Integration on Manifolds

- volume elements
- orientability
- integration on orientable Manifolds
  - partitions of unity
- integration on Manifolds with a metric

Volume in $\mathbb{R}^n$

- volume less restrictive structure than metric
  - → many metrics give same volumes

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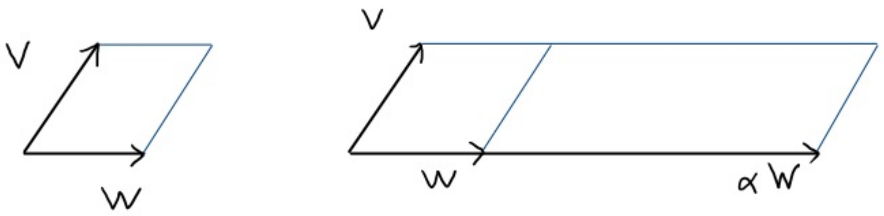
- volume is a linear function on n vectors



$$\omega(\alpha w, v) = \alpha\, \omega(w, v)$$

E.g. $n = 2$

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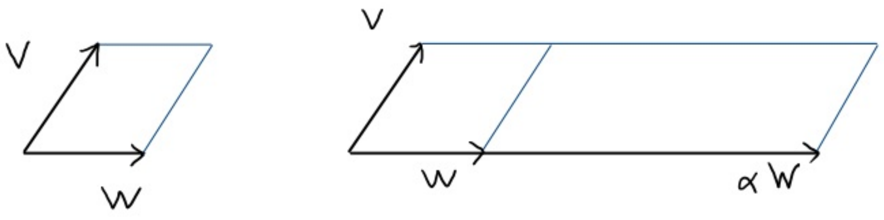

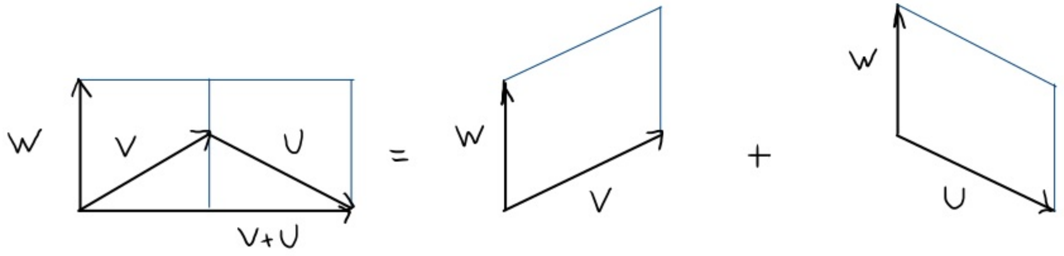

$$\omega(\alpha W, V) = \alpha\, \omega(W, V) \qquad \omega(W, V+U) = \omega(W, V) + \omega(W, U)$$

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$$\omega(V, W, U) = V \cdot (W \times U) = \begin{vmatrix} V_x & V_y & V_z \\ W_x & W_y & W_z \\ U_x & U_y & U_z \end{vmatrix}$$

E.g. $n = 3$

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$$\alpha_1 \omega(V_1, W, U) + \alpha_2 \omega(V_2, W, U)$$

E.g. $n = 3$

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- it is zero if any two vectors are colinear:

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$$\Rightarrow \omega(W,U,\dots) = -\omega(U,W,\dots)$$

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$\Rightarrow$ ω antisymmetric $\Rightarrow \dots \Rightarrow$ completely antisymmetric

Volume in $\mathbb{R}^n$

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- volume is a linear function on n vectors
- it is zero if any two vectors are colinear
- it is a n-form !

Volume in $\mathbb{R}^n$

In $\mathbb{R}^n$, consider cartesian coordinates $(x^1, \dots, x^n)$, and the n-form:

$$\omega = dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$$

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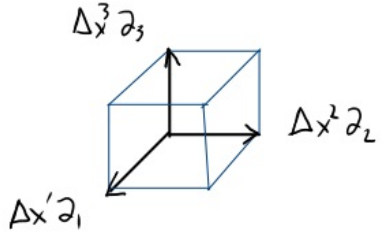

$$\omega(\Delta x^1 \partial_1, \dots, \Delta x^n \partial_n) = \Delta x^1 \dots \Delta x^n = \Delta V$$

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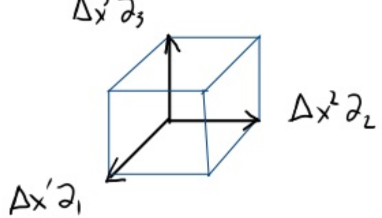

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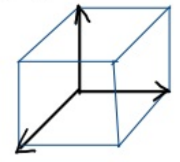

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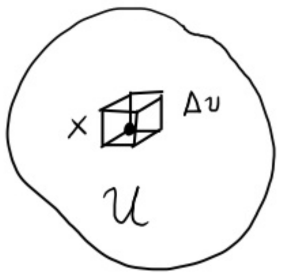
Denote by $x = (x^1, x^2, \dots, x^n)$, then

$$f(x)\,\omega(\Delta x^1 \partial_1, \dots, \Delta x^n \partial_n) = f(x) \cdot \Delta v$$

Volume in $\mathbb{R}^n$

Therefore:

$$\int_{\mathcal{U}} f(x)\, d^n x \approx \sum_{\Delta v} f(x)\, \Delta v = \sum_{\Delta v} f(x)\, \omega(\Delta x^1 \partial_1, \dots, \Delta x^n \partial_n)$$



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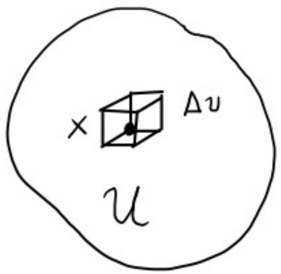

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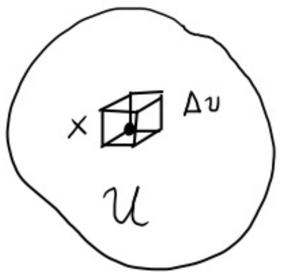

We define:

$$\int_{\mathcal{U}} f\omega = \int_{\mathcal{U}} f(x)\, d^n x$$

Any other n-form $\sigma = f\omega$, for some f, so

$$\int_{\mathcal{U}} \sigma = \int_{\mathcal{U}} f\omega$$

Volume in $\mathbb{R}^n$

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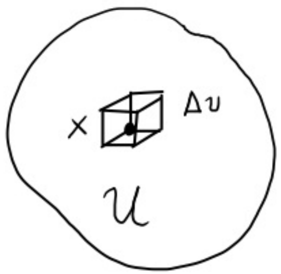

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$\rightsquigarrow$ Integration of functions is integration of n-forms!

Integration on $\mathbb{R}^n$

We have chosen $\omega = dx^1 \wedge \cdots \wedge dx^n$ to be our volume element so that $\int_U \sigma = \int_U f \omega$ for $\sigma = f\omega$.

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But we could have chosen

$$\omega' = dx^2 \wedge dx^1 \wedge \cdots \wedge dx^n = -\omega$$

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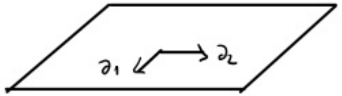
$\Rightarrow$ The overall sign of integral a convention!

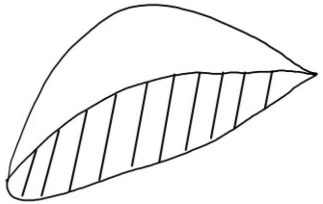
$\Rightarrow$ But we choose ω or ω' and define the integral to be positive w.r.t. to our choice!

Integration on $\mathbb{R}^n$



- there is no natural choice of orientation

"up" or "down"



"inside" or "outside"

$\Rightarrow$ we choose orientation (and then keep it fixed ...)

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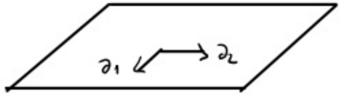
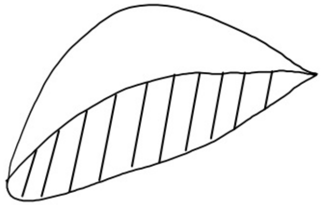
$\rightsquigarrow$ choice of orientation

Integration on $\mathbb{R}^n$





- there is no natural choice of orientation

"up" or "down"

"inside" or "outside"

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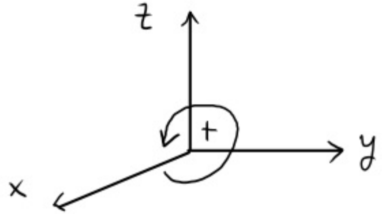
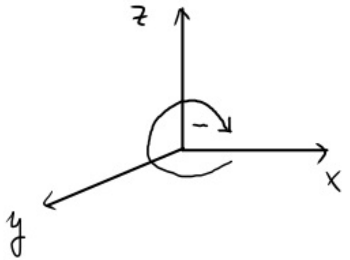
e.g. in $\mathbb{R}^3$



positive orientation



negative orientation

but this is a convention

Integration on $\mathbb{R}^n$

- So, the volume element ω should be:
  - a n-form
  - nowhere vanishing (⇒ no change of sign/orientation)

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Then, there exists coordinate system $(x^1, \dots, x^n)$, s.t.

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with $h(x) > 0 \quad \forall\, x \in \mathbb{R}^n$

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Pick a different coordinate system $(x^{1'}, x^{2'}, \dots, x^{n'})$, then:

Integration on $\mathbb{R}^n$

$$\omega = h(x)\, dx^1 \wedge \cdots \wedge dx^n = \frac{1}{n!} h(x)\, \tilde{\epsilon}_{\mu_1 \cdots \mu_n}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}$$

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$$= h(x)\, J\, dx^{1'} \wedge \cdots \wedge dx^{n'}$$

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If $\{x^\mu\}$ have the same orientation as $\{x^{\mu'}\}$, then $J > 0$

" opposite " $J < 0$

$\omega = h(x) \; J \; dx^{1'} \wedge \dots \wedge dx^{n'}$

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If $\{x^\mu\}$ have the same orientation as $\{x^{\mu'}\}$, then $J > 0$

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We know that, for $J > 0$:

$$\int f(x)\, h(x)\, d^n x = \int f(x')\, h(x')\, J(x')\, d^n x'$$

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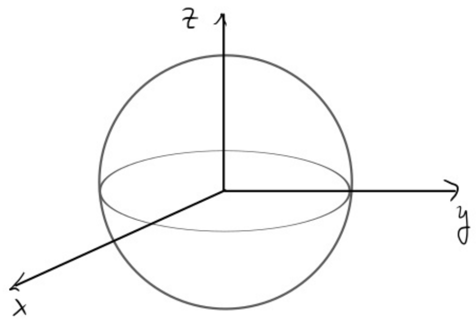
$\Rightarrow \int f \omega$ is independent of choice of coordinates

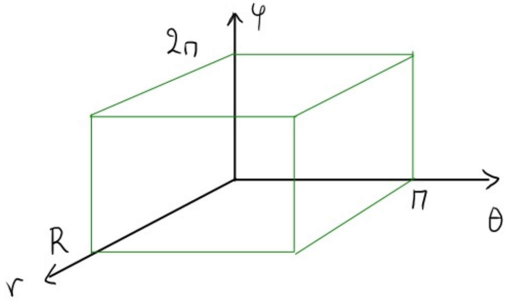
Example: volume of sphere

- coordinates: $0 \le r \le R$, $0 < \theta < \pi$, $0 < \varphi < 2\pi$

$$\omega = dx \wedge dy \wedge dz = r^2 \sin\theta \, dr \wedge d\theta \wedge d\varphi \quad \text{(prove!)}$$

$$h(r, \theta, \varphi) = r^2 \sin\theta$$





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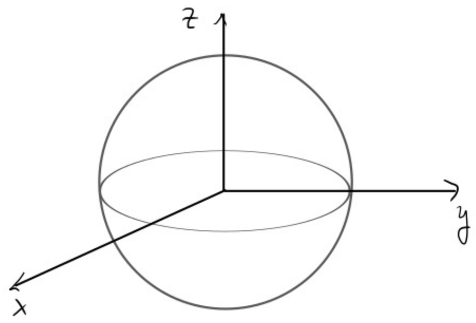

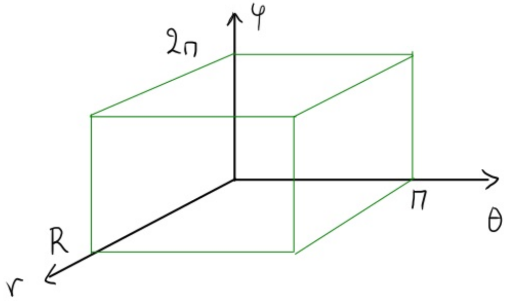

$$\int_{B^3} \omega = \int_{B^3} r^2 \sin\theta \; dr \wedge d\theta \wedge d\varphi$$

