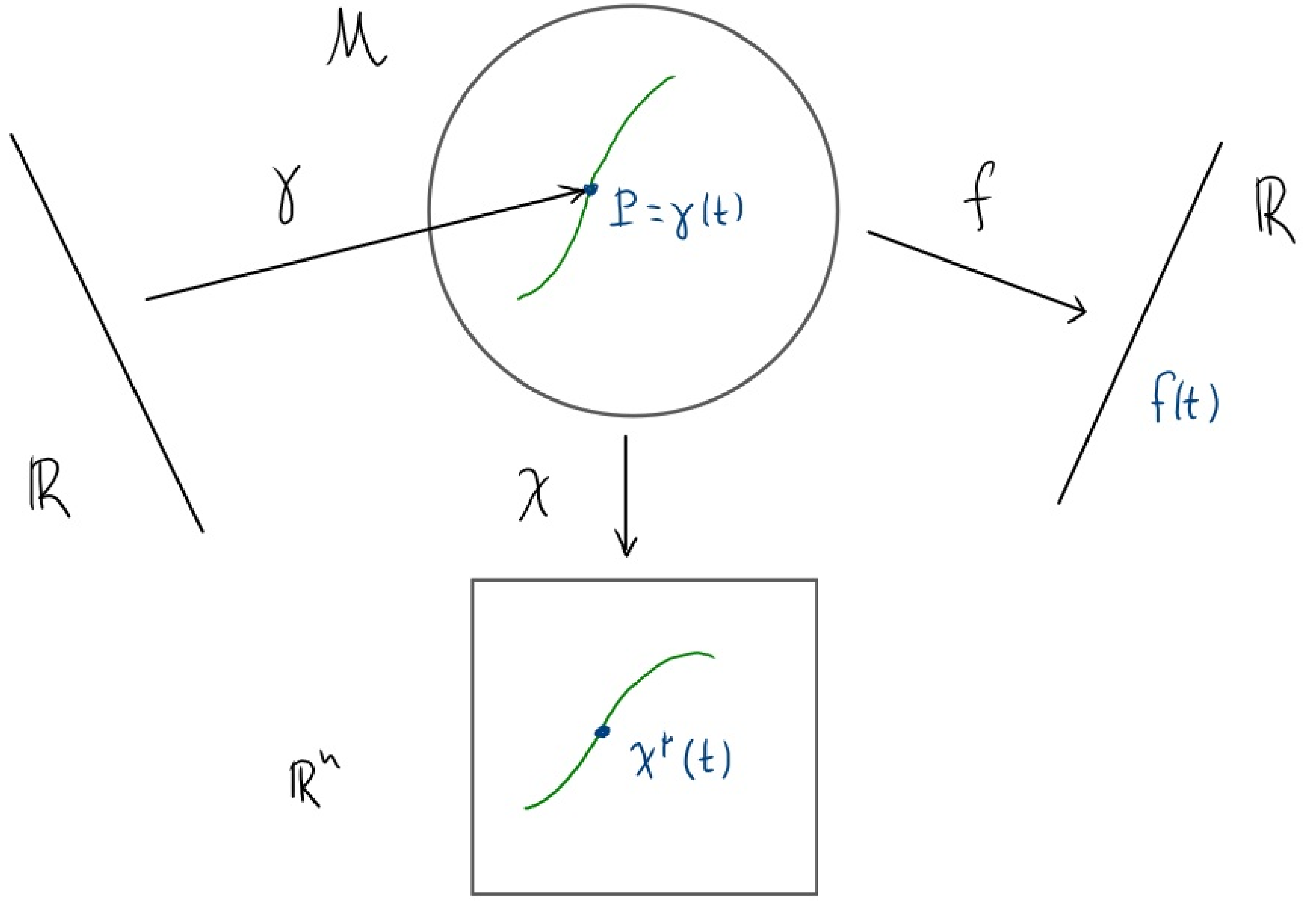
Notation:

$\gamma : \mathbb{R} \to M$ s.t. $t \mapsto \underline{P} = \gamma(t)$

$f : M \to \mathbb{R}$ s.t. $\underline{P} \mapsto f(\underline{P})$

$f(t) \equiv f \circ \gamma(t)$

$\frac{df}{dt} = \frac{d}{dt} f \circ \gamma(t)$



Review of vectors + vector fields

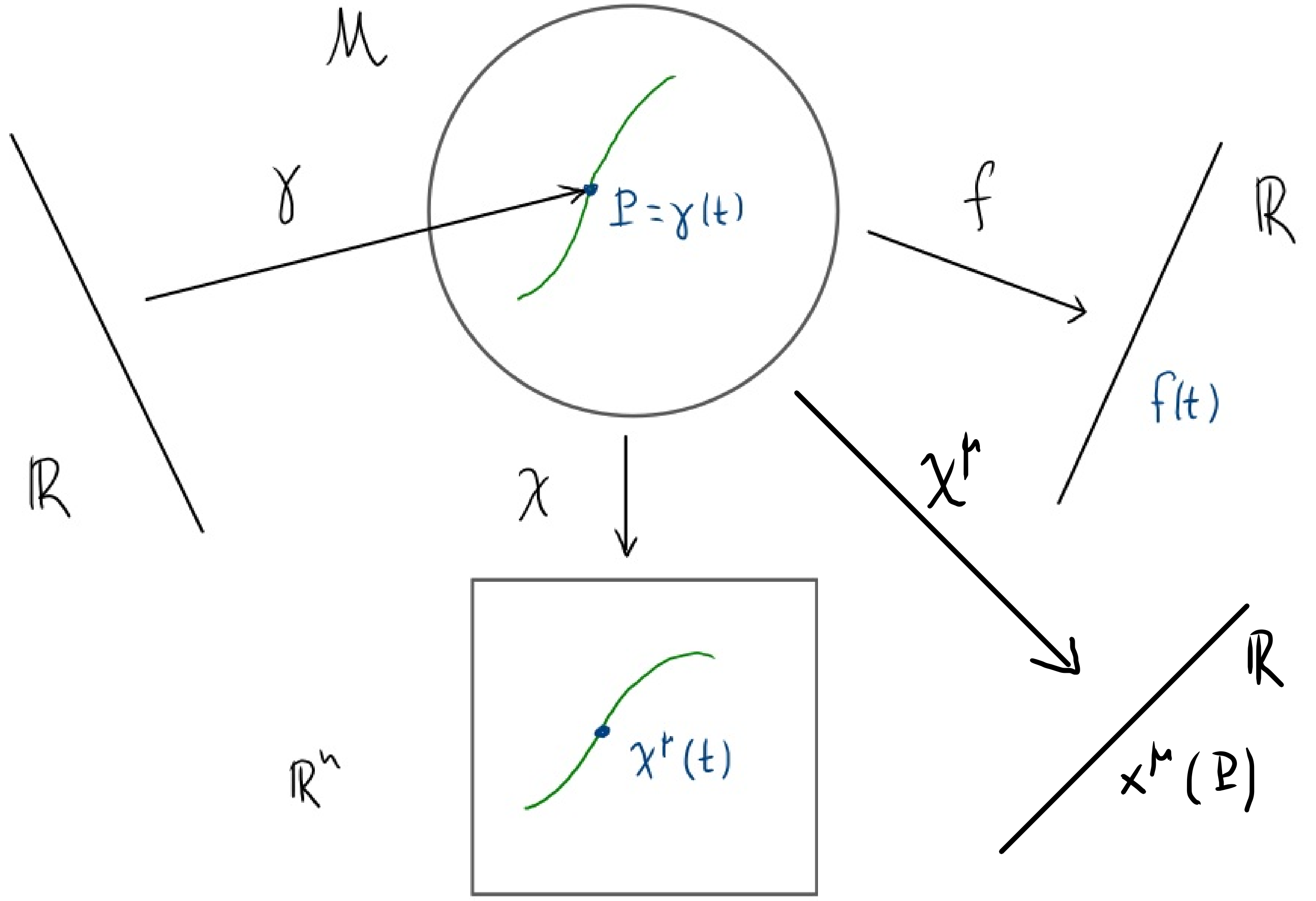
Notation:

$\gamma : \mathbb{R} \to \mathcal{M}$ s.t. $t \mapsto P = \gamma(t)$

$f : \mathcal{M} \to \mathbb{R}$ s.t. $P \mapsto f(P)$

$f(t) \equiv f \circ \gamma(t)$

$\frac{df}{dt} = \frac{d}{dt} f \circ \gamma(t)$



For given μ: X^μ is a function of $\mathcal{M}$

$X^\mu : \mathcal{M} \to \mathbb{R}$ s.t. $P \to x^\mu(P)$

Review of vectors + vector fields

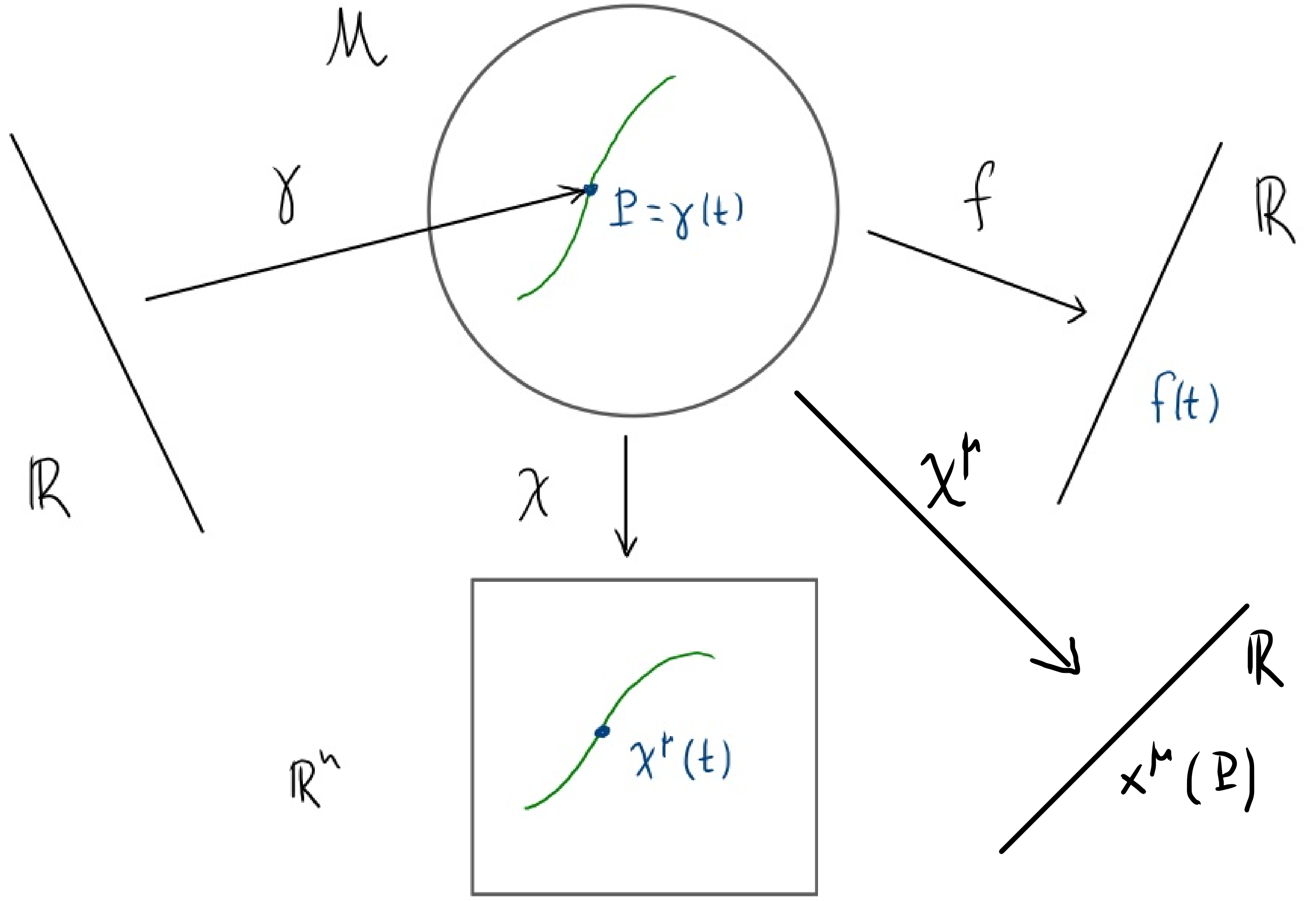
Notation:

$\gamma : \mathbb{R} \to \mathcal{M}$ s.t. $t \mapsto P = \gamma(t)$

$f : \mathcal{M} \to \mathbb{R}$ s.t. $P \mapsto f(P)$

$$f(t) \equiv f \circ \gamma (t)$$

$$\frac{df}{dt} = \frac{d}{dt} f \circ \gamma (t)$$



For given μ: χ^μ is a function of $\mathcal{M}$

$\chi^\mu : \mathcal{M} \to \mathbb{R}$ s.t. $P \to x^\mu(P)$

$x^\mu(t) \equiv \chi \circ \gamma (t)$ coordinates of curve γ

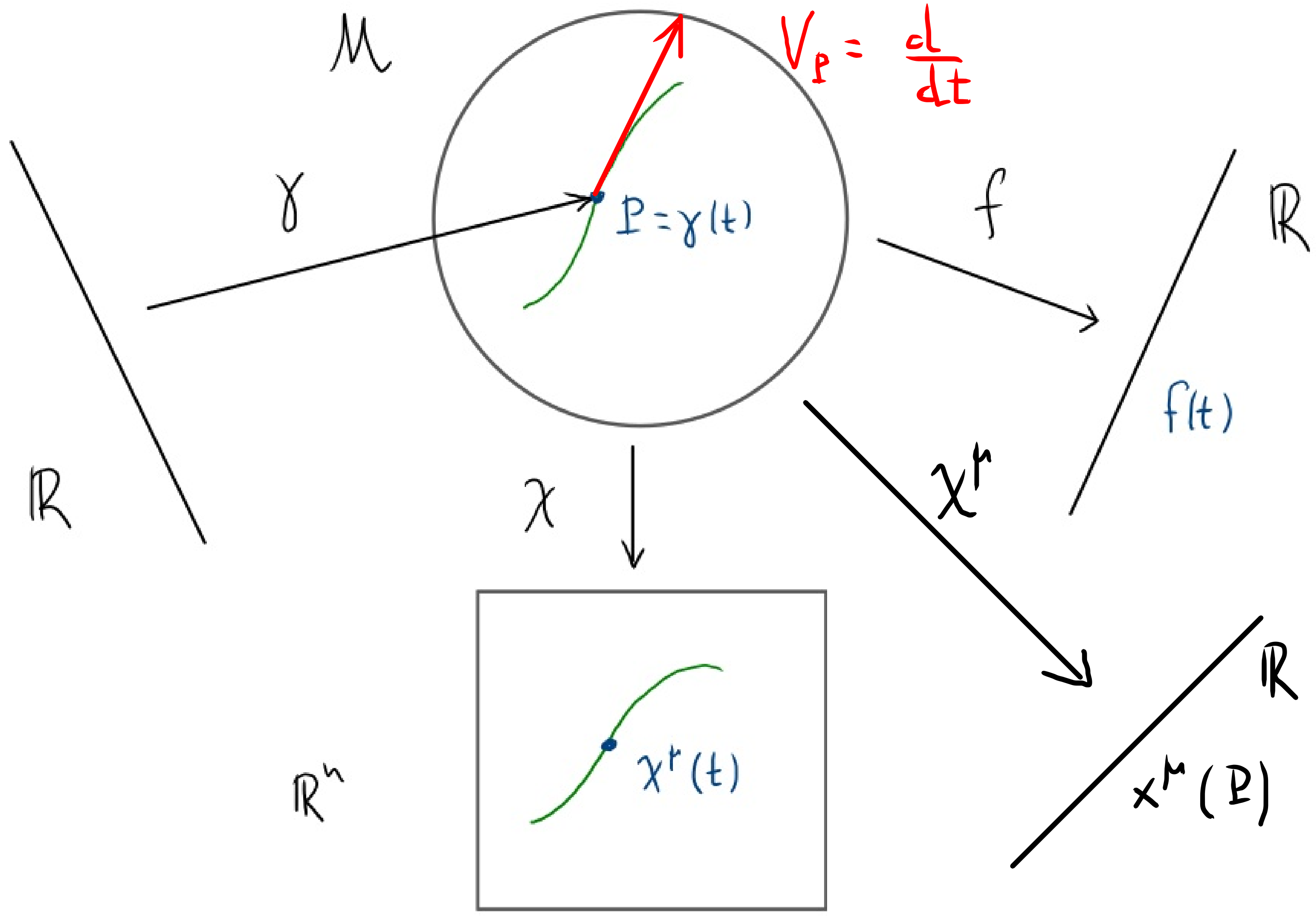
Review of vectors + vector fields

Notation:

vector at P:

$$V_p : \mathcal{F}(M) \longrightarrow \mathbb{R}$$

$$f \mapsto \left.\frac{df}{dt}\right|_p = \frac{d\, f\circ\gamma(t)}{dt}, \quad P = \gamma(t)$$



$x^\mu(t) \equiv \chi \circ \gamma(t)$ coordinates of curve γ

For given μ: x^μ is a function of M

$x^\mu : M \to \mathbb{R}$ s.t. $P \to x^\mu(P)$

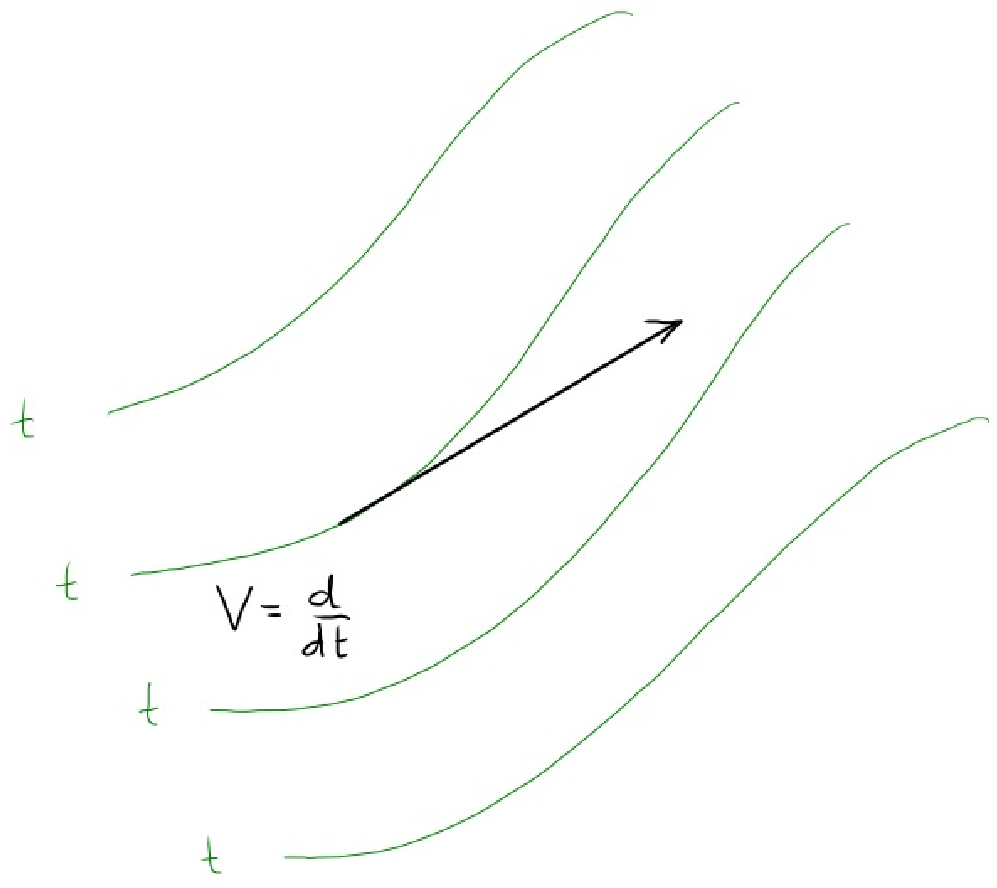
Review of vectors + vector fields

Notation:

vector at p:

$$V_p : \mathcal{F}(M) \longrightarrow \mathbb{R}$$

$$f \mapsto \left.\frac{df}{dt}\right|_p \vdots \frac{d f\circ\gamma(t)}{dt} \quad , \quad p = \gamma(t)$$

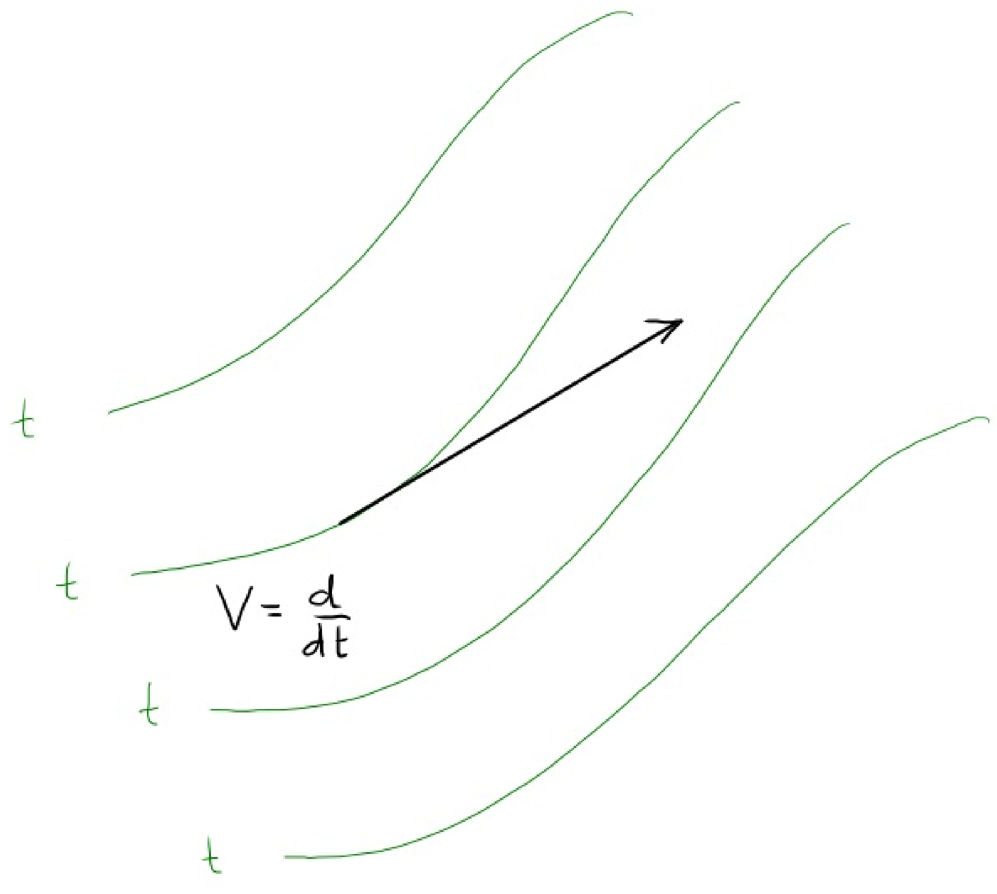


---

vector field:

$V : p \mapsto V_p$ s.t. $V(f) = \frac{df}{dt} \vdots \frac{d f\circ\gamma(t)}{dt}$ where $\gamma(t)$ an integral curve of V

Review of vectors + vector fields



Notation:

vector at p:

$$V_p : \mathcal{F}(M) \longrightarrow \mathbb{R}$$

$$f \mapsto \left.\frac{df}{dt}\right|_p = \frac{d\, f\circ\gamma(t)}{dt}, \quad p = \gamma(t)$$

---

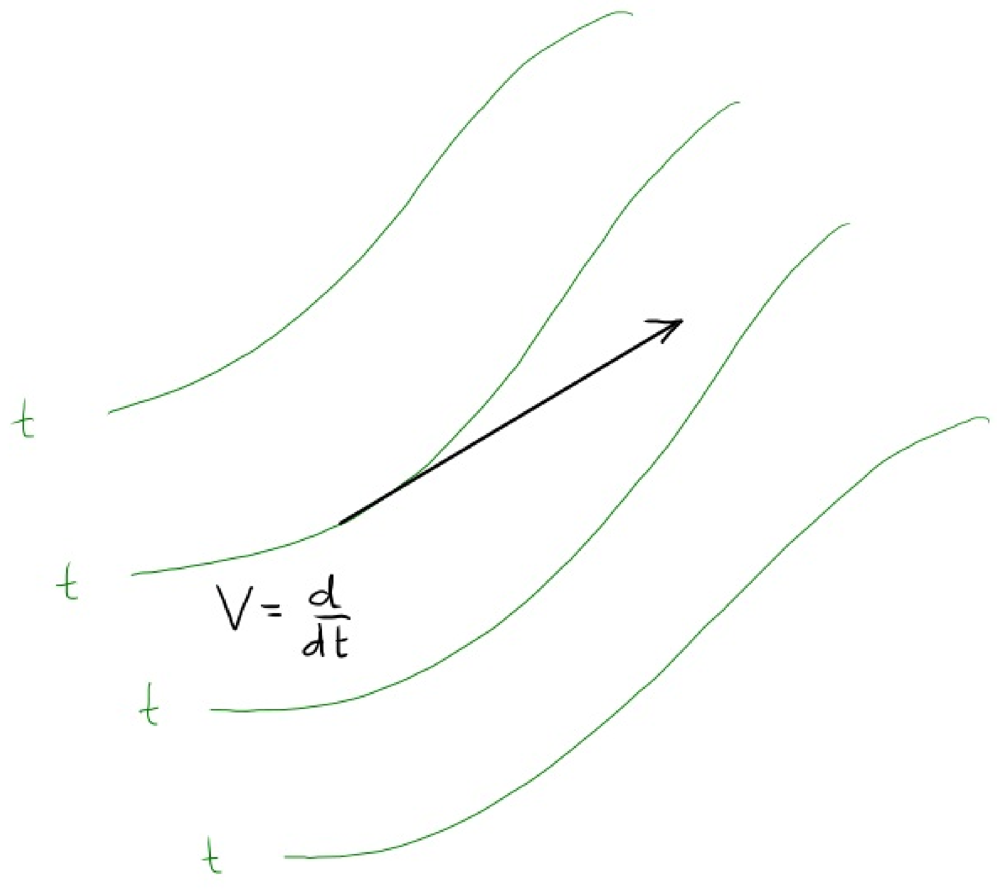
vector field:

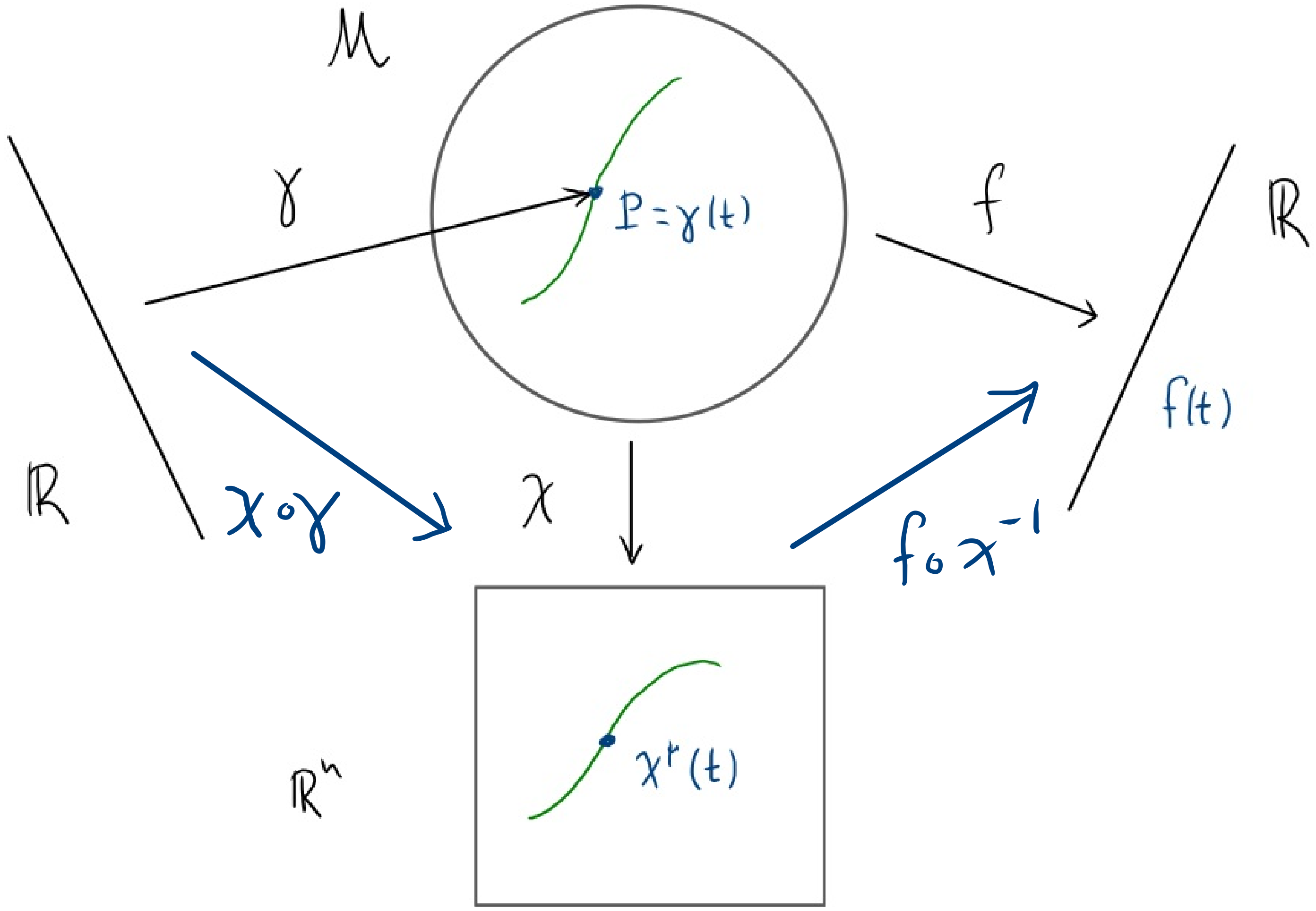
$V: p \mapsto V_p$ s.t. $V(f) = \frac{df}{dt} = \frac{d\, f\circ\gamma(t)}{dt}$ where $\gamma(t)$ an integral curve of V

Integral curves form a congruence: they fill M and don't cross ($V \neq 0$)

Review of vectors + vector fields

$$\frac{df}{dt} = \frac{d}{dt} f\circ\gamma(t) = \frac{d}{dt} \underbrace{f\circ\chi^{-1}}_{f(x^\mu)}\circ\underbrace{\chi\circ\gamma(t)}_{x^\mu(t)} = \frac{dx^\mu(t)}{dt}\,\frac{\partial f(x^\mu)}{\partial x^\mu}$$





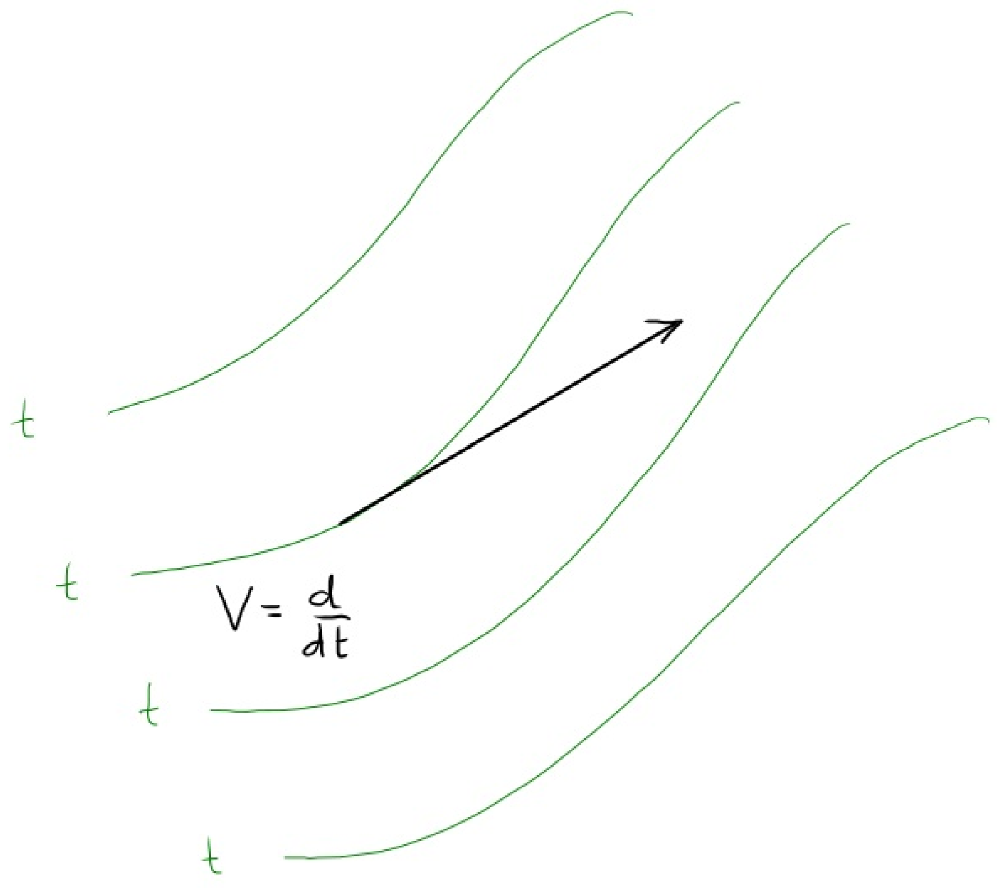
vector f

$V: P \mapsto$ where $\gamma(t)$ an integral curve of V

Integral curves form a congruence. they fill M and don't cross ($V\neq 0$)

Review of vectors + vector fields

$$\frac{df}{dt} = \frac{d}{dt} f \circ \gamma(t) = \frac{d}{dt} f \circ \chi^{-1} \circ \chi \circ \gamma(t) = \frac{dx^\mu(t)}{dt} \frac{\partial f(x^\mu)}{\partial x^\mu}$$



So:

$$V(f) = \frac{df}{dt} = V^\mu \partial_\mu f = \frac{dx^\mu}{dt} \partial_\mu f \quad \Rightarrow \quad V^\mu = \frac{dx^\mu}{dt}$$

---

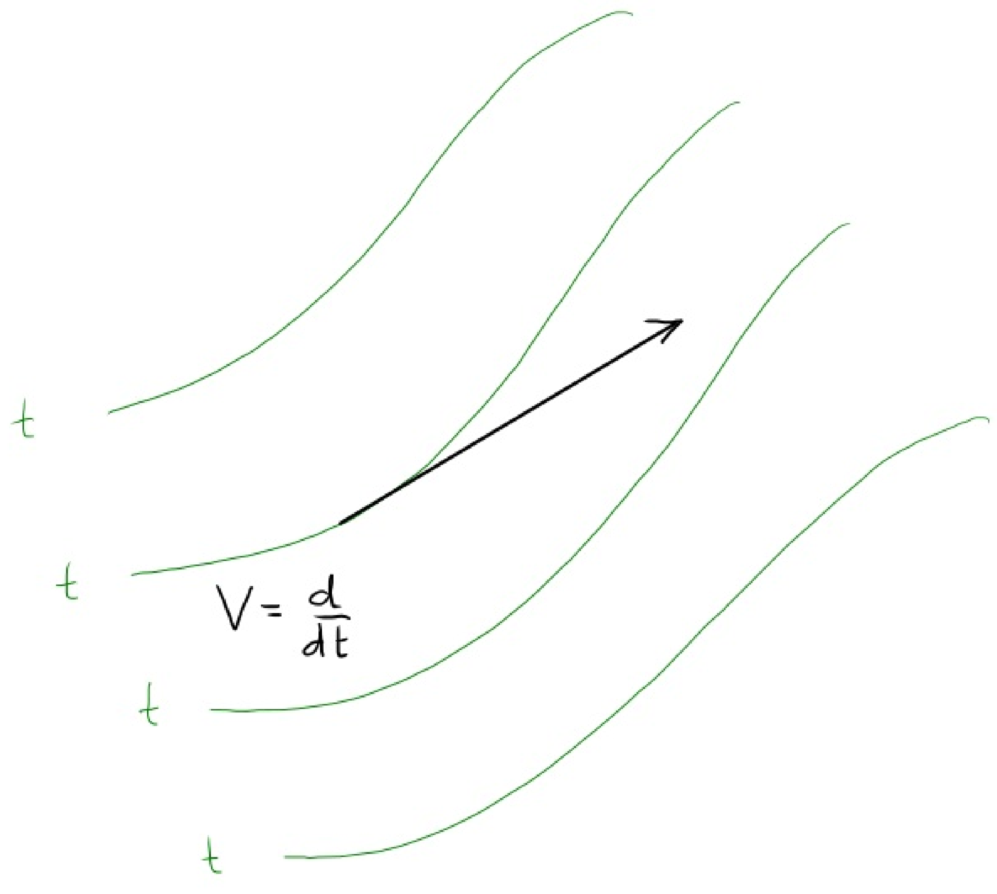
vector field:

$V : P \mapsto V_P$ s.t. $V(f) = \frac{df}{dt} = \frac{d f \circ \gamma(t)}{dt}$ where $\gamma(t)$ an integral curve of V

Integral curves form a congruence: they fill $\mathcal{U}$ and don't cross ($V \neq 0$)

Review of vectors + vector fields

$$\frac{df}{dt} = \frac{d}{dt} f\circ\gamma(t) = \frac{d}{dt} f\circ\chi^{-1}\circ\chi\circ\gamma(t) = \frac{dx^\mu(t)}{dt}\,\frac{\partial f(x^\mu)}{\partial x^\mu}$$



So:

$$V(f) = \frac{df}{dt} = V^\mu \partial_\mu f = \frac{dx^\mu}{dt}\partial_\mu f \quad \Rightarrow \quad V^\mu = \frac{dx^\mu}{dt}$$

In particular: $f = \chi^\mu : P \mapsto x^\mu(P)$ (fixed μ)

$$V(\chi^\mu) \equiv V(x^\mu) = V^\nu \partial_\nu x^\mu = V^\nu \delta_\nu{}^\mu = V^\mu$$

---

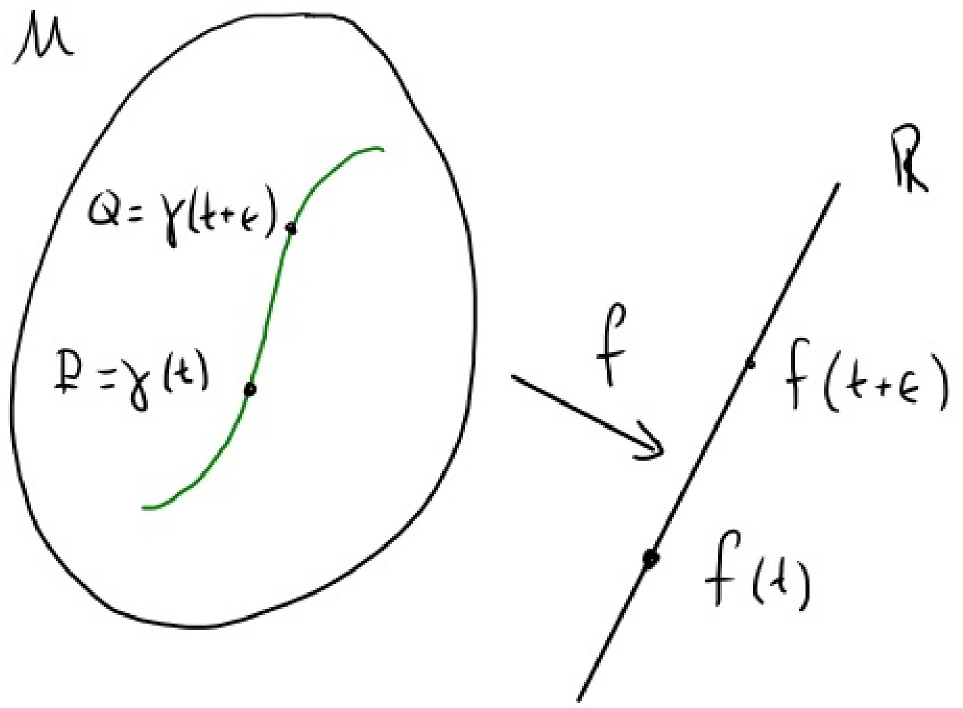
vector field:

$V : P \mapsto V_P$ s.t. $V(f) = \dfrac{df}{dt} \equiv \dfrac{d f\circ\gamma(t)}{dt}$ where $\gamma(t)$ an integral curve of V

Integral curves form a congruence: they fill $\mathcal{U}$ and don't cross ($V \neq 0$)

Exponential Function

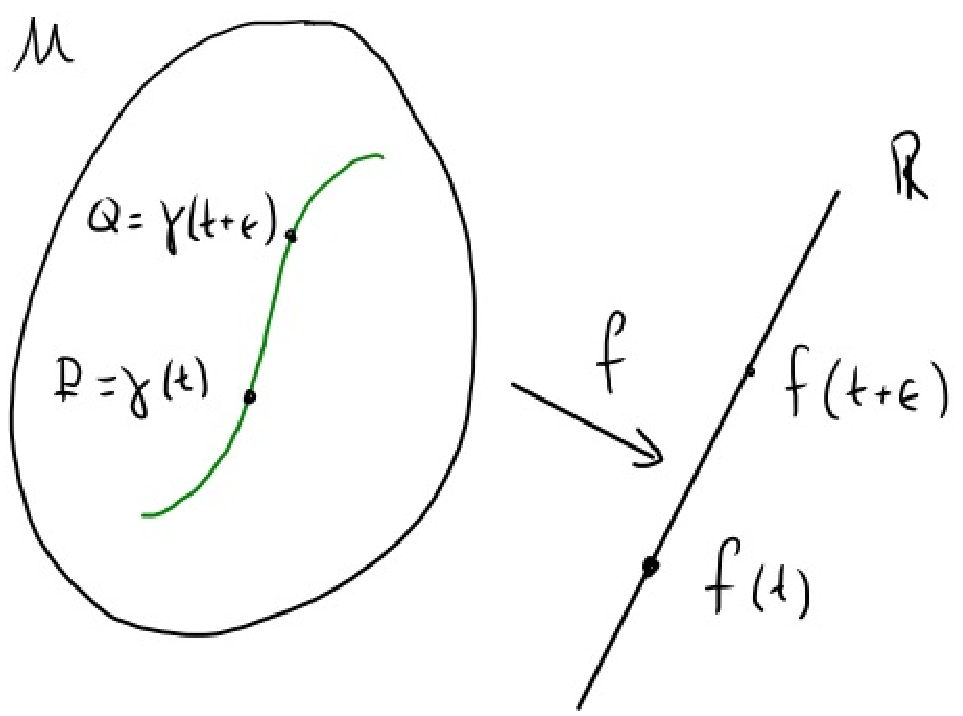
$$f(t+\epsilon) \equiv f \circ \gamma\,(t+\epsilon)$$



Exponential Function

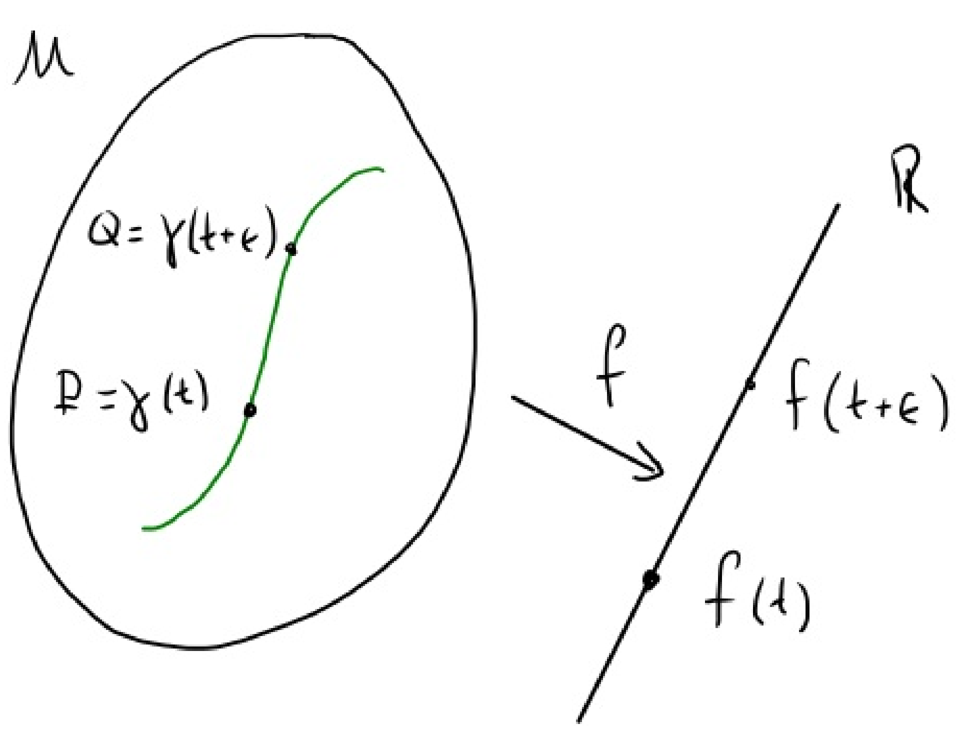
$$f(t+\epsilon) \equiv f \circ \gamma\,(t+\epsilon)$$

$$= f(t) + \epsilon \frac{df}{dt} + \frac{\epsilon^2}{2!} \frac{d^2 f}{dt^2} + \cdots$$



Exponential Function

$$
\begin{aligned}
f(t+\epsilon) &\equiv f \circ \gamma\,(t+\epsilon) \\
&= f(t) + \epsilon \frac{df}{dt} + \frac{\epsilon^2}{2!} \frac{d^2 f}{dt^2} + \dots \\
&= f(t) + \epsilon V(f) + \frac{\epsilon^2}{2} V(\underbrace{V(f)}_{\text{a function on } \mathcal{M}}) + \dots
\end{aligned}
$$



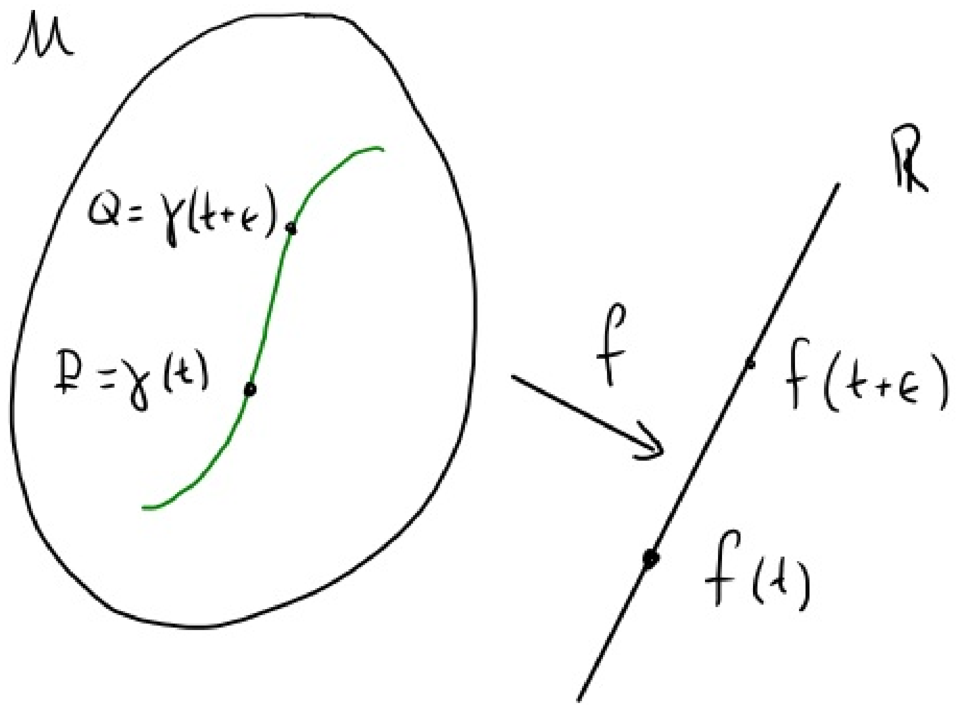
Exponential Function

$$f(t+\epsilon) \equiv f \circ \gamma\,(t+\epsilon)$$

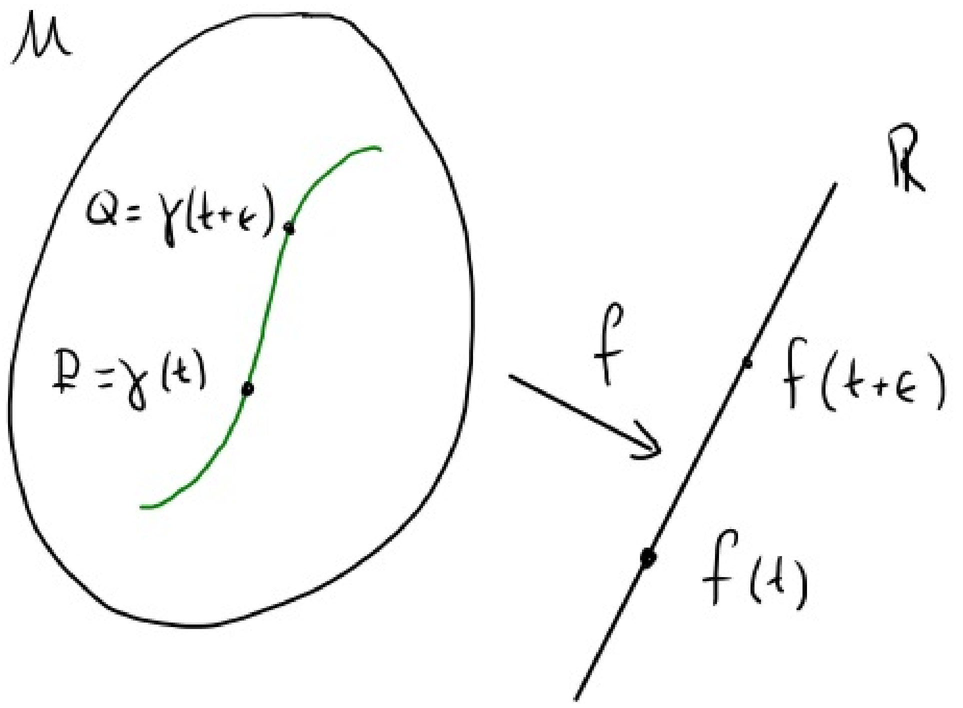
$$= f(t) + \epsilon \frac{df}{dt} + \frac{\epsilon^2}{2!}\frac{d^2 f}{dt^2} + \dots$$

$$= f(t) + \epsilon V(f) + \frac{\epsilon^2}{2} V(V(f)) + \dots$$

$$= e^{\epsilon \frac{d}{dt}} f(t) \equiv e^{\epsilon V(t)} f(t)$$



Exponential Function



$$f(t+\epsilon) \equiv f\circ\gamma\,(t+\epsilon)$$

$$= f(t) + \epsilon\frac{df}{dt} + \frac{\epsilon^2}{2!}\frac{d^2f}{dt^2} + \dots$$

$$= f(t) + \epsilon V(f) + \frac{\epsilon^2}{2} V(V(f)) + \dots$$

$$= e^{\epsilon\frac{d}{dt}} f(t) \equiv e^{\epsilon V(t)} f(t)$$

Clarify: If at specific point $P = \gamma(t)$

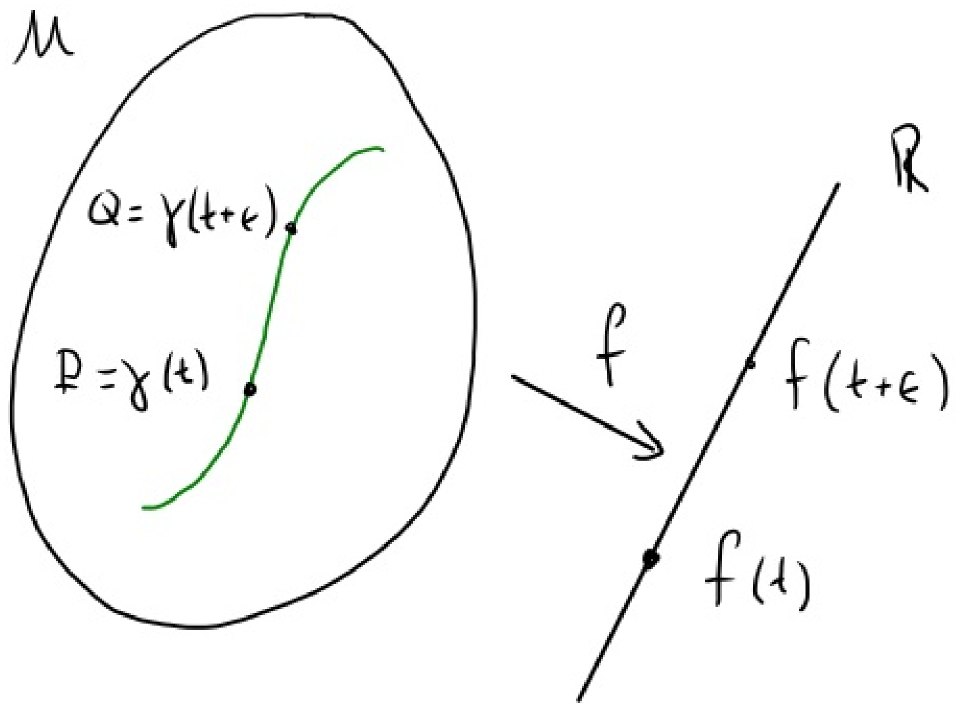
$$f(Q) = f(P) + \epsilon\left.\frac{df}{dt}\right|_P + \frac{\epsilon^2}{2}\left.\frac{d^2f}{dt^2}\right|_P + \dots = f(P) + \epsilon V_P(f) + \frac{\epsilon^2}{2} V_P(V(f)) + \dots$$

vector at P

function on neighborhood of P

→ requires vector field

Exponential Function

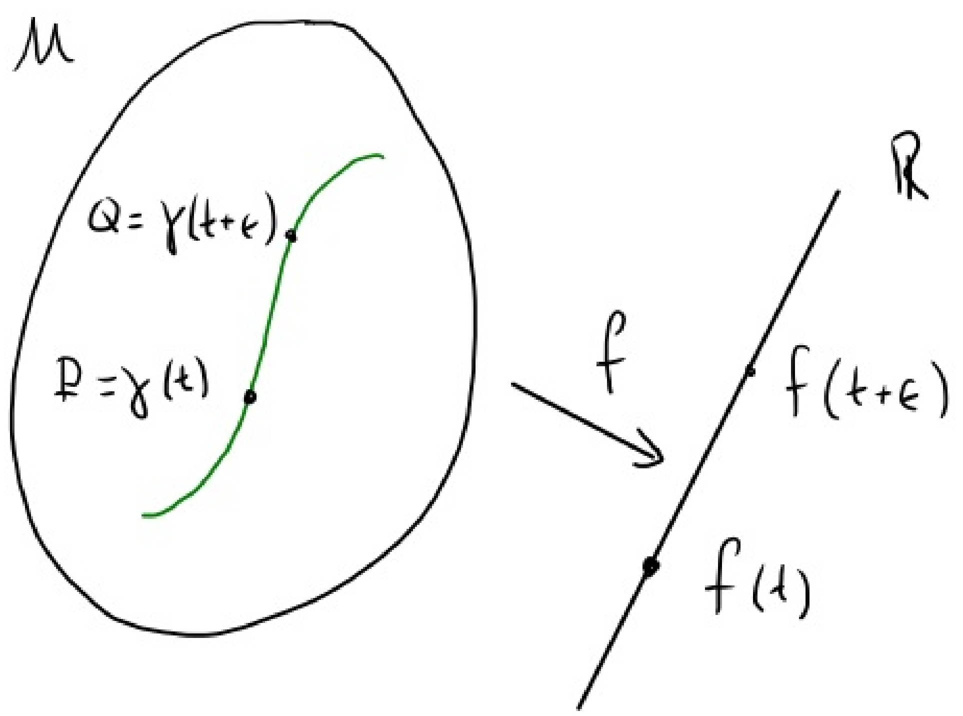


$$
\begin{aligned}
f(t+\epsilon) &\equiv f \circ \gamma\,(t+\epsilon) \\
&= f(t) + \epsilon \frac{df}{dt} + \frac{\epsilon^2}{2!}\frac{d^2 f}{dt^2} + \cdots \\
&= f(t) + \epsilon V(f) + \frac{\epsilon^2}{2} V(V(f)) + \cdots \\
&= e^{\epsilon \frac{d}{dt}} f(t) \equiv e^{\epsilon V(t)} f(t)
\end{aligned}
$$

For given μ, $x^\mu(P)$ a function on $\mathcal{M}$, $x^\mu(t) \equiv \chi^\mu \circ \gamma(t)$

$$
x^\mu(t+\epsilon) = e^{\epsilon V(t)} x^\mu(t) = x^\mu(t) + \epsilon \frac{dx^\mu(t)}{dt} + \cdots = x^\mu(t) + \epsilon V^\mu(x^\nu(t)) + \cdots
$$

Exponential Function



$$
\begin{aligned}
f(t+\epsilon) &\equiv f \circ \gamma\,(t+\epsilon) \\
&= f(t) + \epsilon \frac{df}{dt} + \frac{\epsilon^2}{2!} \frac{d^2 f}{dt^2} + \dots \\
&= f(t) + \epsilon V(f) + \frac{\epsilon^2}{2} V(V(f)) + \dots \\
&= e^{\epsilon \frac{d}{dt}} f(t) \equiv e^{\epsilon V(t)} f(t)
\end{aligned}
$$

For given μ, $x^\mu(P)$ a function on $\mathcal{M}$, $x^\mu(t) \equiv \chi^\mu \circ \gamma(t)$

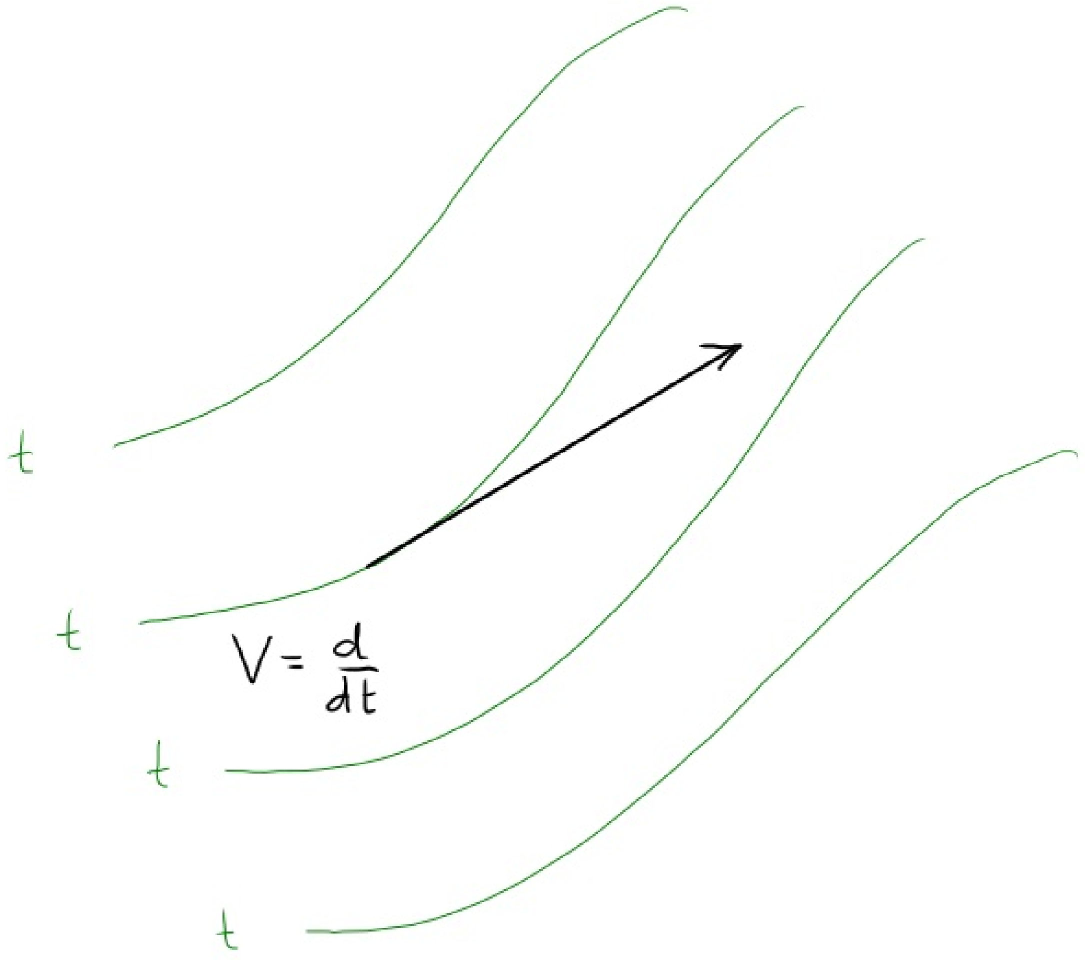
$$
x^\mu(t+\epsilon) = e^{\epsilon V(t)} x^\mu(t) = x^\mu(t) + \epsilon \frac{dx^\mu(t)}{dt} + \dots = x^\mu(t) + \epsilon V^\mu(x^\nu(t)) + \dots
$$

$$
\Rightarrow \frac{dx^\mu}{dt} = V^\mu(x^\nu)
$$

Exponential Function

If $V^\mu(x^\nu)$ known functions of x^ν,
and $x^\mu(0)$ is given, (1) has a unique solution
$x^\mu(t)$: the integral curves of $V = \frac{d}{dt}$

$\Rightarrow$ integral curves never cross $(V \neq 0)$



For given μ, $x^\mu(p)$ a function on M, $x^\mu(t) \equiv \chi^\mu \circ \gamma(t)$

$$x^\mu(t+\epsilon) = e^{\epsilon V(t)} x^\mu(t) = x^\mu(t) + \epsilon \frac{dx^\mu(t)}{dt} + \dots = x^\mu(t) + \epsilon V^\mu(x^\nu(t)) + \dots$$

$$\Rightarrow \frac{dx^\mu}{dt} = V^\mu(x^\nu) \quad (1)$$

Lie bracket of vector fields:

Choose a coordinate system x^μ $\longrightarrow$ coordinate basis ∂_μ

vector fields: $V = \frac{d}{dt} = V^\mu \partial_\mu$ $\qquad V^\mu = \frac{dx^\mu}{dt}$

$W = \frac{d}{d\lambda} = W^\mu \partial_\mu$ $\qquad W^\mu = \frac{dx^\mu}{d\lambda}$

Lie bracket of vector fields:

Choose a coordinate system x^μ $\longrightarrow$ coordinate basis ∂_μ

vector fields: $V = \frac{d}{dt} = V^\mu \partial_\mu \qquad V^\mu = \frac{dx^\mu}{dt}$

$W = \frac{d}{d\lambda} = W^\mu \partial_\mu \qquad W^\mu = \frac{dx^\mu}{d\lambda}$

Lie bracket: $[V, W](f) = V(W(f)) - W(V(f))$ (a vector field)

Note: at point P: $[V, W]_P(f) = V_P\,\underbrace{(W(f))}\; - \;W_P\,\underbrace{(V(f))}$

$\hookrightarrow$ regarded as functions!

$\Rightarrow$ makes sense only for vector fields

Lie bracket of vector fields:

Choose a coordinate system x^μ $\longrightarrow$ coordinate basis ∂_μ

vector fields: $V = \frac{d}{dt} = V^\mu \partial_\mu$ $\qquad V^\mu = \frac{dx^\mu}{dt}$

$W = \frac{d}{d\lambda} = W^\mu \partial_\mu$ $\qquad W^\mu = \frac{dx^\mu}{d\lambda}$

Lie bracket: $[V, W](f) = V(W(f)) - W(V(f))$ (a vector field)

$= V^\mu \partial_\mu (W^\nu \partial_\nu f) - W^\mu \partial_\mu (V^\nu \partial_\nu f)$

Lie bracket of vector fields:

Choose a coordinate system x^μ $\longrightarrow$ coordinate basis ∂_μ

vector fields: $V = \dfrac{d}{dt} = V^\mu \partial_\mu$ $\qquad V^\mu = \dfrac{dx^\mu}{dt}$

$W = \dfrac{d}{d\lambda} = W^\mu \partial_\mu$ $\qquad W^\mu = \dfrac{dx^\mu}{d\lambda}$

Lie bracket: $[V, W](f) = V(W(f)) - W(V(f))$ (a vector field)

$= V^\mu \partial_\mu (W^\nu \partial_\nu f) - W^\mu \partial_\mu (V^\nu \partial_\nu f)$

$= (V^\mu \partial_\mu W^\nu \partial_\nu f + V^\mu W^\nu \partial_\mu \partial_\nu f) - (W^\mu \partial_\mu V^\mu \partial_\nu f + W^\mu V^\mu \partial_\mu \partial_\nu f)$

Lie bracket of vector fields:

Choose a coordinate system x^μ $\longrightarrow$ coordinate basis ∂_μ

vector fields: $V = \dfrac{d}{dt} = V^\mu \partial_\mu \qquad V^\mu = \dfrac{dx^\mu}{dt}$

$W = \dfrac{d}{d\lambda} = W^\mu \partial_\mu \qquad W^\mu = \dfrac{dx^\mu}{d\lambda}$

Lie bracket: $[V, W](f) = V(W(f)) - W(V(f))$ (a vector field)

$$= V^\mu \partial_\mu (W^\nu \partial_\nu f) - W^\mu \partial_\mu (V^\nu \partial_\nu f)$$

$$= (V^\mu \partial_\mu W^\nu \partial_\nu f + \cancel{V^\mu W^\nu \partial_\mu \partial_\nu f}) - (W^\mu \partial_\mu V^\nu \partial_\nu f + \cancel{W^\mu V^\nu \partial_\mu \partial_\nu f})$$

$$= (V^\mu \partial_\mu W^\nu - W^\mu \partial_\mu V^\nu)\,\partial_\nu f = [V, W]^\nu \partial_\nu f$$

Lie bracket of vector fields:

therefore: $[V,W]^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu$

only in a coordinate basis!

Lie bracket: $[V,W](f) = V(W(f)) - W(V(f))$ (a vector field)

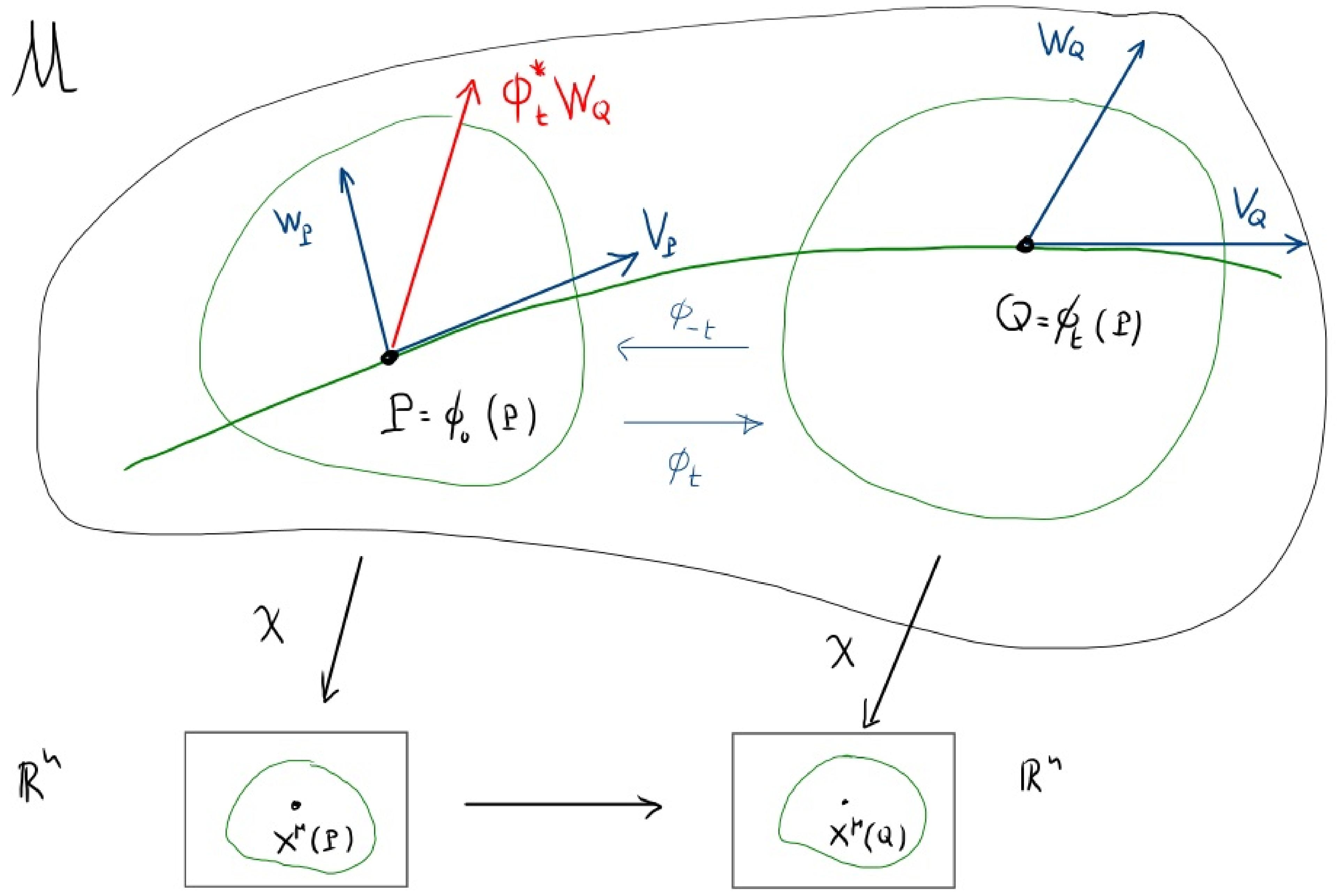
$= V^\mu \partial_\mu (W^\nu \partial_\nu f) - W^\mu \partial_\mu (V^\nu \partial_\nu f)$

$= (V^\mu \partial_\mu W^\nu \partial_\nu f + \cancel{V^\mu W^\nu \partial_\mu} \partial_\nu f) - (W^\mu \partial_\mu V^\mu \partial_\nu f + W^\mu \cancel{V^\mu} \partial_\mu \partial_\nu f)$

$= (V^\mu \partial_\mu W^\nu - W^\mu \partial_\mu V^\nu) \partial_\nu f = [V,W]^\nu \partial_\nu f$

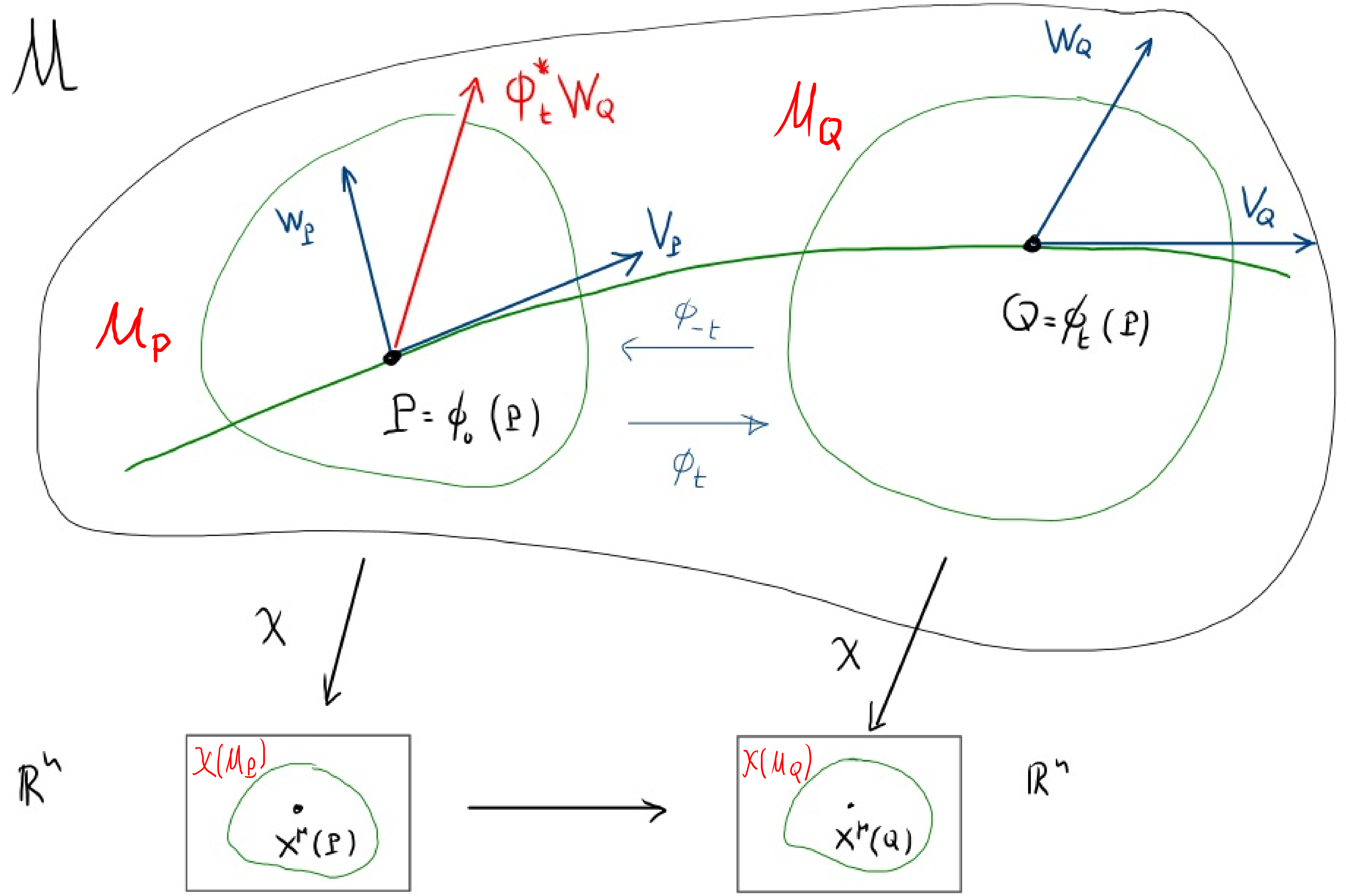
Compute $\mathcal{L}_V W$

- Consider a chart χ containing P + Q



Compute $\mathcal{L}_V W$

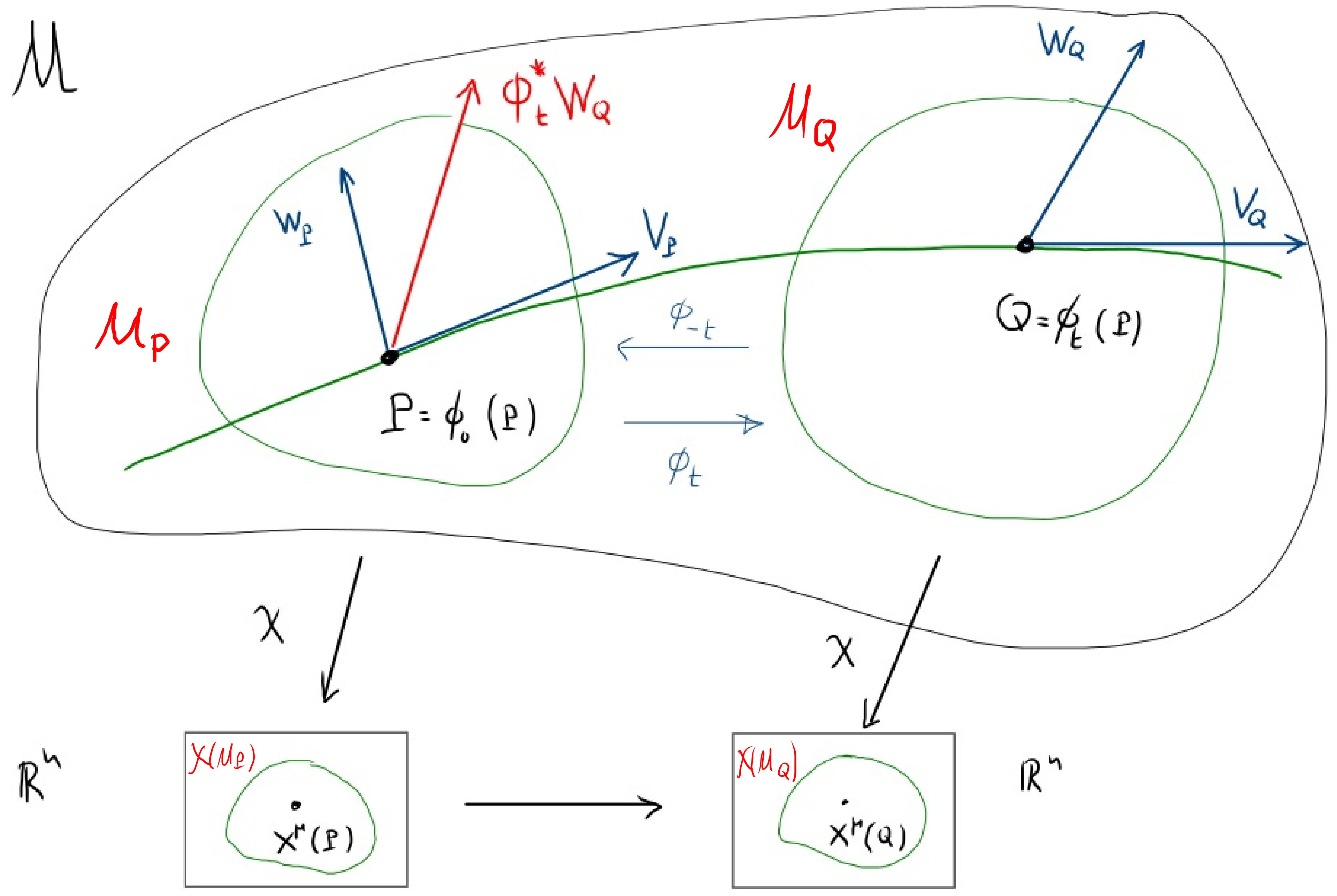
- Consider a chart χ containing $P + Q$
- Restrict χ to disjoint neighborhoods of $P + Q$, such that the open set
  $\mathcal{U}_Q = \phi_t(\mathcal{U}_P)$
  $\mathcal{U}_P$ and $\mathcal{U}_Q$ are diffeomorphic



Compute $\mathcal{L}_V W$

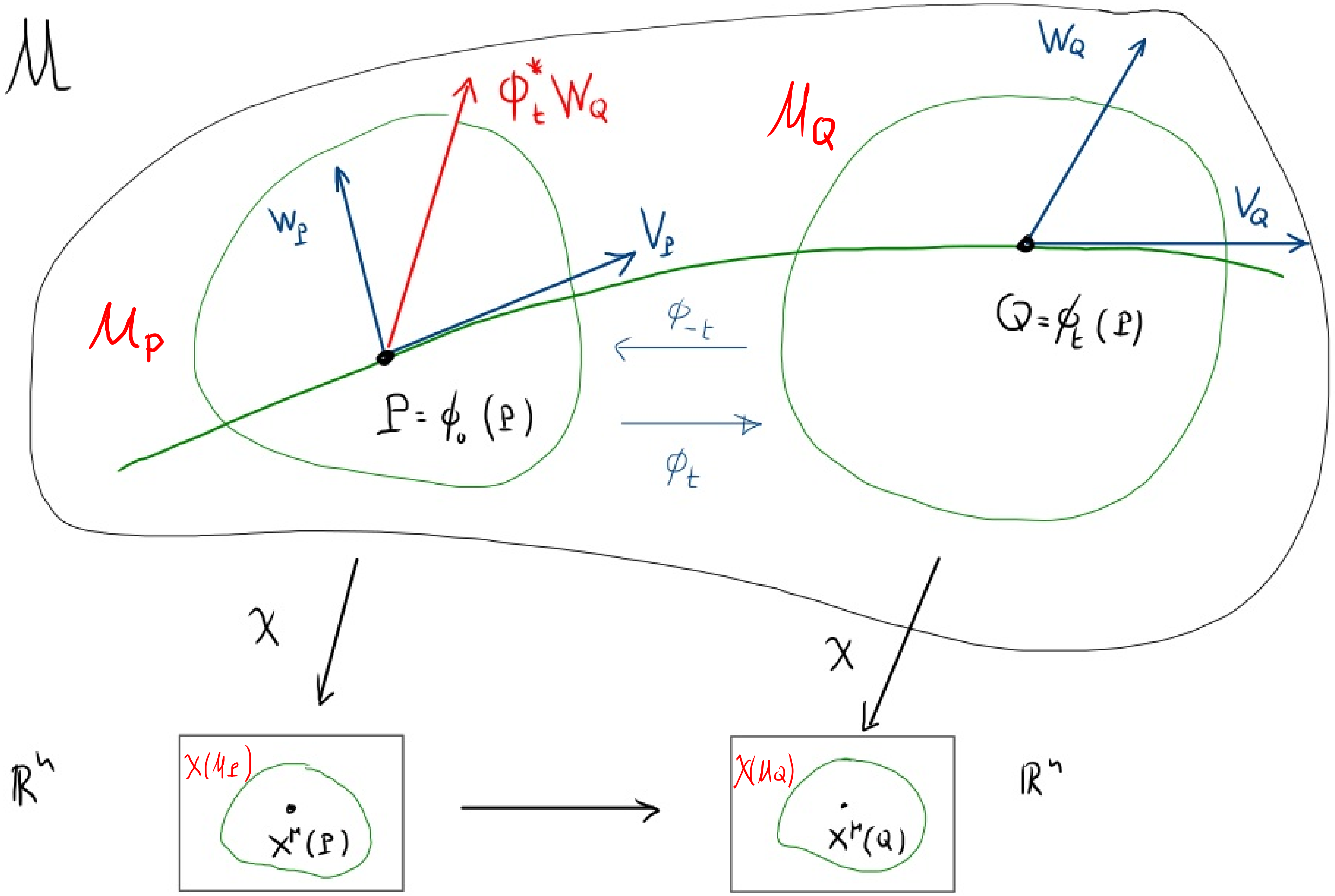
- Consider a chart χ containing $P + Q$
- Restrict χ to disjoint neighborhoods of $P + Q$, such that the open set $\mathcal{U}_Q = \phi_t(\mathcal{U}_P)$
  $\mathcal{U}_P$ and $\mathcal{U}_Q$ are diffeomorphic
- χ restricted to $\mathcal{U}_P$ and $\mathcal{U}_Q$ define charts in $\mathcal{U}_P$ and $\mathcal{U}_Q$

  <u>notation:</u> $x^\mu(0) \equiv x^\mu(P)$
  $x^\mu(t) = x^\mu(Q)$



Compute $\mathcal{L}_V W$

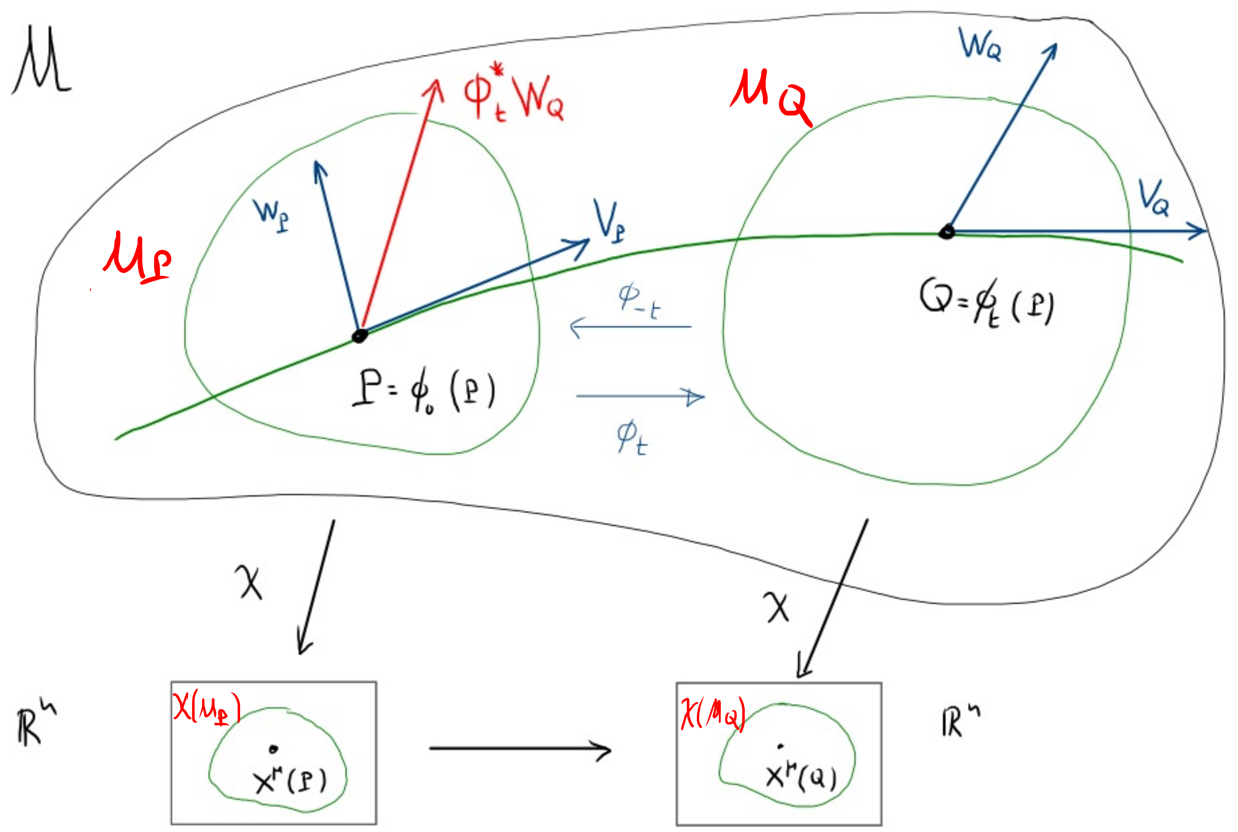
- Consider a chart χ containing $P + Q$
- Restrict χ to disjoint neighborhoods of $P + Q$, such that the open set $\mathcal{U}_Q = \phi_t(\mathcal{U}_P)$
  $\mathcal{U}_P$ and $\mathcal{U}_Q$ are diffeomorphic
- χ restricted to $\mathcal{U}_P$ and $\mathcal{U}_Q$ define charts in $\mathcal{U}_P$ and $\mathcal{U}_Q$
  <u>notation:</u> $x^\mu(0) \equiv x^\mu(P)$
  $x^\mu(t) = x^\mu(Q)$
- then $\phi_t^* W^\mu(0) = \dfrac{\partial x^\mu(0)}{\partial x^\nu(t)} W^\nu(t)$



Compute $\mathcal{L}_V W$

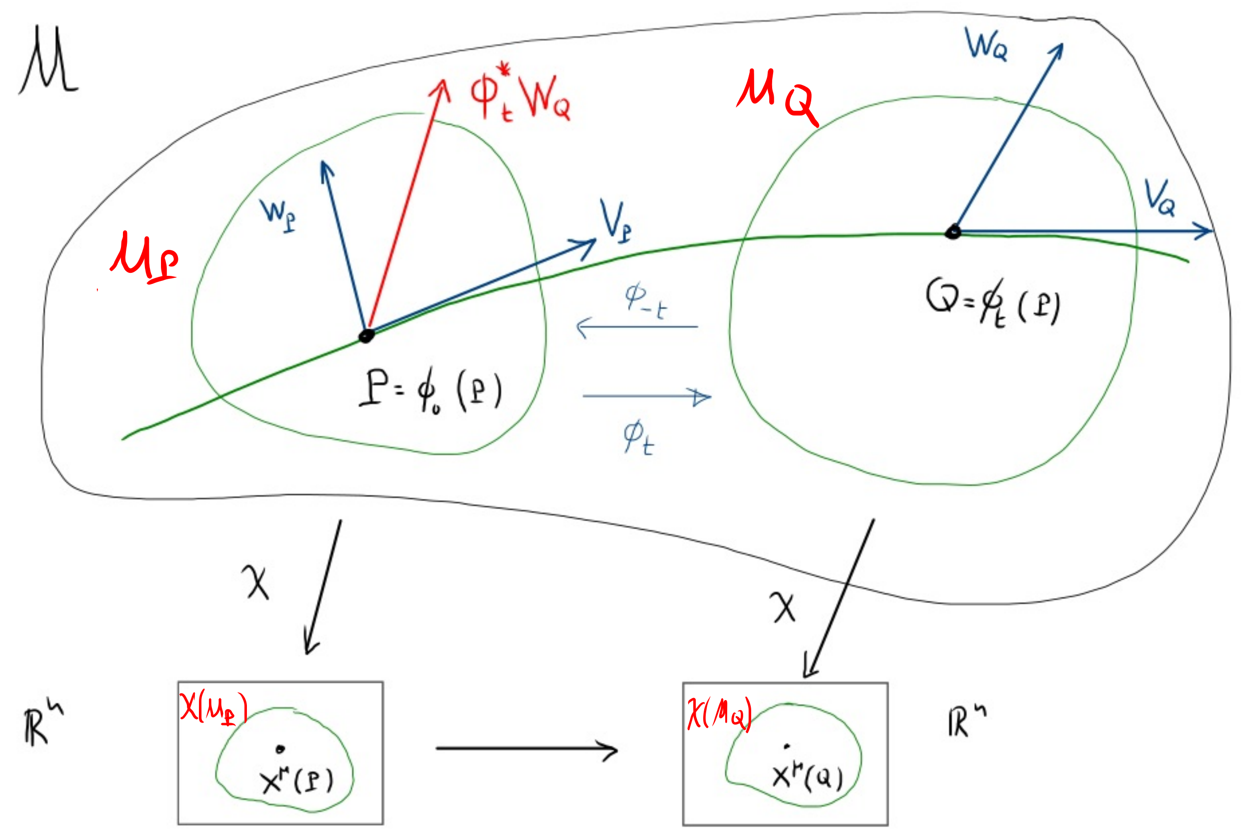
But $x^\mu(0) = e^{-t\frac{d}{dt}} x^\mu(t) = x^\mu(t) - t\,\dfrac{dx^\mu(t)}{dt} + \mathcal{O}(t^2)$

$$= x^\mu(t) - t\,V^\mu(t) + \mathcal{O}(t^2)$$



• then $\phi_t^* W^\mu(0) = \dfrac{\partial x^\mu(0)}{\partial x^\nu(t)} \quad W^\nu(t)$

Compute $\mathcal{L}_V W$



But $x^\mu(0) = e^{-t\frac{d}{dt}} x^\mu(t) = x^\mu(t) - t\,\frac{dx^\mu(t)}{dt} + \mathcal{O}(t^2)$

$$= x^\mu(t) - t\,V^\mu(t) + \mathcal{O}(t^2)$$

$$\Rightarrow \frac{\partial x^\mu(0)}{\partial x^\nu(t)} = \delta_\nu^{\ \mu} - t\,\partial_\nu V^\mu(t) + \mathcal{O}(t^2) \qquad (2)$$