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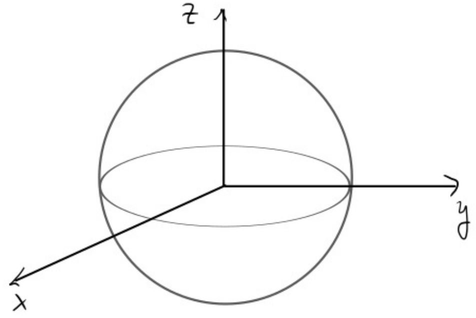

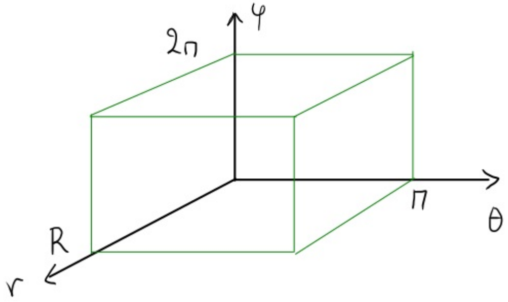

$$\int_{B^3} \omega = \int_{B^3} r^2 \sin\theta \, dr \wedge d\theta \wedge d\varphi = \int_{B^3} r^2 \sin\theta \, dr \, d\theta \, d\varphi$$ (use definition)

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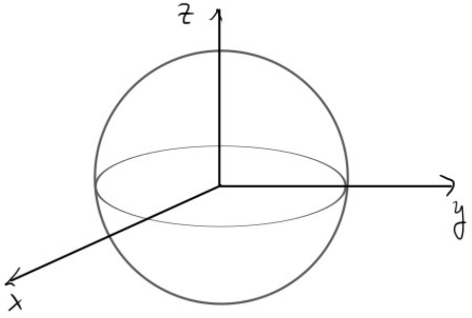

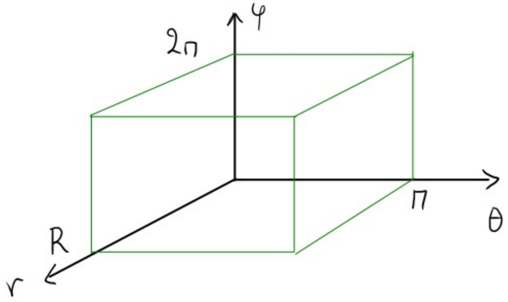

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$$= \int_0^R r^2 \Big[\underbrace{\int_0^\pi \sin\theta \Big(\int_0^{2\pi} d\varphi \Big) d\theta}_{} \Big] dr$$

← order does not matter if done properly

must use $\int_0^\pi d\theta$, $\int_0^{2\pi} d\varphi$, not $\int_\pi^0 d\theta$ or $\int_{2\pi}^0 d\varphi$ ←

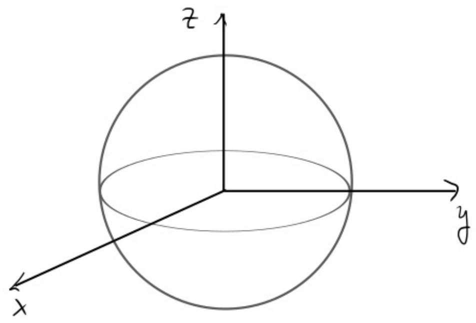
<u>Reason</u>: volume element used in ordinary integrals is <u>by definition</u> positive

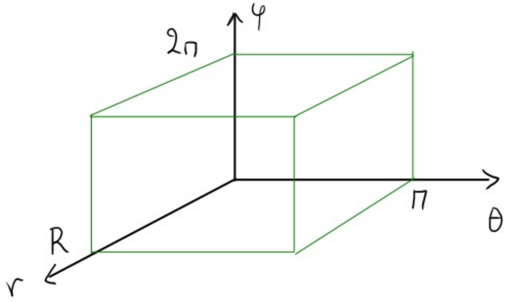
Example: volume of sphere

- coordinates: $0 \leq r \leq R$, $0 < \theta < \pi$, $0 < \varphi < 2\pi$

$$\omega = dx \wedge dy \wedge dz = r^2 \sin\theta \, dr \wedge d\theta \wedge d\varphi \quad \text{(prove!)}$$

$$h(r, \theta, \varphi) = r^2 \sin\theta$$





$$\int_{B^3} \omega = \int_{B^3} r^2 \sin\theta \, dr \wedge d\theta \wedge d\varphi = \int_{B^3} r^2 \sin\theta \, dr \, d\theta \, d\varphi$$

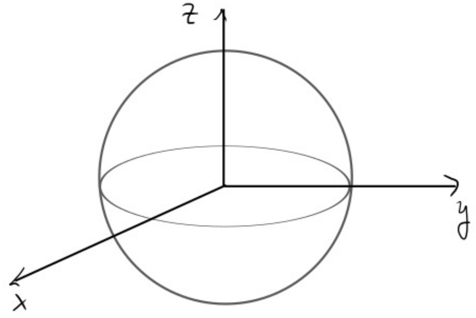
$$= \int_0^R r^2 \left[\int_0^\pi \sin\theta \left(\int_0^{2\pi} d\varphi \right) d\theta \right] dr$$

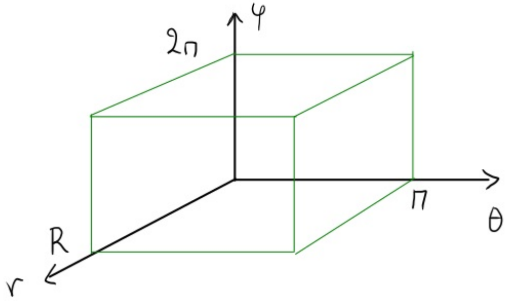
$$= \int_0^R r^2 \quad 2 \quad 2\pi \quad dr = \frac{4}{3} \pi R^3$$

Example: volume of sphere

Under a change of coordinates:

$$(r, \theta, \varphi) \longrightarrow (r, \varphi, \theta)$$





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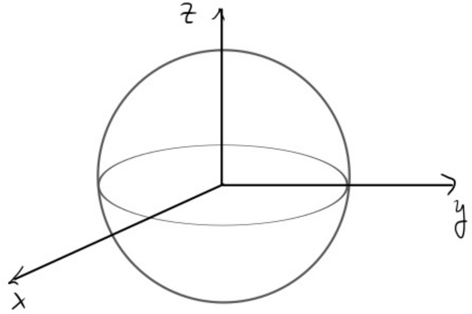

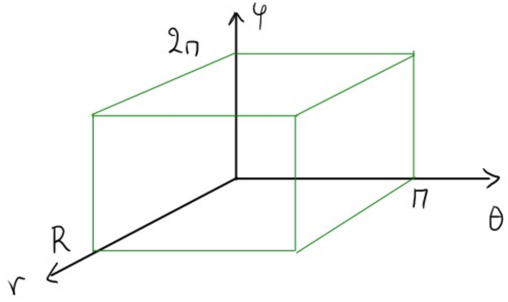

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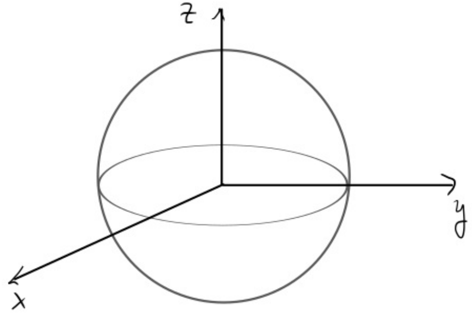

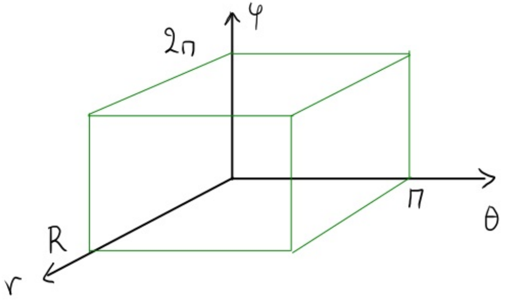

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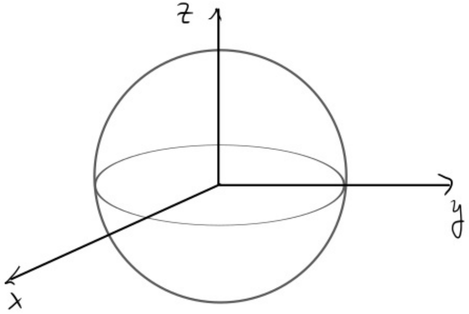

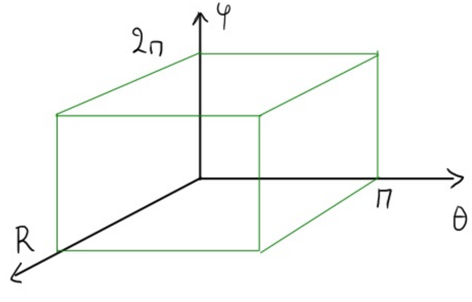

in new coordinates: $h(x') = r^2 \sin\theta = h(x)$, and

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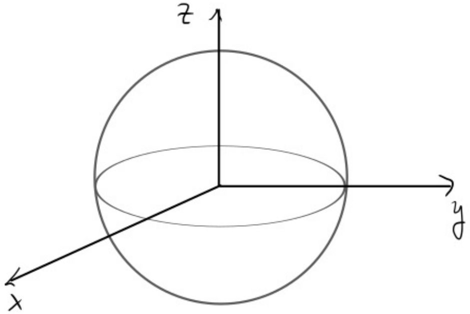

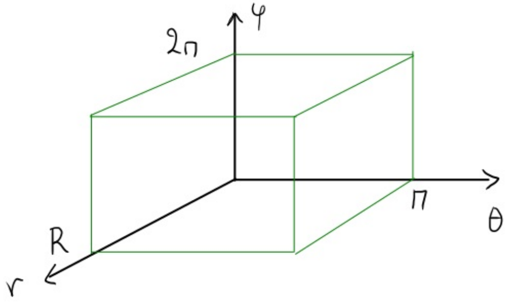

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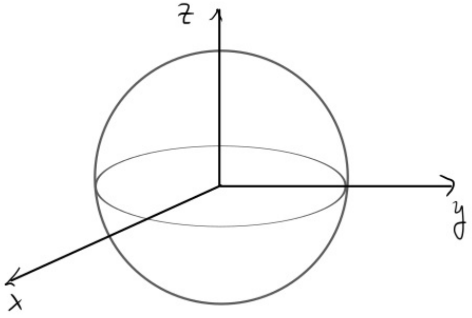

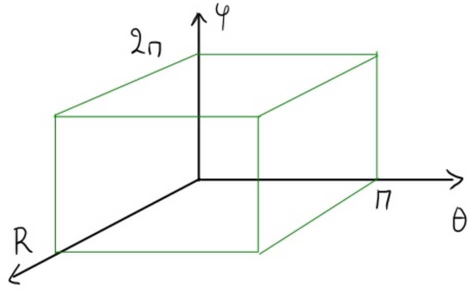

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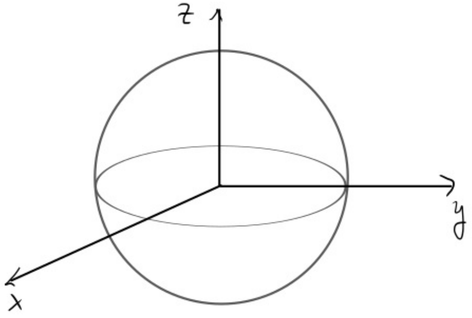

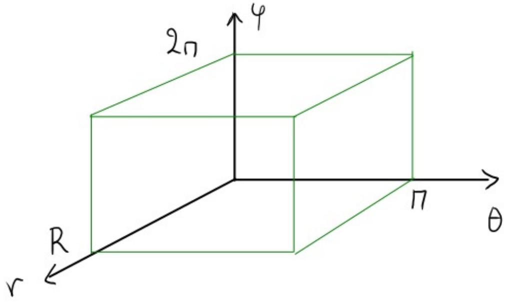

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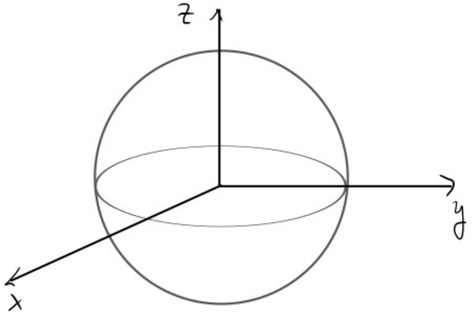

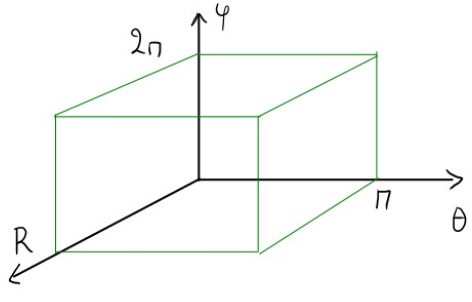

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of course $\omega' = -\omega$, but if this is *defined* to be the positive orientation, then