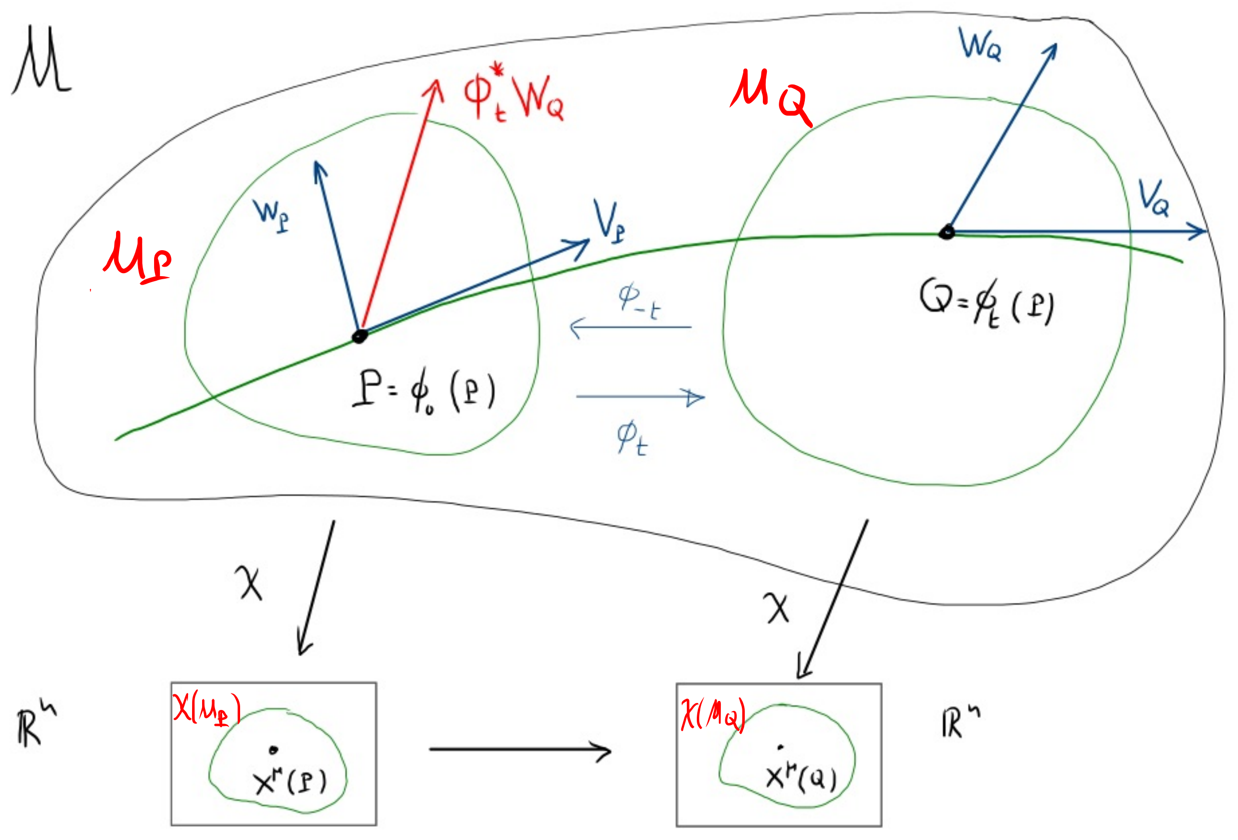
• then $\phi_t^* W^\mu(0) = \dfrac{\partial x^\mu(0)}{\partial x^\nu(t)} \, W^\nu(t) \qquad (1)$

Compute $\mathcal{L}_V W$

But $x^\mu(0) = e^{-t\frac{d}{dt}} x^\mu(t) = x^\mu(t) - t\,\frac{dx^\mu(t)}{dt} + \theta(t^2)$

$= x^\mu(t) - t V^\mu(t) + \theta(t^2)$

$$\Rightarrow \frac{\partial x^\mu(0)}{\partial x^\nu(t)} = \delta_\nu{}^\mu - t\,\partial_\nu V^\mu(t) + \theta(t^2) \quad (2)$$

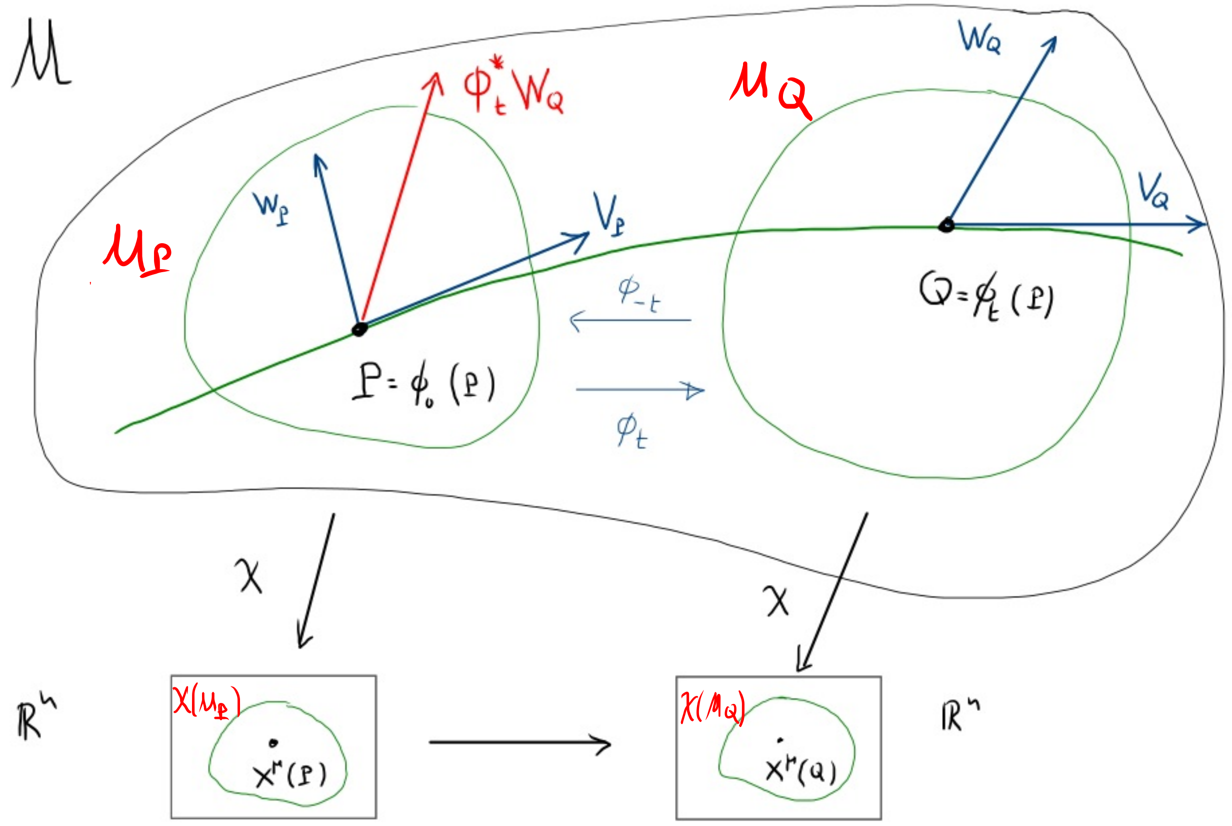


$$(1),(2) \Rightarrow \phi_t^* W^\mu(0) = \left[\delta_\nu{}^\mu - t\,\partial_\nu V^\mu(t) + \theta(t^2)\right] W^\nu(t)$$

$$= W^\mu(t) - t\left(\partial_\nu V^\mu(t)\right) W^\nu(t) + \theta(t^2) \quad (3)$$

• then $\phi_t^* W^\mu(0) = \dfrac{\partial x^\mu(0)}{\partial x^\nu(t)} \; W^\nu(t) \quad (1)$

Compute $\mathcal{L}_V W$



But $x^\mu(0) = e^{-t\frac{d}{dt}} x^\mu(t) = x^\mu(t) - t\,\dfrac{dx^\mu(t)}{dt} + \theta(t^2)$

$$= x^\mu(t) - t V^\mu(t) + \theta(t^2)$$

$$\Rightarrow \frac{\partial x^\mu(0)}{\partial x^\nu(t)} = \delta_\nu{}^\mu - t\,\partial_\nu V^\mu(t) + \theta(t^2) \quad (2)$$

---

(1), (2) $\Rightarrow$ $\phi_t^* W^\mu(0) = \left[\delta_\nu{}^\mu - t\,\partial_\nu V^\mu(t) + \theta(t^2)\right] W^\nu(t)$

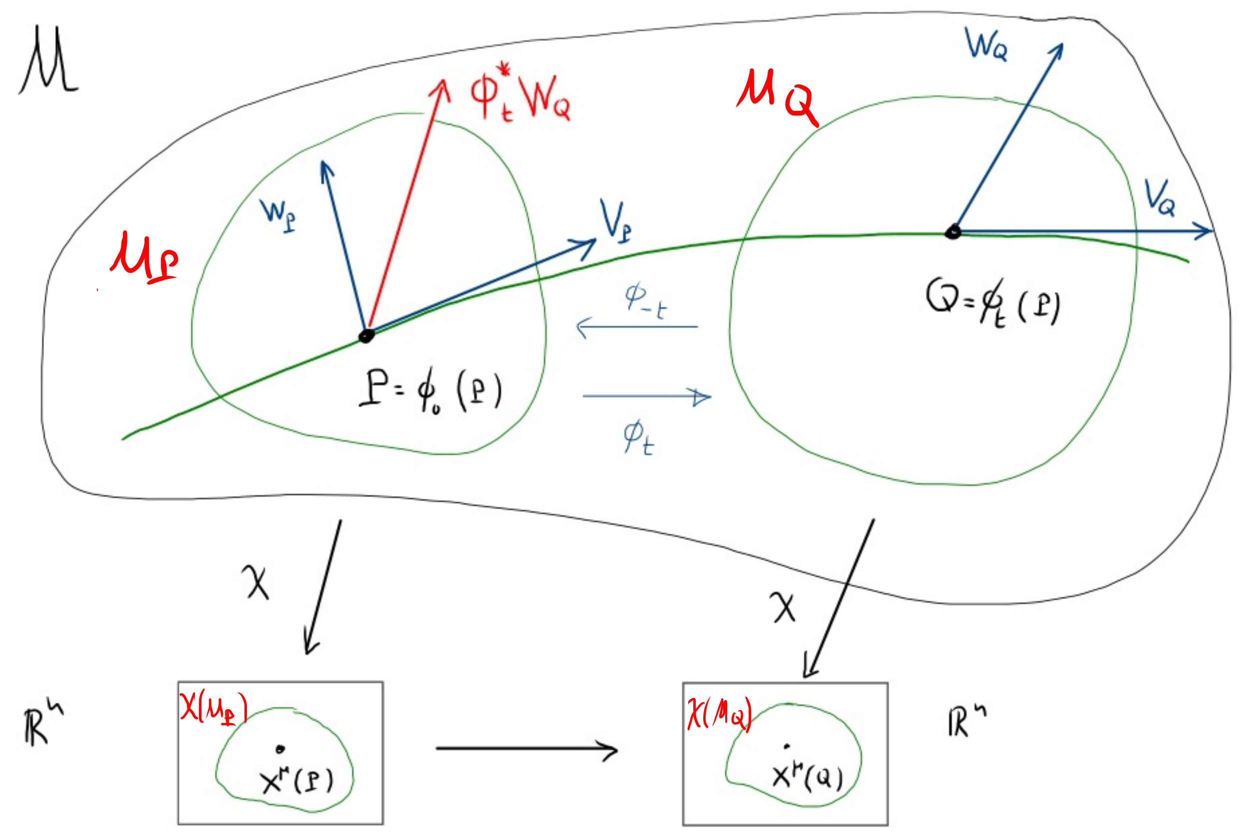
$$= W^\mu(t) - t\left(\partial_\nu V^\mu(t)\right) W^\nu(t) + \theta(t^2) \quad (3)$$

But

$$W^\mu(t) = W^\mu(0) + t\,\frac{d}{dt} W^\mu(0) + \theta(t^2)$$

- then $\phi_t^* W^\mu(0) = \dfrac{\partial x^\mu(0)}{\partial x^\nu(t)}\, W^\nu(t) \quad (1)$

Compute $\mathcal{L}_V W$



But $\quad x^\mu(0) = e^{-t\frac{d}{dt}} x^\mu(t) = x^\mu(t) - t\,\dfrac{dx^\mu(t)}{dt} + \mathcal{O}(t^2)$

$$= x^\mu(t) - t V^\mu(t) + \mathcal{O}(t^2)$$

$$\Rightarrow \frac{\partial x^\mu(0)}{\partial x^\nu(t)} = \delta_\nu{}^\mu - t\,\partial_\nu V^\mu(t) + \mathcal{O}(t^2) \quad (2)$$

$$(1),(2) \Rightarrow \phi_t^* W^\mu(0) = \left[\delta_\nu{}^\mu - t\,\partial_\nu V^\mu(t) + \mathcal{O}(t^2)\right] W^\nu(t)$$

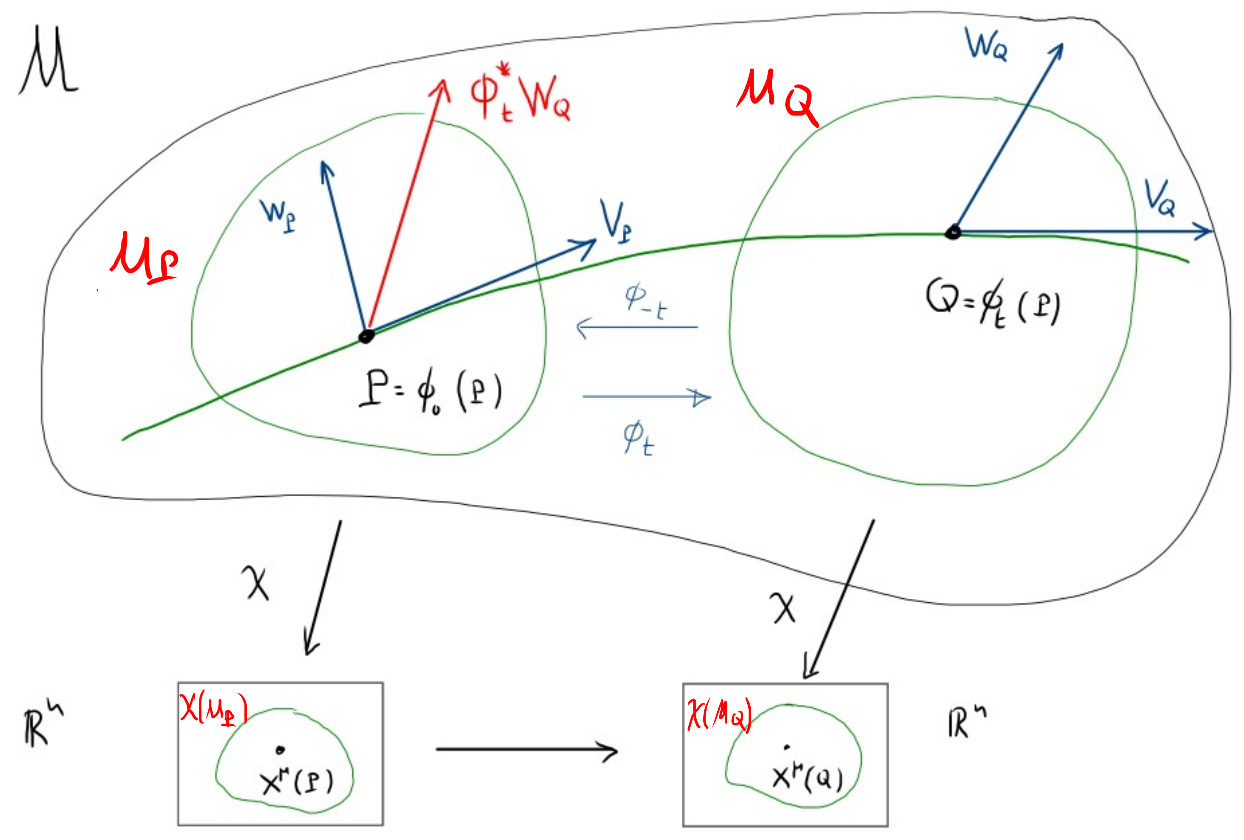
$$= W^\mu(t) - t\left(\partial_\nu V^\mu(t)\right) W^\nu(t) + \mathcal{O}(t^2) \quad (3)$$

But

$$W^\mu(t) = W^\mu(0) + t\,\frac{d}{dt} W^\mu(0) + \mathcal{O}(t^2) = W^\mu(0) + t\,\frac{dx^\nu}{dt}(0)\,\partial_\nu W^\mu(0) + \mathcal{O}(t^2)$$

- then $\quad \phi_t^* W^\mu(0) = \dfrac{\partial x^\mu(0)}{\partial x^\nu(t)}\, W^\nu(t) \quad (1)$

Compute $\mathcal{L}_V W$



But $x^\mu(0) = e^{-t\frac{d}{dt}} x^\mu(t) = x^\mu(t) - t\,\dfrac{dx^\mu(t)}{dt} + \mathcal{O}(t^2)$

$$= x^\mu(t) - t V^\mu(t) + \mathcal{O}(t^2)$$

$$\Rightarrow \frac{\partial x^\mu(0)}{\partial x^\nu(t)} = \delta_\nu{}^\mu - t\,\partial_\nu V^\mu(t) + \mathcal{O}(t^2) \qquad (2)$$

$(1),(2) \Rightarrow \phi_t^* W^\mu(0) = \left[\delta_\nu{}^\mu - t\,\partial_\nu V^\mu(t) + \mathcal{O}(t^2)\right] W^\nu(t)$

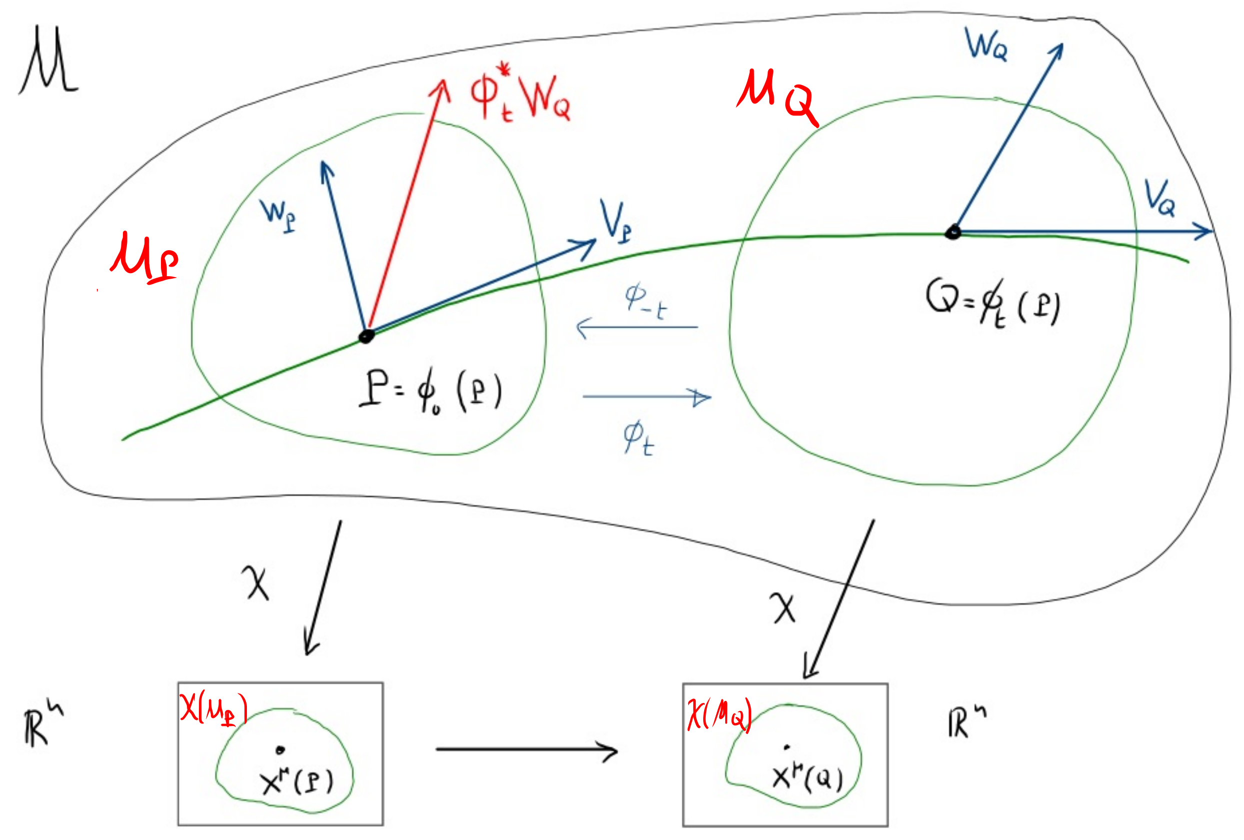
$$= W^\mu(t) - t\left(\partial_\nu V^\mu(t)\right) W^\nu(t) + \mathcal{O}(t^2) \qquad (3)$$

But

$$W^\mu(t) = W^\mu(0) + t\frac{d}{dt}W^\mu(0) + \mathcal{O}(t^2) = W^\mu(0) + t\frac{dx^\nu(0)}{dt}\partial_\nu W^\mu(0) + \mathcal{O}(t^2) = W^\mu(0) + tV^\nu(0)\,\partial_\nu W^\mu(0) + \mathcal{O}(t^2)$$

• then $\phi_t^* W^\mu(0) = \dfrac{\partial x^\mu(0)}{\partial x^\nu(t)} \, W^\nu(t) \qquad (1)$

Compute $\mathcal{L}_V W$



But $x^\mu(0) = e^{-t\frac{d}{dt}} x^\mu(t) = x^\mu(t) - t\,\dfrac{dx^\mu(t)}{dt} + \mathcal{O}(t^2)$

$= x^\mu(t) - t V^\mu(t) + \mathcal{O}(t^2)$

$$\Rightarrow \frac{\partial x^\mu(0)}{\partial x^\nu(t)} = \delta_\nu{}^\mu - t\,\partial_\nu V^\mu(t) + \mathcal{O}(t^2) \quad (2)$$

$(1),(2) \Rightarrow \phi_t^* W^\mu(0) = \left[\delta_\nu{}^\mu - t\,\partial_\nu V^\mu(t) + \mathcal{O}(t^2)\right] W^\nu(t)$

$= W^\mu(t) - t\left(\partial_\nu V^\mu(t)\right) W^\nu(t) + \mathcal{O}(t^2) \quad (3)$

But

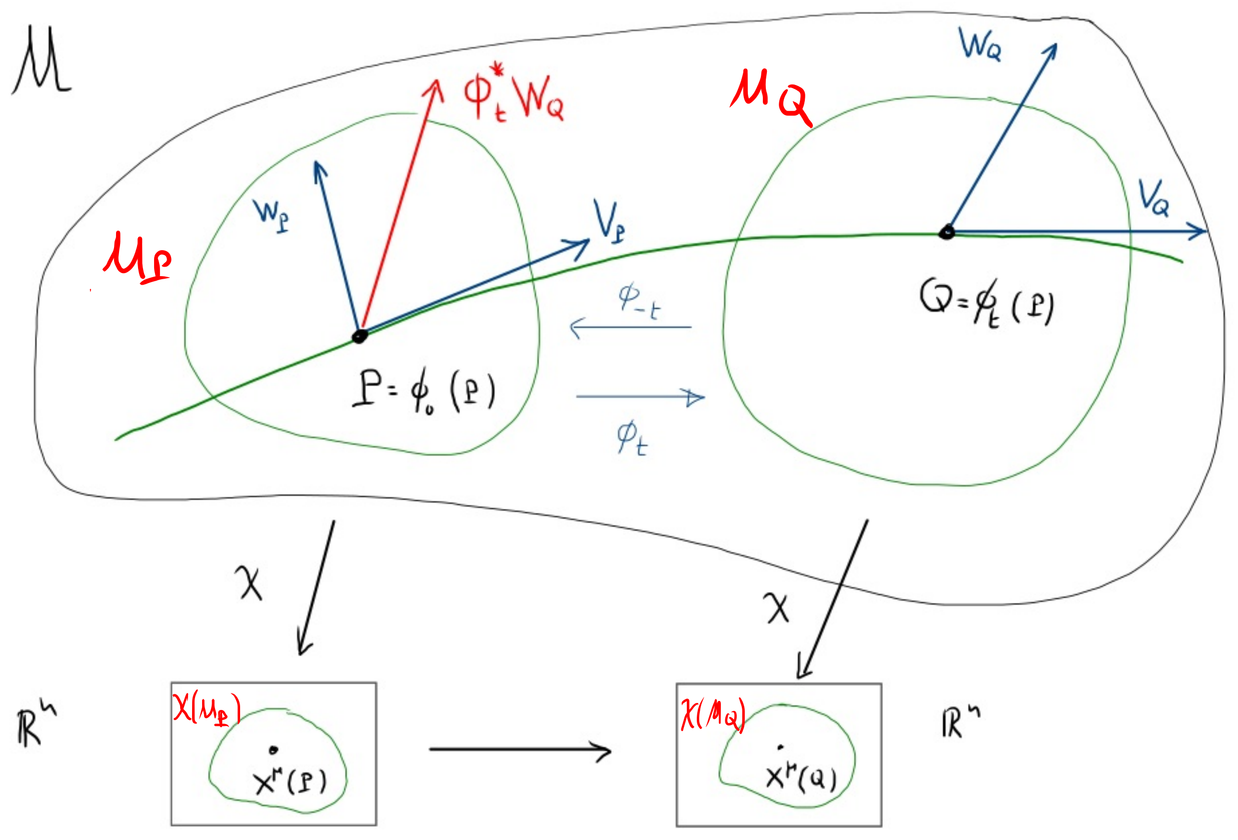
$W^\mu(t) = W^\mu(0) + t\,\dfrac{d}{dt} W^\mu(0) + \mathcal{O}(t^2) = W^\mu(0) + t\,\dfrac{dx^\nu(0)}{dt}\partial_\nu W^\mu(0) + \mathcal{O}(t^2) = W^\mu(0) + t V^\nu(0)\,\partial_\nu W^\mu(0) + \mathcal{O}(t^2)$

and

$-t\,\partial_\nu V^\mu(t) = -t\,\partial_\nu V^\mu(0) + \mathcal{O}(t^2)$

- then $\phi_t^* W^\mu(0) = \dfrac{\partial x^\mu(0)}{\partial x^\nu(t)} W^\nu(t) \quad (1)$

Compute $\mathcal{L}_V W$



But $x^\mu(0) = e^{-t\frac{d}{dt}} x^\mu(t) = x^\mu(t) - t\,\dfrac{dx^\mu(t)}{dt} + \theta(t^2)$

$= x^\mu(t) - t V^\mu(t) + \theta(t^2)$

$\Rightarrow \dfrac{\partial x^\mu(0)}{\partial x^\nu(t)} = \delta_\nu{}^\mu - t\,\partial_\nu V^\mu(t) + \theta(t^2)$ (2)

---

(1), (2) $\Rightarrow$ $\phi_t^* W^\mu(0) = \left[\delta_\nu{}^\mu - t\,\partial_\nu V^\mu(t) + \theta(t^2)\right] W^\nu(t)$

$= W^\mu(t) - t\left(\partial_\nu V^\mu(t)\right) W^\nu(t) + \theta(t^2)$ (3)

But

$W^\mu(t) = W^\mu(0) + t\dfrac{d}{dt}W^\mu(0) + \theta(t^2) = W^\mu(0) + t\dfrac{dx^\nu(0)}{dt}\partial_\nu W^\mu(0) + \theta(t^2) = W^\mu(0) + t V^\nu(0)\,\partial_\nu W^\mu(0) + \theta(t^2)$

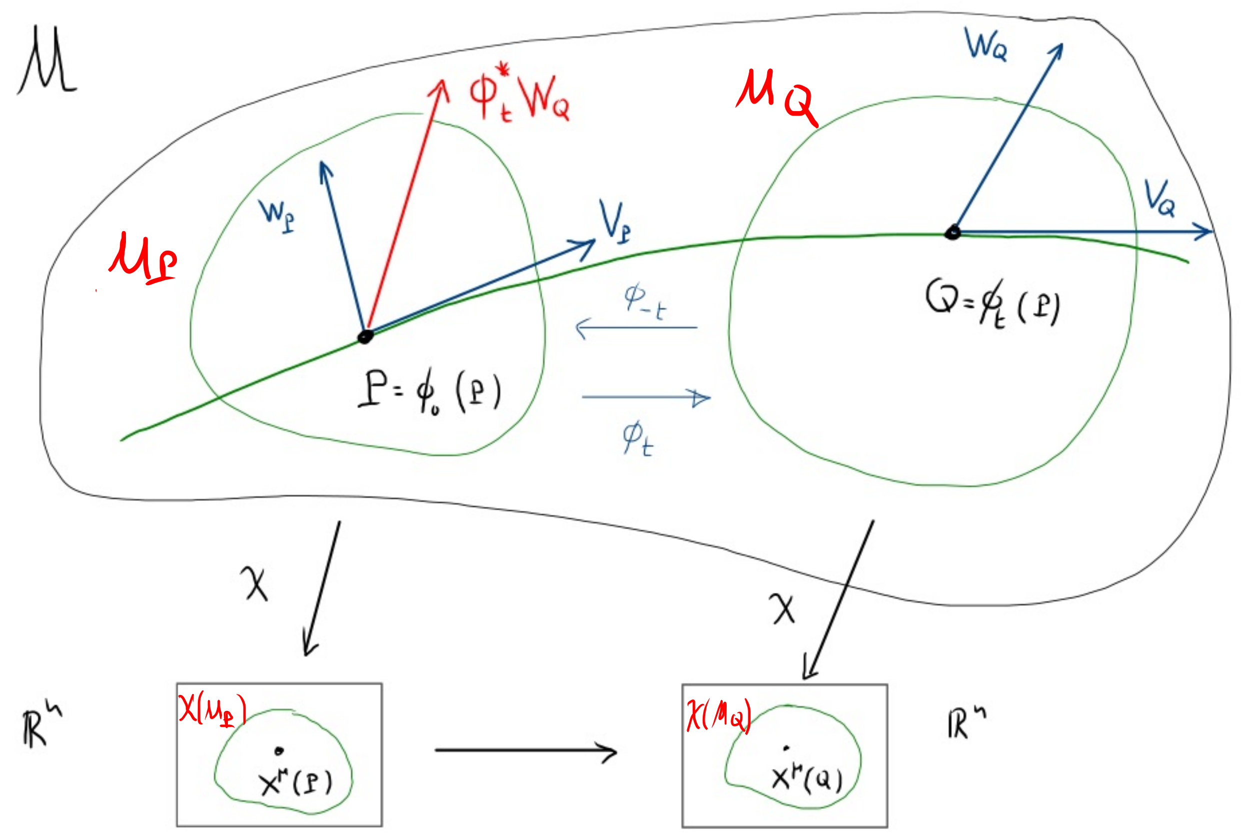
and

$-t\,\partial_\nu V^\mu(t) = -t\,\partial_\nu V^\mu(0) + \theta(t^2)$

Substituting to (3):

$\phi^* W^\mu(0) = \left(W^\mu(0) + t V^\nu \partial_\nu W^\mu(0) + \theta(t^2)\right) - t\left(\partial_\nu V^\mu(0) + \theta(t)\right)\left(W^\nu(0) + \theta(t)\right) + \theta(t^2)$

Compute $\mathcal{L}_V W$



$$\Rightarrow \phi^* W^\mu(0) = W^\mu(0) + t\,(V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu)(0) + O(t^2)$$

$$(1),(2) \Rightarrow \phi_t^* W^\mu(0) = \left[\delta_\nu{}^\mu - t\,\partial_\nu V^\mu(t) + O(t^2)\right] W^\nu(t)$$

$$= W^\mu(t) - t\,(\partial_\nu V^\mu(t))\,W^\nu(t) + O(t^2) \qquad (3)$$

But

$$W^\mu(t) = W^\mu(0) + t\,\frac{d}{dt} W^\mu(0) + O(t^2) = W^\mu(0) + t\,\frac{dx^\nu}{dt}(0)\,\partial_\nu W^\mu(0) + O(t^2) = W^\mu(0) + t\,V^\nu(0)\,\partial_\nu W^\mu(0) + O(t^2)$$

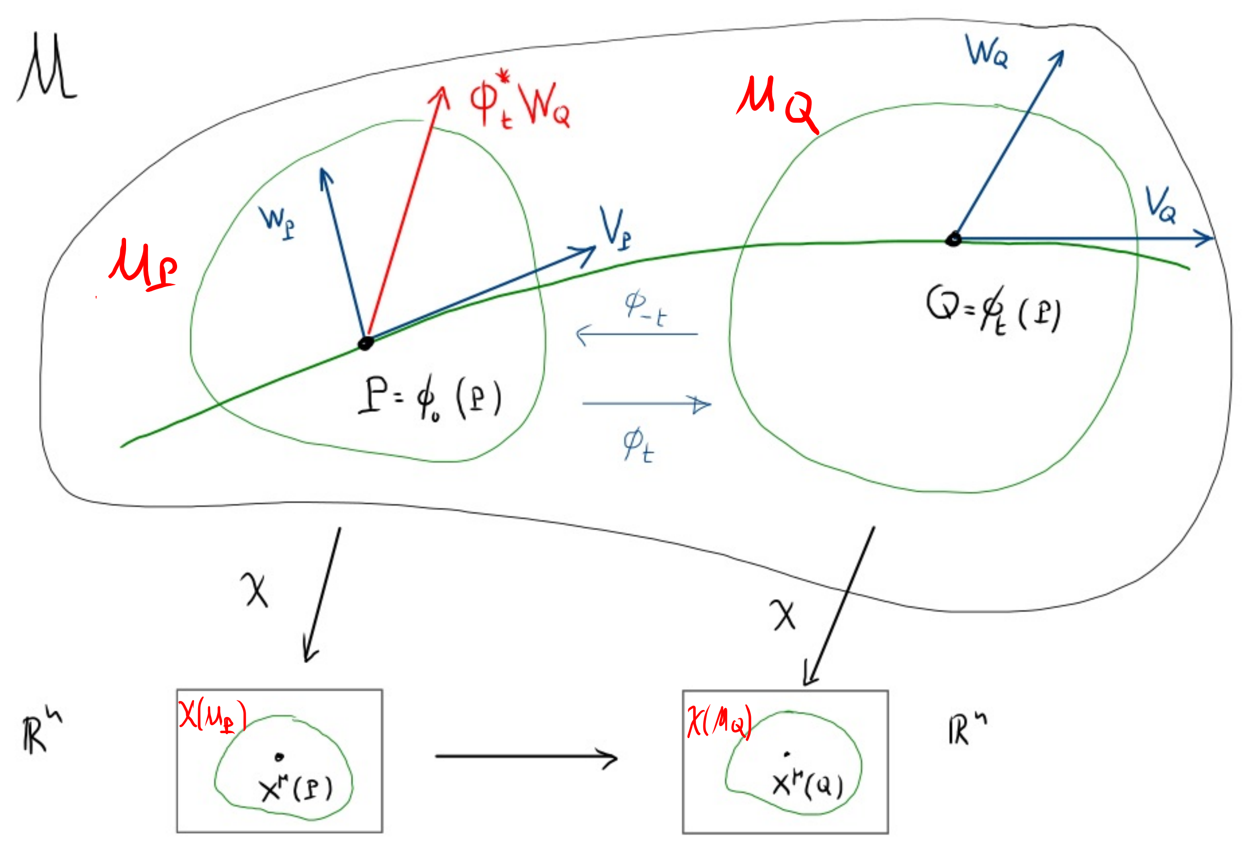
and

$$-t\,\partial_\nu V^\mu(t) = -t\,\partial_\nu V^\mu(0) + O(t^2)$$

Substituting to (3):

$$\phi^* W^\mu(0) = \left(W^\mu(0) + t\,V^\nu \partial_\nu W^\mu(0) + O(t^2)\right) - t\,\left(\partial_\nu V^\mu(0) + O(t)\right)\left(W^\nu(0) + O(t)\right) + O(t^2)$$

Compute $\mathcal{L}_V W$



$$\Rightarrow \phi^* W^\mu(0) = W^\mu(0) + t\left(V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu\right)(0) + \mathcal{O}(t^2)$$

$$\Rightarrow \frac{1}{t}\left[\phi^* W^\mu(0) - W^\mu(0)\right] = \left(V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu\right)(0) + \mathcal{O}(t)$$

$$(1),(2) \Rightarrow \phi_t^* W^\mu(0) = \left[\delta_\nu{}^\mu - t\,\partial_\nu V^\mu(t) + \mathcal{O}(t^2)\right] W^\nu(t)$$

$$= W^\mu(t) - t\left(\partial_\nu V^\mu(t)\right) W^\nu(t) + \mathcal{O}(t^2) \qquad (3)$$

But

$$W^\mu(t) = W^\mu(0) + t\frac{d}{dt}W^\mu(0) + \mathcal{O}(t^2) = W^\mu(0) + t\frac{dx^\nu(0)}{dt}\partial_\nu W^\mu(0) + \mathcal{O}(t^2) = W^\mu(0) + tV^\nu(0)\,\partial_\nu W^\mu(0) + \mathcal{O}(t^2)$$

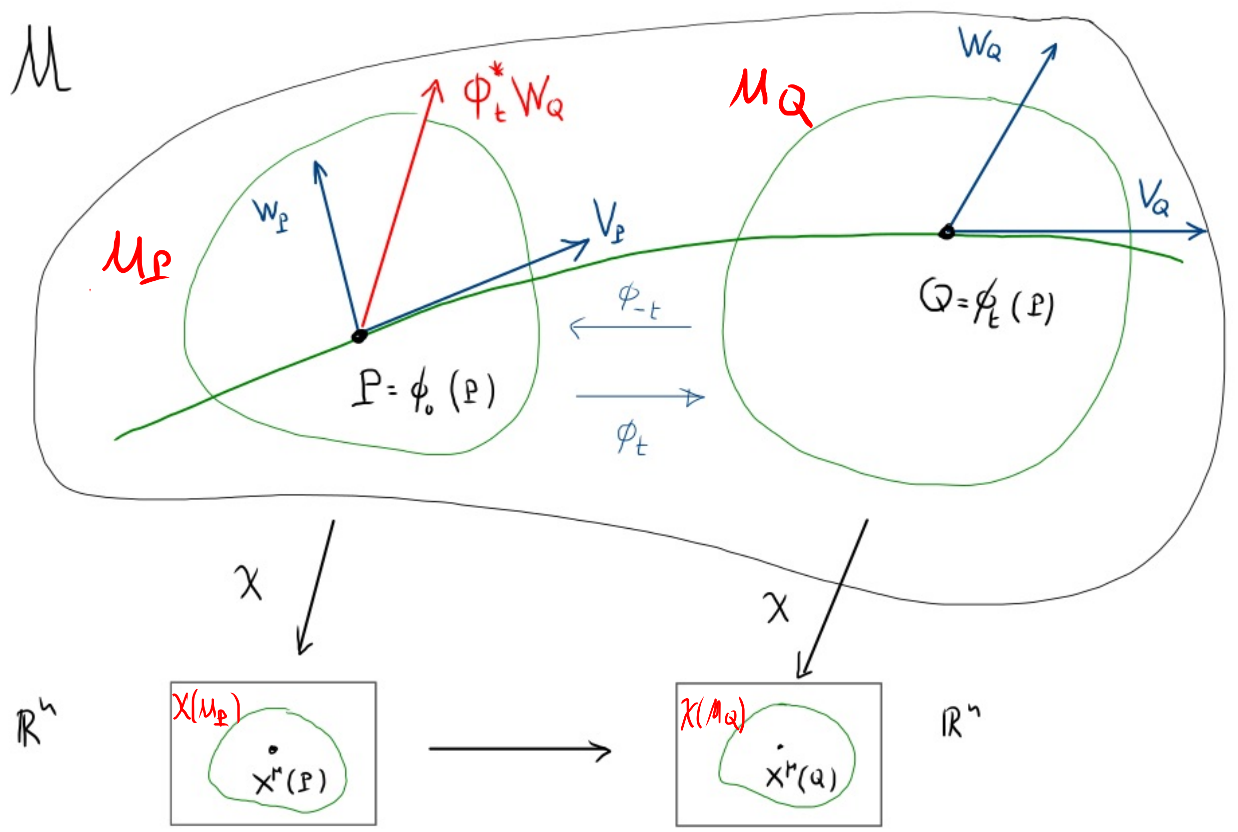
and

$$-t\,\partial_\nu V^\mu(t) = -t\,\partial_\nu V^\mu(0) + \mathcal{O}(t^2)$$

Substituting to (3):

$$\phi^* W^\mu(0) = \left(W^\mu(0) + t\,V^\nu \partial_\nu W^\mu(0) + \mathcal{O}(t^2)\right) - t\left(\partial_\nu V^\mu(0) + \mathcal{O}(t)\right)\left(W^\nu(0) + \mathcal{O}(t)\right) + \mathcal{O}(t^2)$$

Compute $\mathcal{L}_V W$



$$\Rightarrow \phi^* W^\mu(0) = W^\mu(0) + t\,\left(V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu\right)(0) + \mathcal{O}(t^2)$$

$$\Rightarrow \frac{1}{t}\left[\phi^* W^\mu(0) - W^\mu(0)\right] = \left(V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu\right)(0) + \mathcal{O}(t)$$

$$\Rightarrow \lim_{t\to 0} \frac{1}{t}\left[\phi^* W^\mu(0) - W^\mu(0)\right] = [V, W]^\mu(0)$$

$$(1),(2) \Rightarrow \phi_t^* W^\mu(0) = \left[\delta_\nu{}^\mu - t\,\partial_\nu V^\mu(t) + \mathcal{O}(t^2)\right] W^\nu(t)$$

$$= W^\mu(t) - t\left(\partial_\nu V^\mu(t)\right) W^\nu(t) + \mathcal{O}(t^2) \qquad (3)$$

But

$$W^\mu(t) = W^\mu(0) + t\,\frac{d}{dt} W^\mu(0) + \mathcal{O}(t^2) = W^\mu(0) + t\,\frac{dx^\nu(0)}{dt}\partial_\nu W^\mu(0) + \mathcal{O}(t^2) = W^\mu(0) + t V^\nu(0)\partial_\nu W^\mu(0) + \mathcal{O}(t^2)$$

and

$$-t\,\partial_\nu V^\mu(t) = -t\,\partial_\nu V^\mu(0) + \mathcal{O}(t^2)$$

Substituting to (3):

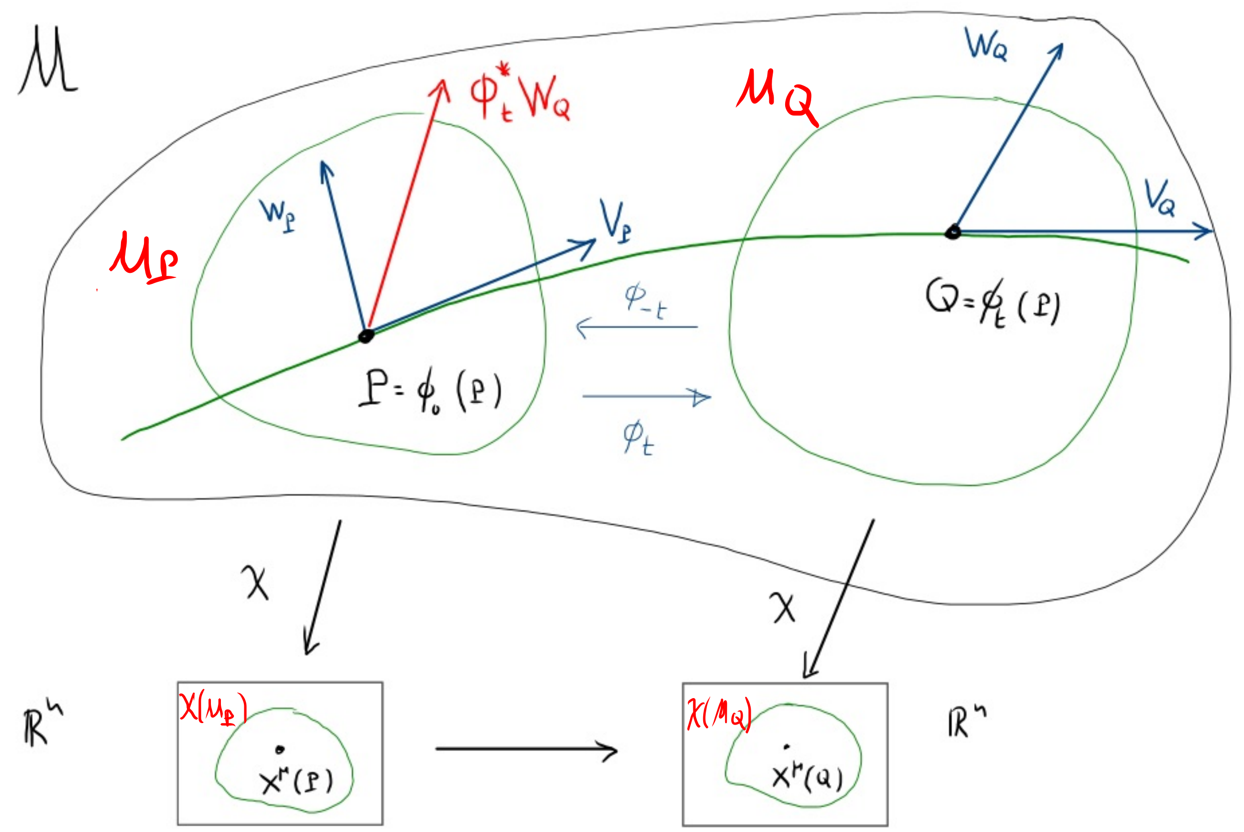
$$\phi^* W^\mu(0) = \left(W^\mu(0) + t\,V^\nu \partial_\nu W^\mu(0) + \mathcal{O}(t^2)\right) - t\left(\partial_\nu V^\mu(0) + \mathcal{O}(t)\right)\left(W^\nu(0) + \mathcal{O}(t)\right) + \mathcal{O}(t^2)$$

Compute $\mathcal{L}_V W$

$$\Rightarrow \phi^* W^\mu(0) = W^\mu(0) + t\left(V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu\right)(0) + \mathcal{O}(t^2)$$

$$\Rightarrow \frac{1}{t}\left[\phi^* W^\mu(0) - W^\mu(0)\right] = \left(V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu\right)(0) + \mathcal{O}(t)$$

$$\Rightarrow \lim_{t\to 0} \frac{1}{t}\left[\phi^* W^\mu(0) - W^\mu(0)\right] = [V, W]^\mu(0)$$



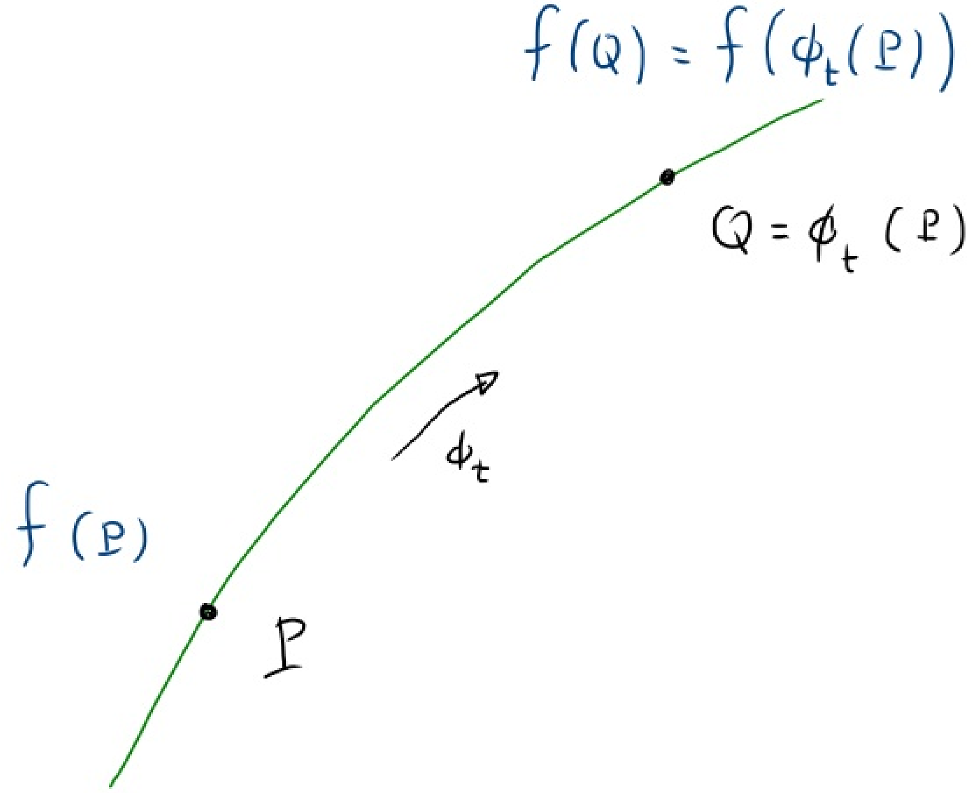
$$\Rightarrow \quad \mathcal{L}_V W = [V, W]$$

$$\mathcal{L}_V W^\mu = V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu \qquad \text{(coordinate basis only)}$$

Compute $\mathcal{L}_v f$

Consider $Q = \phi_t(P)$ and $f : M \to \mathbb{R}$

$$f(Q) = f(\phi_t(P)) = f \circ \phi_t (P) = \phi_t^* f (P)$$

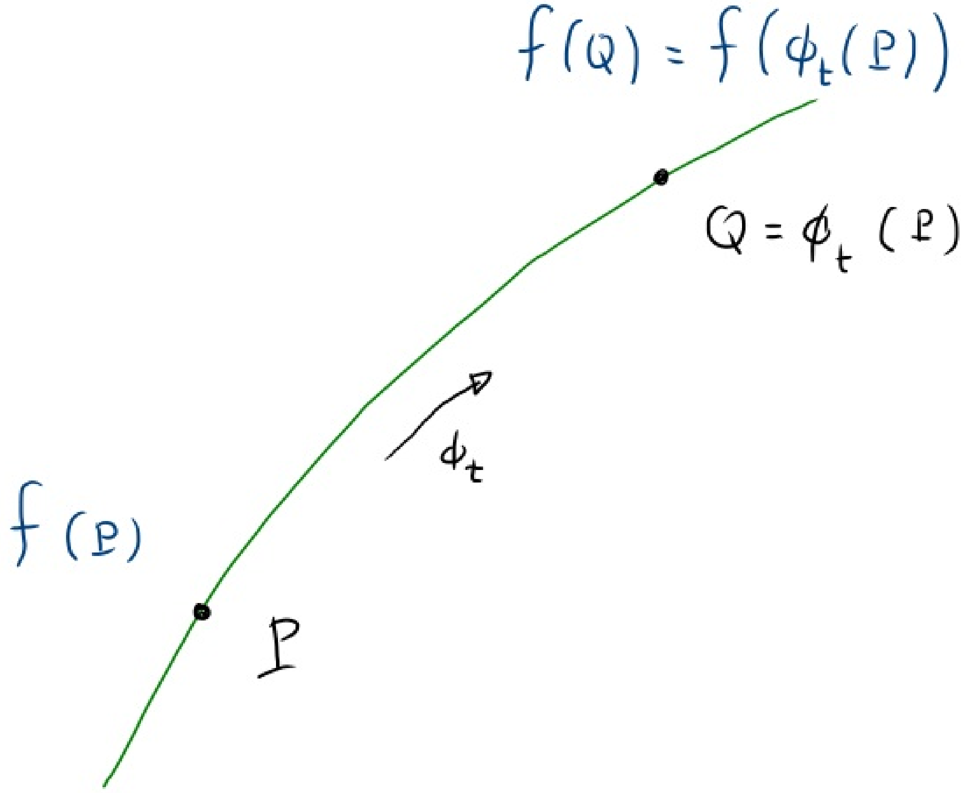


Compute $\mathcal{L}_v f$

Consider $Q = \phi_t(P)$ and $f : M \to \mathbb{R}$

$$f(Q) = f(\phi_t(P)) = f \circ \phi_t \,(P) = \phi_t^* f \,(P)$$

$$(\mathcal{L}_v f)_P = \lim_{t \to 0} \frac{1}{t} \left(\phi_t^* f(P) - f(P) \right) \qquad \text{(definition)}$$



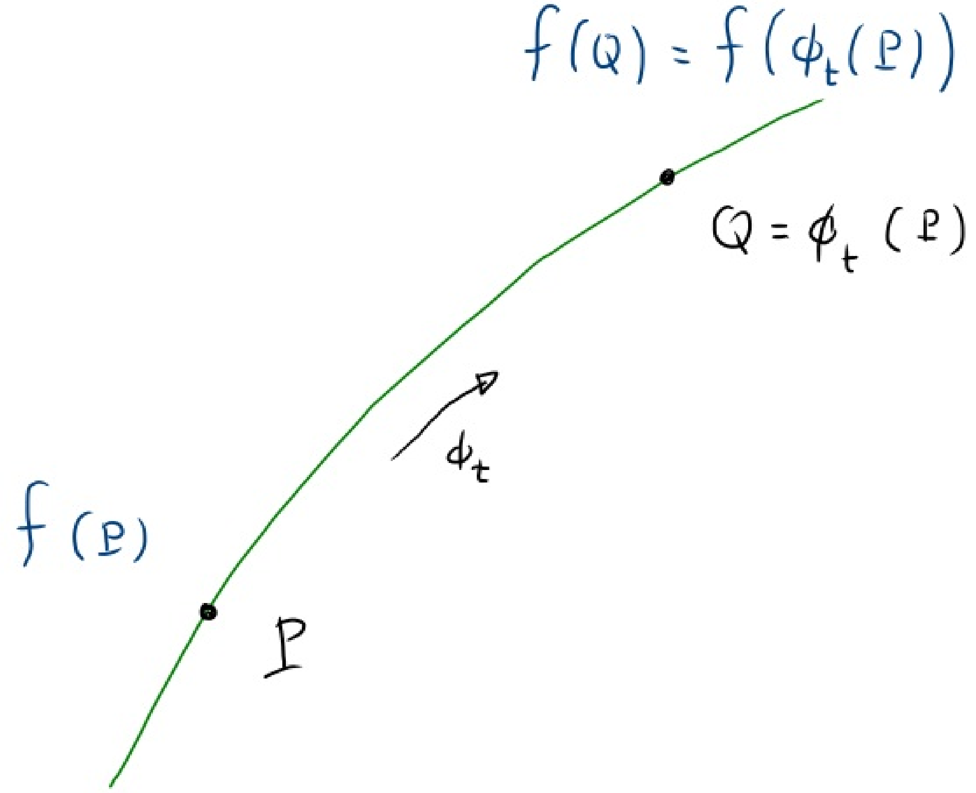
Compute $\mathcal{L}_v f$

Consider $Q = \phi_t(P)$ and $f : M \to \mathbb{R}$

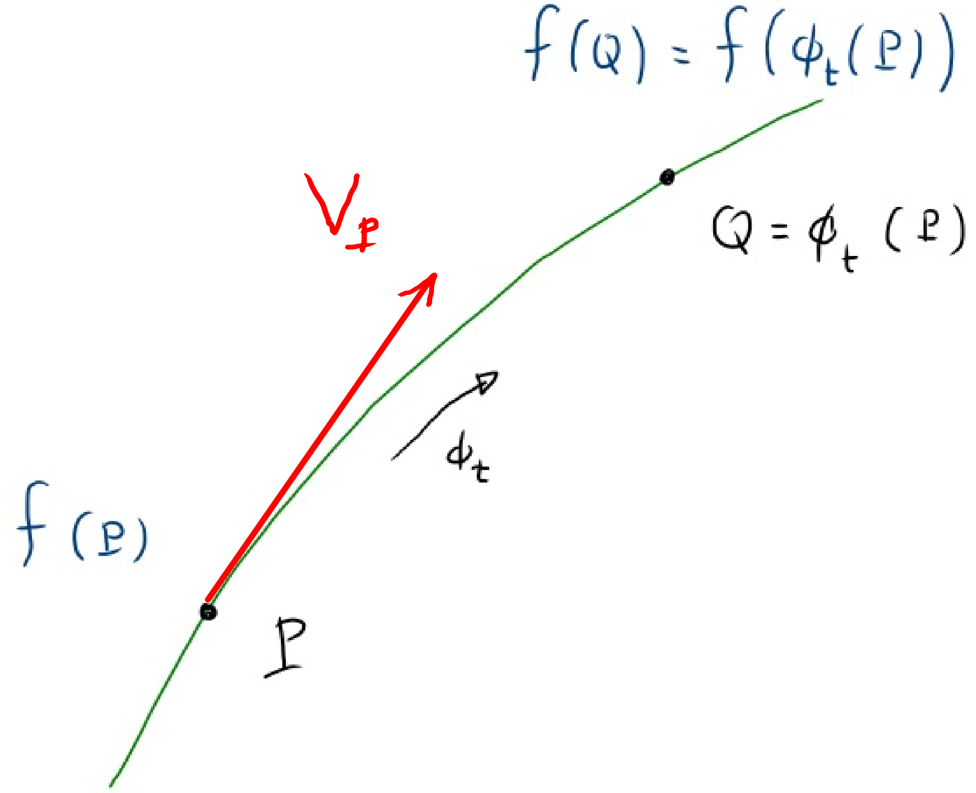
$$f(Q) = f(\phi_t(P)) = f \circ \phi_t\,(P) = \phi_t^* f\,(P)$$

$$(\mathcal{L}_v f)_P = \lim_{t \to 0} \frac{1}{t}\left(\phi_t^* f(P) - f(P)\right) \qquad \text{(definition)}$$

$$= \lim_{t \to 0} \frac{1}{t}\left(f(Q) - f(P)\right)$$



<u>Compute $\mathcal{L}_V f$</u>



Consider $Q = \phi_t(P)$ and $f : M \to \mathbb{R}$

$$f(Q) = f(\phi_t(P)) = f \circ \phi_t\,(P) = \phi_t^* f\,(P)$$

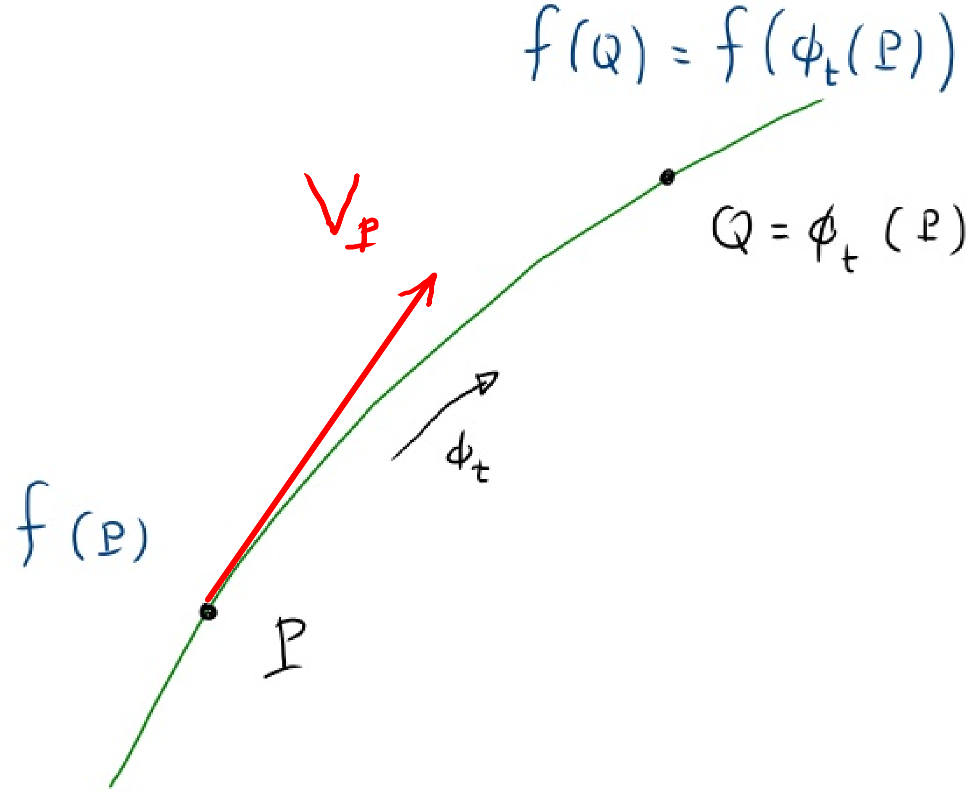
$$(\mathcal{L}_V f)_P = \lim_{t \to 0} \frac{1}{t} \left(\phi_t^* f(P) - f(P) \right) \qquad \text{(definition)}$$

$$= \lim_{t \to 0} \frac{1}{t} \left(f(Q) - f(P) \right) \qquad (1)$$

Consider the curve $\gamma(t) = \phi_t(P)$ s.t. $\gamma(0) = P$, $\gamma(t) = Q$, $V_P = \frac{d}{dt}$

Compute $\mathcal{L}_V f$

Consider $Q = \phi_t(P)$ and $f : M \to \mathbb{R}$



$$f(Q) = f(\phi_t(P)) = f \circ \phi_t(P) = \phi_t^* f(P)$$

$$(\mathcal{L}_V f)_P = \lim_{t \to 0} \frac{1}{t}\left(\phi_t^* f(P) - f(P)\right) \qquad \text{(definition)}$$

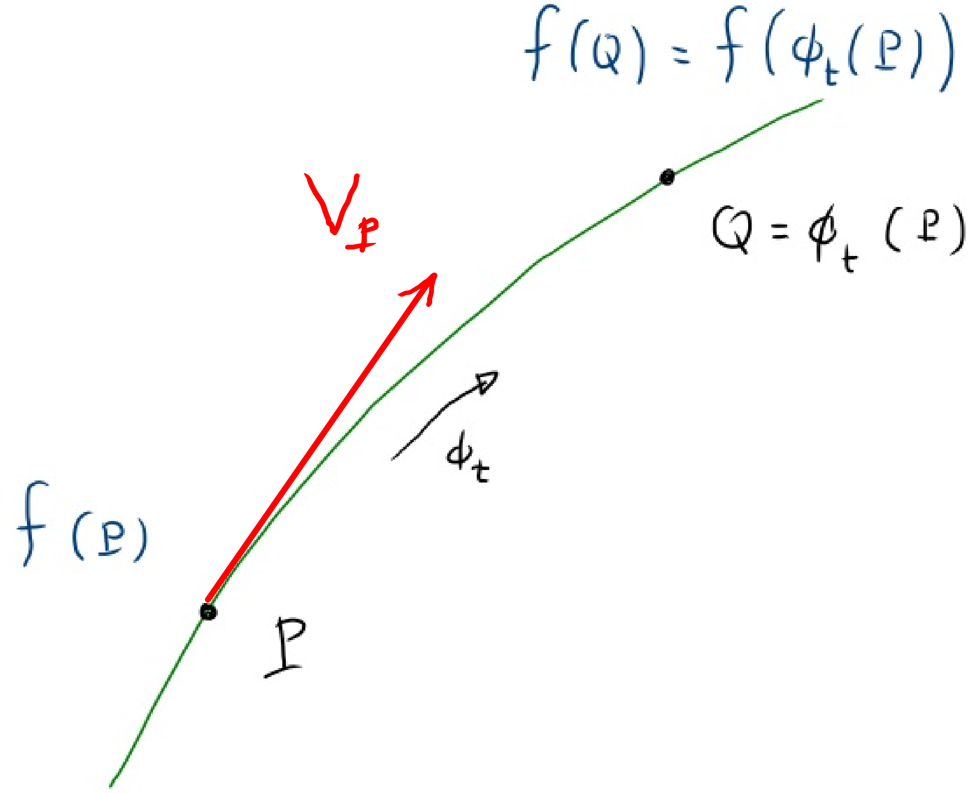
$$= \lim_{t \to 0} \frac{1}{t}\left(f(Q) - f(P)\right) \qquad (1)$$

Consider the curve $\gamma(t) = \phi_t(P)$ s.t. $\gamma(0) = P$, $\gamma(t) = Q$, $V_P = \frac{d}{dt}$

then: $V_P(f) = \frac{d}{dt} f(0) = \lim_{t \to 0} \frac{1}{t}\left(f \circ \gamma(t) - f \circ \gamma(0)\right) = \lim_{t \to 0} \frac{1}{t}\left(f(Q) - f(P)\right)$

Compute $\mathcal{L}_V f$

Consider $Q = \phi_t(P)$ and $f : M \to \mathbb{R}$



$$f(Q) = f(\phi_t(P)) = f \circ \phi_t(P) = \phi_t^* f(P)$$

$$(\mathcal{L}_V f)_P = \lim_{t \to 0} \frac{1}{t} \left(\phi_t^* f(P) - f(P) \right) \qquad \text{(definition)}$$

$$= \lim_{t \to 0} \frac{1}{t} \left(f(Q) - f(P) \right) \qquad (1)$$

Consider the curve $\gamma(t) = \phi_t(P)$ s.t. $\gamma(0) = P$, $\gamma(t) = Q$, $V_P = \frac{d}{dt}$

then: $V_P(f) = \frac{d}{dt} f(0) = \lim_{t \to 0} \frac{1}{t} \left(f \circ \gamma(t) - f \circ \gamma(0) \right) = \lim_{t \to 0} \frac{1}{t} \left(f(Q) - f(P) \right) \qquad (2)$

$$\Rightarrow \quad \mathcal{L}_V f = V(f) = \frac{df}{dt}$$

Compute $\mathcal{L}_V \omega$

Will show that:

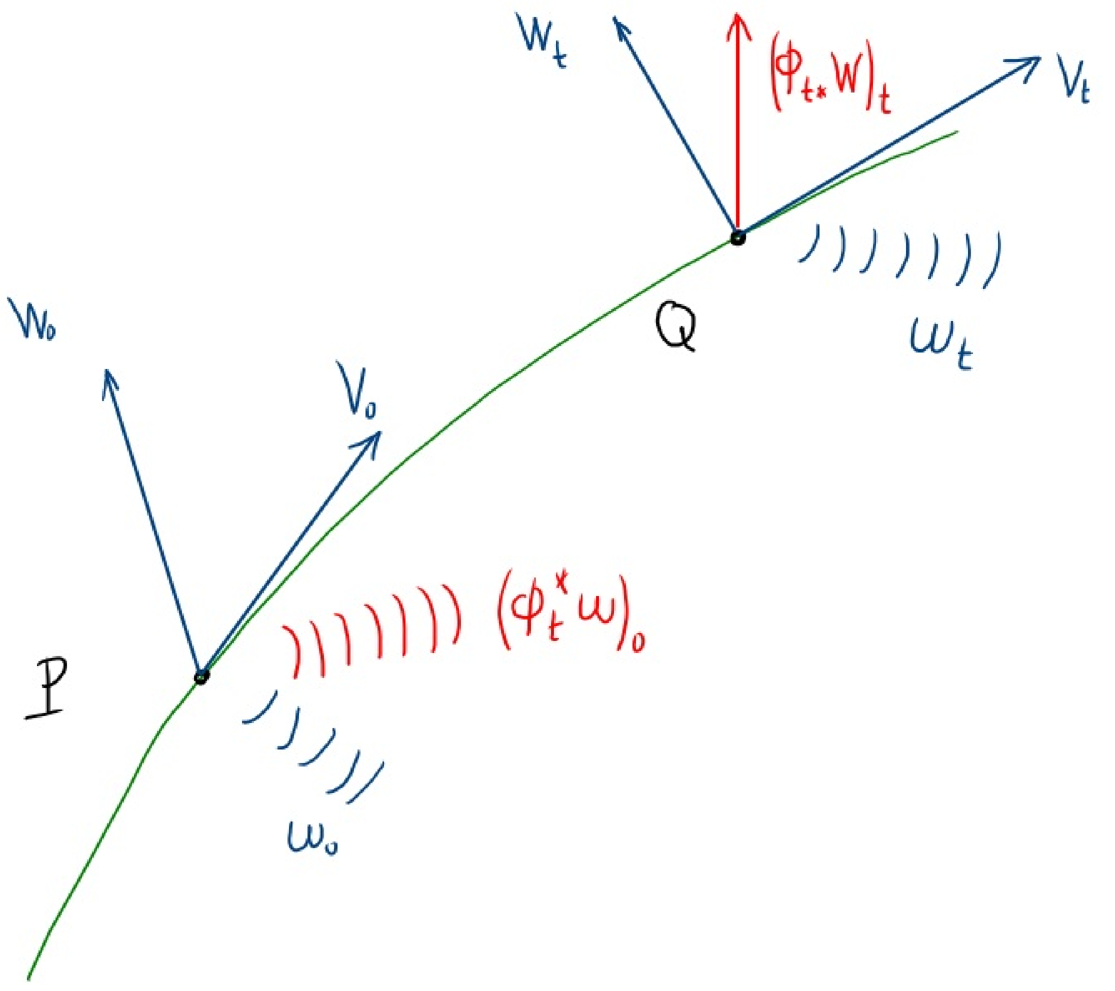
$$\mathcal{L}_V(\omega(W)) = \mathcal{L}_V \omega\,(W) + \omega(\mathcal{L}_V(W))$$

Leibniz-like rule!

a function: we know how to compute: $\frac{d}{dt}(\omega(W))$

defines a one form by its action on any W

Now we know: $[V, W]$

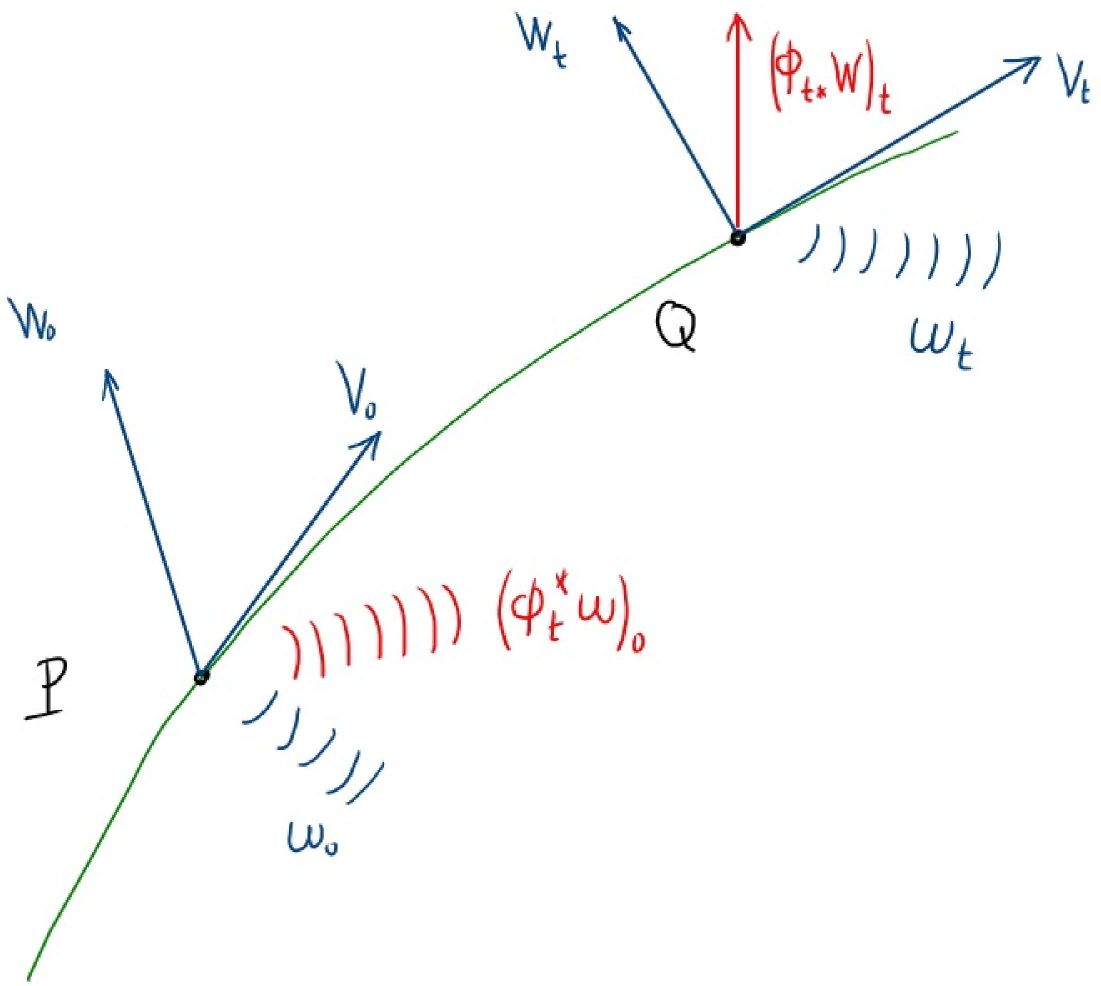


Compute $\mathcal{L}_V \omega$

Will show that:

$$\mathcal{L}_V(\omega(W)) = \mathcal{L}_V\omega(W) + \omega(\mathcal{L}_V(W))$$

Leibnitz-like rule!



a function: we know how to compute: $\frac{d}{dt}(\omega(W))$

defines a one form by its action on any W

Now we know: $[V, W]$

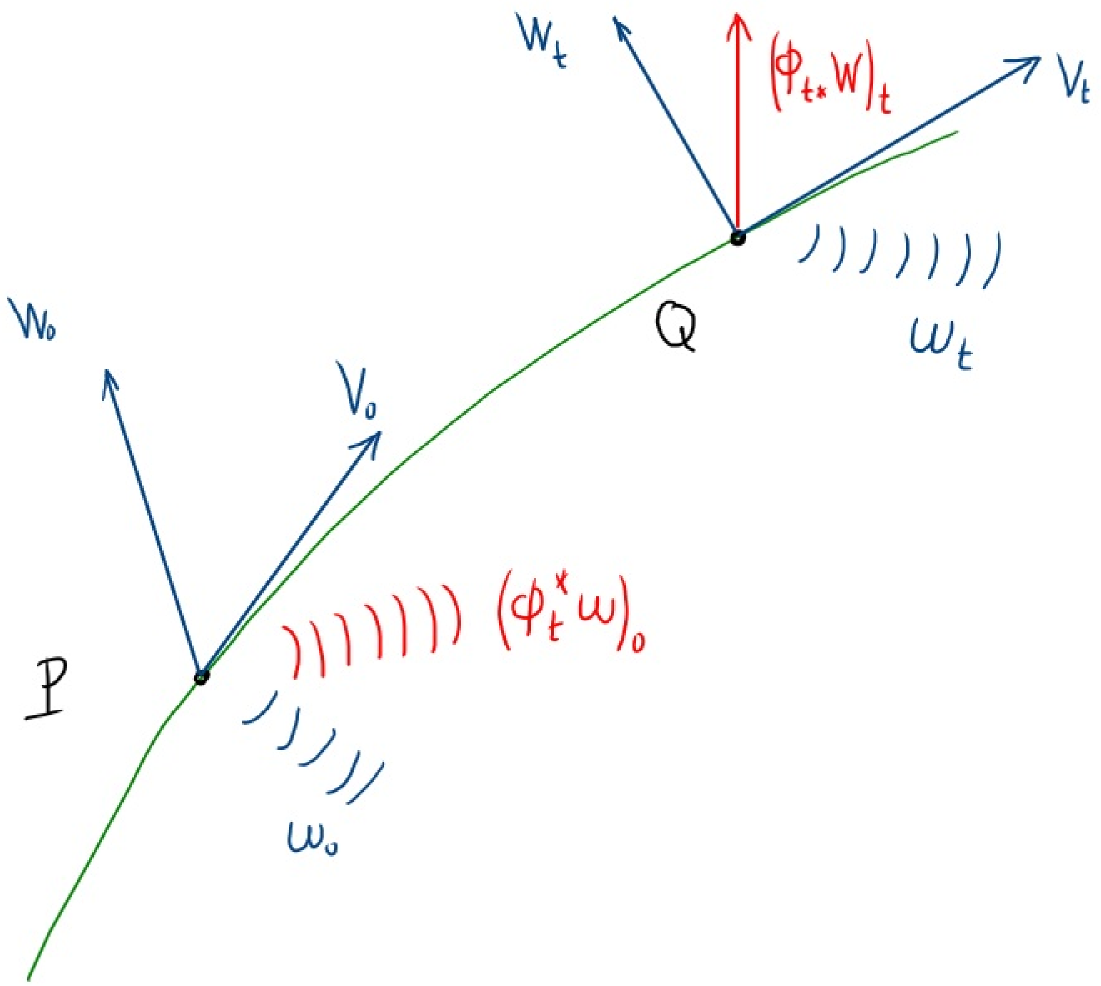
- From this relation we computed: (previous video) $(\mathcal{L}_V\omega)_\mu = V^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu V^\nu$

<u>Compute $\mathcal{L}_V \omega$</u>

$$\mathcal{L}_V(\omega(W)) = \mathcal{L}_V\omega(W) + \omega(\mathcal{L}_V W)$$

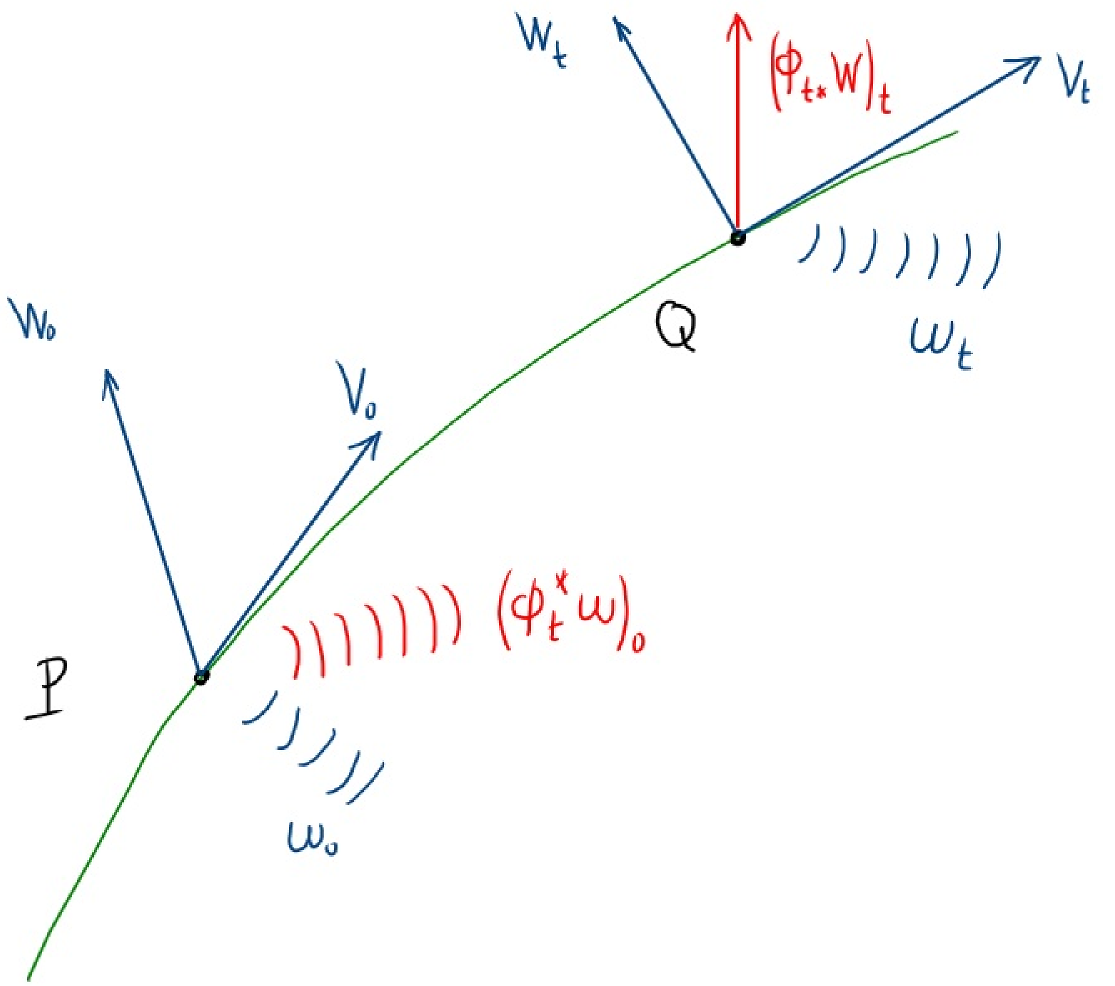
$$(\mathcal{L}_V\omega)_0 = \lim_{t\to 0}\frac{1}{t}\left[(\phi_t^*\omega)_0 - \omega_0\right] \Rightarrow$$

$$\mathcal{L}_V\omega(W)\Big|_0 = \lim_{t\to 0}\frac{1}{t}\left[(\phi_t^*\omega)_0(W_0) - \omega_0(W_0)\right]$$



Compute $\mathcal{L}_V \omega$

$$\mathcal{L}_V(\omega(W)) = \mathcal{L}_V\omega(W) + \omega(\mathcal{L}_V W)$$



$$(\mathcal{L}_V\omega)_0 = \lim_{t\to 0}\frac{1}{t}\left[(\phi_t^*\omega)_0 - \omega_0\right] \Rightarrow$$

$$\mathcal{L}_V\omega(W)\Big|_0 = \lim_{t\to 0}\frac{1}{t}\left[(\phi_t^*\omega)_0(W_0) - \omega_0(W_0)\right]$$

$$= \lim_{t\to 0}\frac{1}{t}\left[\ \omega_t\big((\phi_{*t}W)_t\big) - \omega_0(W_0)\right]$$

Compute $\mathcal{L}_V \omega$

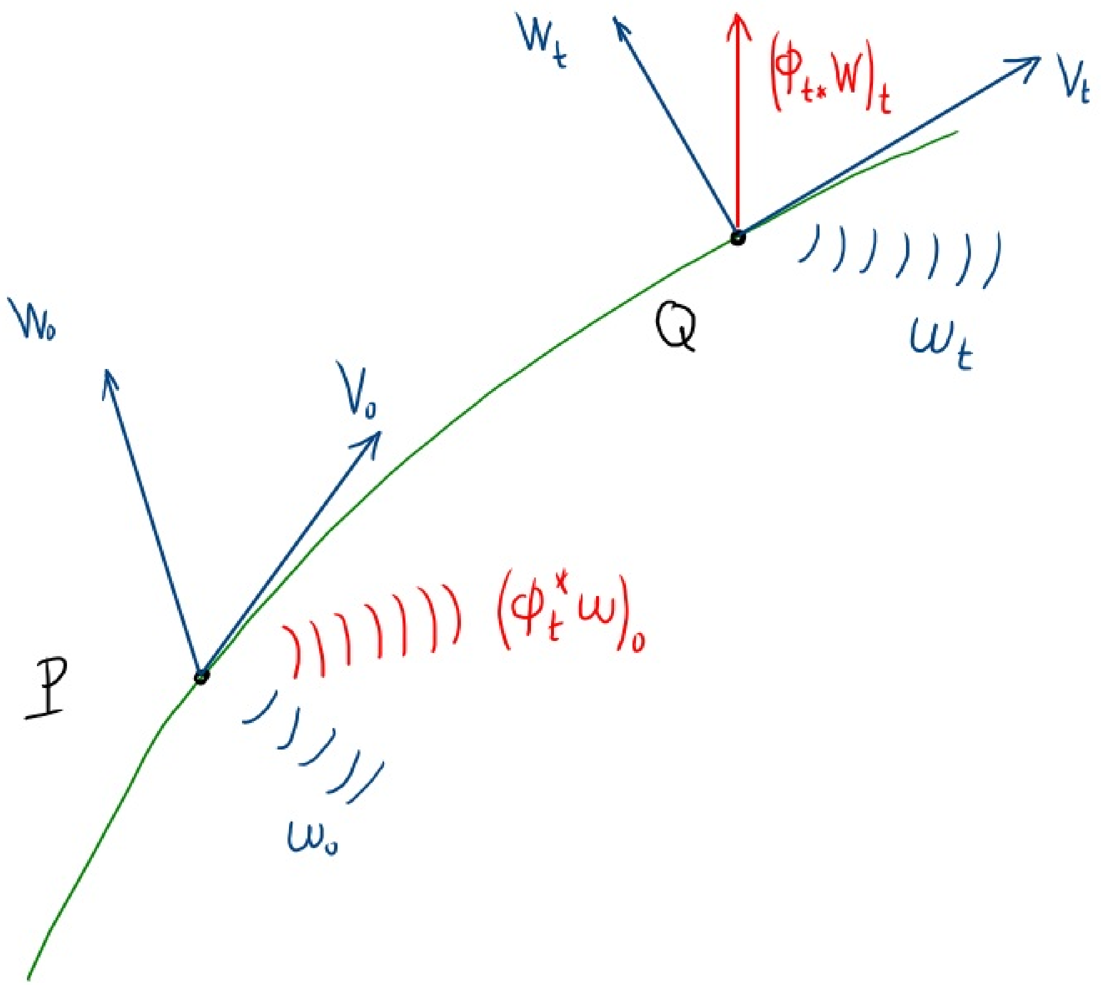
$$\mathcal{L}_V(\omega(W)) = \mathcal{L}_V\omega(W) + \omega(\mathcal{L}_V W)$$

$$(\mathcal{L}_V\omega)_0 = \lim_{t\to 0}\frac{1}{t}\left[(\phi_t^*\omega)_0 - \omega_0\right] \Rightarrow$$

$$\mathcal{L}_V\omega(W)\big|_0 = \lim_{t\to 0}\frac{1}{t}\left[(\phi_t^*\omega)_0(W_0) - \omega_0(W_0)\right]$$

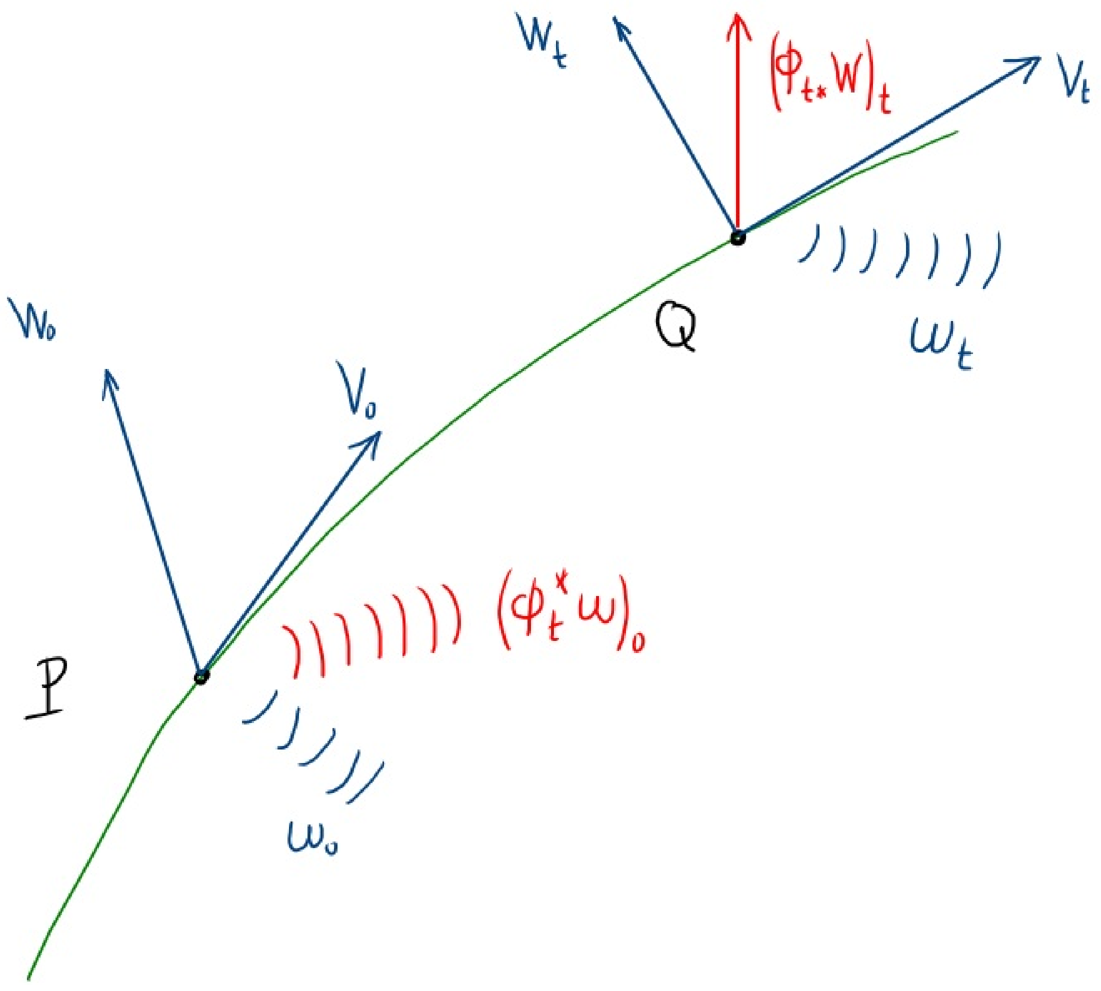
$$= \lim_{t\to 0}\frac{1}{t}\left[\omega_t((\phi_{*t}W)_t) - \omega_0(W_0)\right]$$

$$= \lim_{t\to 0}\frac{1}{t}\left[\omega_t((\phi_{-t}^* W)_t) - \omega_t(W_t) + \omega_t(W_t) - \omega_0(W_0)\right]$$



Compute $\mathcal{L}_V \omega$

$$\mathcal{L}_V(\omega(W)) = \mathcal{L}_V\omega(W) + \omega(\mathcal{L}_V W)$$



$$(\mathcal{L}_V \omega)_0 = \lim_{t\to 0} \frac{1}{t}\left[(\phi_t^*\omega)_0 - \omega_0\right] \Rightarrow$$

$$\mathcal{L}_V\omega(W)\Big|_0 = \lim_{t\to 0} \frac{1}{t}\left[(\phi_t^*\omega)_0(W_0) - \omega_0(W_0)\right]$$

$$= \lim_{t\to 0} \frac{1}{t}\left[\omega_t((\phi_{*t}W)_t) - \omega_0(W_0)\right]$$

$$= \lim_{t\to 0} \frac{1}{t}\left[\omega_t((\phi^*_{-t}W)_t) - \omega_t(W_t) + \omega_t(W_t) - \omega_0(W_0)\right]$$

$$= \lim_{t\to 0} \omega_t\left(\frac{1}{t}\left[(\phi^*_{-t}W)_t - W_t\right]\right) + \lim_{t\to 0}\frac{1}{t}\left[\omega_t(W_t) - \omega_0(W_0)\right]$$

used linearity

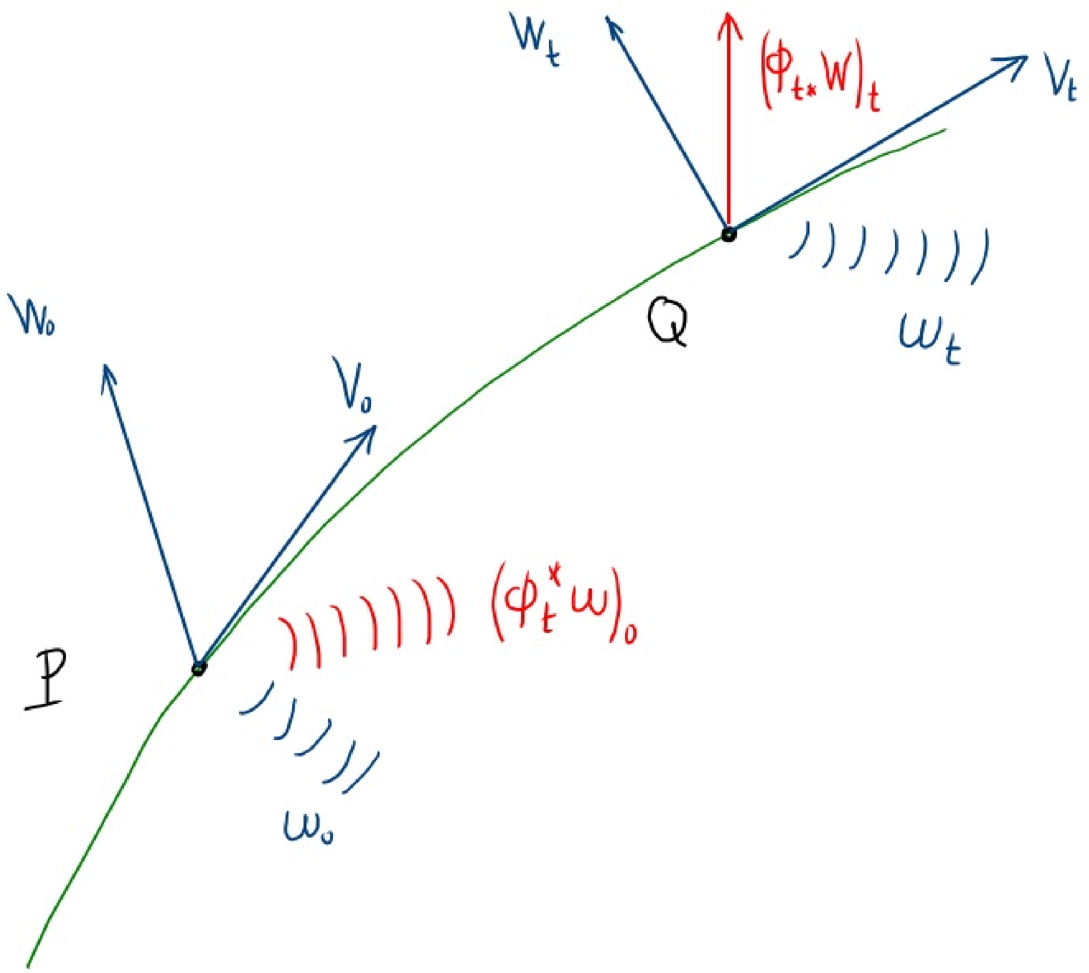
$\omega(\alpha V + \beta W) = \alpha\,\omega(V) + \beta\,\omega(W)$