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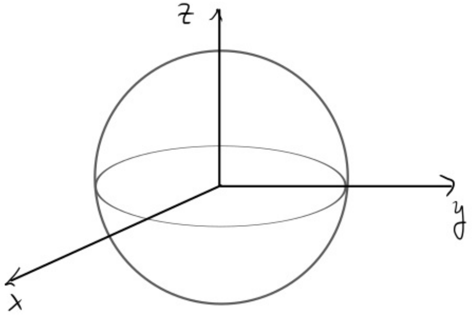

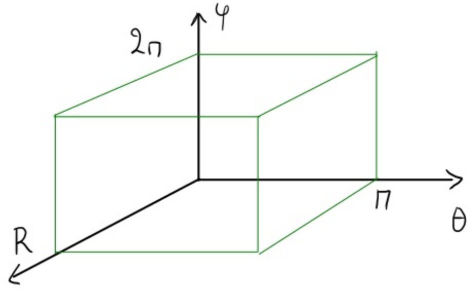

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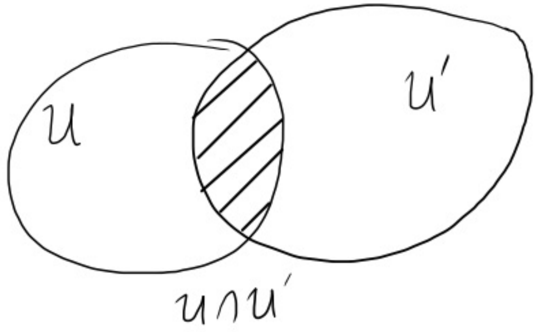

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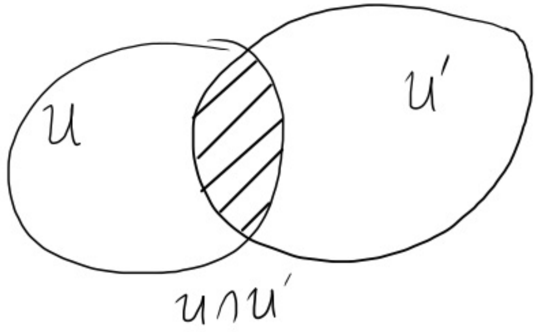

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Since $\partial_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu$, $dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} dx^\mu$, the $\{\partial_\mu\}, \{\partial_{\mu'}\}, \{dx^\mu\}, \{dx^{\mu'}\}$ bases are of the same orientation

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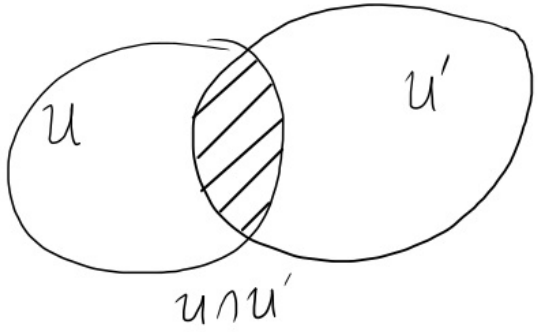

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- If $h > 0$, ω' and ω are called equivalent "right handed"
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How to define ω : (example)

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$$\omega = h(p)\, dx^1 \wedge \dots \wedge dx^n = \frac{1}{n!} h(p)\, \tilde{\epsilon}_{\mu_1 \dots \mu_n} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}$$

with $\omega_{\mu_1 \dots \mu_n} = \tilde{\epsilon}_{\mu_1 \dots \mu_n} h(p)$

How to define ω: (example)

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- choose a coordinate system U ("right handed" by choice) and define the n-form on U:

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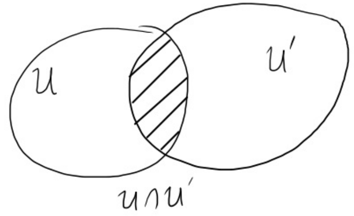


- extend the definition to a chart U' of the same orientation $(J>0)$

$$\omega = h(p)\, J(x')\, dx^{1'} \wedge dx^{2'} \wedge \dots \wedge dx^{n'}$$

where $J(x')$ is analytically continued from $U \cap U'$ to all of U'

How to define ω: (example)

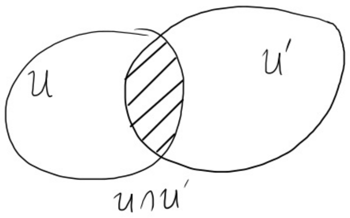
- repeat until you cover $\mathcal{M}$

Then ω is a n-form by construction, because on $U \cap U'$ transforms as:

$$\omega_{\mu_1' \dots \mu_n'} = \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}} \, \omega_{\mu_1 \dots \mu_n} =$$

$$= \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}} \, \tilde{\epsilon}_{\mu_1 \dots \mu_n} \, h(x) =$$

$$= J(x') \, h(x') \quad \tilde{\epsilon}_{\mu_1' \dots \mu_n'}$$



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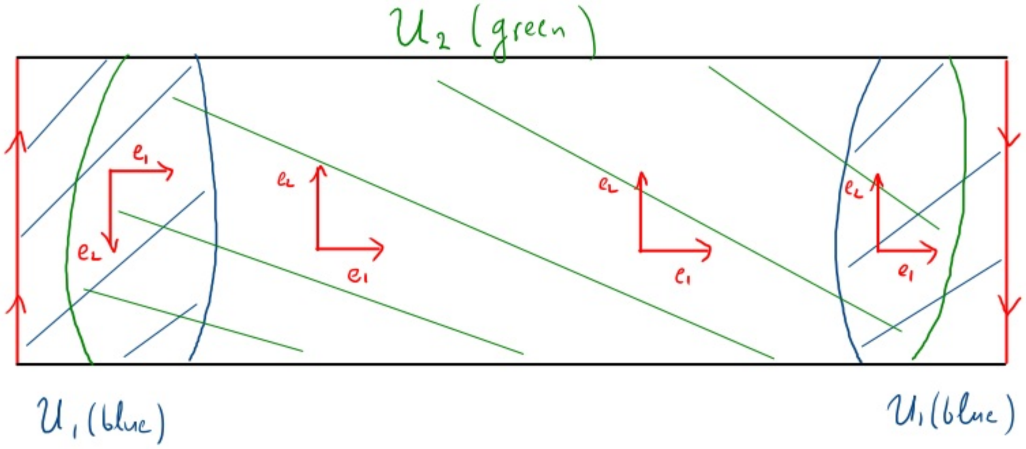
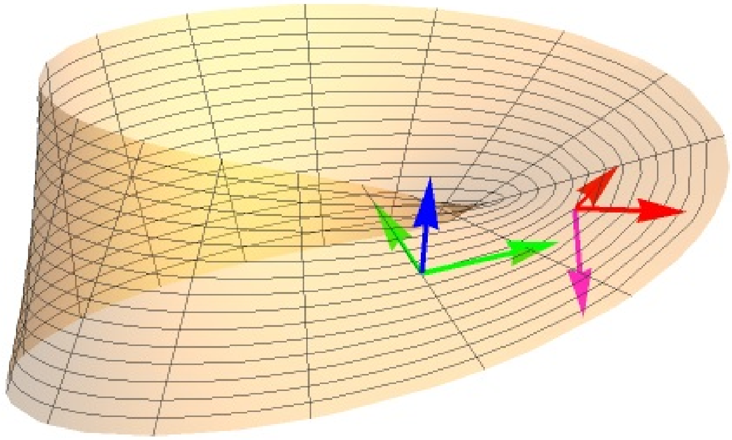
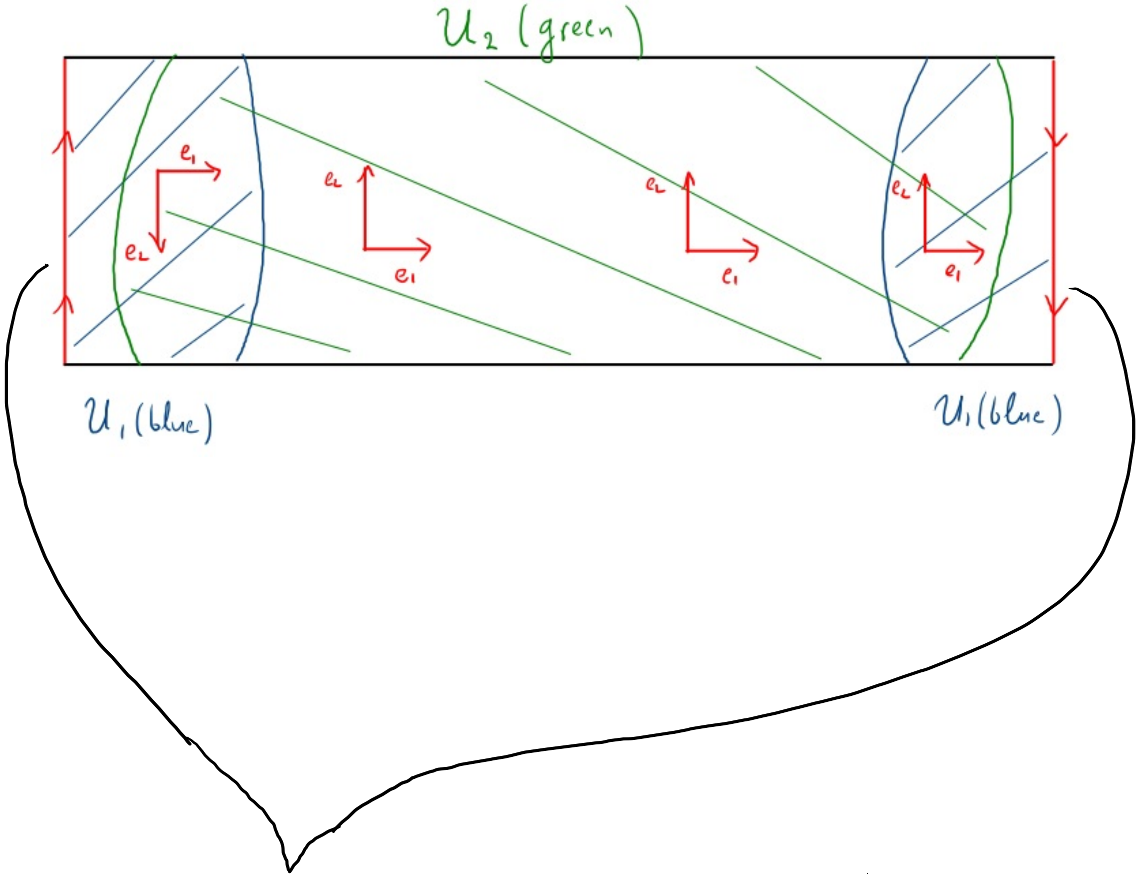
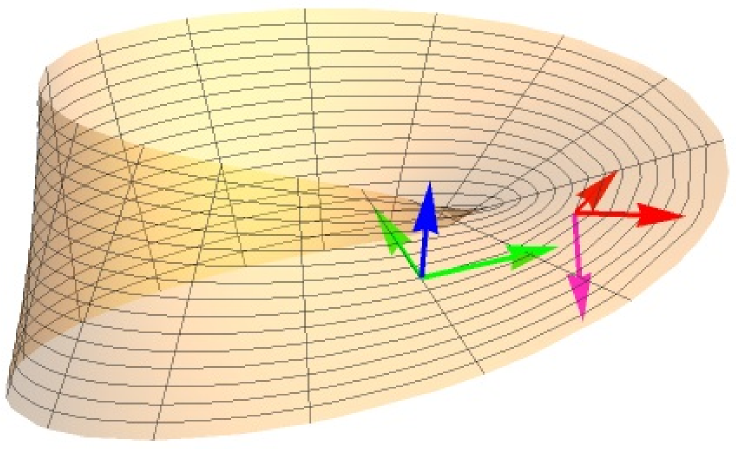