↳ a function

$= \mathcal{L}_V\,\omega(W)$

Compute $\mathcal{L}_V \omega$

$$\mathcal{L}_V(\omega(W)) = \mathcal{L}_V\omega(W) + \omega(\mathcal{L}_V W)$$



$$(\mathcal{L}_V \omega)_0 = \lim_{t\to 0} \frac{1}{t}\left[(\phi_t^*\omega)_0 - \omega_0\right] \Rightarrow$$

$$\mathcal{L}_V\omega(W)\Big|_0 = \lim_{t\to 0} \frac{1}{t}\left[(\phi_t^*\omega)_0(W_0) - \omega_0(W_0)\right]$$

$$= \lim_{t\to 0} \frac{1}{t}\left[\omega_t((\phi_{*t}W)_t) - \omega_0(W_0)\right]$$

$$= \lim_{t\to 0} \frac{1}{t}\left[\omega_t((\phi_{-t}^* W)_t) - \omega_t(W_t) + \omega_t(W_t) - \omega_0(W_0)\right]$$

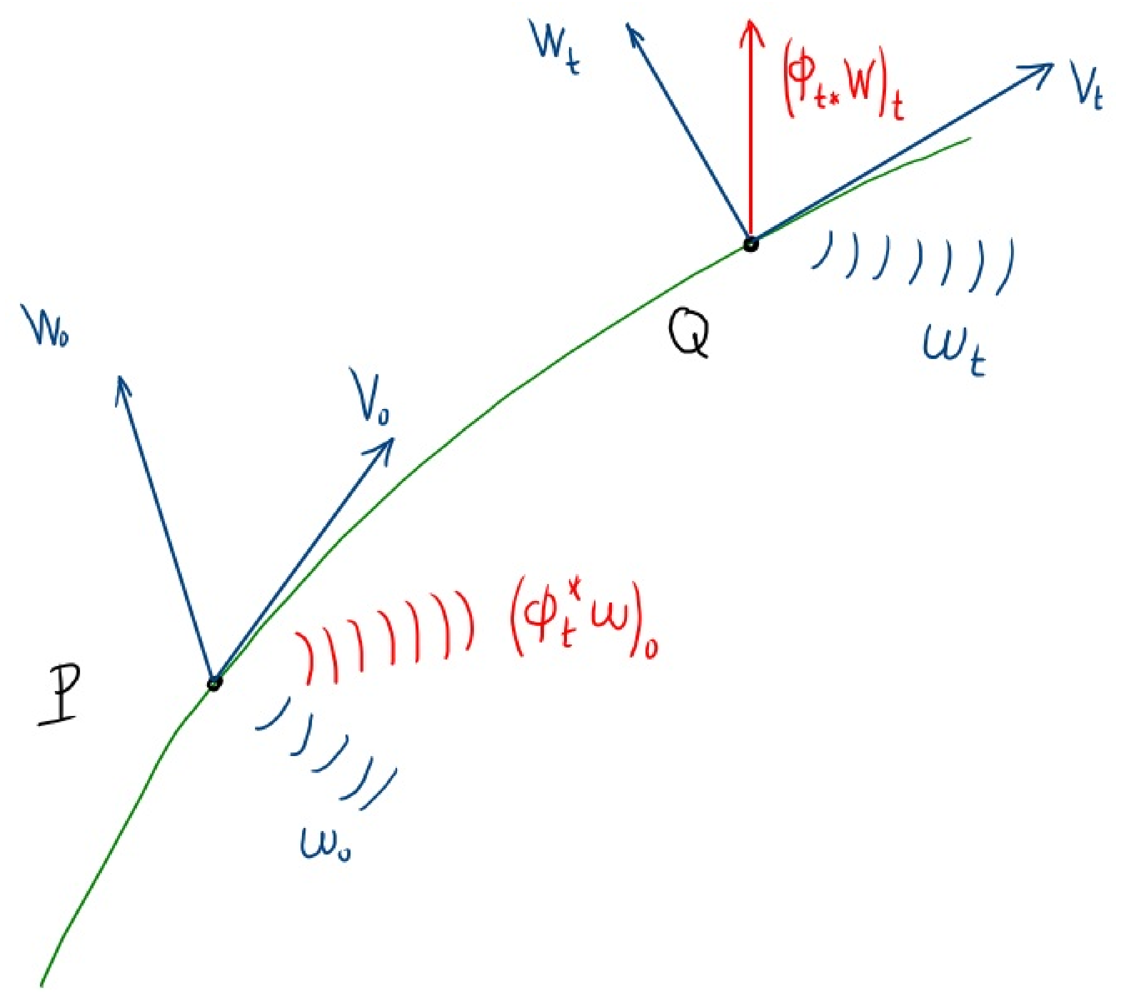
$$= \lim_{t\to 0} \omega_t\left(\frac{1}{t}\left[(\phi_{-t}^* W)_t - W_t\right]\right) + \lim_{t\to 0}\frac{1}{t}\left[\omega_t(W_t) - \omega_0(W_0)\right]$$

- $\lim_{t\to 0} \frac{1}{t}\left[\omega_t(W_t) - \omega_0(W_0)\right] = \frac{d}{dt}\omega(W)\Big|_0 = (\mathcal{L}_V(\omega(W)))_0$
- $\lim_{t\to 0} \omega_t = \omega_0$
- $\lim_{t\to 0} \frac{1}{t}\left[(\phi_{-t}^* W)_t - W_t\right] = (\mathcal{L}_{-V} W)_0 = -(\mathcal{L}_V W)_0$

$\hookrightarrow \mathcal{L}_{-V} W = [-V, W] = -[V, W] = -\mathcal{L}_V W$

Compute $\mathcal{L}_V \omega$

$$\mathcal{L}_V(\omega(W)) = \mathcal{L}_V\omega(W) + \omega(\mathcal{L}_V W)$$



$$(\mathcal{L}_V\omega)_0 = \lim_{t\to 0}\frac{1}{t}\left[(\phi_t^*\omega)_0 - \omega_0\right] \Rightarrow$$

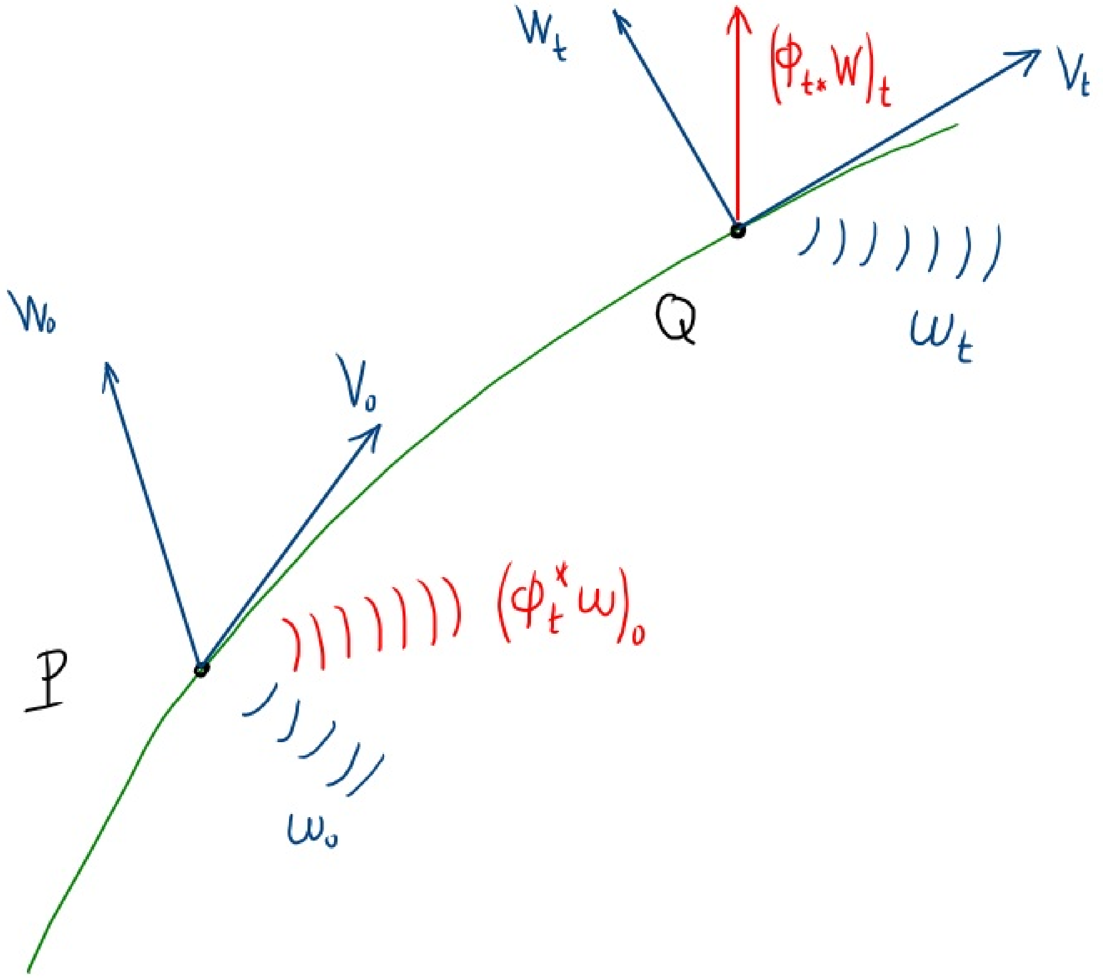
$$\mathcal{L}_V\omega(W)\big|_0 = -\omega_0\left((\mathcal{L}_V W)_0\right) + \left(\mathcal{L}_V(\omega(W))\right)_0$$

$$= \lim_{t\to 0}\omega_t\left(\frac{1}{t}\left[(\phi_{-t}^*W)_t - W_t\right]\right) + \lim_{t\to 0}\frac{1}{t}\left[\omega_t(W_t) - \omega_0(W_0)\right]$$

- $\lim_{t\to 0}\frac{1}{t}\left[\omega_t(W_t) - \omega_0(W_0)\right] = \frac{d}{dt}\omega(W)\Big|_0 = \left(\mathcal{L}_V(\omega(W))\right)_0$
- $\lim_{t\to 0}\omega_t = \omega_0$
- $\lim_{t\to 0}\frac{1}{t}\left[(\phi_{-t}^*W)_t - W_t\right] = (\mathcal{L}_{-V}W)_0 = -(\mathcal{L}_V W)_0$

Compute $\mathcal{L}_V \omega$

$$\mathcal{L}_V(\omega(W)) = \mathcal{L}_V\omega(W) + \omega(\mathcal{L}_V W)$$



$$(\mathcal{L}_V\omega)_0 = \lim_{t\to 0}\frac{1}{t}\left[(\phi_t^*\omega)_0 - \omega_0\right] \Rightarrow$$

$$\mathcal{L}_V\omega(W)\Big|_0 = -\omega_0\left((\mathcal{L}_V W)_0\right) + \left(\mathcal{L}_V(\omega(W))\right)_0 \quad \Rightarrow$$

$$\mathcal{L}_V(\omega(W)) = \mathcal{L}_V\omega(W) + \omega(\mathcal{L}_V W)$$

$$= \lim_{t\to 0}\omega_t\left(\frac{1}{t}\left[(\phi_{-t}^*W)_t - W_t\right]\right) + \lim_{t\to 0}\frac{1}{t}\left[\omega_t(W_t) - \omega_0(W_0)\right]$$

- $\lim\limits_{t\to 0}\frac{1}{t}\left[\omega_t(W_t) - \omega_0(W_0)\right] = \frac{d}{dt}\omega(W)\Big|_0 = \left(\mathcal{L}_V(\omega(W))\right)_0$
- $\lim\limits_{t\to 0}\omega_t = \omega_0$
- $\lim\limits_{t\to 0}\frac{1}{t}\left[(\phi_{-t}^*W)_t - W_t\right] = (\mathcal{L}_{-V}W)_0 = -(\mathcal{L}_V W)_0$

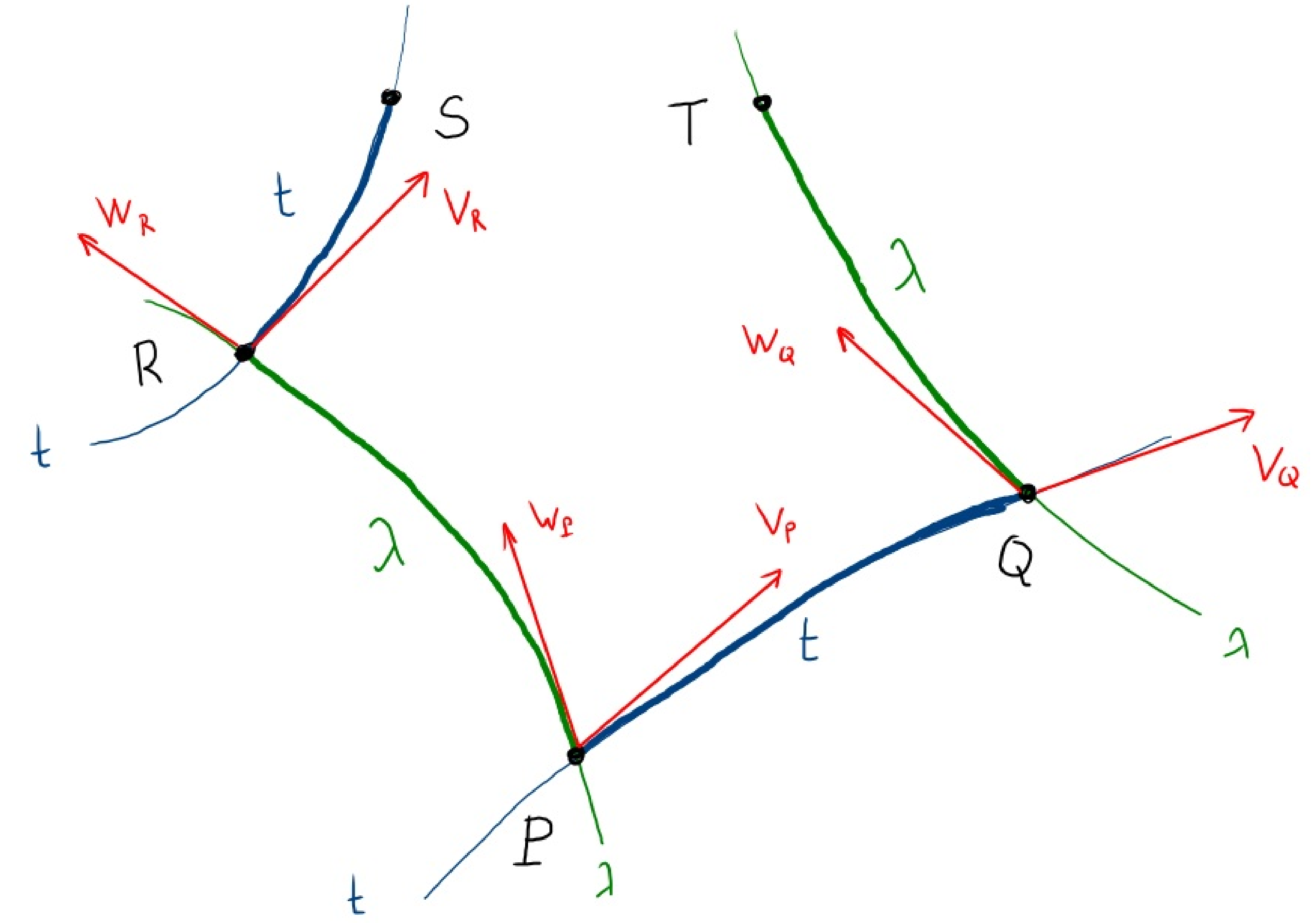
Geometric Interpretation of $\mathcal{L}_V W = [V, W]$

First consider $[V, W]$

- vector fields V, W

  V : t - lines

  W : λ - lines



- move along integral lines of $V + W$:

  (1) $P \to Q \to T$

  (2) $P \to R \to S$

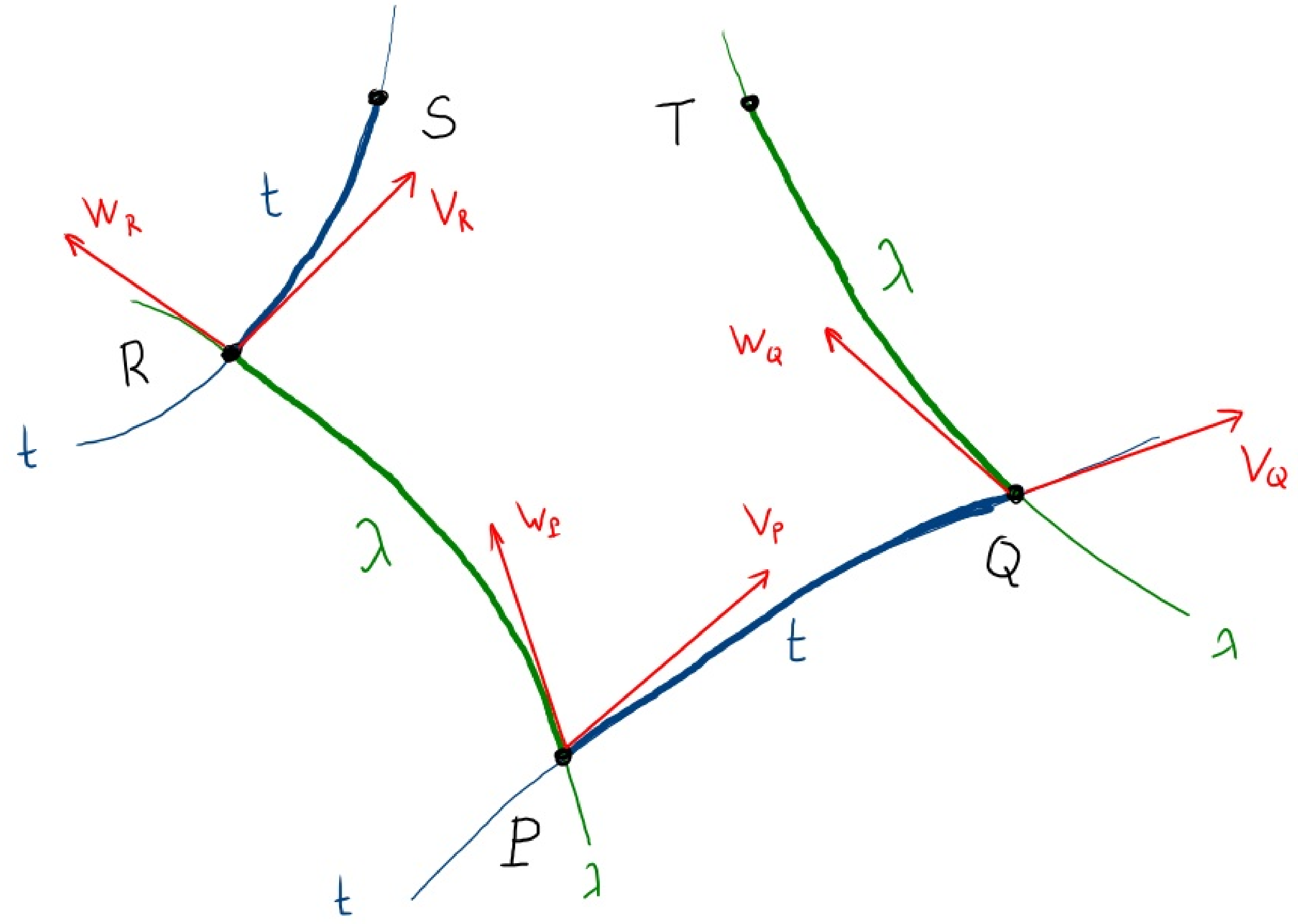
- will show that for $\epsilon = \lambda = t$

$$\begin{cases} f(T) - f(S) = \epsilon^2 [V, W]_P(f) + \mathcal{O}(\epsilon^3) \\ x^\mu(T) - x^\mu(S) = \epsilon^2 [V, W]^\mu_P + \mathcal{O}(\epsilon^3) \end{cases}$$

Geometric Interpretation of $\mathcal{L}_V W = [V, W]$

$Q = \phi_t(P)$ $\quad$ $T = \chi_\lambda(Q)$

$R = \chi_\lambda(P)$ $\quad$ $S = \phi_t(R)$

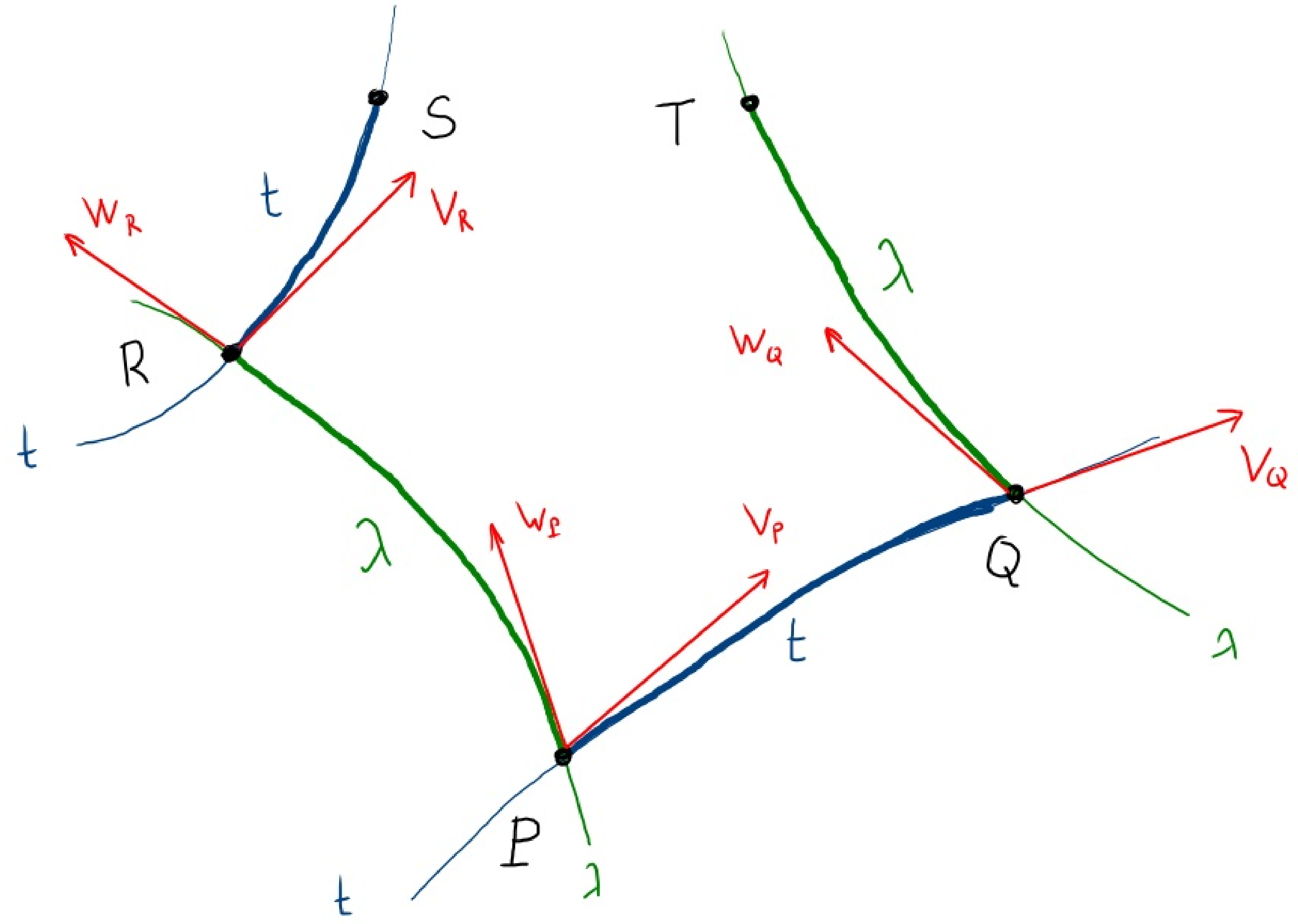


Geometric Interpretation of $\mathcal{L}_V W = [V, W]$

$Q = \phi_t(P) \qquad T = \chi_\lambda(Q)$

$R = \chi_\lambda(P) \qquad S = \phi_t(R)$

$$f(S) = e^{tV_R} e^{\lambda W_P} f(P) = e^{tV_R} f(R)$$



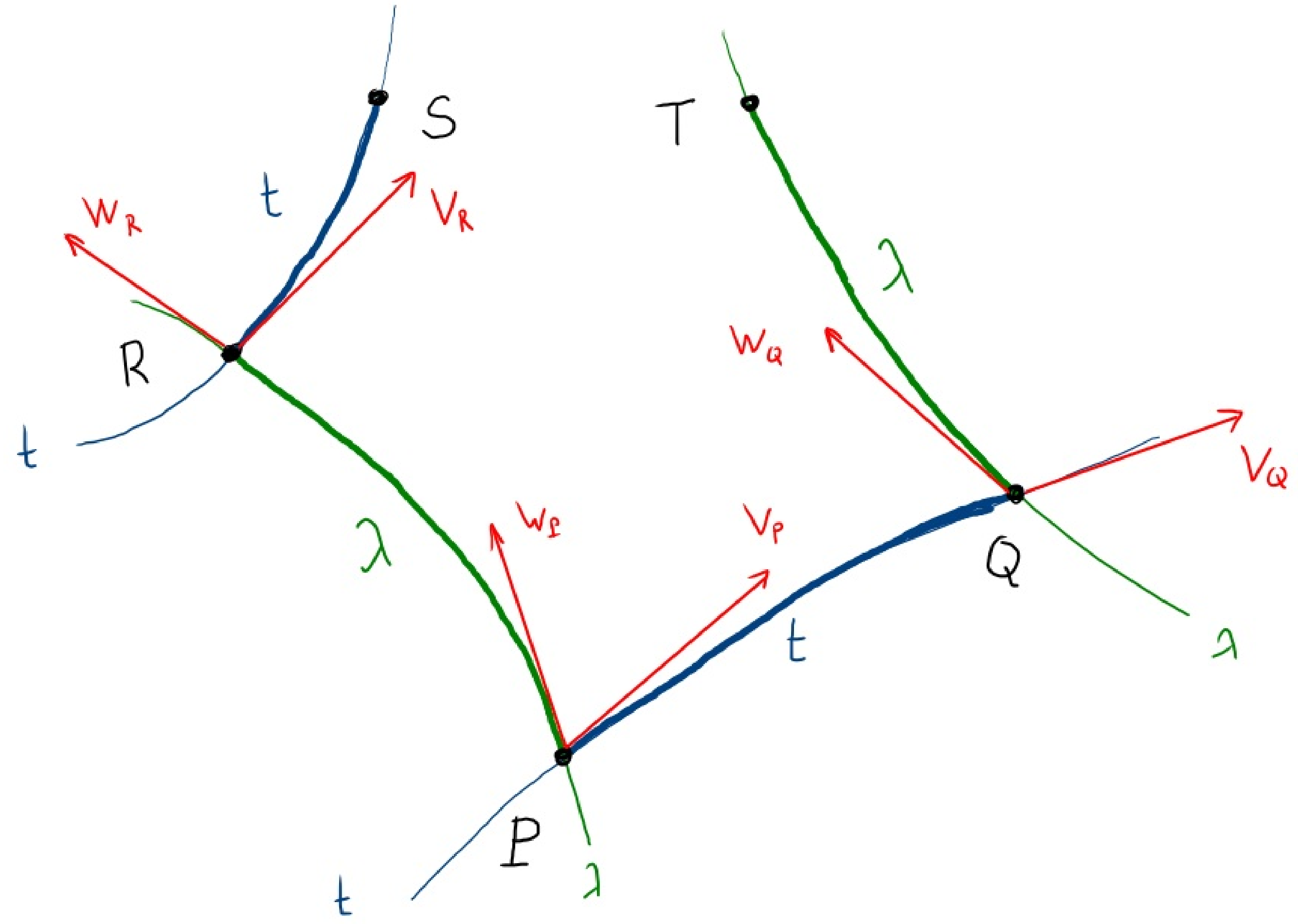
Geometric Interpretation of $\mathcal{L}_V W = [V,W]$

$Q = \phi_t(P) \qquad T = \chi_\lambda(Q)$

$R = \chi_\lambda(P) \qquad S = \phi_t(R)$

$$f(S) = e^{tV_R}\, e^{\lambda W_P} f(P) = e^{tV_R} f(R)$$

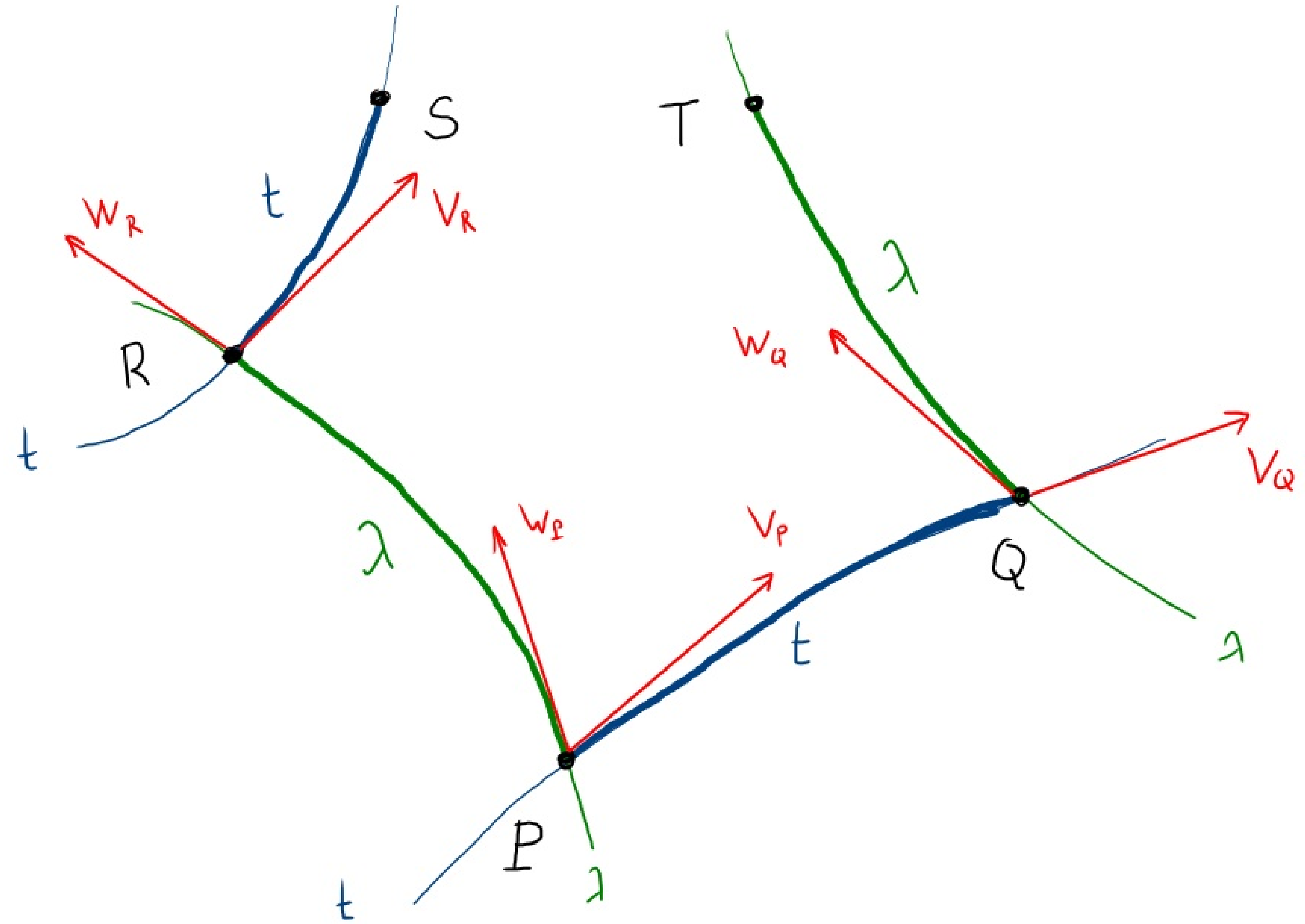
$$= f(R) + tV_R(f) + \frac{t^2}{2} \underbrace{V_R\big(\underset{\text{function}}{V(f)}\big)}_{\text{number}} + O(t^3)$$



Geometric Interpretation of $\mathcal{L}_V W = [V, W]$

$Q = \phi_t(P)$ $\qquad T = \chi_\lambda(Q)$

$R = \chi_\lambda(P)$ $\qquad S = \phi_t(R)$



$$f(S) = e^{tV_R} e^{\lambda W_P} f(P) = e^{tV_R} f(R)$$

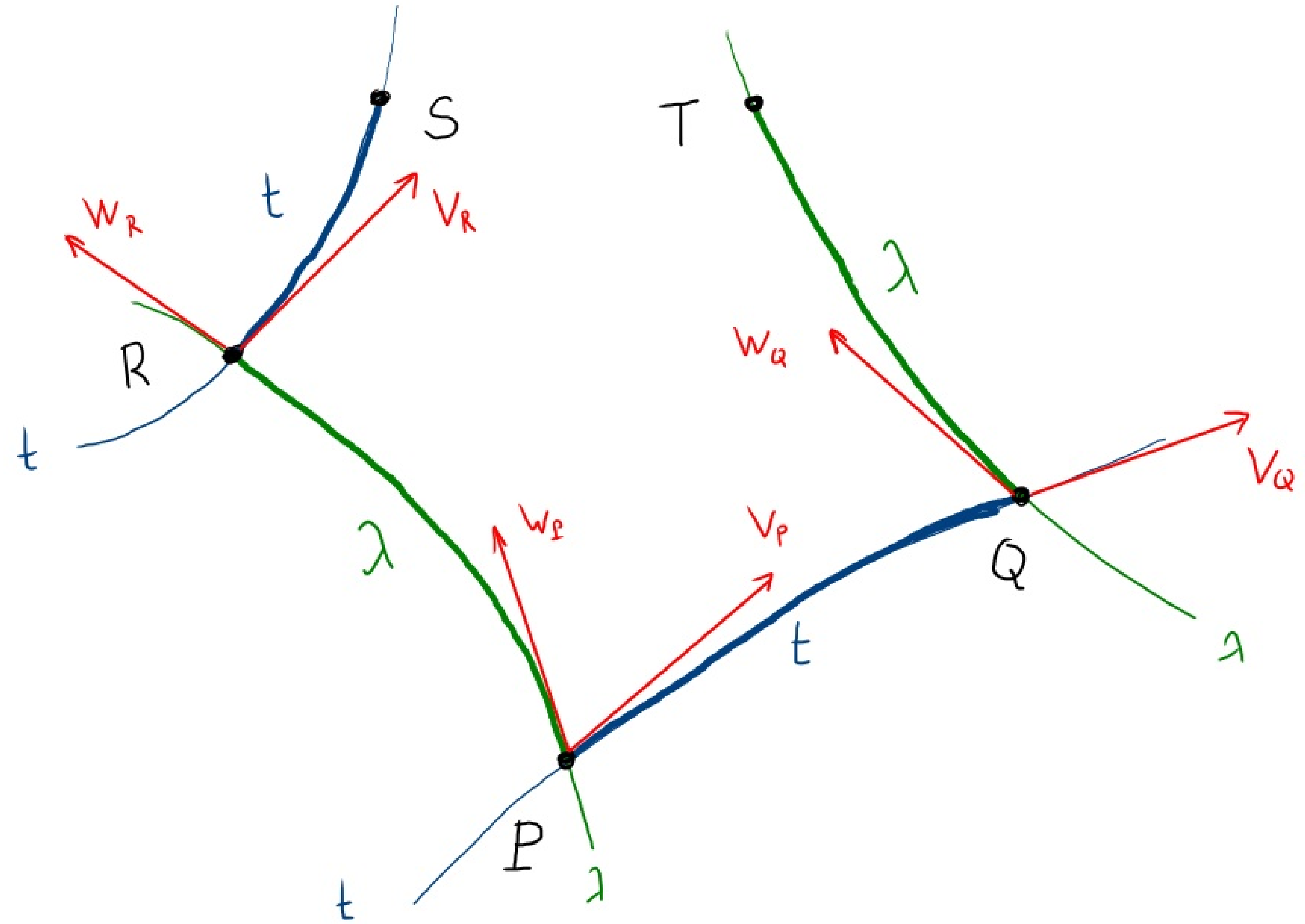
$$= f(R) + t V_R(f) + \frac{t^2}{2} V_R(V(f)) + \mathcal{O}(t^3)$$

$$= f(R) + t \frac{df}{dt}\Big|_R + \frac{t^2}{2} \frac{d}{dt}\left(\frac{df}{dt}\right)\Big|_R + \mathcal{O}(t^3)$$

Geometric Interpretation of $\mathcal{L}_V W = [V, W]$

$Q = \phi_t(P)$ $\quad T = \chi_\lambda(Q)$

$R = \chi_\lambda(P)$ $\quad S = \phi_t(R)$



$$f(S) = e^{tV_R} e^{\lambda W_P} f(P) = e^{tV_R} f(R)$$

$$= f(R) + t V_R(f) + \frac{t^2}{2} V_R(V(f)) + O(t^3)$$

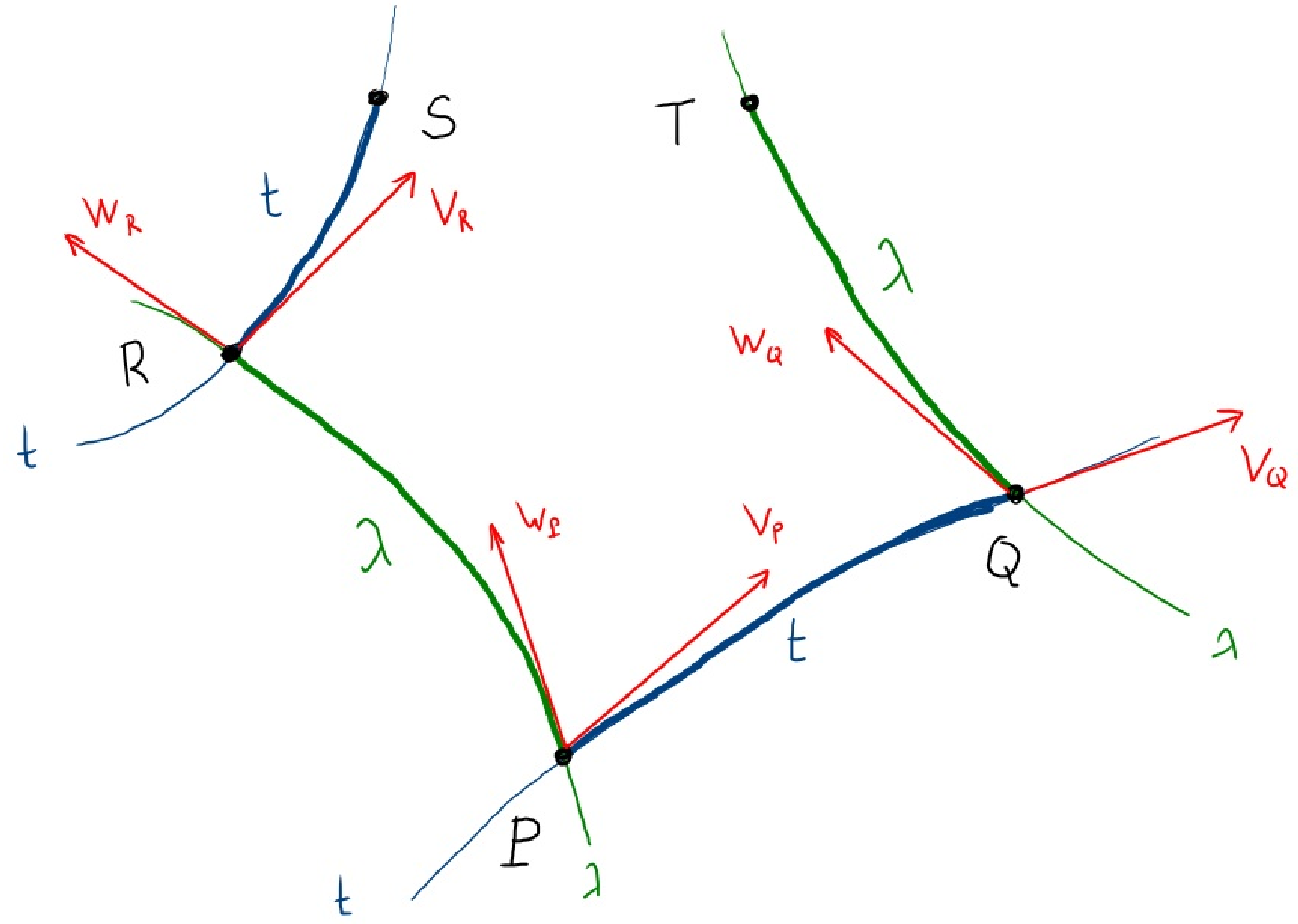
$$= f(R) + t \frac{df}{dt}\Big|_R + \frac{t^2}{2} \frac{d}{dt}\left(\frac{df}{dt}\right)\Big|_R + O(t^3)$$

- $f(R) = e^{\lambda W_P} f(P) = f(P) + \lambda W_P(f) + \frac{\lambda^2}{2} W_P(W(f)) + O(\lambda^3)$

Geometric Interpretation of $\mathcal{L}_V W = [V, W]$

$Q = \phi_t(P) \qquad T = \chi_\lambda(Q)$

$R = \chi_\lambda(P) \qquad S = \phi_t(R)$



$$f(S) = e^{tV_R} e^{\lambda W_P} f(P) = e^{tV_R} f(R)$$

$$= f(R) + t V_R(f) + \frac{t^2}{2} V_R(V(f)) + O(t^3)$$

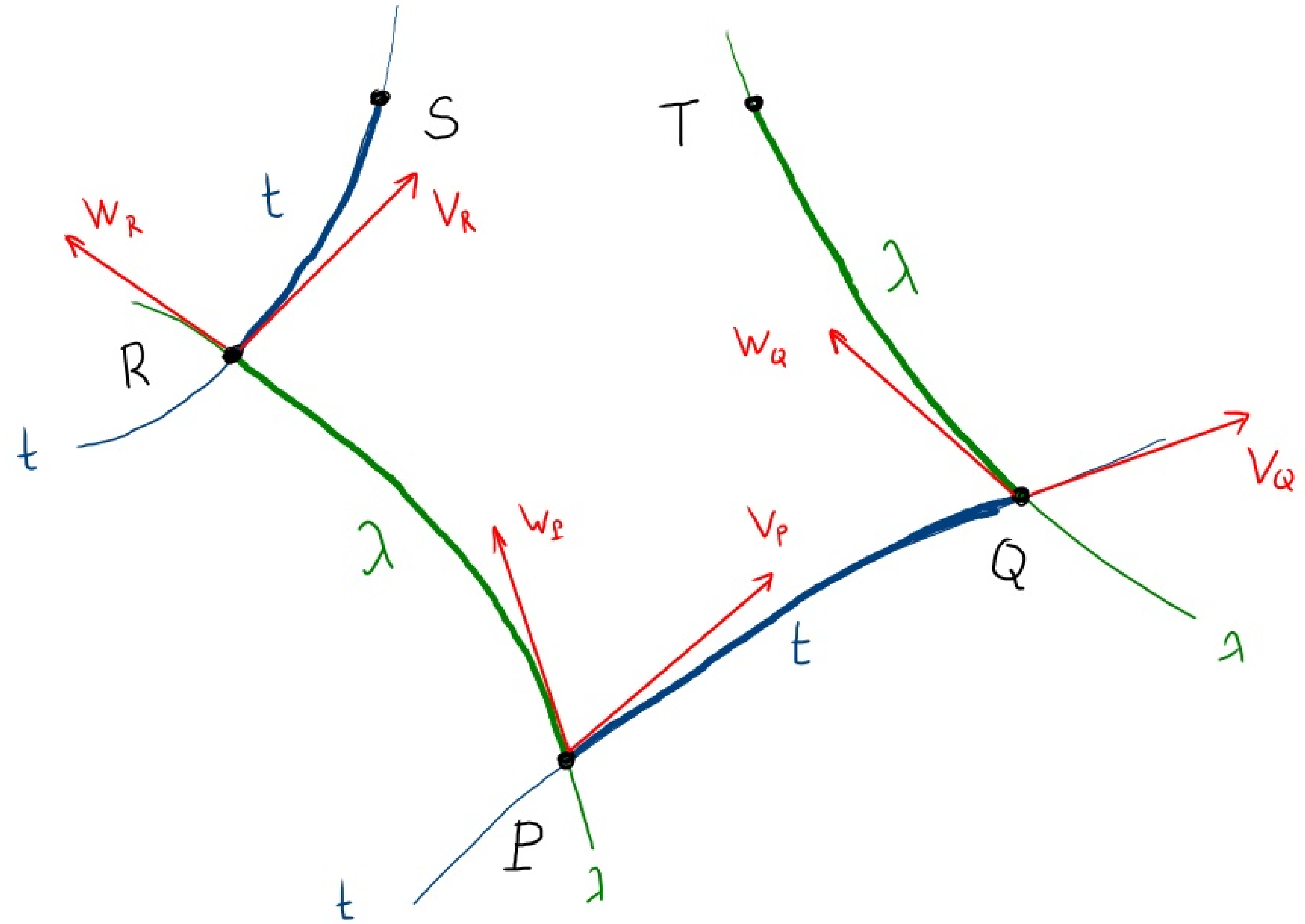
$$= f(R) + t \left.\frac{df}{dt}\right|_R + \frac{t^2}{2} \frac{d}{dt}\left(\frac{df}{dt}\right)\Big|_R + O(t^3)$$

- $f(R) = e^{\lambda W_P} f(P) = f(P) + \lambda W_P(f) + \frac{\lambda^2}{2} W_P(W(f)) + O(\lambda^3)$
- $V_R(f) = \left.\frac{df}{dt}\right|_R = e^{\lambda W_P} \left.\frac{df}{dt}\right|_P = \left.\frac{df}{dt}\right|_P + \lambda W_P\left(\frac{df}{dt}\right) + O(\lambda^2)$

Geometric Interpretation of $\mathcal{L}_V W = [V, W]$

$Q = \phi_t(P) \qquad T = \chi_\lambda(Q)$

$R = \chi_\lambda(P) \qquad S = \phi_t(R)$



$$f(S) = e^{tV_R} e^{\lambda W_P} f(P) = e^{tV_R} f(R)$$
$$= f(R) + tV_R(f) + \frac{t^2}{2} V_R(V(f)) + \mathcal{O}(t^3)$$
$$= f(R) + t \left.\frac{df}{dt}\right|_R + \frac{t^2}{2} \frac{d}{dt}\left(\frac{df}{dt}\right)\Big|_R + \mathcal{O}(t^3)$$

- $f(R) = e^{\lambda W_P} f(P) = f(P) + \lambda W_P(f) + \frac{\lambda^2}{2} W_P(W(f)) + \mathcal{O}(\lambda^3)$

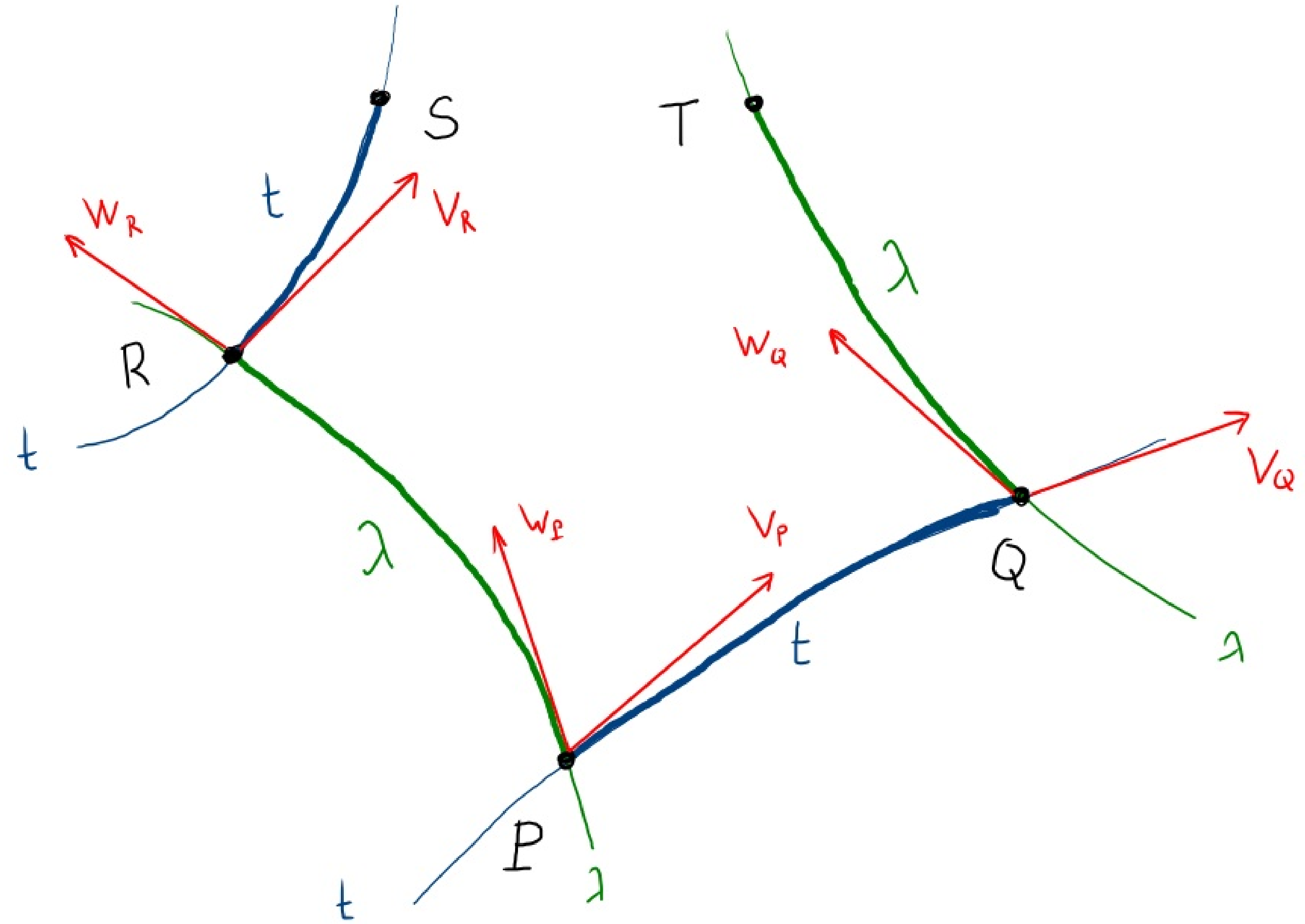
- $V_R(f) = \left.\frac{df}{dt}\right|_R = e^{\lambda W_P} \left.\frac{df}{dt}\right|_P = \left.\frac{df}{dt}\right|_P + \lambda W_P\left(\frac{df}{dt}\right) + \mathcal{O}(\lambda^2)$

$$= V_P(f) + \lambda W_P(V(f)) + \mathcal{O}(\lambda^2)$$

Geometric Interpretation of $\mathcal{L}_V W = [V, W]$

$$Q = \phi_t(P) \qquad T = \chi_\lambda(Q)$$

$$R = \chi_\lambda(P) \qquad S = \phi_t(R)$$



$$f(S) = e^{tV_R} e^{\lambda W_P} f(P) = e^{tV_R} f(R)$$

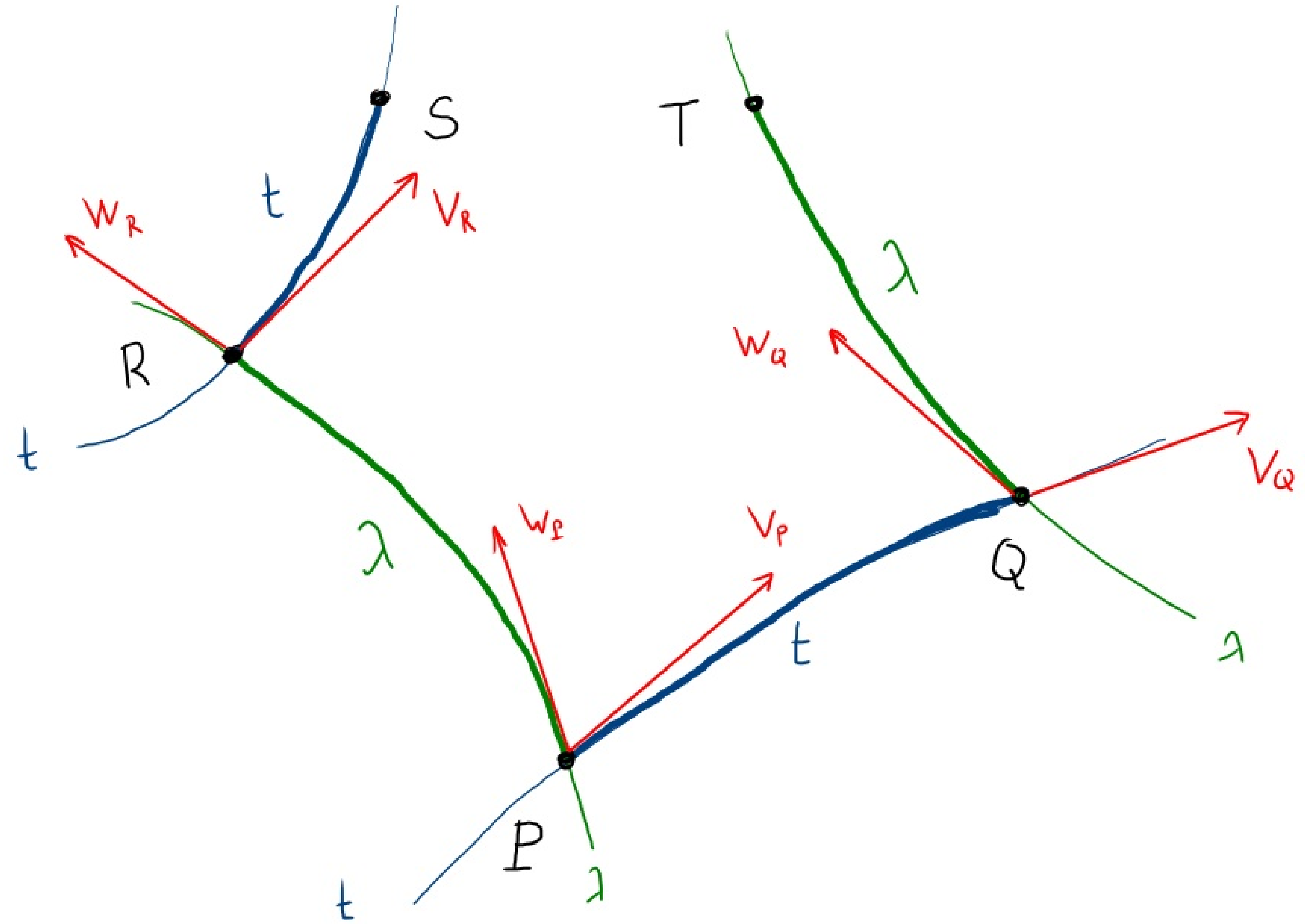
$$= f(R) + t V_R(f) + \frac{t^2}{2} V_R(V(f)) + \mathcal{O}(t^3)$$

$$= f(R) + t \left.\frac{df}{dt}\right|_R + \frac{t^2}{2} \frac{d}{dt}\left(\frac{df}{dt}\right)\Big|_R + \mathcal{O}(t^3)$$

- $f(R) = e^{\lambda W_P} f(P) = f(P) + \lambda W_P(f) + \frac{\lambda^2}{2} W_P(W(f)) + \mathcal{O}(\lambda^3)$
- $V_R(f) = \left.\frac{df}{dt}\right|_R = e^{\lambda W_P} \left.\frac{df}{dt}\right|_P = \left.\frac{df}{dt}\right|_P + \lambda W_P\left(\frac{df}{dt}\right) + \mathcal{O}(\lambda^2)$

  $= V_P(f) + \lambda W_P(V(f)) + \mathcal{O}(\lambda^2)$
- $V_R(V(f)) = \left.\frac{d^2 f}{dt^2}\right|_R = e^{\lambda W_P} \left.\frac{d^2 f}{dt^2}\right|_P = \left.\frac{d^2 f}{dt^2}\right|_P + \mathcal{O}(\lambda) = V_P(V(f)) + \mathcal{O}(\lambda)$

Geometric Interpretation of $\mathcal{L}_V W = [V,W]$



$$\Rightarrow f(S) = f(P) + \lambda W_P(f) + \frac{\lambda^2}{2} W_P(W(f))$$
$$+ t V_P(f) + t\lambda W_P(V(f))$$
$$+ \frac{t^2}{2} V_P(V(f)) + (\text{cubic in } \lambda, t)$$

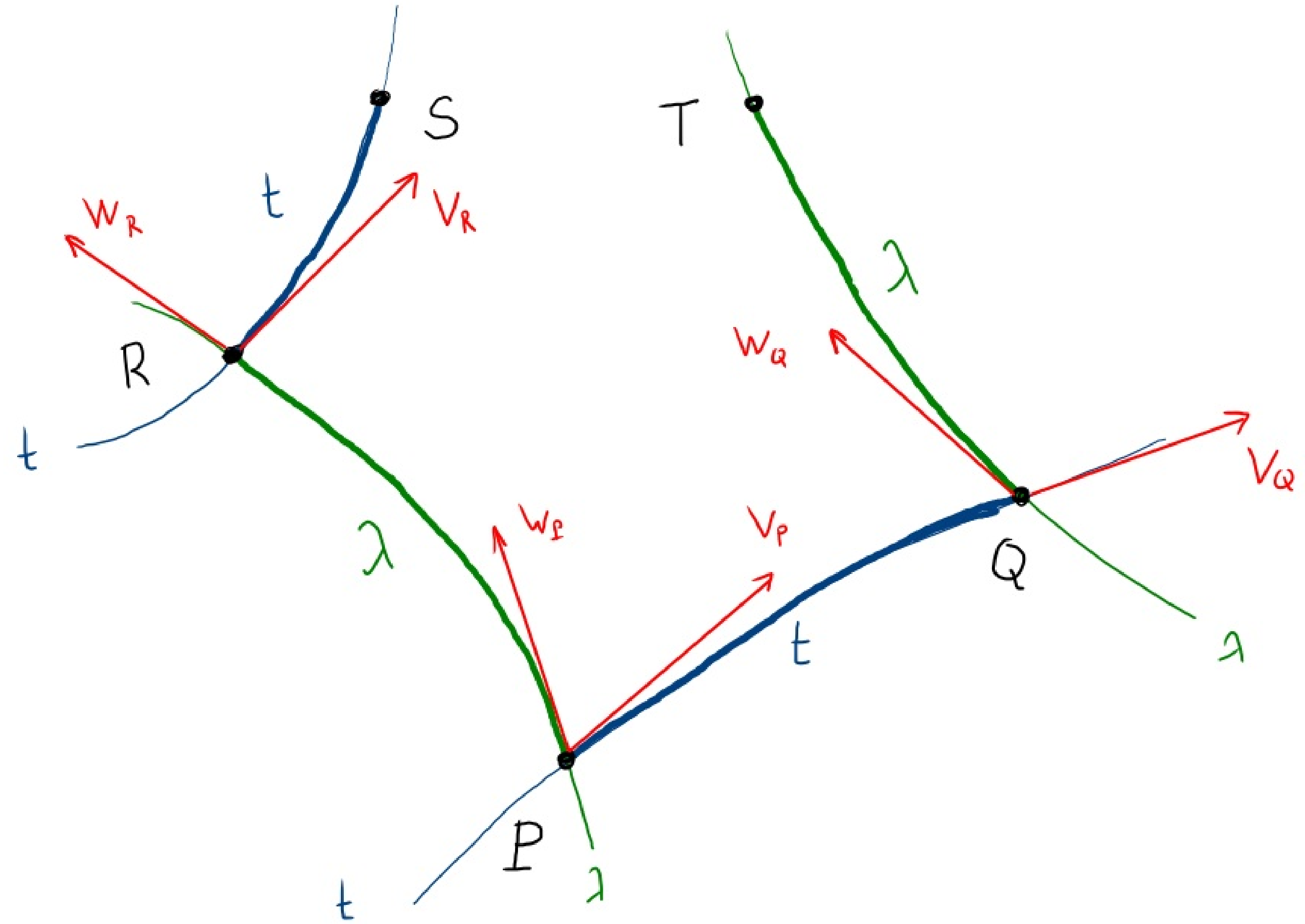
$$f(S) = e^{tV_R} e^{\lambda W_P} f(P) = e^{tV_R} f(R)$$
$$= f(R) + tV_R(f) + \frac{t^2}{2} V_R(V(f)) + \mathcal{O}(t^3)$$
$$= f(R) + t \frac{df}{dt}\Big|_R + \frac{t^2}{2} \frac{d}{dt}\left(\frac{df}{dt}\right)\Big|_R + \mathcal{O}(t^3)$$

- $f(R) = e^{\lambda W_P} f(P) = f(P) + \lambda W_P(f) + \frac{\lambda^2}{2} W_P(W(f)) + \mathcal{O}(\lambda^3)$
- $V_R(f) = \frac{df}{dt}\Big|_R = e^{\lambda W_P} \frac{df}{dt}\Big|_P = \frac{df}{dt}\Big|_P + \lambda W_P\left(\frac{df}{dt}\right) + \mathcal{O}(\lambda^2)$
  $= V_P(f) + \lambda W_P(V(f)) + \mathcal{O}(\lambda^2)$
- $V_R(V(f)) = \frac{d^2 f}{dt^2}\Big|_R = e^{\lambda W_P} \frac{d^2 f}{dt^2}\Big|_P = \frac{d^2 f}{dt^2}\Big|_P + \mathcal{O}(\lambda) = V_P(V(f)) + \mathcal{O}(\lambda)$

Geometric Interpretation of $\mathcal{L}_V W = [V,W]$

$$\Rightarrow f(S) = f(P) + \lambda W_P(f) + \frac{\lambda^2}{2} W_P(W(f))$$
$$+ t V_P(f) + t\lambda W_P(V(f))$$
$$+ \frac{t^2}{2} V_P(V(f)) + (\text{cubic in } \lambda, t)$$

$$f(T) = e^{\lambda W_Q} e^{tV_P} f(P) = e^{\lambda W_Q} f(Q)$$
$$= f(Q) + \lambda W_Q(f) + \frac{\lambda^2}{2} W_Q(W(f)) + O(\lambda^3)$$

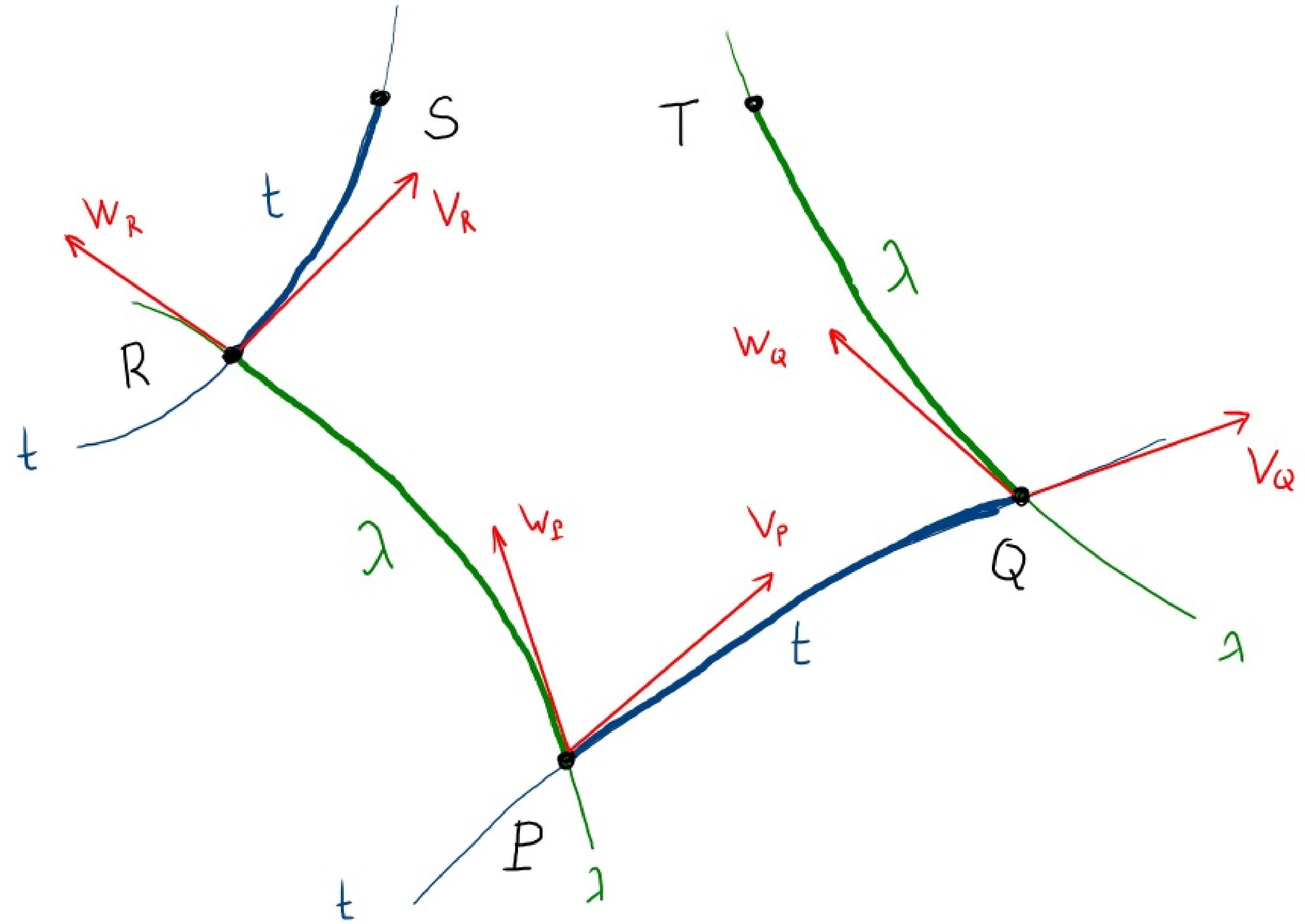


Geometric Interpretation of $\mathcal{L}_V W = [V,W]$

$$\Rightarrow f(S) = f(P) + \lambda W_P(f) + \frac{\lambda^2}{2} W_P(W(f))$$
$$+ t V_P(f) + t\lambda W_P(V(f))$$
$$+ \frac{t^2}{2} V_P(V(f)) \quad + (\text{cubic in } \lambda, t)$$

$$f(T) = e^{\lambda W_Q} e^{t V_P} f(P) = e^{\lambda W_Q} f(Q)$$
$$= f(Q) + \lambda W_Q(f) + \frac{\lambda^2}{2} W_Q(W(f)) + O(\lambda^3)$$

- $f(Q) = e^{t V_P} f(P) = f(P) + t V_P(f) + \frac{t^2}{2} V_P(V(f)) + O(t^3)$



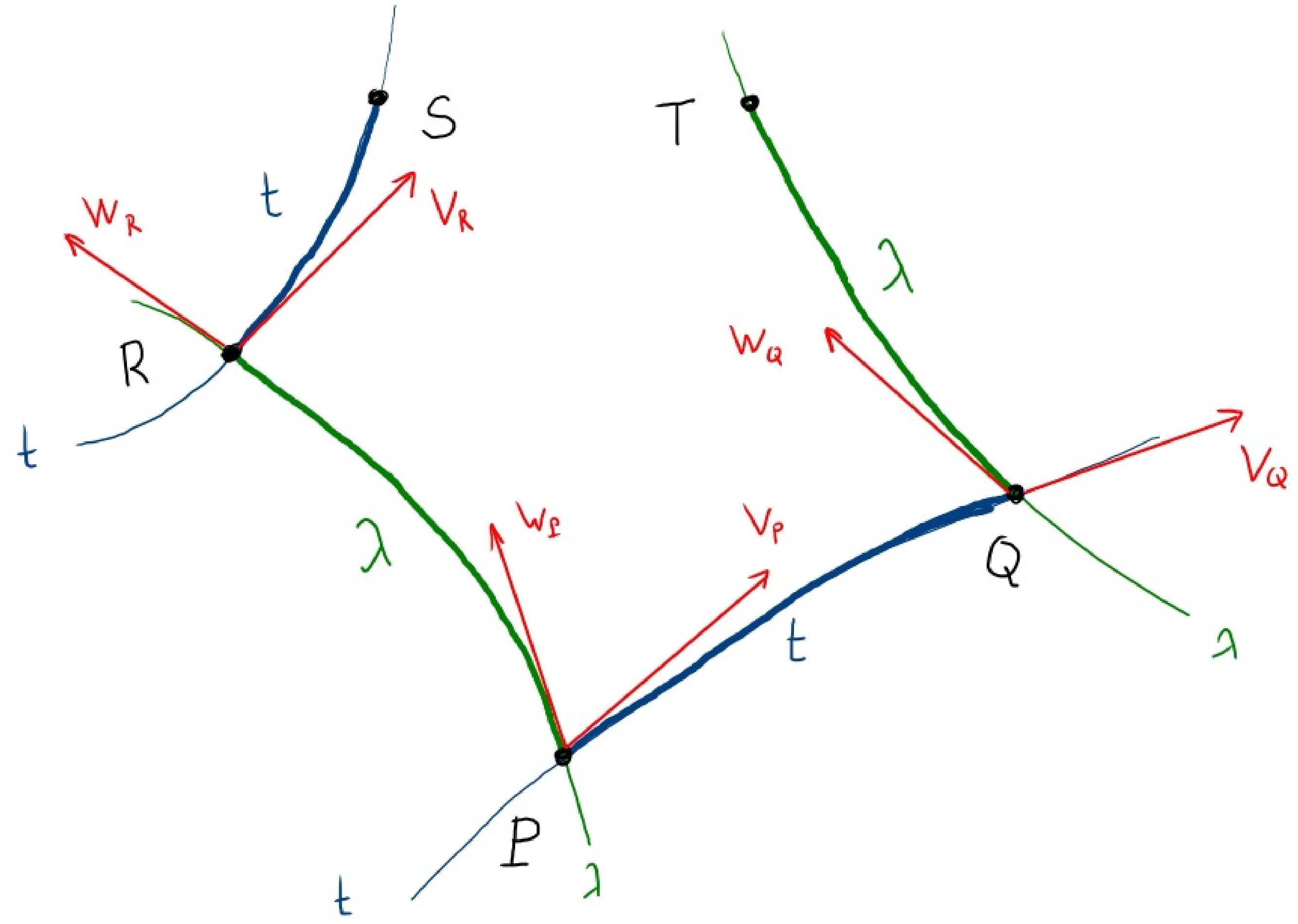
Geometric Interpretation of $\mathcal{L}_V W = [V,W]$

$$\Rightarrow f(S) = f(P) + \lambda W_P(f) + \frac{\lambda^2}{2} W_P(W(f))$$
$$+ t V_P(f) + t\lambda W_P(V(f))$$
$$+ \frac{t^2}{2} V_P(V(f)) \quad + (\text{cubic in } \lambda, t)$$

$$f(T) = e^{\lambda W_Q} e^{t V_P} f(P) = e^{\lambda W_Q} f(Q)$$
$$= f(Q) + \lambda W_Q(f) + \frac{\lambda^2}{2} W_Q(W(f)) + O(\lambda^3)$$

- $f(Q) = e^{tV_P} f(P) = f(P) + t V_P(f) + \frac{t^2}{2} V_P(V(f)) + O(t^3)$
- $W_Q(f) = \frac{df}{d\lambda}\Big|_Q = e^{tV_P} \frac{df}{d\lambda}\Big|_P = \frac{df}{d\lambda}\Big|_P + t V_P\left(\frac{df}{d\lambda}\right) + O(t^2)$

$$= W_P(f) + t V_P(W(f)) + O(t^2)$$



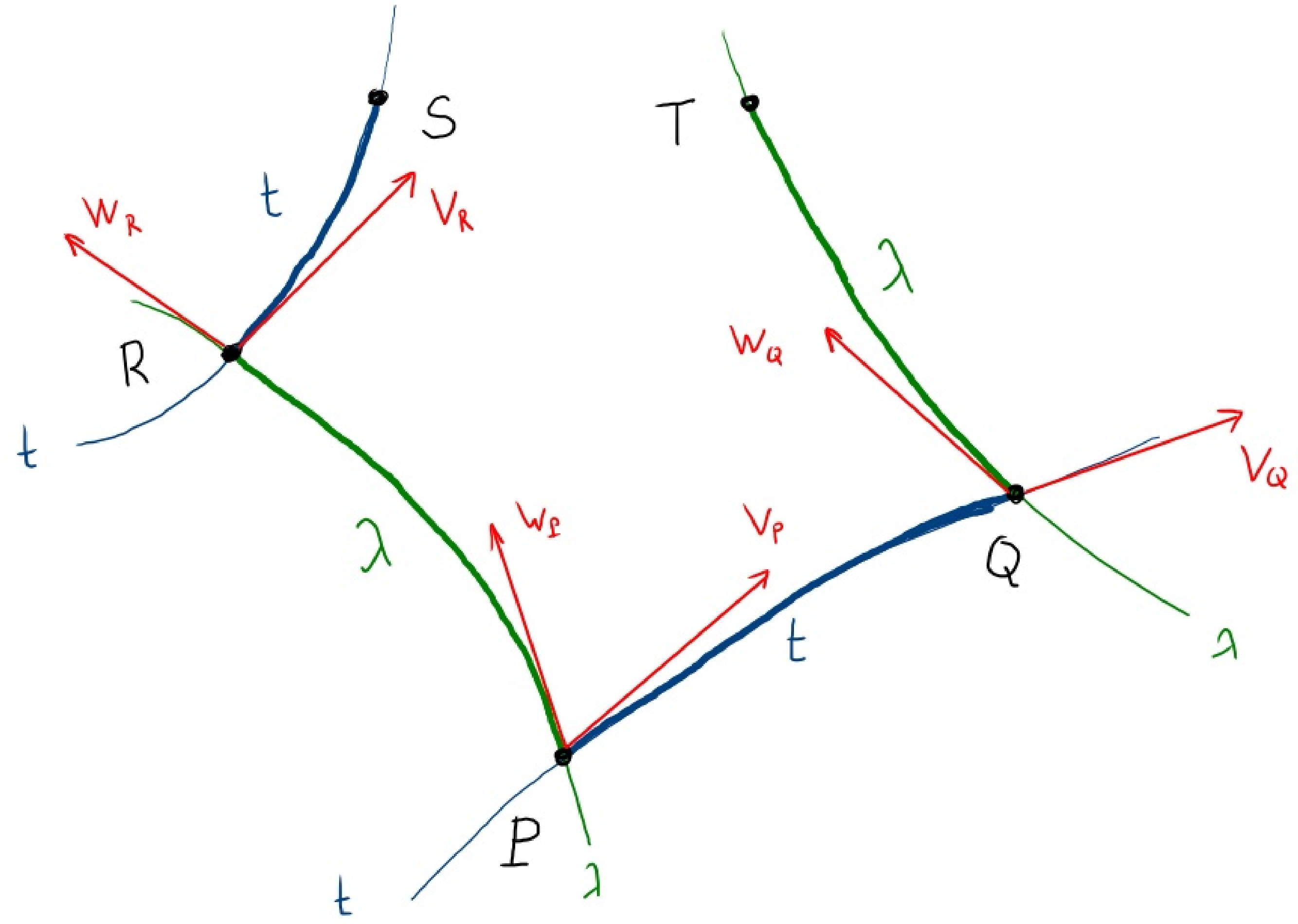
Geometric Interpretation of $\mathcal{L}_V W = [V,W]$

$$\Rightarrow f(S) = f(P) + \lambda W_P(f) + \frac{\lambda^2}{2} W_P(W(f))$$
$$+ t V_P(f) + t\lambda W_P(V(f))$$
$$+ \frac{t^2}{2} V_P(V(f)) + (\text{cubic in } \lambda, t)$$

$$f(T) = e^{\lambda W_Q} e^{t V_P} f(P) = e^{\lambda W_Q} f(Q)$$
$$= f(Q) + \lambda W_Q(f) + \frac{\lambda^2}{2} W_Q(W(f)) + \mathcal{O}(\lambda^3)$$

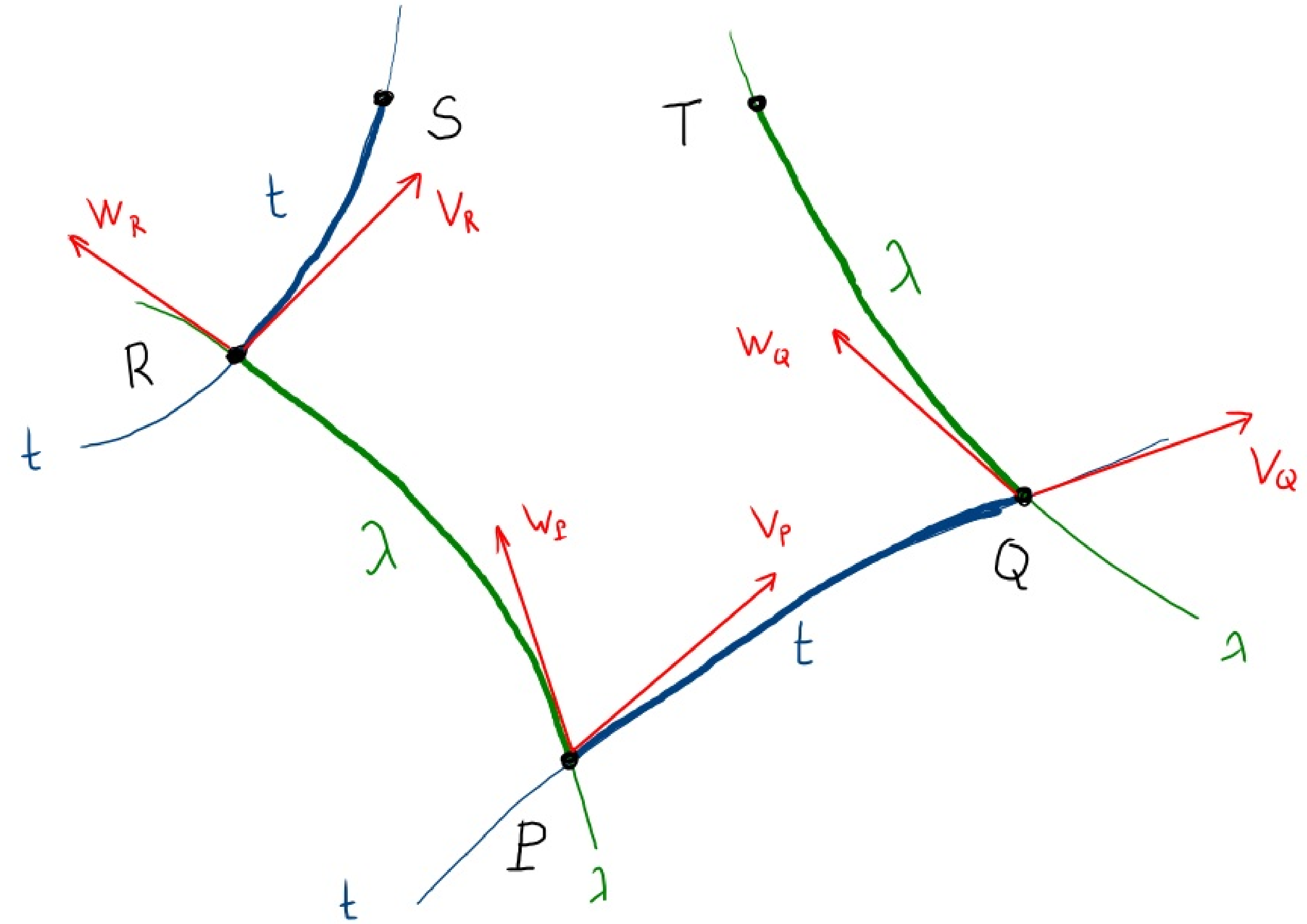
- $f(Q) = e^{tV_P} f(P) = f(P) + t V_P(f) + \frac{t^2}{2} V_P(V(f)) + \mathcal{O}(t^3)$
- $W_Q(f) = \frac{df}{d\lambda}\Big|_Q = e^{tV_P} \frac{df}{d\lambda}\Big|_P = \frac{df}{d\lambda}\Big|_P + t V_P\left(\frac{df}{d\lambda}\right) + \mathcal{O}(t^2)$

  $= W_P(f) + t V_P(W(f)) + \mathcal{O}(t^2)$
- $W_Q(W(f)) = \frac{d^2 f}{d\lambda^2}\Big|_Q = e^{tV_P} \frac{d^2 f}{d\lambda^2}\Big|_P = \frac{d^2 f}{d\lambda^2}\Big|_P + \mathcal{O}(t) = W_P(W(f)) + \mathcal{O}(t)$



Geometric Interpretation of $\mathcal{L}_V W = [V,W]$

$$\Rightarrow f(S) = f(P) + \lambda W_P(f) + \frac{\lambda^2}{2} W_P(W(f))$$
$$+ t V_P(f) + t\lambda W_P(V(f))$$
$$+ \frac{t^2}{2} V_P(V(f)) + (\text{cubic in } \lambda, t)$$



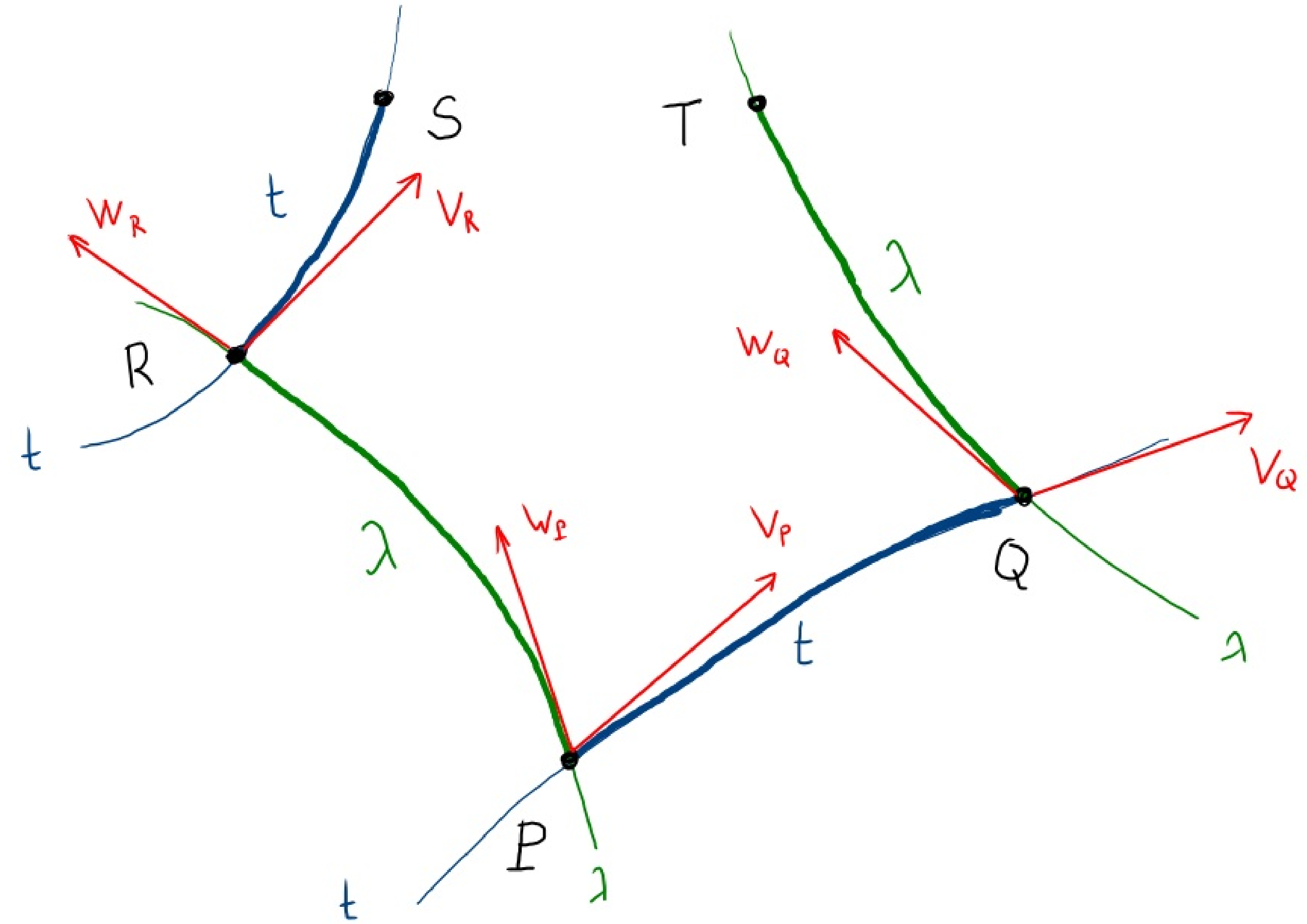
$$f(T) = e^{\lambda W_Q} e^{t V_P} f(P) = e^{\lambda W_Q} f(Q)$$
$$= f(Q) + \lambda W_Q(f) + \frac{\lambda^2}{2} W_Q(W(f)) + O(\lambda^3)$$

- $f(Q) = e^{tV_P} f(P) = f(P) + t V_P(f) + \frac{t^2}{2} V_P(V(f)) + O(t^3)$
- $W_Q(f) = \frac{df}{d\lambda}\Big|_Q = e^{tV_P} \frac{df}{d\lambda}\Big|_P = \frac{df}{d\lambda}\Big|_P + t V_P\left(\frac{df}{d\lambda}\right) + O(t^2)$

  $= W_P(f) + t V_P(W(f)) + O(t^2)$
- $W_Q(W(f)) = \frac{d^2 f}{d\lambda^2}\Big|_Q = e^{tV_P} \frac{d^2 f}{d\lambda^2}\Big|_P = \frac{d^2 f}{d\lambda^2}\Big|_P + O(t) = W_P(W(f)) + O(t)$

$$\Rightarrow f(T) = f(P) + t V_P(f) + \frac{t^2}{2} V_P(V(f))$$
$$+ \lambda W_P(f) + \lambda t\, V_P(W(f))$$
$$+ \frac{\lambda^2}{2} W_P(W(f)) + (\text{cubic})$$

Geometric Interpretation of $\mathcal{L}_V W = [V,W]$



$$\Rightarrow f(S) = \cancel{f(P)} + \lambda \cancel{W_P(f)} + \frac{\lambda^2}{2} \cancel{W_P(W(f))} + t\cancel{V_P(f)} + t\lambda W_P(V(f)) + \frac{t^2}{2} \cancel{V_P(V(f))} + (\text{cubic in } \lambda, t) \qquad (1)$$

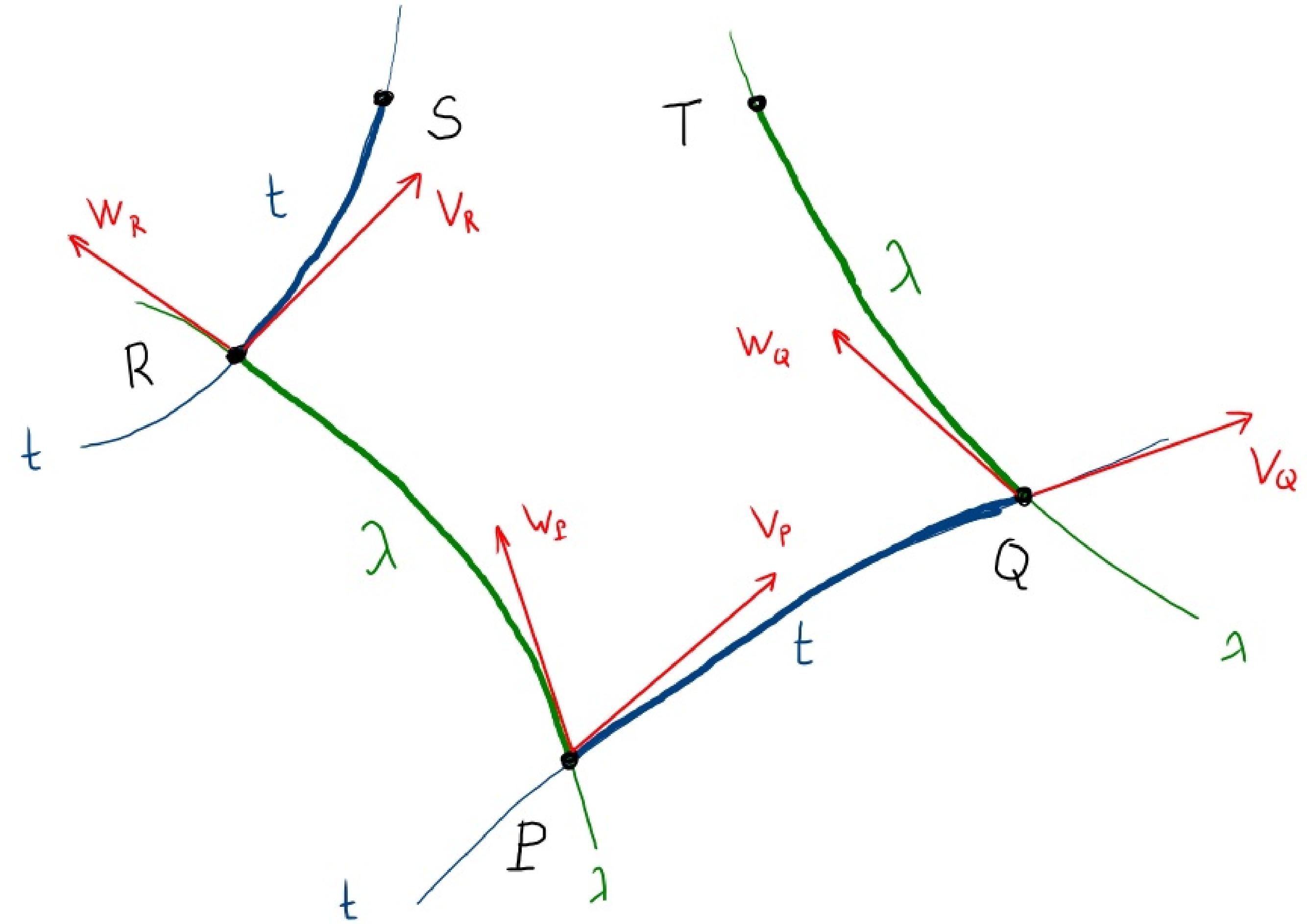
$(2) - (1) \Rightarrow$

$$f(T) - f(S) = t\lambda \left[V_P(W(f)) - W_P(V(f)) \right] + (\text{cubic})$$

$$\Rightarrow f(T) = \cancel{f(P)} + t\cancel{V_P(f)} + \frac{t^2}{2} \cancel{V_P(V(f))} + \lambda \cancel{W_P(f)} + \lambda t\, V_P(W(f)) + \frac{\lambda^2}{2} \cancel{W_P(W(f))} + (\text{cubic}) \qquad (2)$$