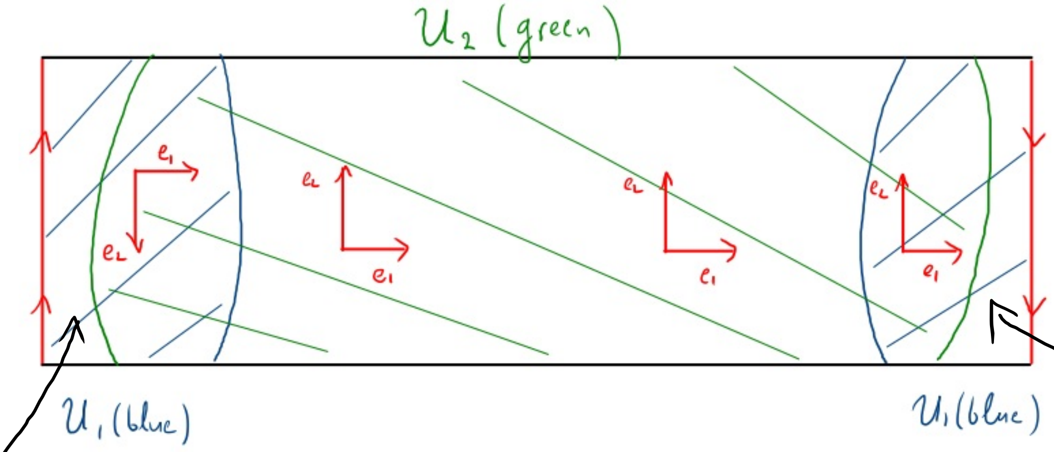


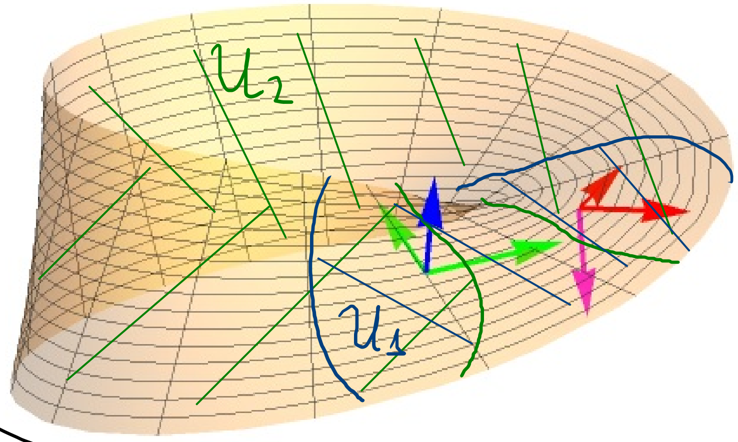
Not all manifolds orientable: e.g. Möbius strip



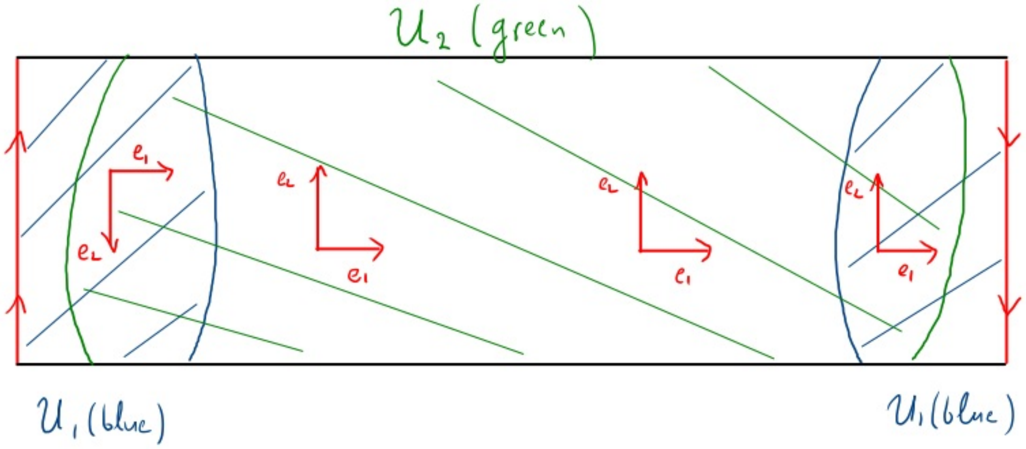


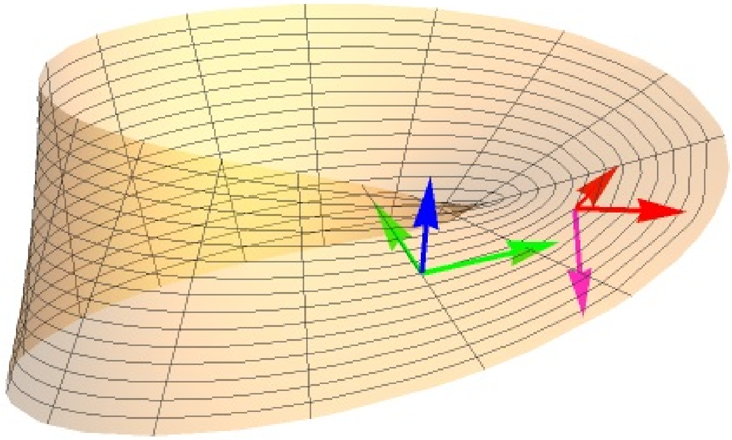
glue these segments the opposite way, as indicated by arrows



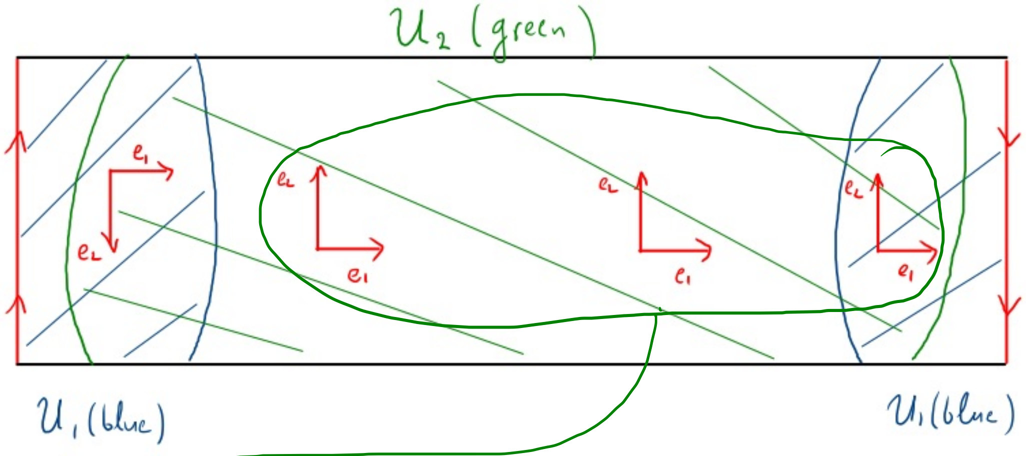


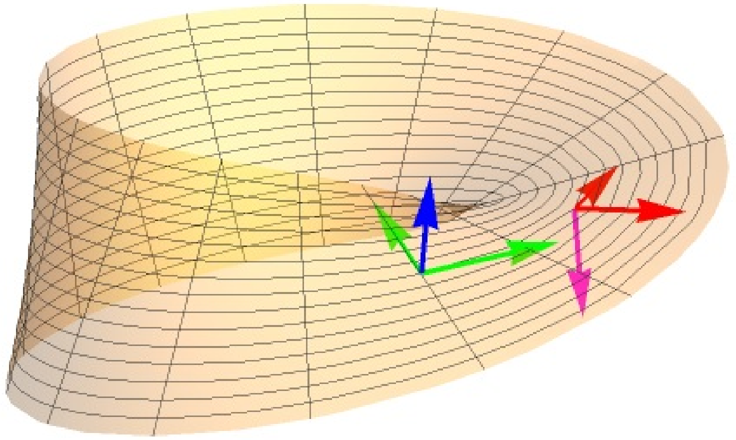
$\mathcal{U}_1$ and $\mathcal{U}_2$ an atlas
($\mathcal{U}_1$ is connected, those are glued together!)





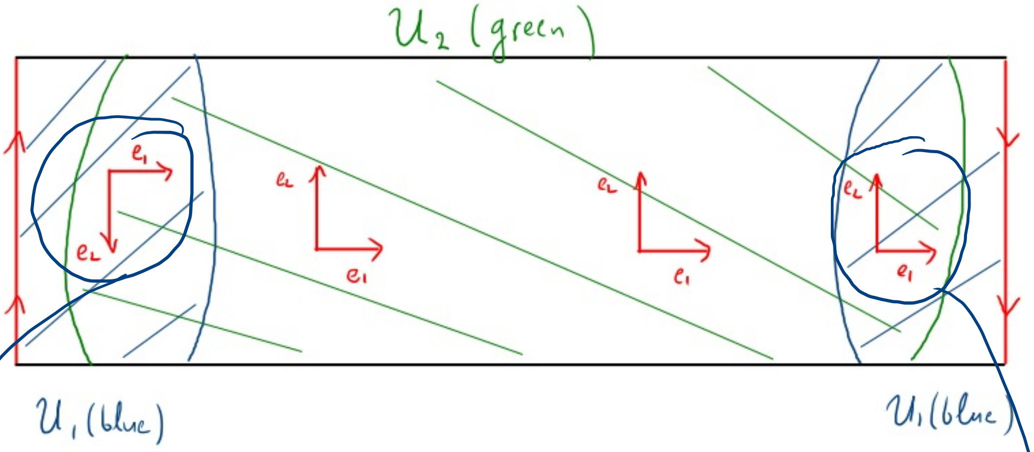
A coordinate basis is defined on each chart

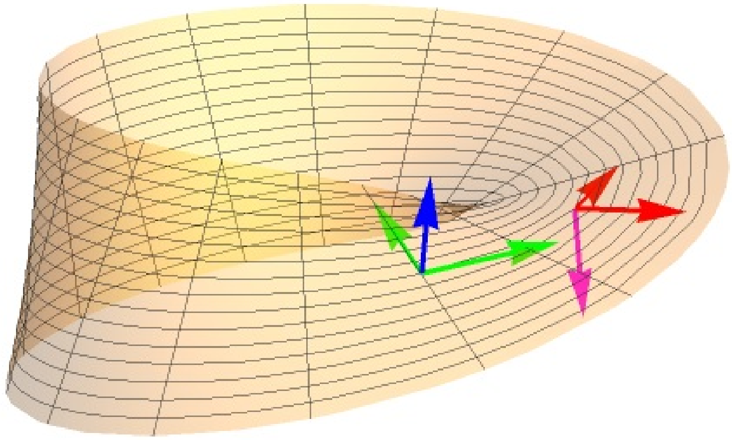




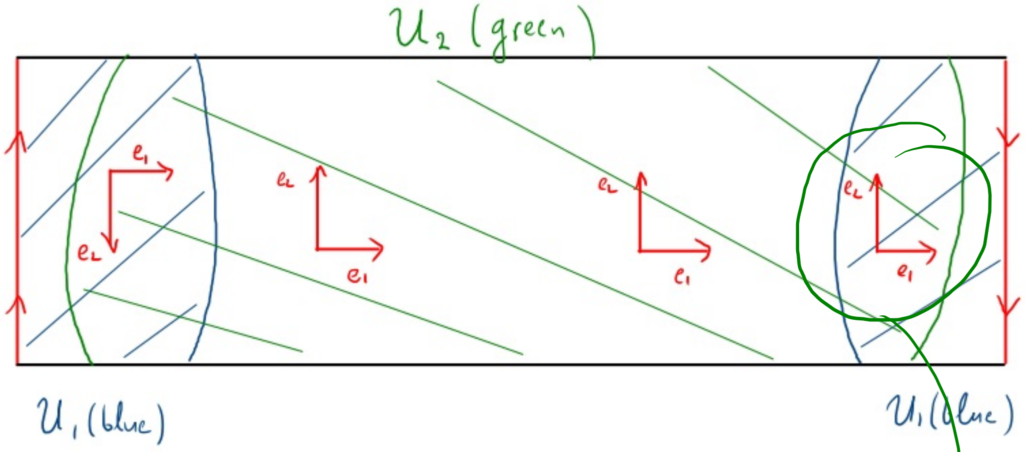
A coordinate basis is defined on each chart

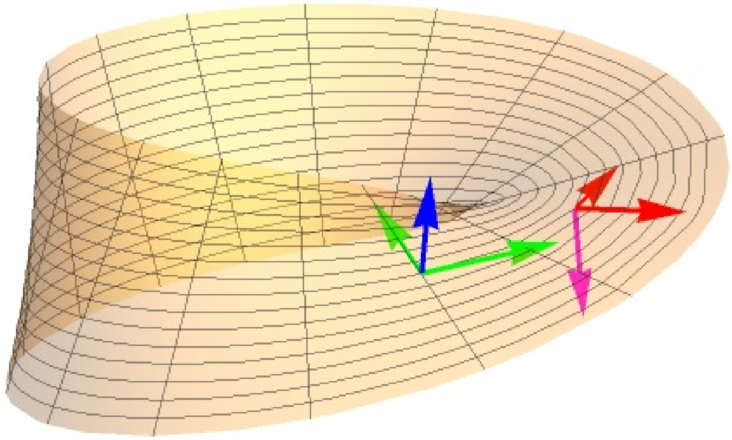
on U_2





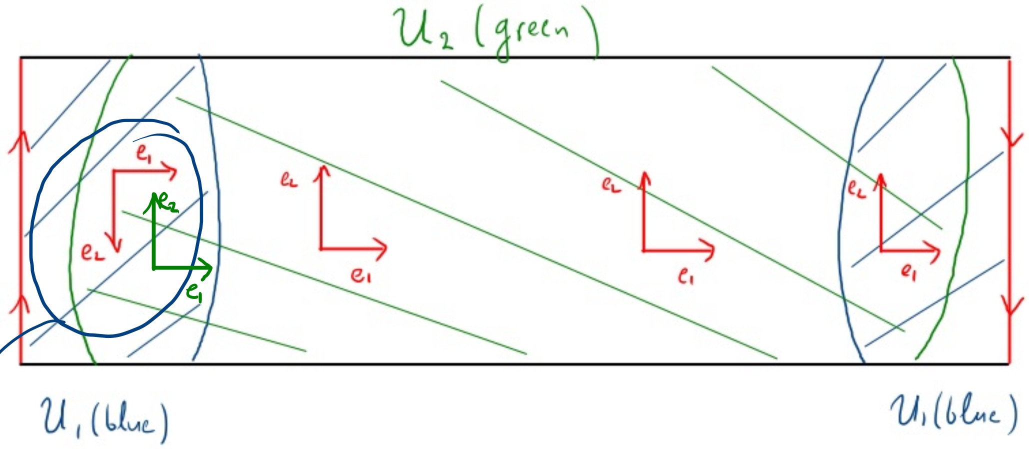
A coordinate basis is defined on $\mathcal{U}_1$