Geometric Interpretation of $\mathcal{L}_V W = [V,W]$

$$\Rightarrow f(S) = \cancel{f(P)} + \lambda \cancel{W_P}(f) + \frac{\lambda^2}{2} \cancel{W_P}(W(f))$$
$$+ t \cancel{V_P}(f) + t\lambda W_P(V(f)) \qquad (1)$$
$$+ \frac{t^2}{2} V_P(\cancel{V}(f)) + (\text{cubic in } \lambda, t)$$



$(2)-(1) \Rightarrow$

$$f(T) - f(S) = t\lambda\left[V_P(W(f)) - W_P(V(f))\right] + (\text{cubic})$$

set $\epsilon = t = \lambda \Rightarrow$

$$f(T) - f(S) = \epsilon^2 [V, W]_P(f) + \mathcal{O}(\epsilon^3)$$

$$\Rightarrow f(T) = \cancel{f(P)} + t \cancel{V_P}(f) + \frac{t^2}{2} \cancel{V_P}(V(f))$$
$$+ \lambda \cancel{W_P}(f) + \lambda t\, V_P(W(f)) \qquad (2)$$
$$+ \frac{\lambda^2}{2} \cancel{W_P}(W(f)) + (\text{cubic})$$

Geometric Interpretation of $\mathcal{L}_V W = [V, W]$

$$\Rightarrow f(S) = \cancel{f(P)} + \lambda \cancel{W_P(f)} + \frac{\lambda^2}{2} \cancel{W_P(W(f))}$$
$$+\, t \cancel{V_P(f)} + t\lambda\, W_P(V(f)) \qquad (1)$$
$$+\, \frac{t^2}{2} \cancel{V_P(V(f))} + (\text{cubic in } \lambda, t)$$

$(2) - (1) \Rightarrow$

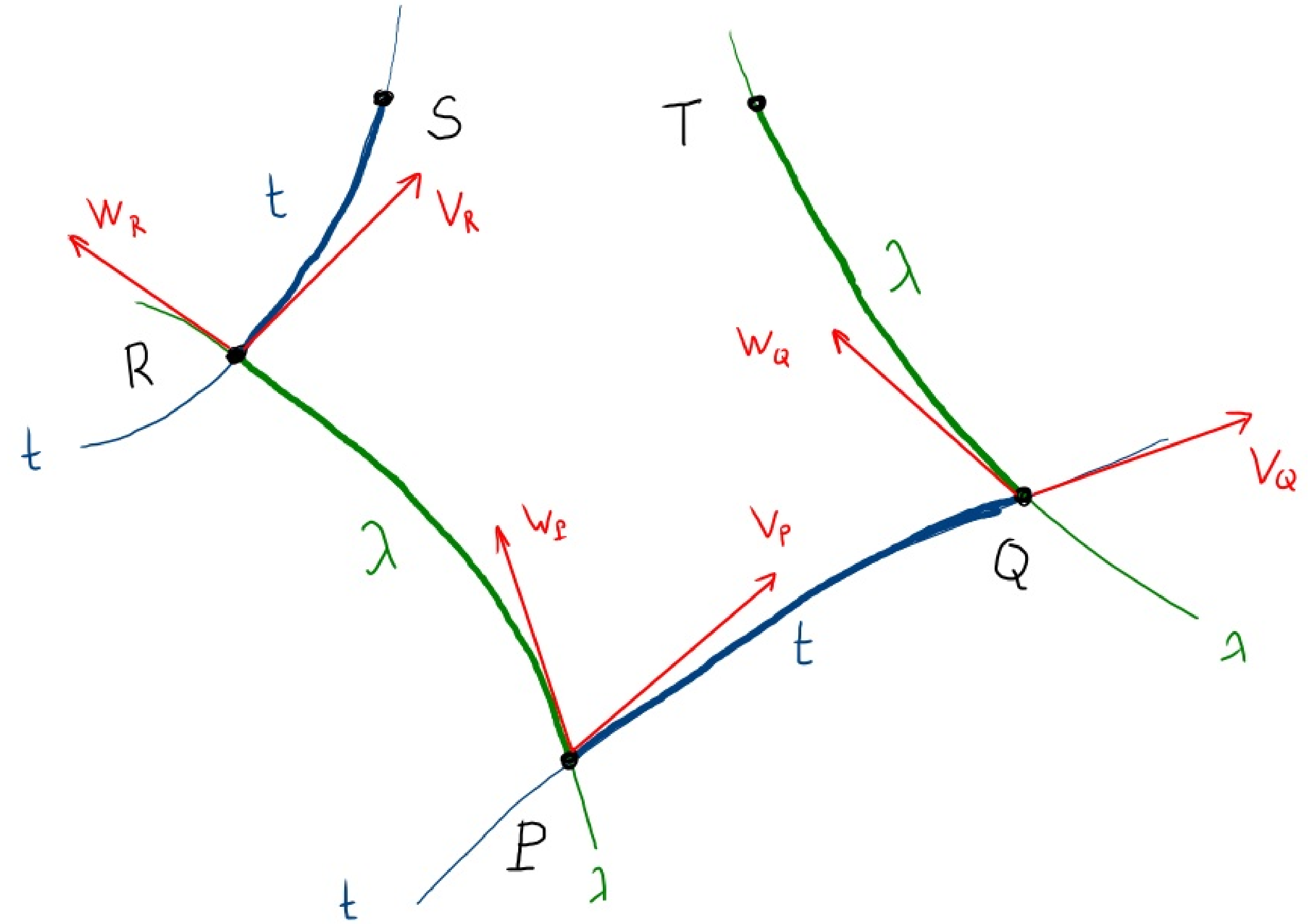
$$f(T) - f(S) = t\lambda \left[V_P(W(f)) - W_P(V(f)) \right] + (\text{cubic})$$

set $\epsilon = t = \lambda \Rightarrow$

$$f(T) - f(S) = \epsilon^2 [V, W]_P(f) + \mathcal{O}(\epsilon^3)$$

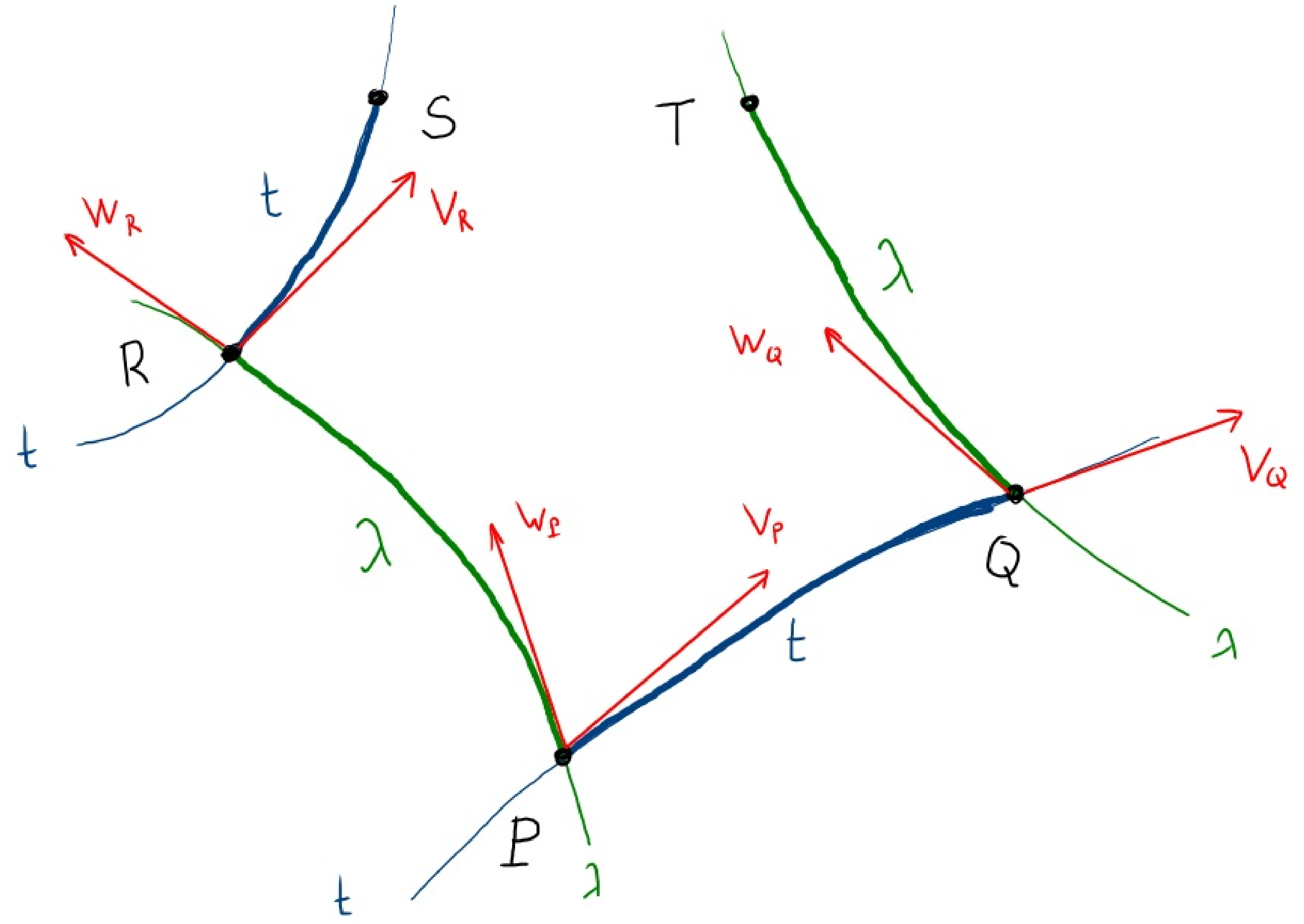
$\Rightarrow$ coordinate vectors must commute, because curve should close ($S \equiv T$)

$$\Rightarrow f(T) = \cancel{f(P)} + t \cancel{V_P(f)} + \frac{t^2}{2} \cancel{V_P(V(f))} \qquad (2)$$
$$+\, \lambda \cancel{W_P(f)} + \lambda t\, V_P(W(f))$$
$$+\, \frac{\lambda^2}{2} \cancel{W_P(W(f))} + (\text{cubic})$$



Geometric Interpretation of $\mathcal{L}_V W = [V, W]$

$$\Rightarrow f(S) = f(\cancel{P}) + \lambda W_{\cancel{P}}(f) + \frac{\lambda^2}{2} W_{\cancel{P}}(W(f))$$
$$+ \, t V_{\cancel{P}}(f) + t\lambda W_P(V(f)) \qquad (1)$$
$$+ \frac{t^2}{2} V_P(\cancel{V}(f)) + (\text{cubic in } \lambda, t)$$



$(2) - (1) \Rightarrow$

$$f(T) - f(S) = t\lambda\left[V_P(W(f)) - W_P(V(f))\right] + (\text{cubic})$$

set $\epsilon = t = \lambda \Rightarrow$

$$f(T) - f(S) = \epsilon^2 [V, W]_P(f) + \mathcal{O}(\epsilon^3)$$

Set $f = x^\mu$ so that $x^\mu(S) = x^\mu(S)$, $x^\mu(T) = x^\mu(T)$, $[V, W]_P(x^\mu) = [V, W]^\mu$

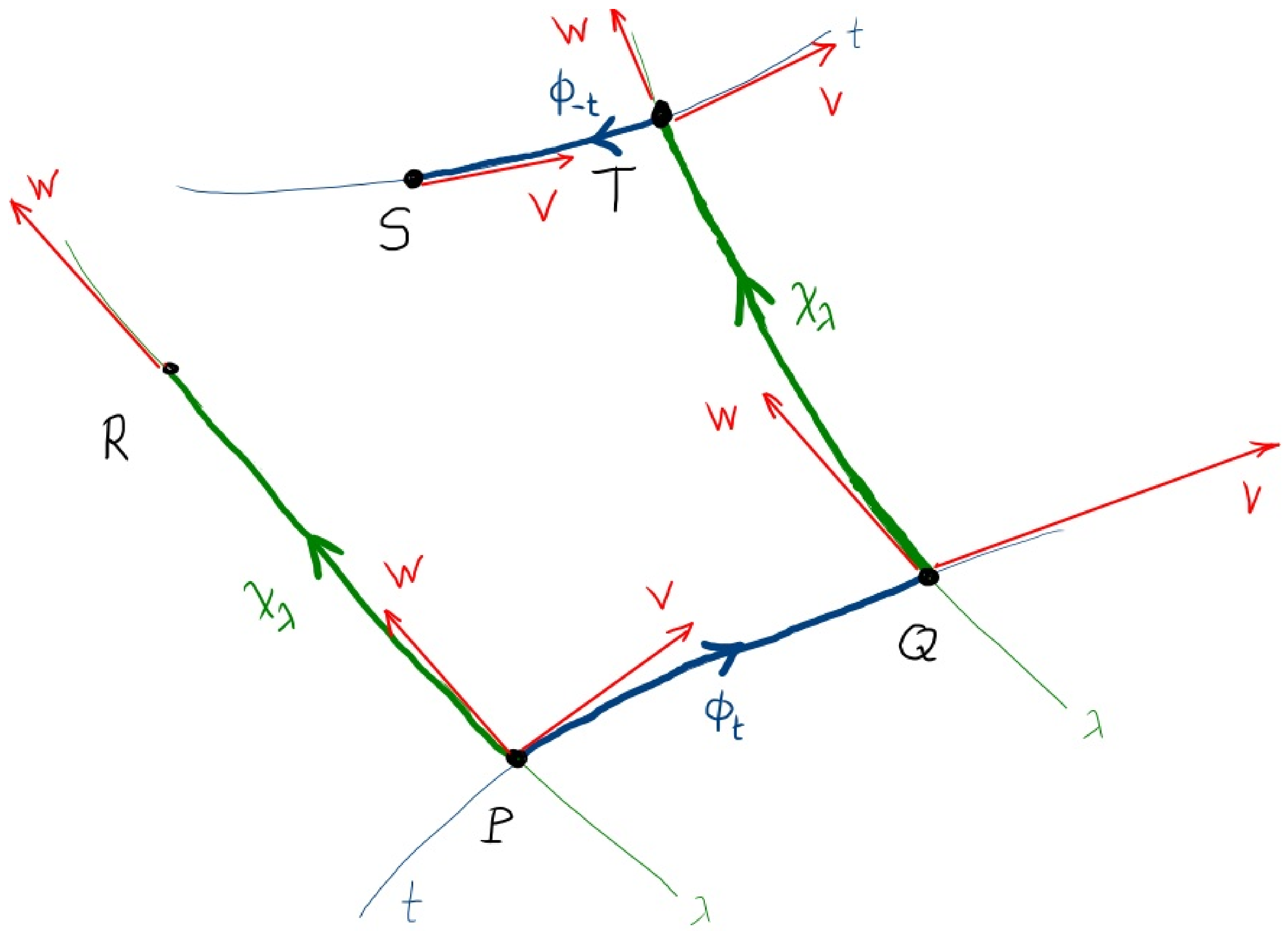
$$x^\mu(T) - x^\mu(S) = \epsilon^2 [V, W]^\mu + \mathcal{O}(\epsilon^3)$$

Geometric Interpretation of $\mathcal{L}_V W$

Consider the points

$Q = \phi_t(P) \quad T = \chi_\lambda(Q) \quad S = \phi_{-t}(T)$

$R = \chi_\lambda(P)$

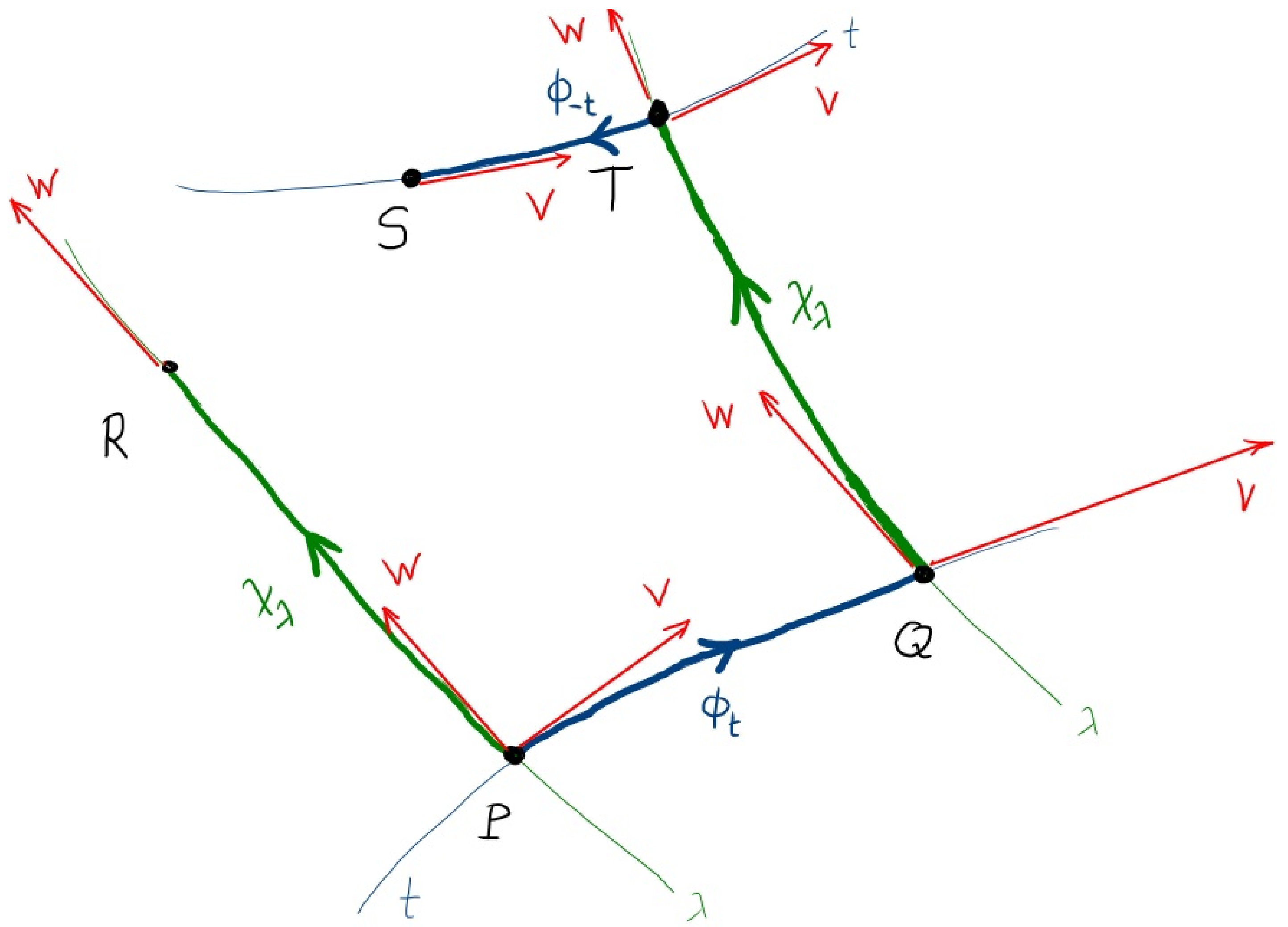


Geometric Interpretation of $\mathcal{L}_V W$

Consider the points

$Q = \phi_t(P)$ $\quad T = \chi_\lambda(Q)$ $\quad S = \phi_{-t}(T)$

$R = \chi_\lambda(P)$



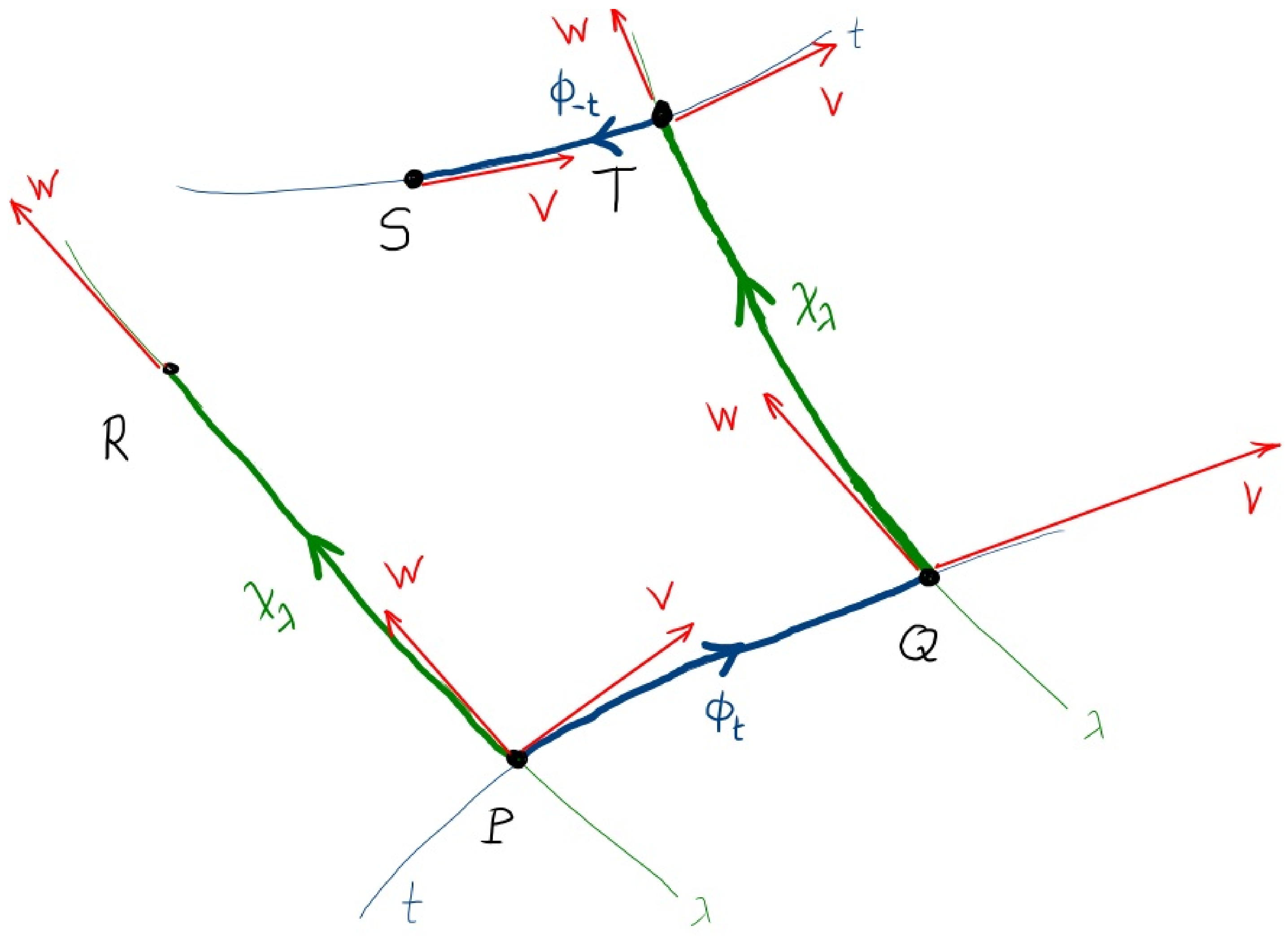
$$(\mathcal{L}_V W)_P = \lim_{t \to 0} \left[\phi_t^* W_Q - W_P \right]$$

Geometric Interpretation of $\mathcal{L}_V W$

Consider the points

$Q = \phi_t(P) \quad T = \chi_\lambda(Q) \quad S = \phi_{-t}(T)$

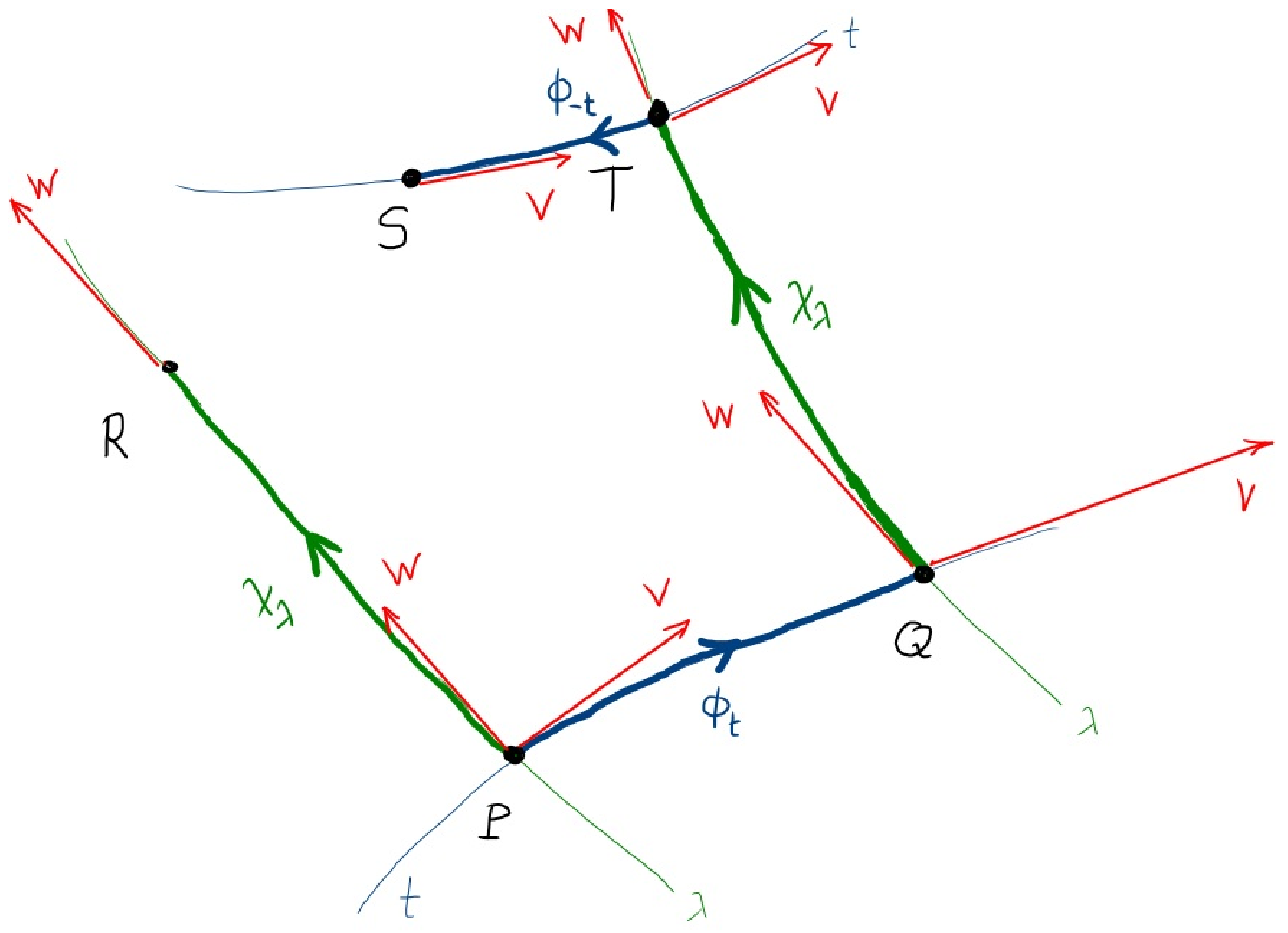
$R = \chi_\lambda(P)$



$$(\mathcal{L}_V W)_P = \lim_{t \to 0} \left[\phi_t^* W_P - W_P \right] \Rightarrow$$

$$\phi_t^* W_P(f) = W_P(f) + t\,(\mathcal{L}_V W)_P + \mathcal{O}(t^2)$$

Geometric Interpretation of $\mathcal{L}_V W$



Consider the points

$Q = \phi_t(P) \qquad T = \chi_\lambda(Q) \qquad S = \phi_{-t}(T)$

$R = \chi_\lambda(P)$

$$(\mathcal{L}_V W)_P = \lim_{t \to 0} \left[\phi_t^* W_P - W_P \right] \Rightarrow$$

$$\phi_t^* W_P(f) = W_P(f) + t\,(\mathcal{L}_V W)_P + \mathcal{O}(t^2)$$

But:

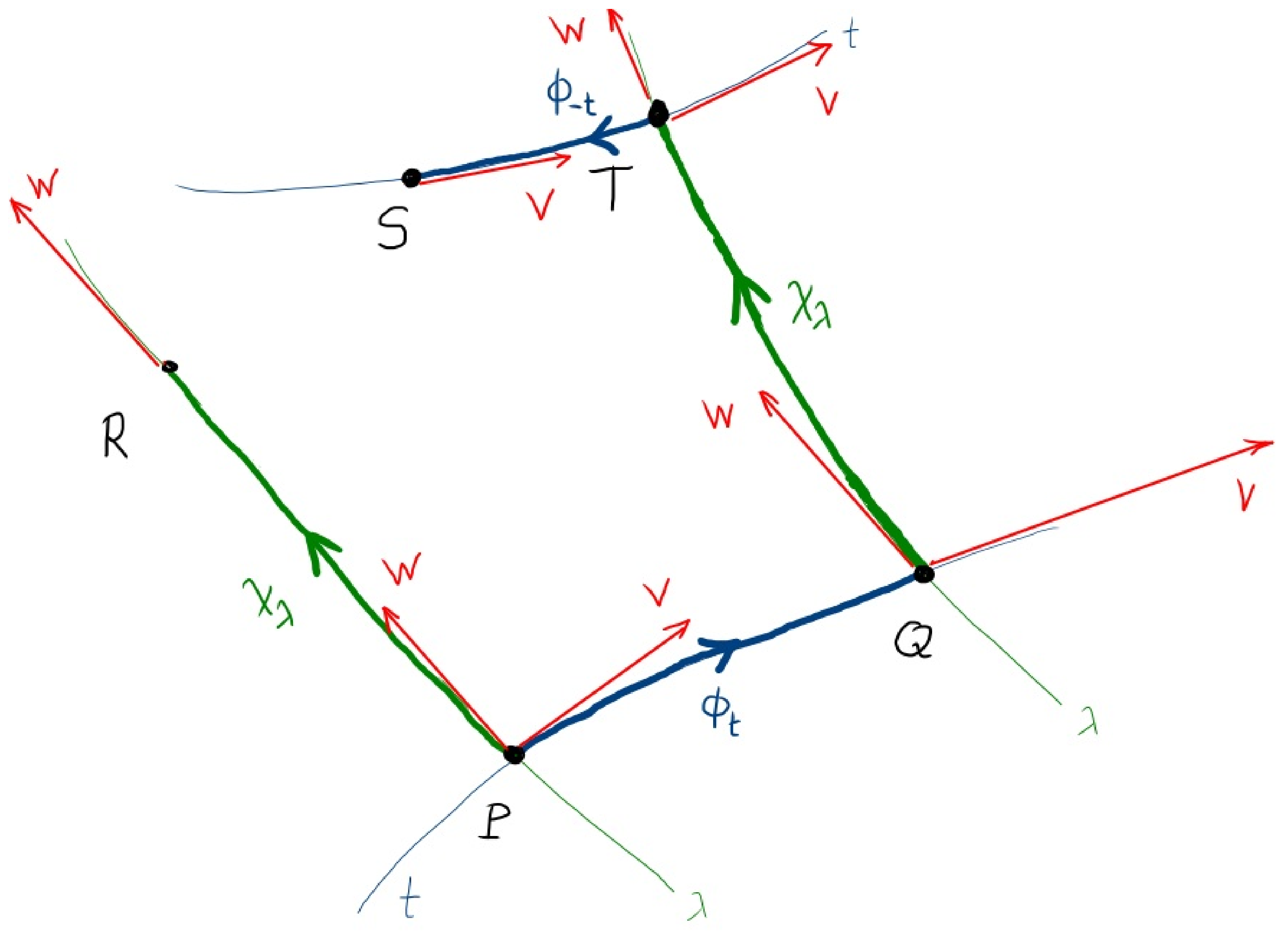
$$\phi_t^* W_P(f) = \phi_{-t*} W_P(f) = W_Q(\phi_{-t}^* f) = W_Q(f \circ \phi_{-t})$$

Geometric Interpretation of $\mathcal{L}_V W$

Consider the points

$Q = \phi_t(P) \quad T = \chi_\lambda(Q) \quad S = \phi_{-t}(T)$

$R = \chi_\lambda(P)$



$$(\mathcal{L}_V W)_P = \lim_{t \to 0} \left[\phi_t^* W_P - W_P \right] \Rightarrow$$

$$\phi_t^* W_P(f) = W_P(f) + t\,(\mathcal{L}_V W)_P(f) + \mathcal{O}(t^2)$$

But:

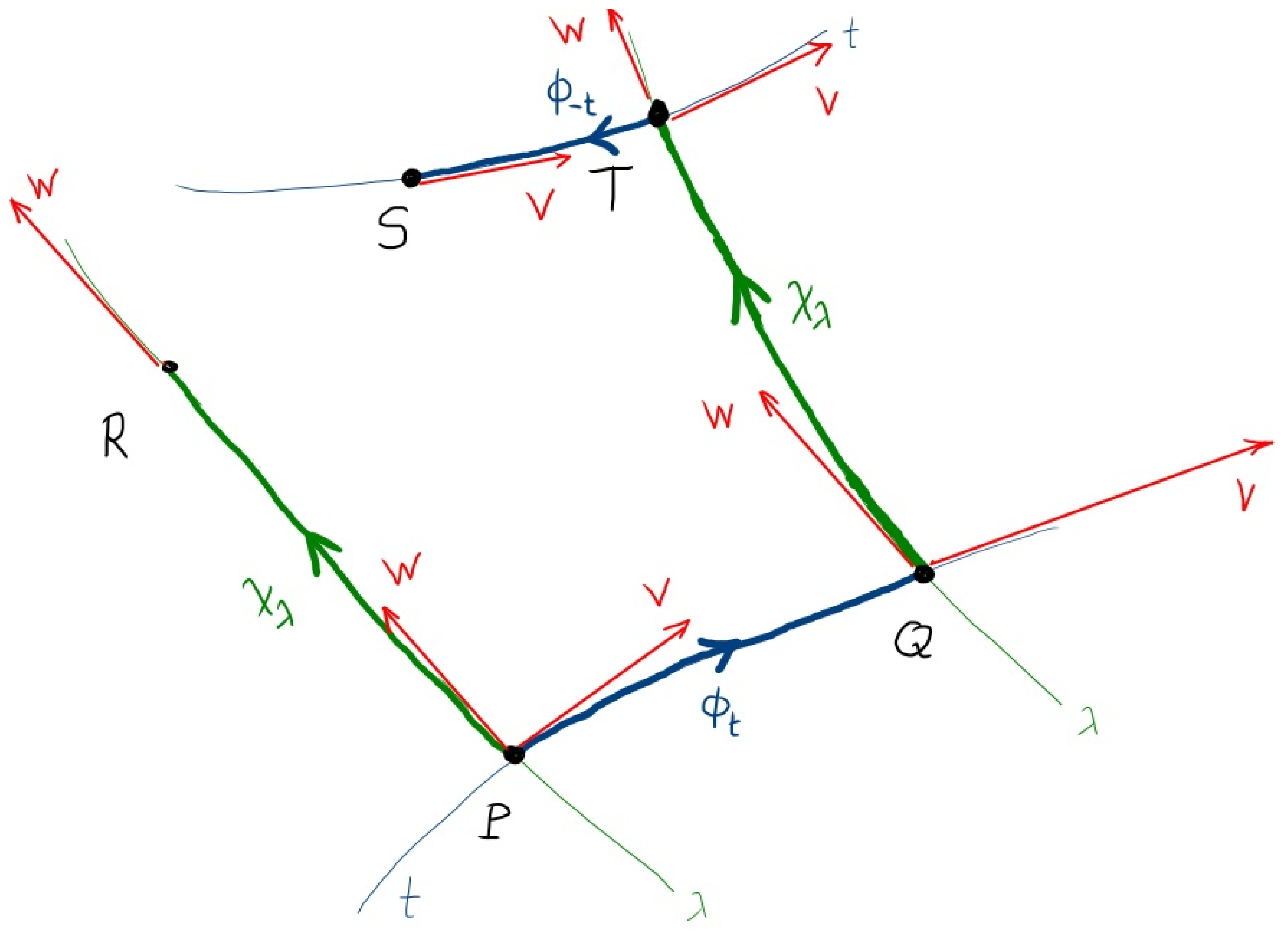
$$\phi_t^* W_P(f) = \phi_{-t*} W_P(f) = W_Q(\phi_{-t}^* f) = W_Q(f \circ \phi_{-t}) \,, \quad \text{therefore}$$

$$W_Q(f \circ \phi_{-t}) = W_P(f) + t\,(\mathcal{L}_V W)_P(f) + \mathcal{O}(t^2)$$

Geometric Interpretation of $\mathcal{L}_V W$

For any function f:

$$f(T) = e^{\lambda W_Q} f(Q) = f(Q) + \lambda W_Q(f) + \frac{\lambda^2}{2} W_Q(W(f)) + O(\lambda^3)$$



$$(\mathcal{L}_V W)_P = \lim_{t \to 0} \left[\phi_t^* W_P - W_P \right] \Rightarrow$$

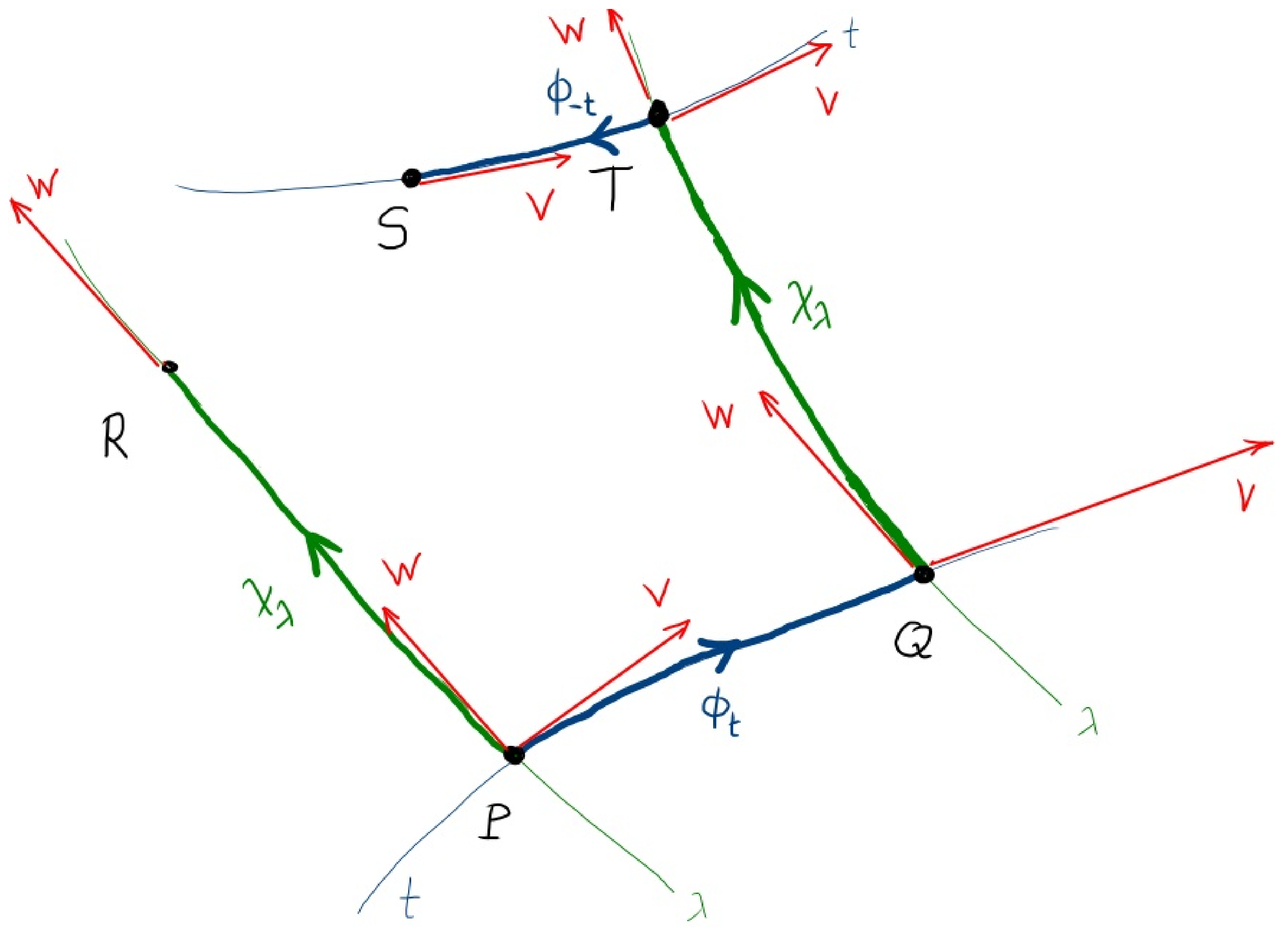
$$\phi_t^* W_P(f) = W_P(f) + t\,(\mathcal{L}_V W)_P(f) + O(t^2)$$

But:

$$\phi_t^* W_P(f) = \phi_{-t*} W_P(f) = W_Q(\phi_{-t}^* f) = W_Q(f \circ \phi_{-t}), \quad \text{therefore}$$

$$W_Q(f \circ \phi_{-t}) = W_P(f) + t\,(\mathcal{L}_V W)_P(f) + O(t^2)$$

Geometric Interpretation of $\mathcal{L}_V W$



For any function f:

$$f(T) = e^{\lambda W_Q} f(Q) = f(Q) + \lambda W_Q(f) + \frac{\lambda^2}{2} W_Q(W(f)) + O(\lambda^3)$$

for $f \to f\circ\phi_{-t}$ we have

$$f\circ\phi_{-t}(T) = f\circ\phi_{-t}(Q) + \lambda W_Q(f\circ\phi_{-t}) + \frac{\lambda^2}{2} W_Q(W(f\circ\phi_{-t})) + O(\lambda^3)$$

$$(\mathcal{L}_V W)_P = \lim_{t\to 0}\left[\phi_t^* W_P - W_P\right] \Rightarrow$$

$$\phi_t^* W_P(f) = W_P(f) + t(\mathcal{L}_V W)_P(f) + O(t^2)$$

But:

$$\phi_t^* W_P(f) = \phi_{-t*} W_P(f) = W_Q(\phi_{-t}^* f) = W_Q(f\circ\phi_{-t}) \quad , \quad \text{therefore}$$

$$W_Q(f\circ\phi_{-t}) = W_P(f) + t(\mathcal{L}_V W)_P(f) + O(t^2)$$

Geometric Interpretation of $\mathcal{L}_V W$

For any function f:

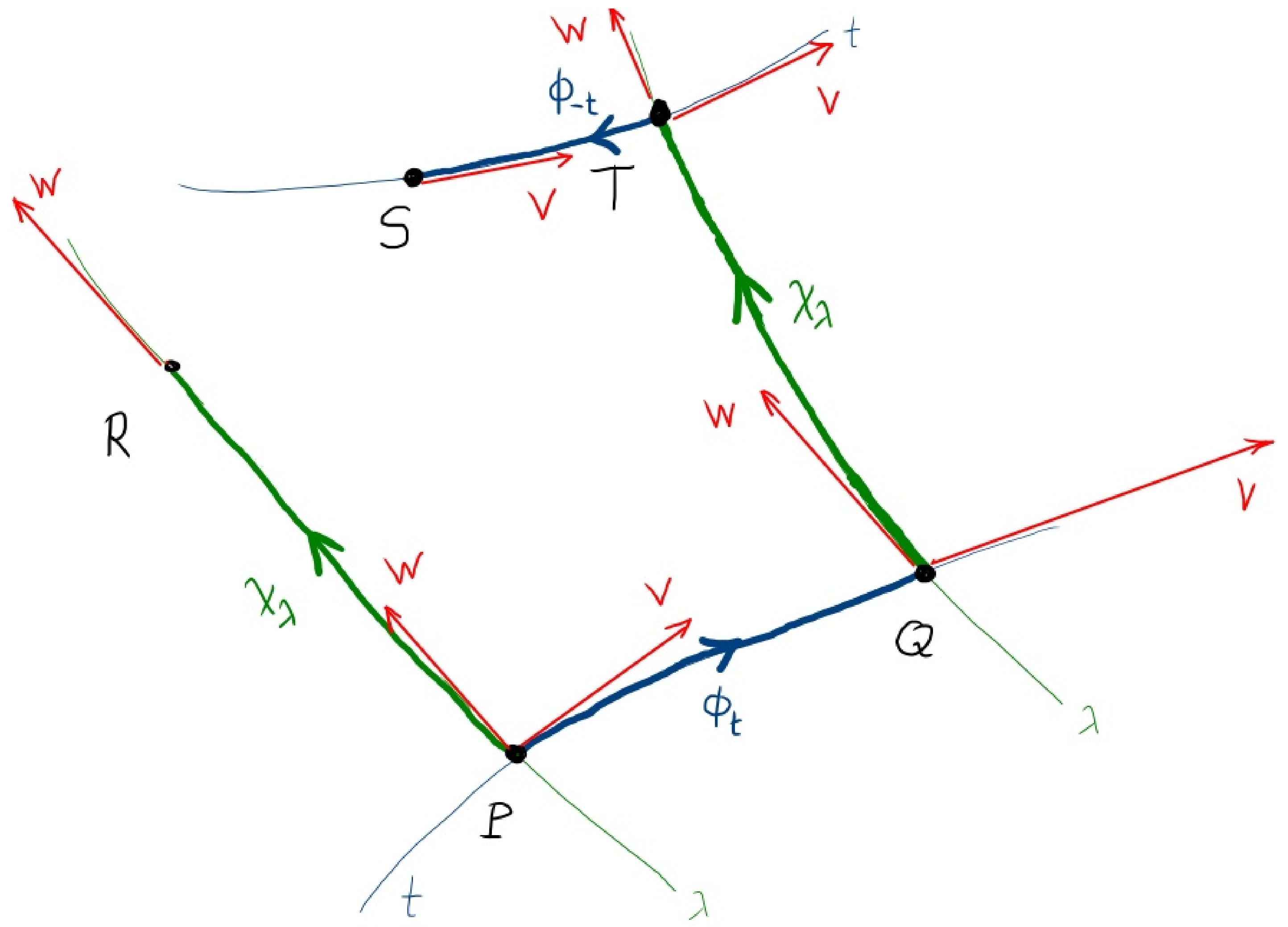
$$f(T) = e^{\lambda W_Q} f(Q) = f(Q) + \lambda W_Q(f) + \frac{\lambda^2}{2} W_Q(W(f)) + O(\lambda^3)$$

for $f \to f\circ\phi_{-t}$ we have

$$\underbrace{f\circ\phi_{-t}(T)}_{f(S)} = \underbrace{f\circ\phi_{-t}}_{f(P)}(Q) + \lambda W_Q(f\circ\phi_{-t}) + \frac{\lambda^2}{2} W_Q(W(f\circ\phi_{-t})) + O(\lambda^3)$$

$$W_Q(W(f\circ\phi_{-t})) = \frac{d^2 f\circ\phi_{-t}}{d\lambda^2}\Big|_Q \overset{f\circ\phi_{-t}(Q)=f(P)}{=\!=\!=} \frac{d^2 f}{d\lambda^2}\Big|_P$$

$$= W_P(W(f))$$



$$W_Q(f\circ\phi_{-t}) = W_P(f) + t\,(\mathcal{L}_V W)_P(f) + O(t^2)$$

Geometric Interpretation of $\mathcal{L}_V W$

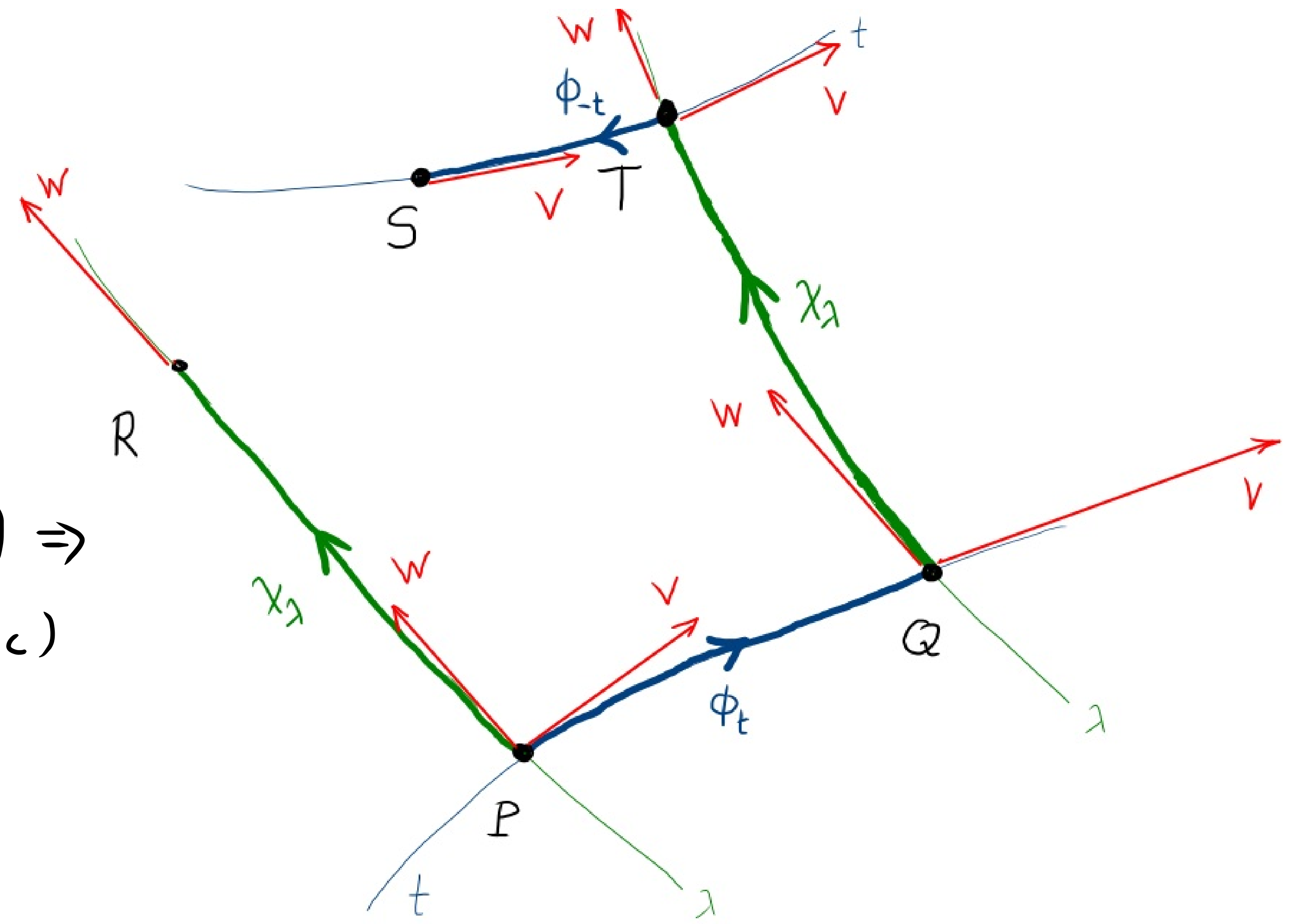
For any function f:

$$f(T) = e^{\lambda W_Q} f(Q) = f(Q) + \lambda W_Q(f) + \frac{\lambda^2}{2} W_Q(W(f)) + O(\lambda^3)$$

for $f \to f\circ\phi_{-t}$ we have

$$f\circ\phi_{-t}(T) = f\circ\phi_{-t}(Q) + \lambda W_Q(f\circ\phi_{-t}) + \frac{\lambda^2}{2} W_Q(W(f\circ\phi_{-t})) + O(\lambda^3) \Rightarrow$$

$$f(S) = f(P) + \lambda\left[W_P(f) + t(\mathcal{L}_V W)_P(f)\right] + \frac{\lambda^2}{2} W_P(W(f)) + (\text{cubic})$$



$$W_Q(f\circ\phi_{-t}) = W_P(f) + t(\mathcal{L}_V W)_P(f) + O(t^2)$$

Geometric Interpretation of $\mathcal{L}_V W$

For any function f:

$$f(T) = e^{\lambda W_Q} f(Q) = f(Q) + \lambda W_Q(f) + \frac{\lambda^2}{2} W_Q(W(f)) + O(\lambda^3)$$

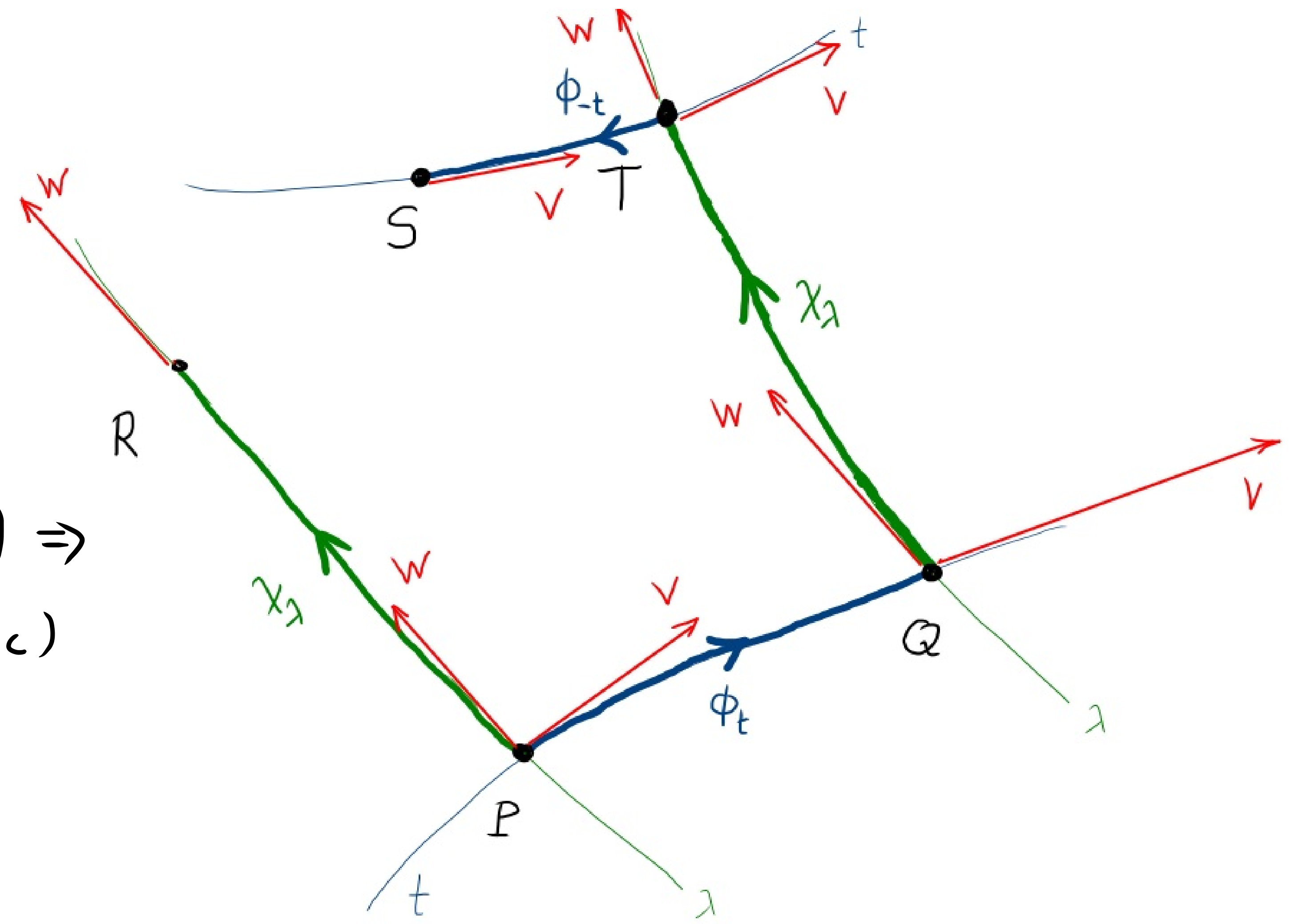
for $f \to f\circ\phi_{-t}$ we have

$$f\circ\phi_{-t}(T) = f\circ\phi_{-t}(Q) + \lambda W_Q(f\circ\phi_{-t}) + \frac{\lambda^2}{2} W_Q(W(f\circ\phi_{-t})) + O(\lambda^3) \Rightarrow$$

$$f(S) = f(P) + \lambda\left[W_P(f) + t(\mathcal{L}_V W)_P(f)\right] + \frac{\lambda^2}{2} W_P(W(f)) + (\text{cubic})$$

set $\lambda = t = \epsilon$:

$$f(S) = f(P) + \epsilon W_P(f) + \epsilon^2 (\mathcal{L}_V W)_P(f) + \frac{\epsilon^2}{2} W_P(W(f)) + O(\epsilon^3)$$



$$W_Q(f\circ\phi_{-t}) = W_P(f) + t(\mathcal{L}_V W)_P(f) + O(t^2)$$

Geometric Interpretation of $\mathcal{L}_V W$

For any function f:

$$f(T) = e^{\lambda W_Q} f(Q) = f(Q) + \lambda W_Q(f) + \frac{\lambda^2}{2} W_Q(W(f)) + O(\lambda^3)$$

for $f \to f\circ\phi_{-t}$ we have

$$f\circ\phi_{-t}(T) = f\circ\phi_{-t}(Q) + \lambda W_Q(f\circ\phi_{-t}) + \frac{\lambda^2}{2} W_Q(W(f\circ\phi_{-t})) + O(\lambda^3) \Rightarrow$$

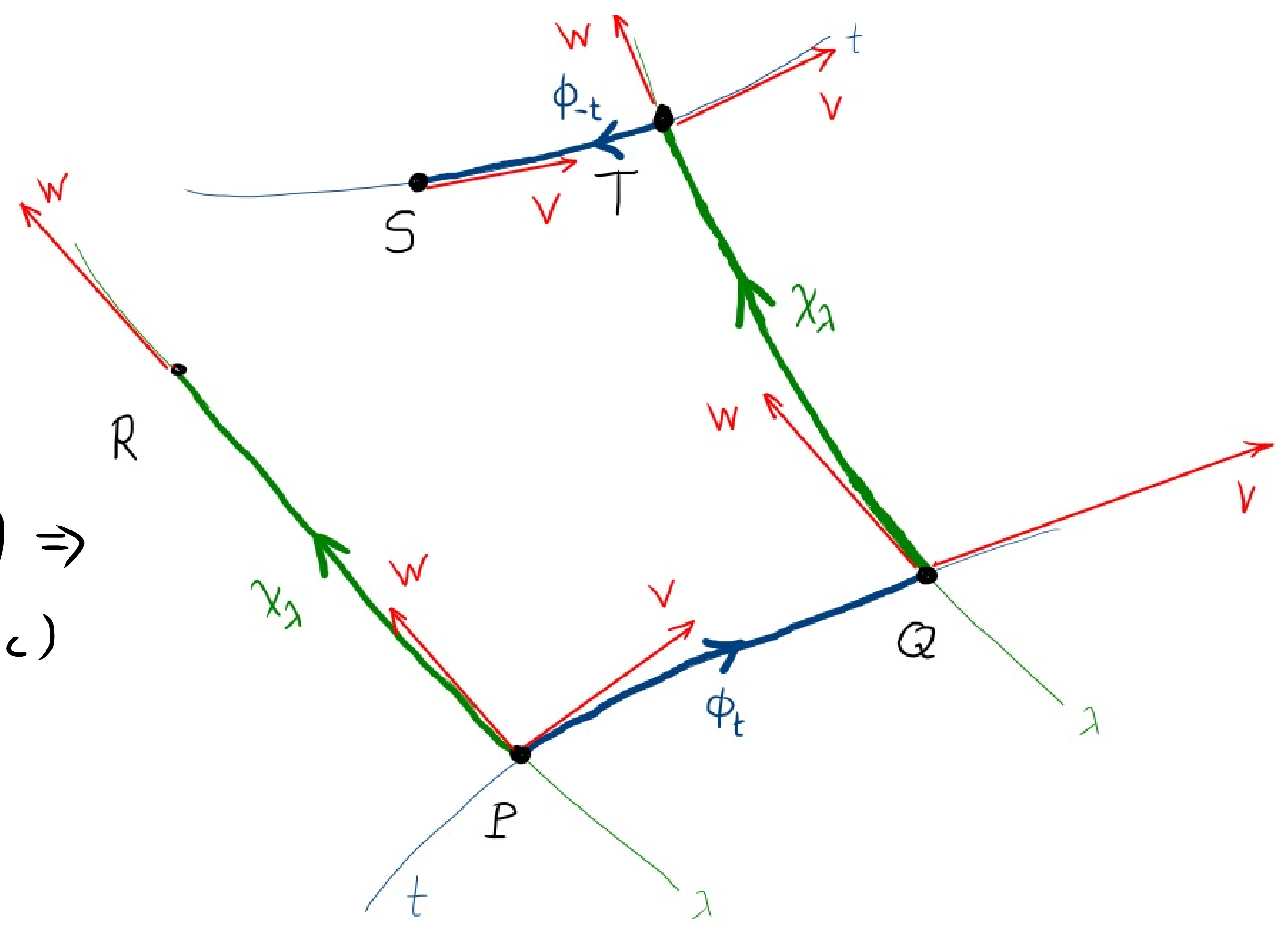
$$f(S) = f(P) + \lambda\left[W_P(f) + t(\mathcal{L}_V W)_P(f)\right] + \frac{\lambda^2}{2} W_P(W(f)) + (\text{cubic})$$

set $\lambda = t = \epsilon$:

$$f(S) = f(P) + \epsilon W_P(f) + \epsilon^2 (\mathcal{L}_V W)_P(f) + \frac{\epsilon^2}{2} W_P(W(f)) + O(\epsilon^3)$$

but:

$$f(R) = e^{\epsilon W_P} f(P) = f(P) + \epsilon W_P(f) + \frac{\epsilon^2}{2} W_P(W(f)) + O(\epsilon^3)$$



Geometric Interpretation of $\mathcal{L}_V W$

For any function f:

$$f(T) = e^{\lambda W_Q} f(Q) = f(Q) + \lambda W_Q(f) + \frac{\lambda^2}{2} W_Q(W(f)) + O(\lambda^3)$$

for $f \to f\circ\phi_{-t}$ we have

$$f\circ\phi_{-t}(T) = f\circ\phi_{-t}(Q) + \lambda W_Q(f\circ\phi_{-t}) + \frac{\lambda^2}{2} W_Q(W(f\circ\phi_{-t})) + O(\lambda^3) \Rightarrow$$

$$f(S) = f(P) + \lambda\left[W_P(f) + t(\mathcal{L}_V W)_P(f)\right] + \frac{\lambda^2}{2} W_P(W(f)) + (\text{cubic})$$

set $\lambda = t = \epsilon$:

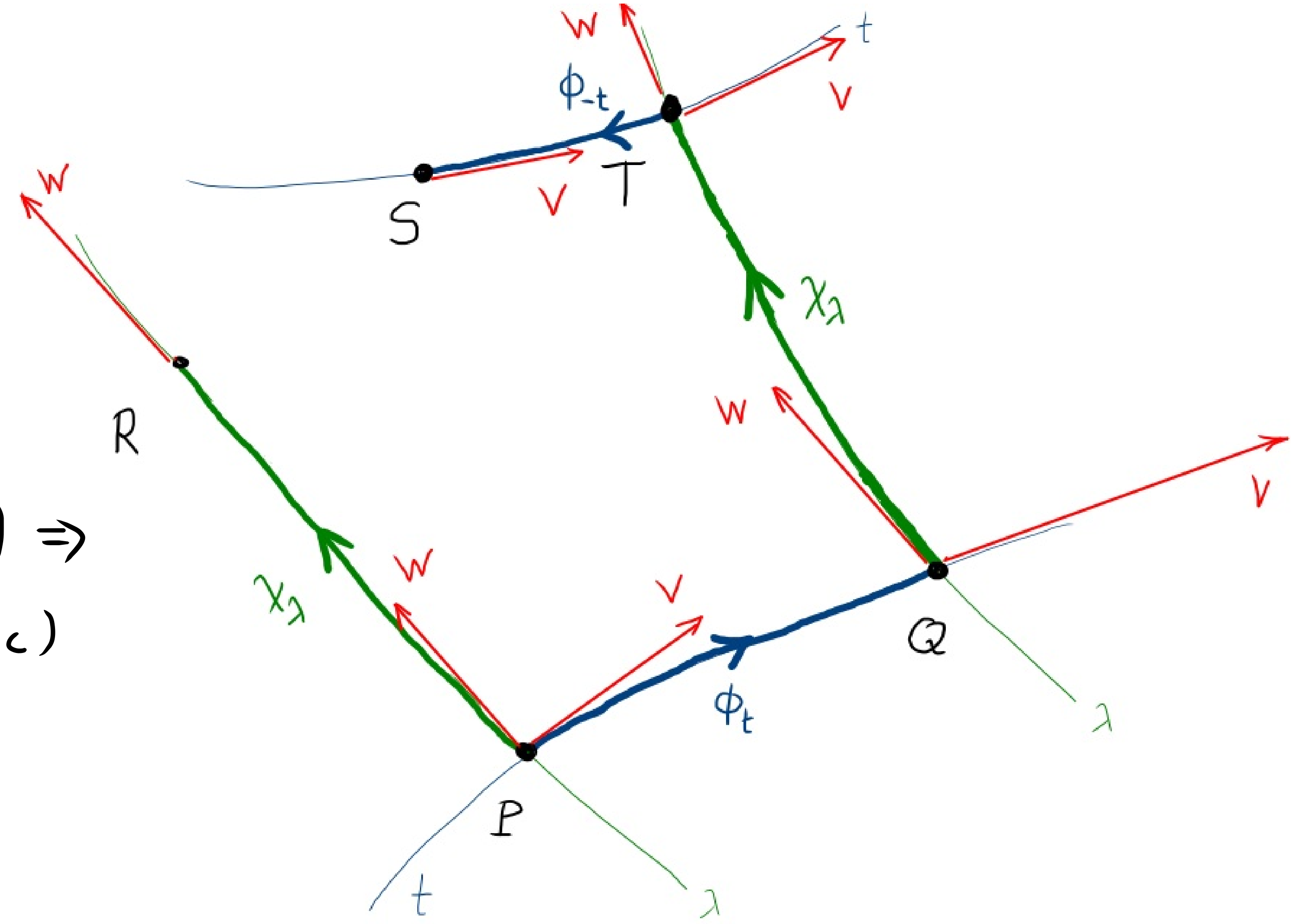
$$f(S) = \cancel{f(P)} + \epsilon \cancel{W_P(f)} + \epsilon^2 (\mathcal{L}_V W)_P(f) + \frac{\epsilon^2}{2} \cancel{W_P(W(f))} + O(\epsilon^3)$$

but:

$$f(R) = e^{\epsilon W_P} f(P) = \cancel{f(P)} + \epsilon \cancel{W_P(f)} + \frac{\epsilon^2}{2} \cancel{W_P(W(f))} + O(\epsilon^3)$$

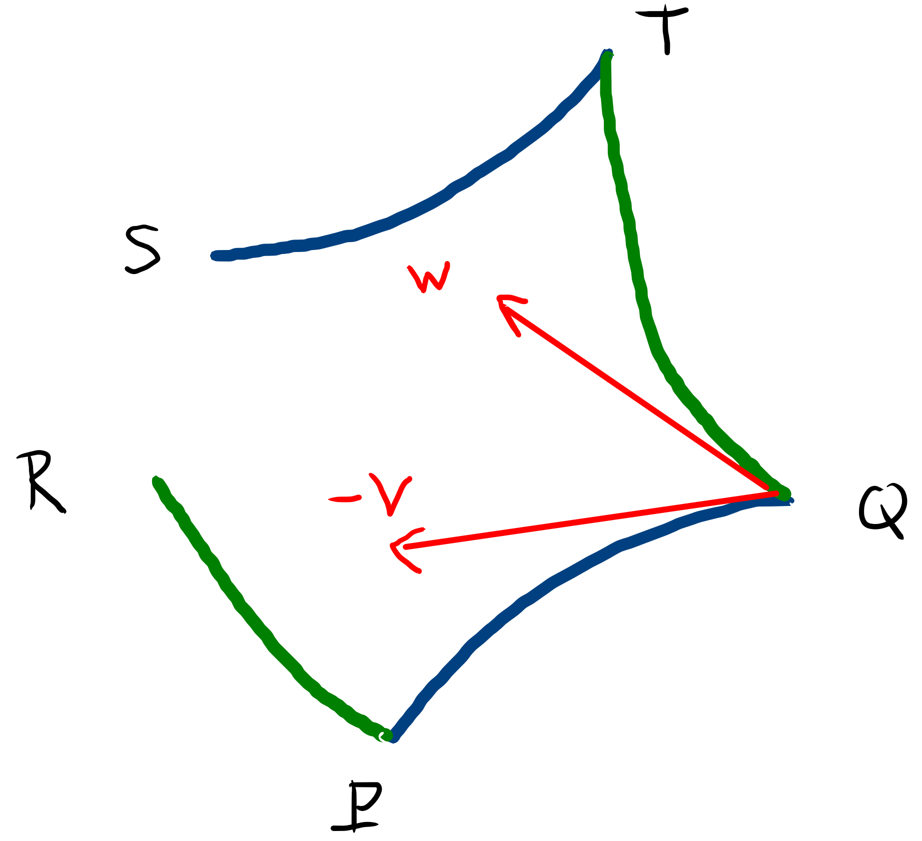
so:

$$f(S) - f(R) = \epsilon^2 (\mathcal{L}_V W)_P(f) + O(\epsilon^3)$$

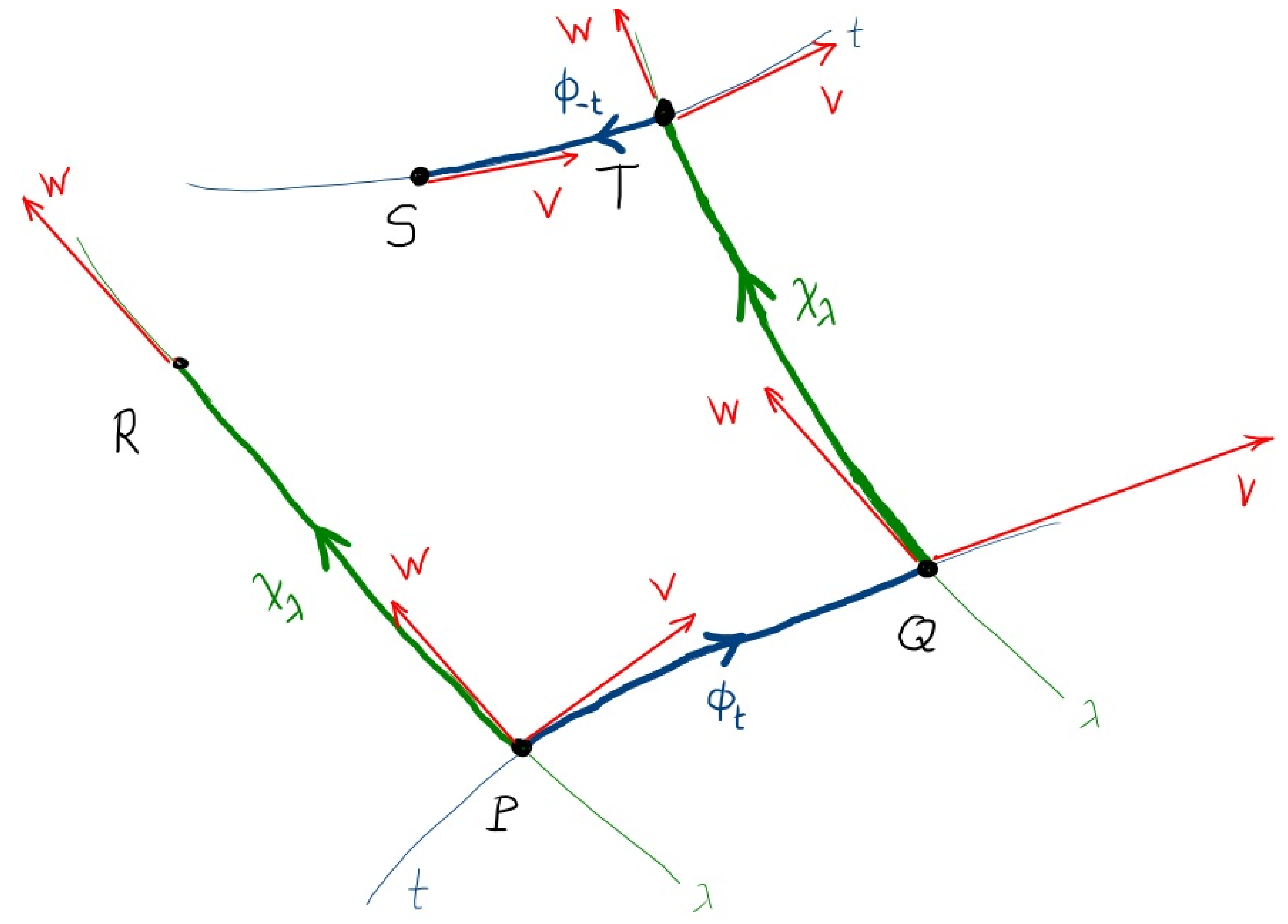


Geometric Interpretation of $\mathcal{L}_V W$

$$f(S) - f(R) = \epsilon^2 (\mathcal{L}_V W)_P (f) + \mathcal{O}(\epsilon^3)$$

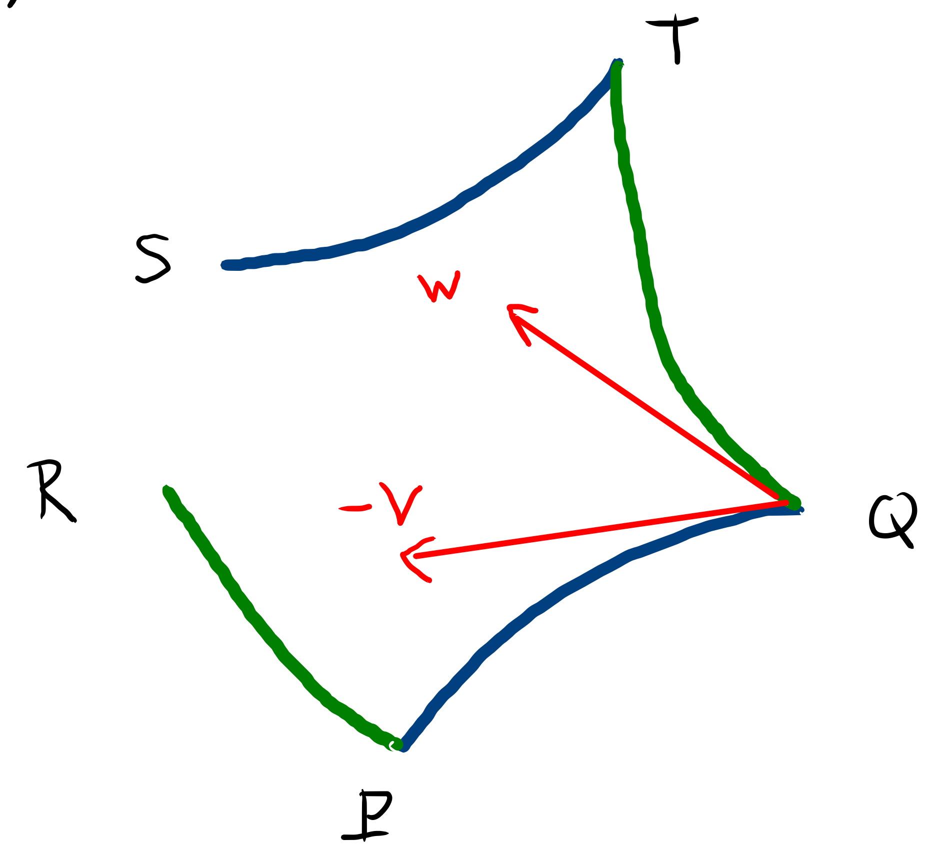


Look from Q

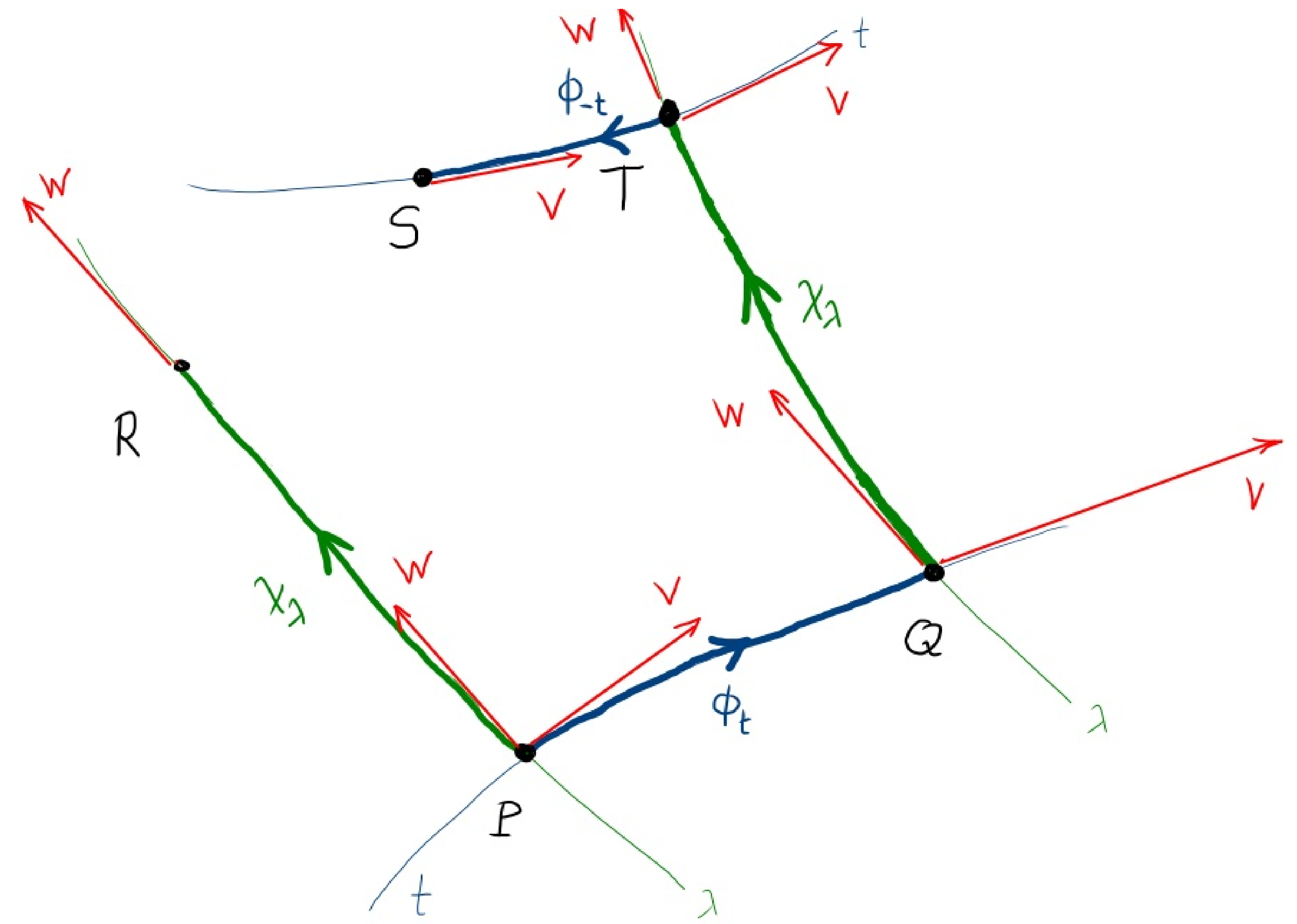


Geometric Interpretation of $\mathcal{L}_V W$

$$f(S) - f(R) = \epsilon^2 (\mathcal{L}_V W)_P(f) + O(\epsilon^3)$$



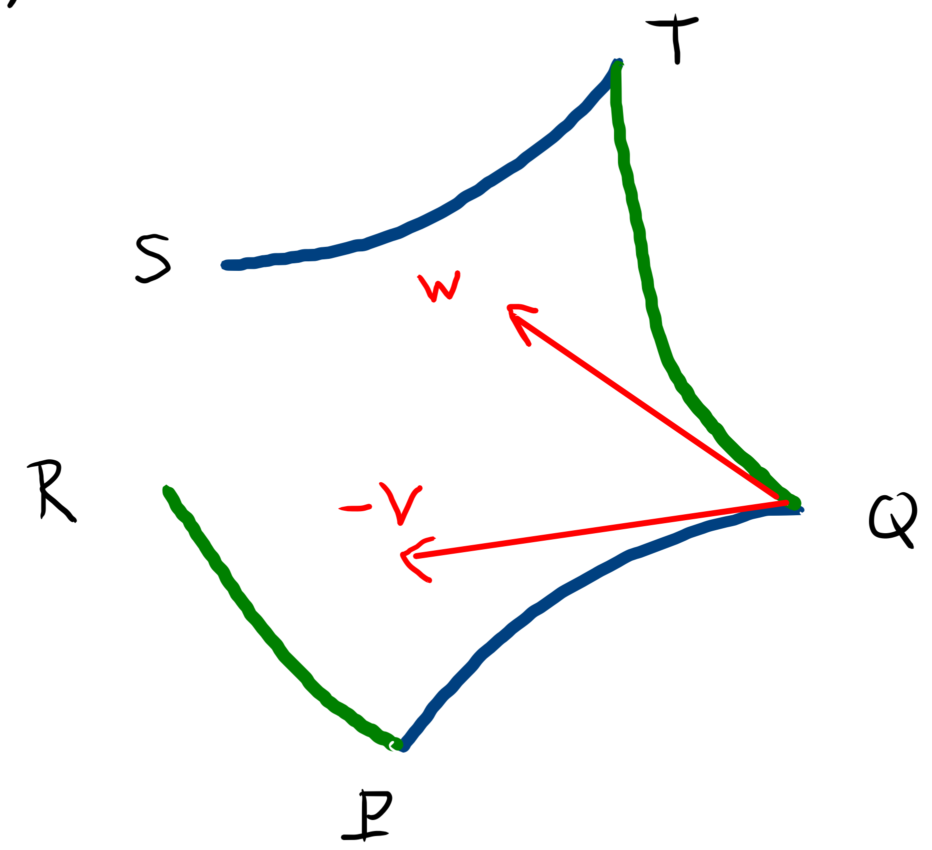
Look from Q



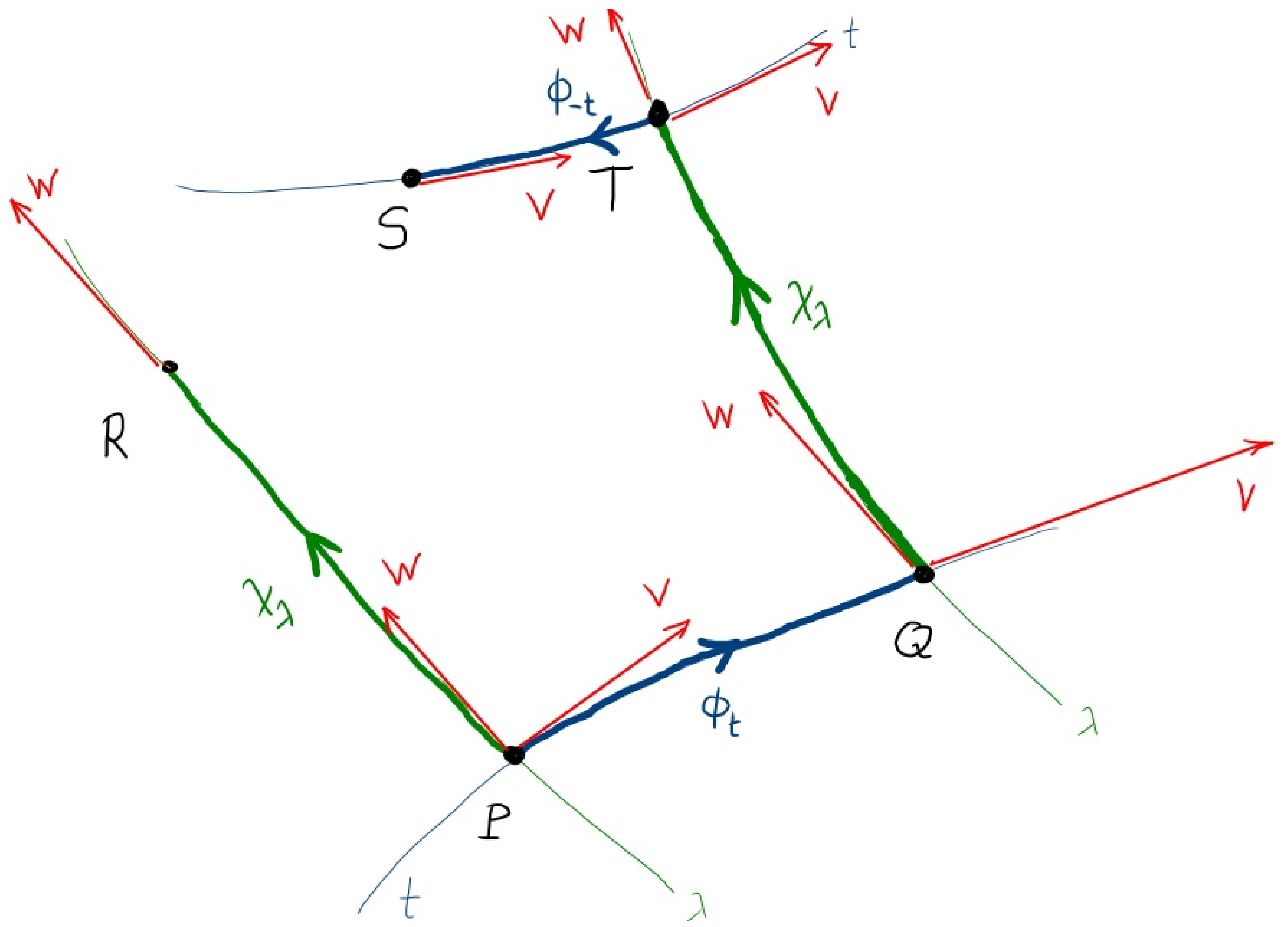
$$f(S) - f(R) = \epsilon^2 [W, -V]_Q (f) + O(\epsilon^3) = \epsilon^2 [V, W]_Q (f) + O(\epsilon^3)$$

Geometric Interpretation of $\mathcal{L}_V W$

$$f(S)-f(R)=\epsilon^2(\mathcal{L}_V W)_P(f)+\mathcal{O}(\epsilon^3)$$



Look from Q



$$f(S)-f(R)=\epsilon^2[W,-V]_Q(f)+\mathcal{O}(\epsilon^3)=\epsilon^2[V,W]_Q(f)+\mathcal{O}(\epsilon^3)$$

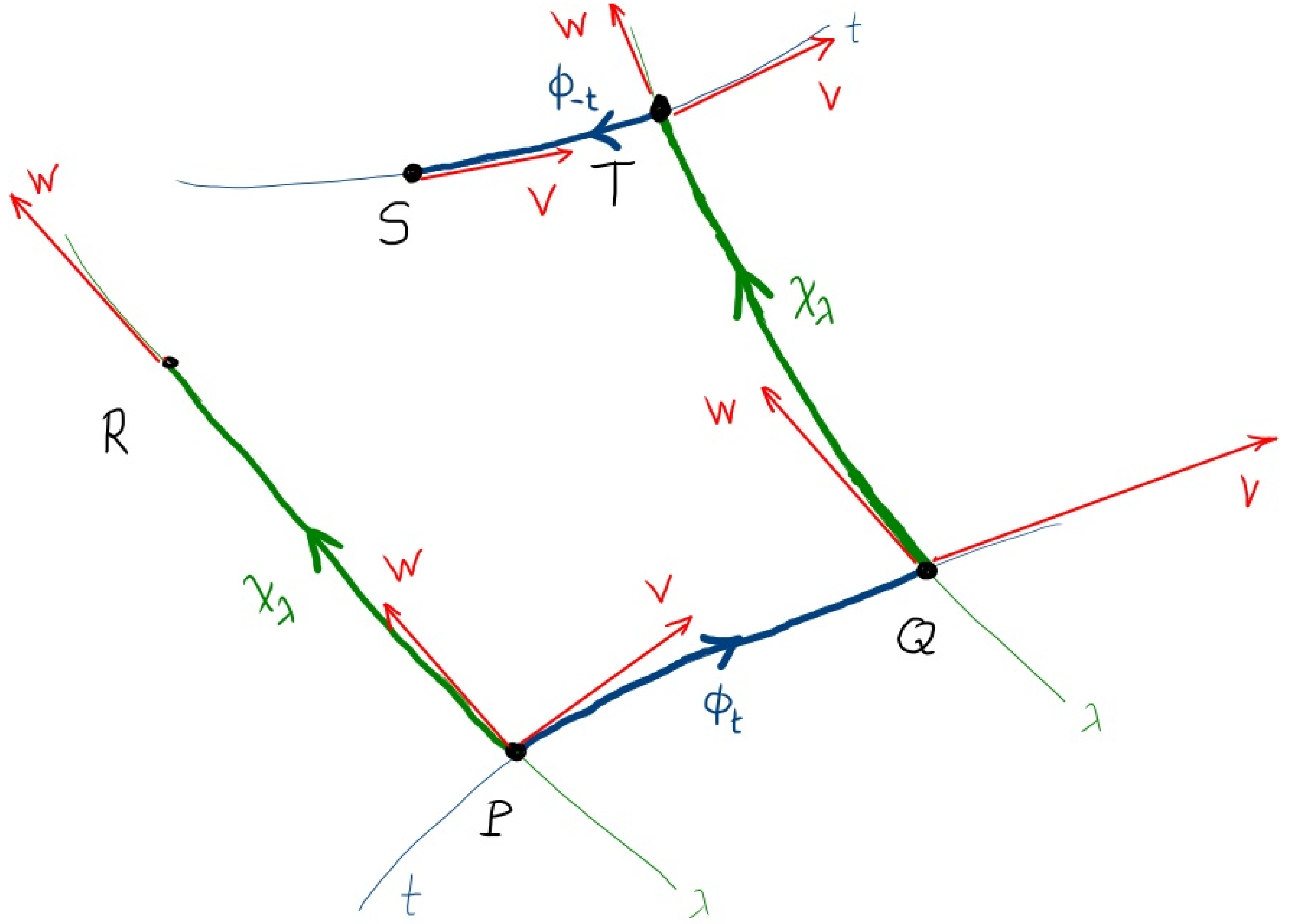
$$=\epsilon^2\Big([V,W]_P(f)+\mathcal{O}(\epsilon)\Big)+\mathcal{O}(\epsilon^3)$$

$$=\epsilon^2[V,W]_P(f)+\mathcal{O}(\epsilon^3)$$

Geometric Interpretation of $\mathcal{L}_V W$

$$f(S) - f(R) = \epsilon^2 (\mathcal{L}_V W)_P(f) + \mathcal{O}(\epsilon^3)$$

$$f(S) - f(R) = \epsilon^2 [V, W]_P(f) + \mathcal{O}(\epsilon^3)$$



so $\displaystyle \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} [f(S) - f(R)] \Rightarrow$

$$(\mathcal{L}_V W)_P(f) = [V, W]_P(f) \qquad \forall f$$

$$\forall P$$

Geometric Interpretation of $\mathcal{L}_V W$

$$f(S) - f(R) = \epsilon^2 (\mathcal{L}_V W)_P(f) + \mathcal{O}(\epsilon^3)$$

$$f(S) - f(R) = \epsilon^2 [V, W]_P(f) + \mathcal{O}(\epsilon^3)$$

so $\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} [f(S) - f(R)] \Rightarrow$

$$(\mathcal{L}_V W)_P(f) = [V, W]_P(f) \qquad \forall f \quad \forall P$$

$$\Rightarrow \quad \mathcal{L}_V W = [V, W]$$