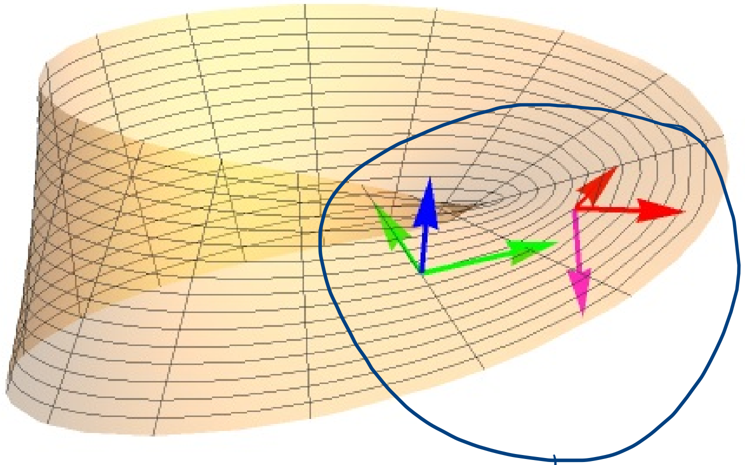




A coordinate basis is defined on each chart

on $(\mathcal{U}_1 \cap \mathcal{U}_2)_{\text{right}}$: same orientation





A coordinate basis is defined on each chart

$(U_1 \cap U_2)_{left}$:

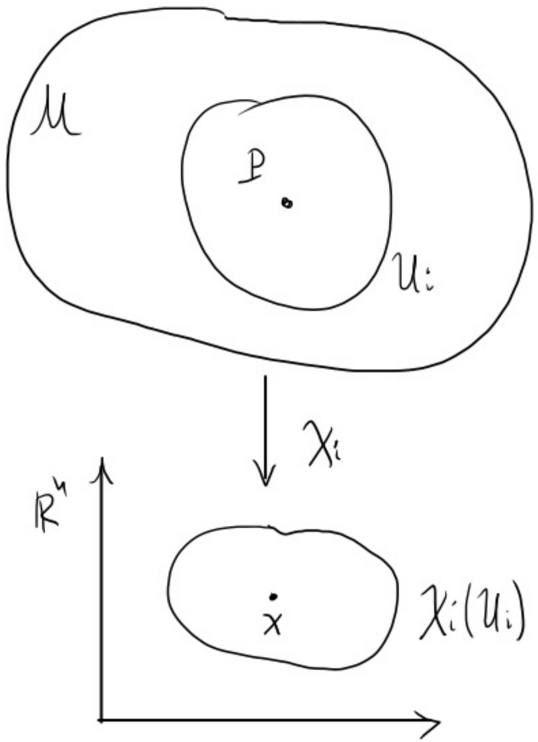
on U_1 orientation opposite from green base of U_2

Integration on Differentiable Manifolds

- Integration on a chart

let $h(P)$ a positive function on M, and

$$\omega = h(P)\, dx^1 \wedge \cdots \wedge dx^n$$



Integration on Differentiable Manifolds

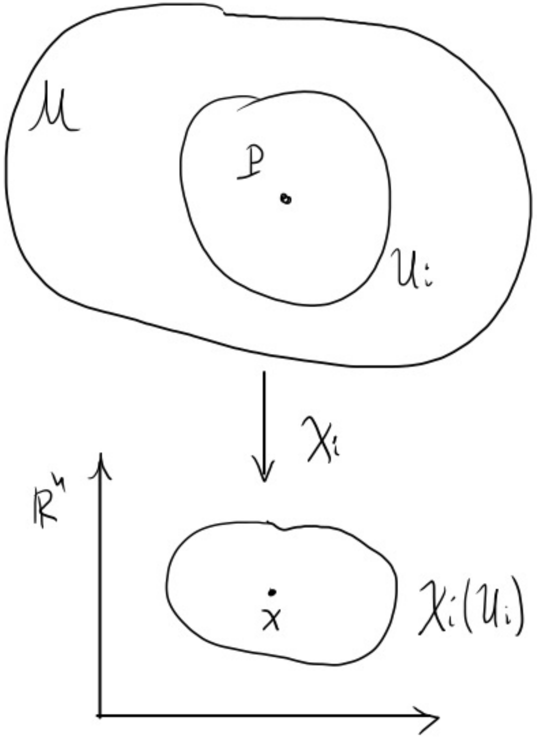
- Integration on a chart

let $h(P)$ a positive function on M, and

$$\omega = h(P)\, dx^1 \wedge \cdots \wedge dx^n \quad ,$$

then:

$$\int_{U_i} f\,\omega = \int_{\chi_i(U_i)} f(x)\, h(x)\, d^n x$$



$x = (x^1, \dots, x^n)$

$f(x) = f \circ \chi_i(x)$

$h(x) = h \circ \chi_i^{-1}(x)$

$P = \chi_i^{-1}(x)$

Integration on Differentiable Manifolds

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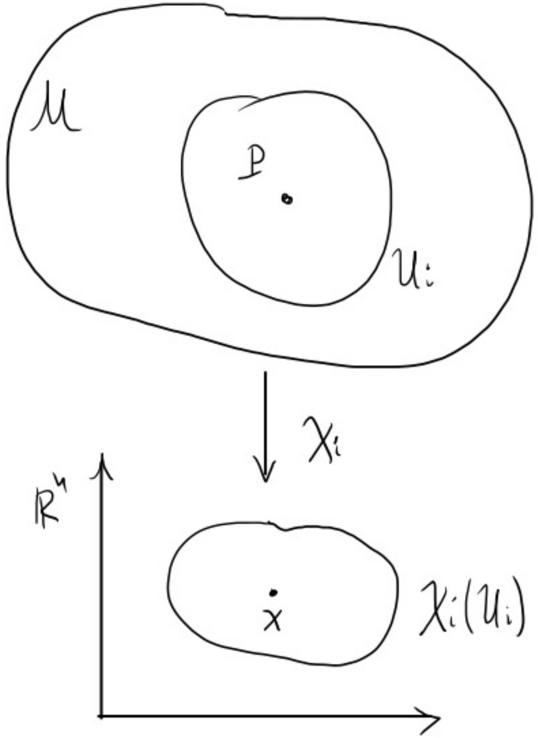


$x = (x^1, \dots, x^n)$

$f(x) = f \circ \chi_i^{-1}(x)$

$h(x) = h \circ \chi_i^{-1}(x)$

$P = \chi_i^{-1}(x)$

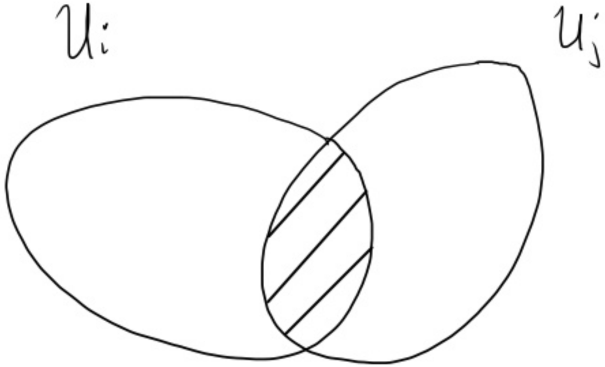
$\int_{\chi_i(U_i)} f(x)\, h(x)\, d^n x$ is an ordinary integral on $\mathbb{R}^n$. It is defined by a positive volume element in $\mathbb{R}^n$

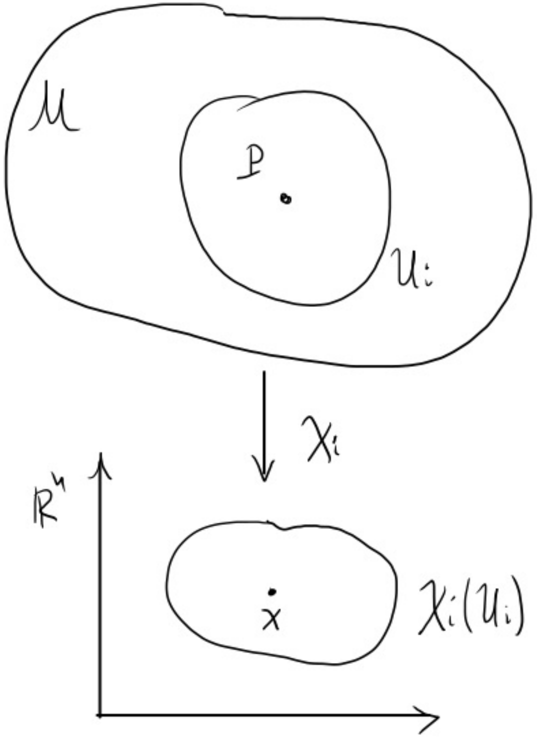
Integration on Differentiable Manifolds

- extend definition on neighboring chart U_j of the same orientation.

$$\int_{U_i} f\omega = \int_{\chi_i(U_i)} f(x)\, h(x)\, d^n x$$

$$\int_{U_j} f\omega = \int_{\chi_j(U_j)} f(x')\, h(x')\, J(x')\, d^n x'$$





Integration on Differentiable Manifolds

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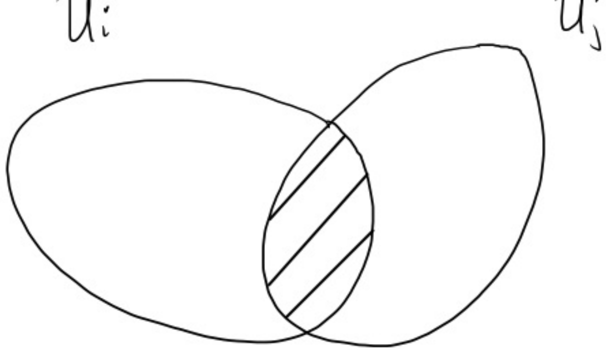
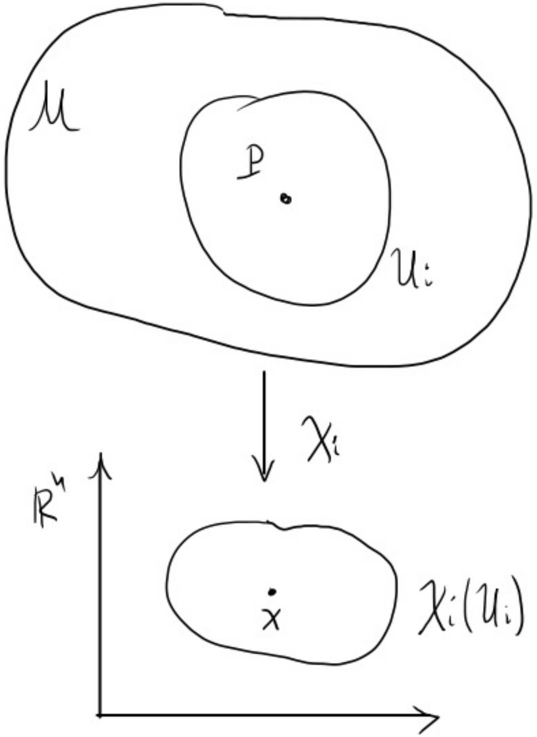
$$\int_{U_i} f\omega = \int_{\chi_i(U_i)} f(x)\, h(x)\, d^n x$$

$$\int_{U_j} f\omega = \int_{\chi_j(U_j)} f(x')\, h(x')\, J(x')\, d^n x'$$

On $U_i \cap U_j$ they are equal:

$$\int_{U_i \cap U_j} f\omega = \int_{\chi_i(U_i \cap U_j)} f(x)\, h(x)\, d^n x = \int_{\chi_j(U_i \cap U_j)} f(x')\, h(x')\, J(x')\, d^n x'$$

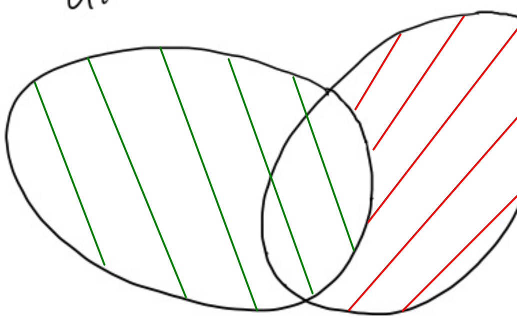




Integration on Differentiable Manifolds

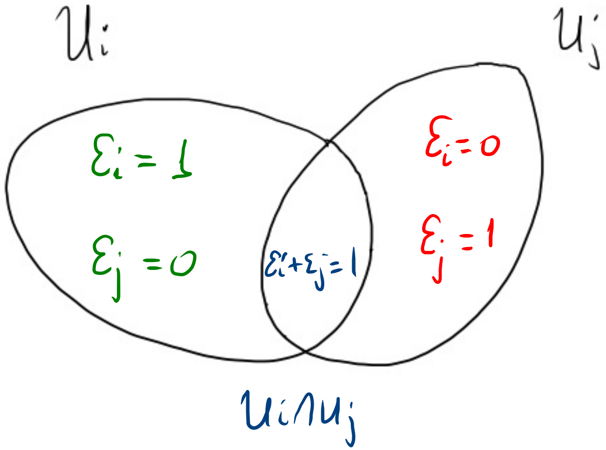
We may define:

$$\int_{U_i \cup U_j} f\omega = \int_{U_i} f\omega + \int_{U_j \setminus (U_i \cap U_j)} f\omega$$



$$\int_{U_i \cap U_j} f\omega = \int_{x_i(U_i \cap U_j)} f(x)\, h(x)\, d^n x = \int_{x_j(U_i \cap U_j)} f(x')\, h(x')\, J(x')\, d^n x'$$

Integration on Differentiable Manifolds



We may define:

$$\int_{U_i \cup U_j} f\omega = \int_{U_i} f\omega + \int_{U_j \setminus (U_i \cap U_j)} f\omega$$

But, if we pick two functions $\varepsilon_i, \varepsilon_j$ on M s.t.:

$$0 \leq \varepsilon_i, \varepsilon_j \leq 1$$

$$\varepsilon_i(p) = 0 \quad \text{for } p \notin U_i$$

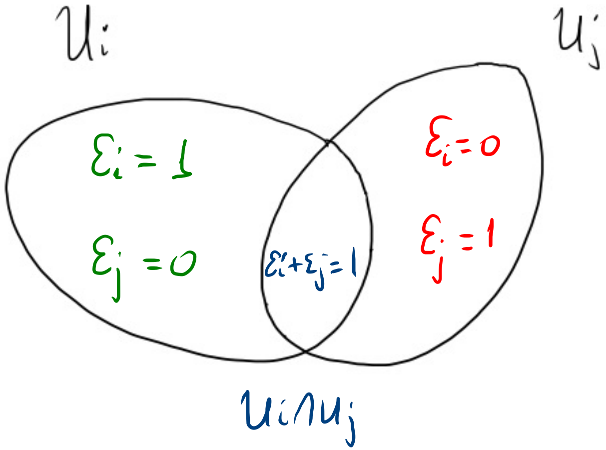
$$\varepsilon_j(p) = 0 \quad \text{for } p \notin U_j$$

$$\varepsilon_i(p) + \varepsilon_j(p) = 1 \quad \text{for } p \in U_i \cap U_j$$

we can also define:

Integration on Differentiable Manifolds

$$\int_{U_i \cup U_j} f\omega = \int_{U_i} \epsilon_i f\omega + \int_{U_j} \epsilon_j f\omega$$



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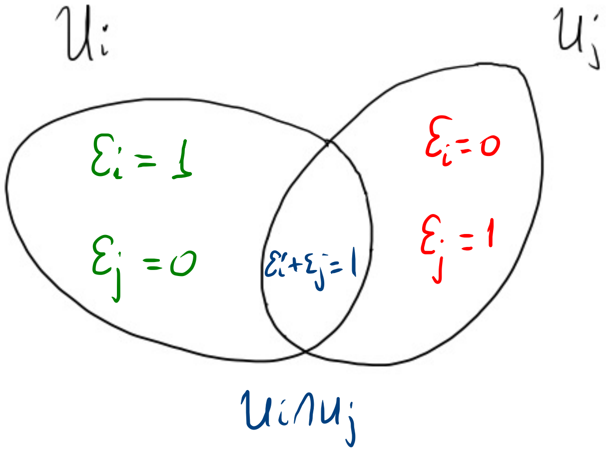
we can also define:

Integration on Differentiable Manifolds

$$\int_{U_i \cup U_j} f\omega = \int_{U_i} \varepsilon_i f\omega + \int_{U_j} \varepsilon_j f\omega \quad , \text{ because}$$

$$\int_{U_i} \varepsilon_i f\omega + \int_{U_j} \varepsilon_j f\omega = \int_{U_i \setminus (U_i \cap U_j)} (\varepsilon_i + 0) f\omega + \int_{U_i \cap U_j} (\varepsilon_i + \varepsilon_j) f\omega + \int_{U_j \setminus (U_i \cap U_j)} (0 + \varepsilon_j) f\omega$$



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Integration on Differentiable Manifolds

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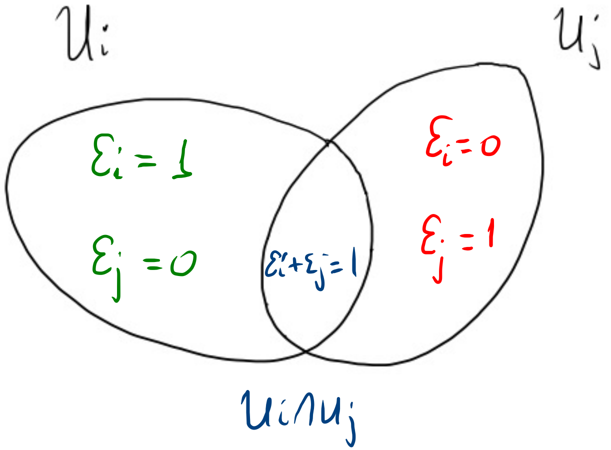

$$\int_{U_i} \varepsilon_i f\omega + \int_{U_j} \varepsilon_j f\omega = \int_{U_i \setminus (U_i \cap U_j)} (\underbrace{\varepsilon_i + 0}_{=1}) f\omega + \int_{U_i \cap U_j} (\underbrace{\varepsilon_i + \varepsilon_j}_{=1}) f\omega + \int_{U_j \setminus (U_i \cap U_j)} (\underbrace{0 + \varepsilon_j}_{=1}) f\omega$$

$$= \int_{U_i \setminus (U_i \cap U_j)} f\omega + \int_{U_i \cap U_j} f\omega + \int_{U_j \setminus (U_i \cap U_j)} f\omega = \int_{U_i \cup U_j} f\omega$$

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Integration on Differentiable Manifolds

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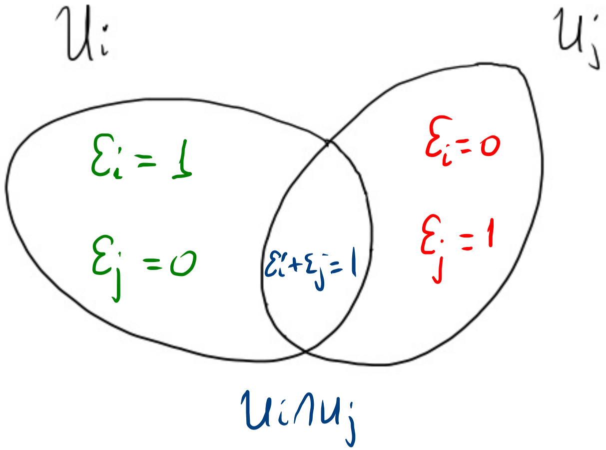

$$\int_{\mathcal{U}_i} \varepsilon_i f\omega + \int_{\mathcal{U}_j} \varepsilon_j f\omega = \int_{\mathcal{U}_i \setminus (\mathcal{U}_i \cap \mathcal{U}_j)} \overset{1}{\overset{\shortparallel}{(\varepsilon_i + 0)}} f\omega + \int_{\mathcal{U}_i \cap \mathcal{U}_j} \overset{1}{\overset{\shortparallel}{(\varepsilon_i + \varepsilon_j)}} f\omega + \int_{\mathcal{U}_j \setminus (\mathcal{U}_i \cap \mathcal{U}_j)} \overset{1}{\overset{\shortparallel}{(0 + \varepsilon_j)}} f\omega$$

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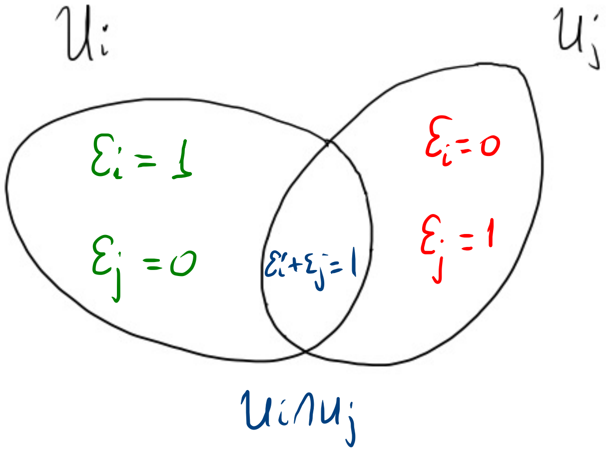
But:

$$\left.\begin{aligned} \int_{\mathcal{U}_i} \varepsilon_i f\omega &= \int_{x_i(\mathcal{U}_i)} \varepsilon_i(x)\, f(x)\, h(x)\, d^n x \\ \int_{\mathcal{U}_j} \varepsilon_j f\omega &= \int_{x_j(\mathcal{U}_j)} \varepsilon_j(x')\, f(x')\, h(x')\, J(x')\, d^n x' \end{aligned}\right\} \text{computable...}$$

Integration on Differentiable Manifolds



$$\int_{U_i \cup U_j} f\omega = \int_{U_i} \epsilon_i f\omega + \int_{U_j} \epsilon_j f\omega \quad , \text{ because}$$

$$= \int_{U_i} f_i \omega + \int_{U_j} f_j \omega$$

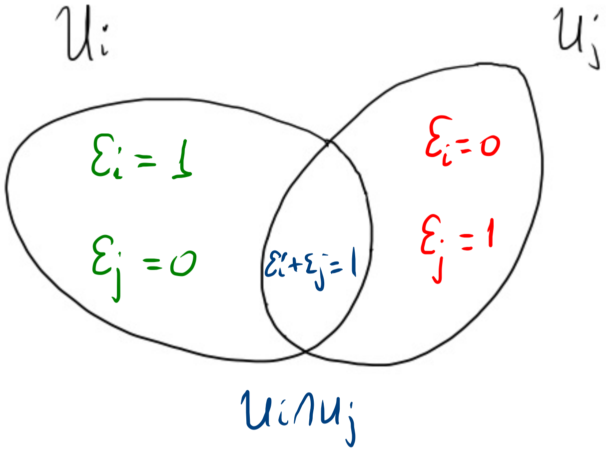
$$f_i = \epsilon_i f \qquad f_j = \epsilon_j f$$

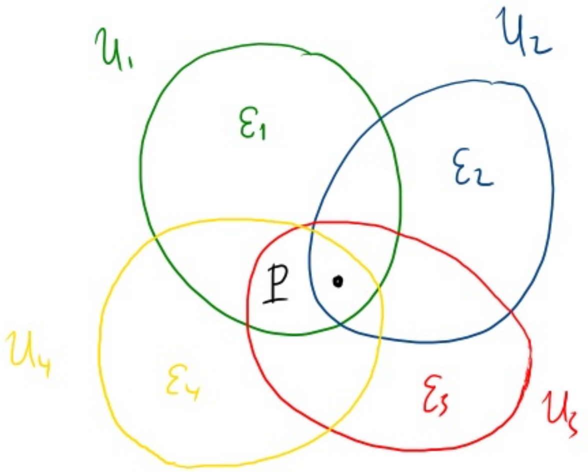
But:

$$\left.\begin{aligned} \int_{U_i} \varepsilon_i f\omega &= \int_{\chi_i(U_i)} \varepsilon_i(x)\, f(x)\, h(x)\, d^n x \\ \int_{U_j} \varepsilon_j f\omega &= \int_{\chi_j(U_j)} \varepsilon_j(x')\, f(x')\, h(x')\, J(x')\, d^n x' \end{aligned}\right\} \text{computable...}$$

Integration on Differentiable Manifolds

For an atlas of M, we may have more complicated structure





Integration on Differentiable Manifolds

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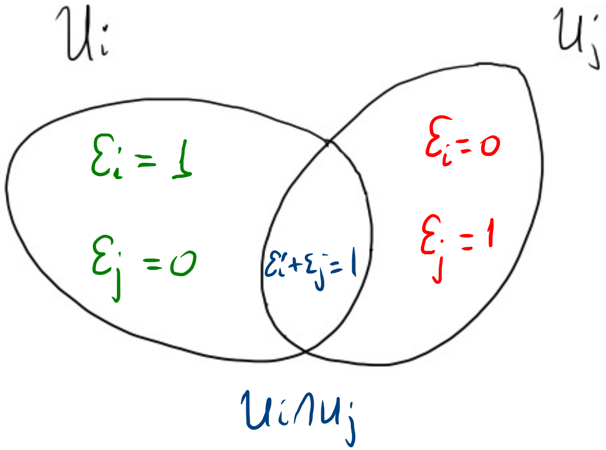
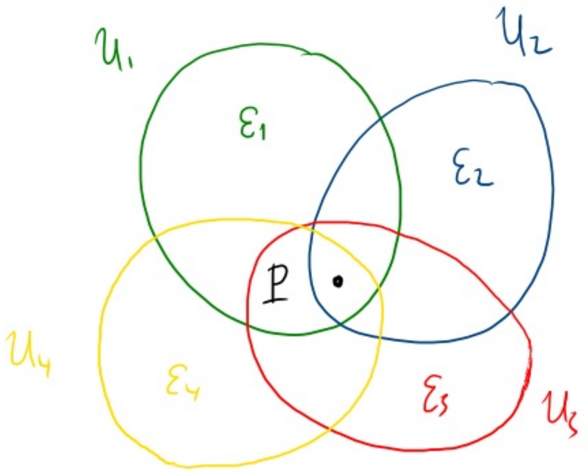


We assume that M is paracompact:
There exists an atlas where each point p is covered by a finite number of charts



Integration on Differentiable Manifolds

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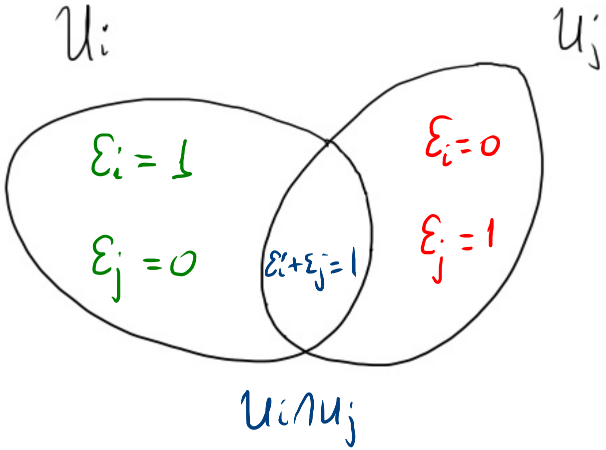

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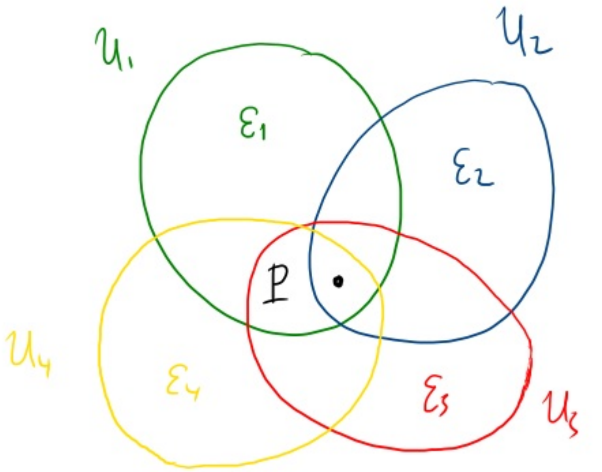
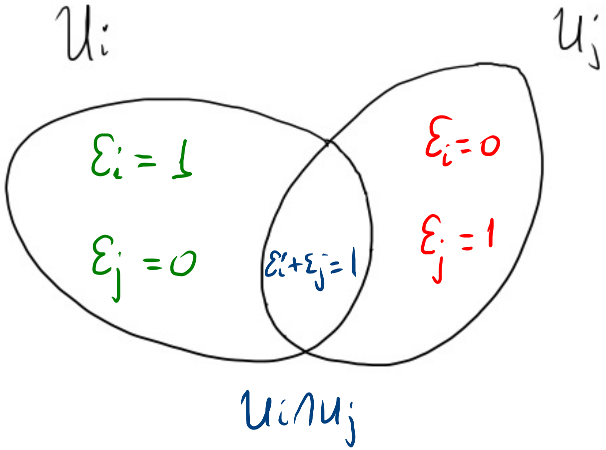
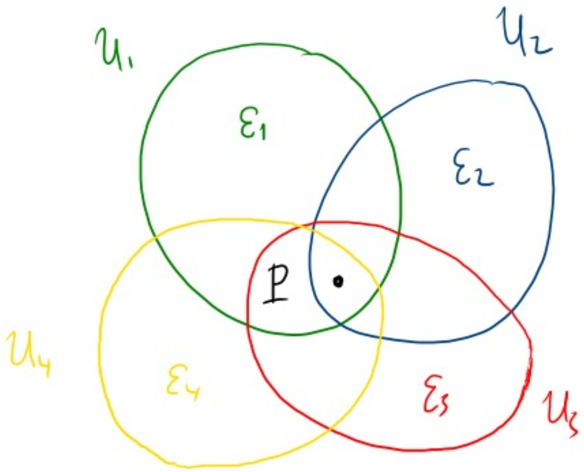


A partition of unity subordinate to $\{U_i\}$ is a family of differentiable functions on M $\{\varepsilon_i(P)\}$ s.t.:

Integration on Differentiable Manifolds

(i) $0 \leq \varepsilon_i(P) \leq 1$





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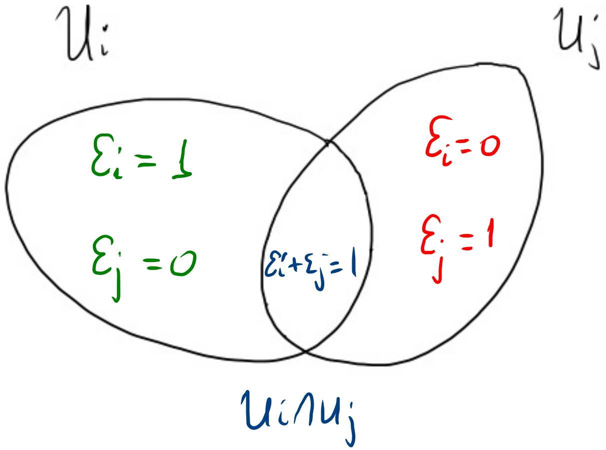
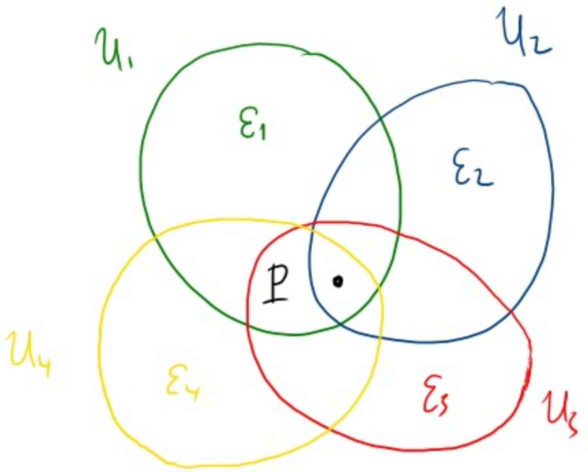
Integration on Differentiable Manifolds

(i) $0 \leq \varepsilon_i(P) \leq 1$

(ii) $\varepsilon_i(P) = 0$ for $P \notin U_i$





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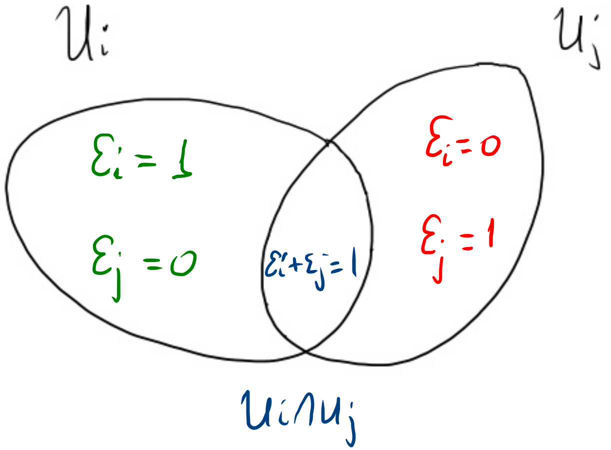
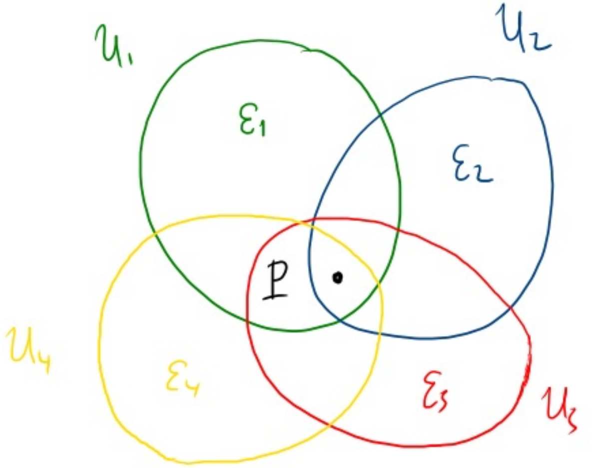
(ii) $\varepsilon_i(p) = 0$ for $p \notin U_i$

(iii) $\sum_i \varepsilon_i(p) = 1 \quad \forall\, p \in M$

$\hookrightarrow$ finite sum! (due to paracompactness)



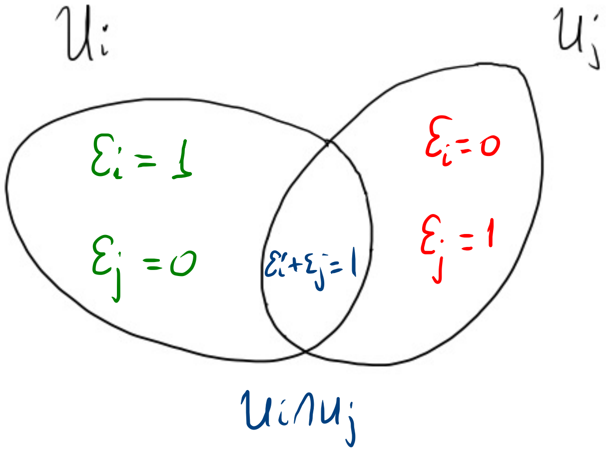


Integration on Differentiable Manifolds

(i) $\quad 0 \leq \varepsilon_i(p) \leq 1$

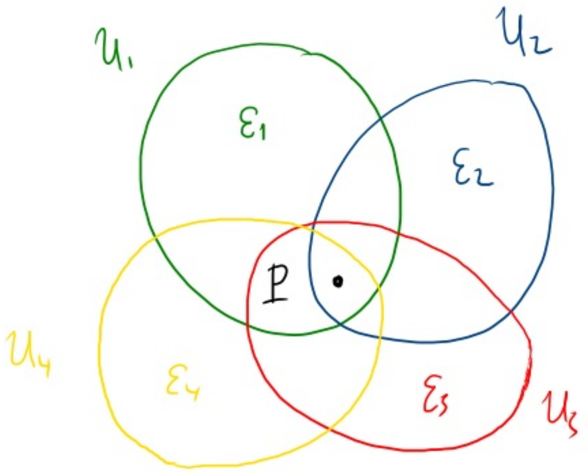
(ii) $\quad \varepsilon_i(p) = 0 \quad$ for $p \notin U_i$

(iii) $\quad \sum_i \varepsilon_i(p) = 1 \quad \forall\, p \in M$



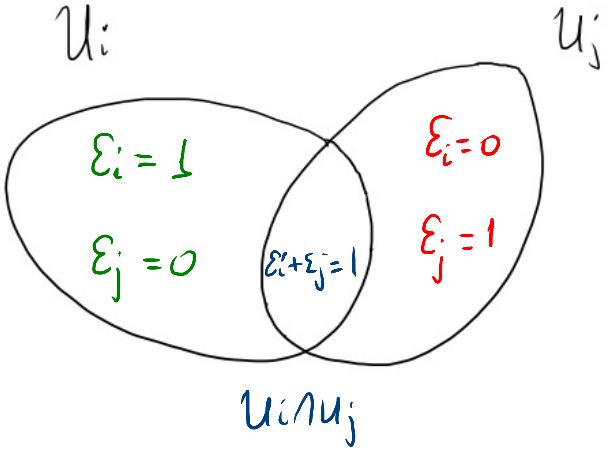
(iii) $\Rightarrow \quad f(p) = \sum_i \varepsilon_i(p) f(p) = \sum_i f_i(p)$

$$f_i(p) \equiv \varepsilon_i(p)\, f(p)$$



A partition of unity subordinate to $\{U_i\}$ is a family of differentiable functions on M $\{\varepsilon_i(p)\}$ s.t.:

Integration on Differentiable Manifolds



(i) $0 \le \varepsilon_i(P) \le 1$

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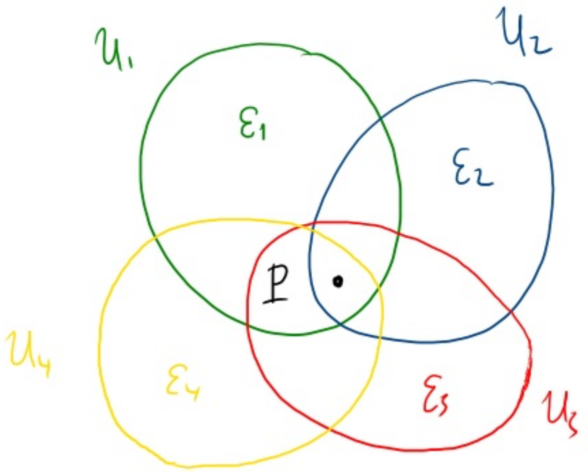
(iii) $\Rightarrow$ $f(P) = \sum_i \varepsilon_i(P) f(P) = \sum_i f_i(P)$

$f_i(P) \equiv \varepsilon_i(P) f(P)$



Define:

$$\int_M f\omega = \sum_i \int_{U_i} f_i \omega$$

independent of the choice of atlas

Example:

$\mathcal{U}_1 = S^1 \setminus \{E\}$

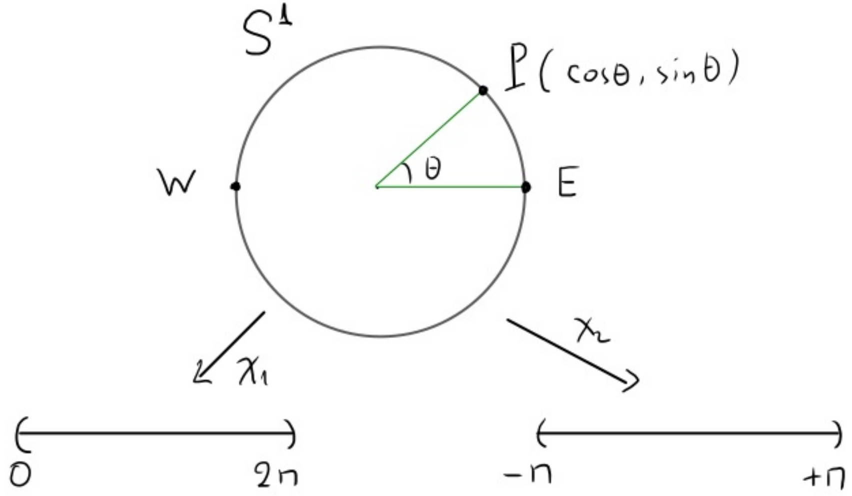
$\chi_1^{-1} : \theta \mapsto (\cos\theta, \sin\theta)$

$0 < \theta < 2\pi$

$\mathcal{U}_2 = S^1 \setminus \{W\}$

$\chi_2^{-1} : \theta \mapsto (\cos\theta, \sin\theta)$

$-\pi < \theta < \pi$



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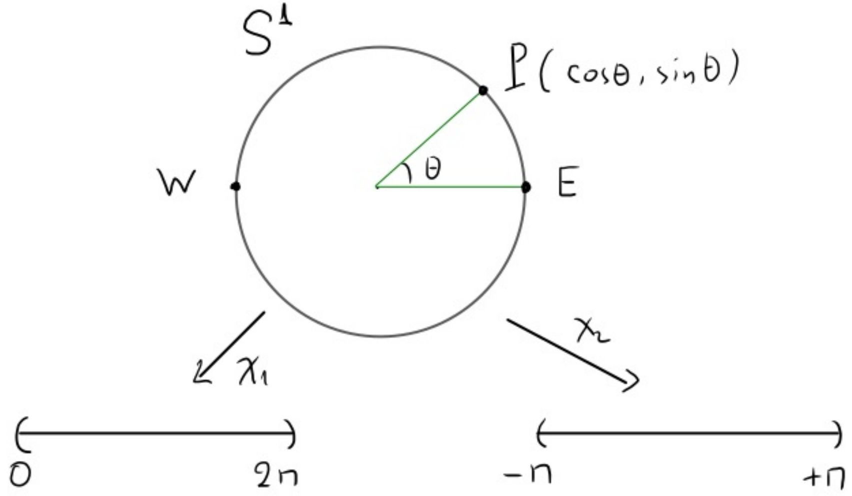

$\omega = d\theta \quad f(\theta) = \cos^2\theta$, compute $\int_{S^1} f\omega$:

Example:

$$\mathcal{U}_1 = S^1 \setminus \{E\}$$

$$\chi_1^{-1} : \theta \mapsto (\cos\theta, \sin\theta)$$

$$0 < \theta < 2\pi$$

$$\mathcal{U}_2 = S^1 \setminus \{W\}$$

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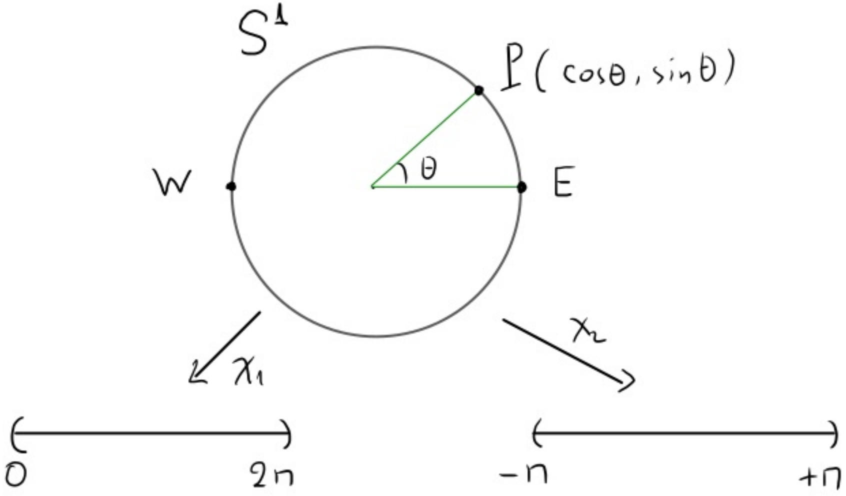

$\omega = d\theta \qquad f(\theta) = \cos^2\theta$, compute $\int_{S^1} f\omega$:

Partition of unity:

$$\varepsilon_1(\theta) = \sin^2\frac{\theta}{2} \qquad 0 < \theta < 2\pi$$

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Example:

$U_1 = S^1 \setminus \{E\}$

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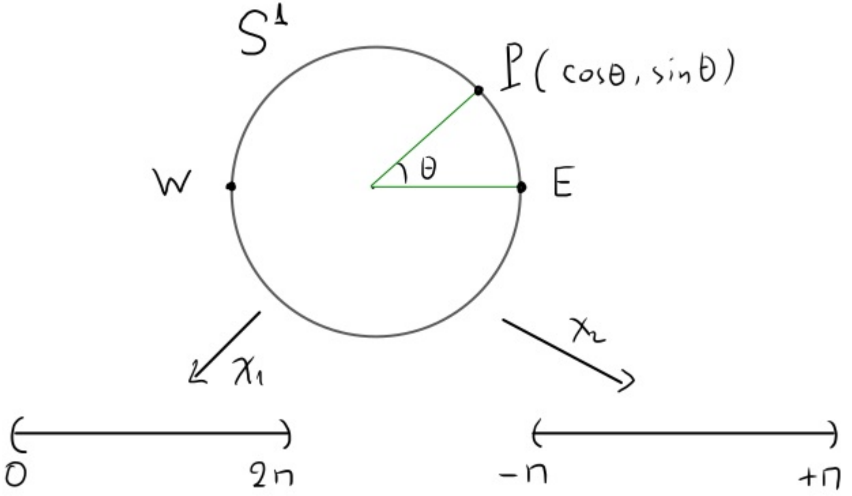

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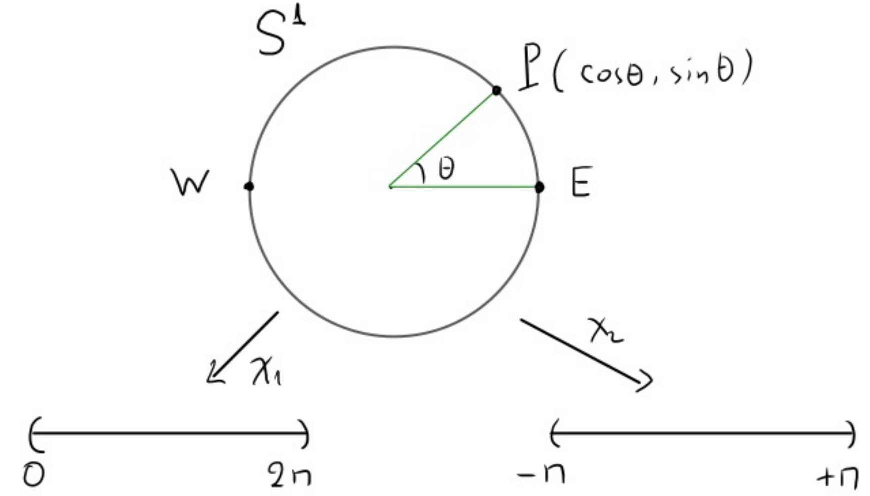

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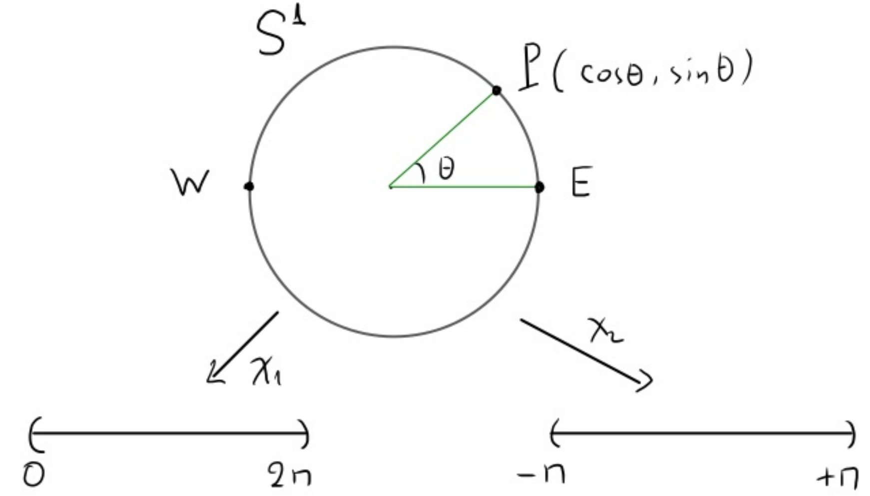

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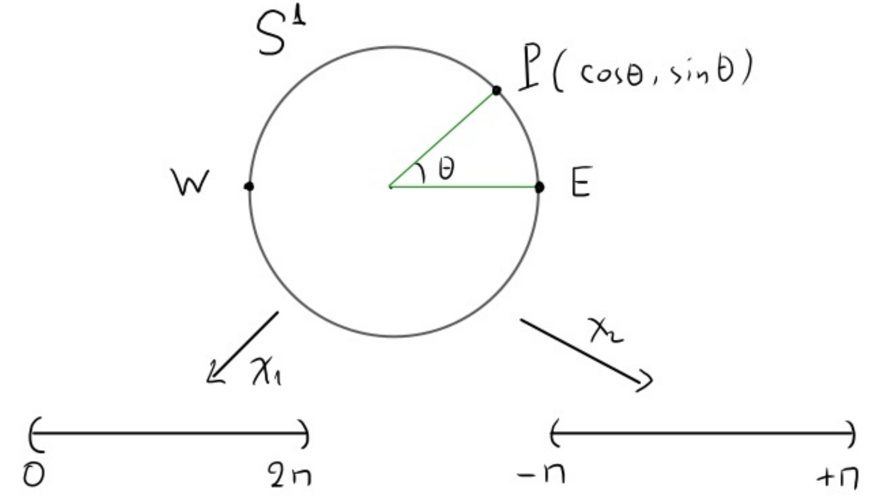

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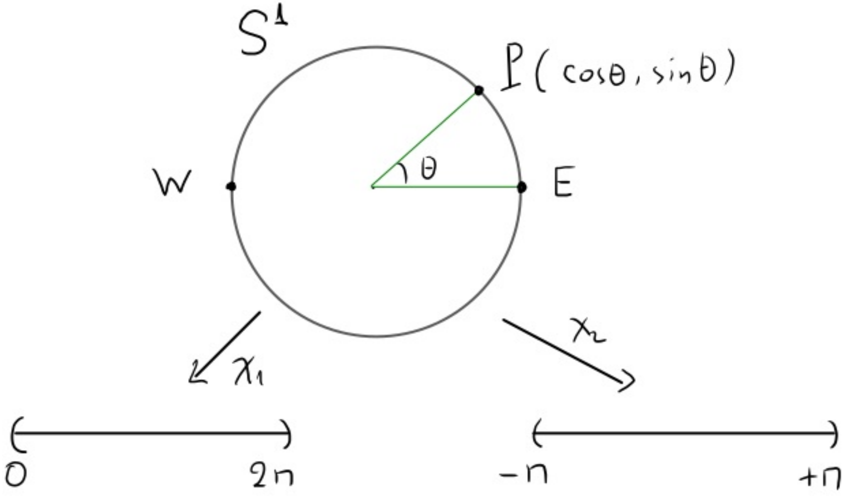

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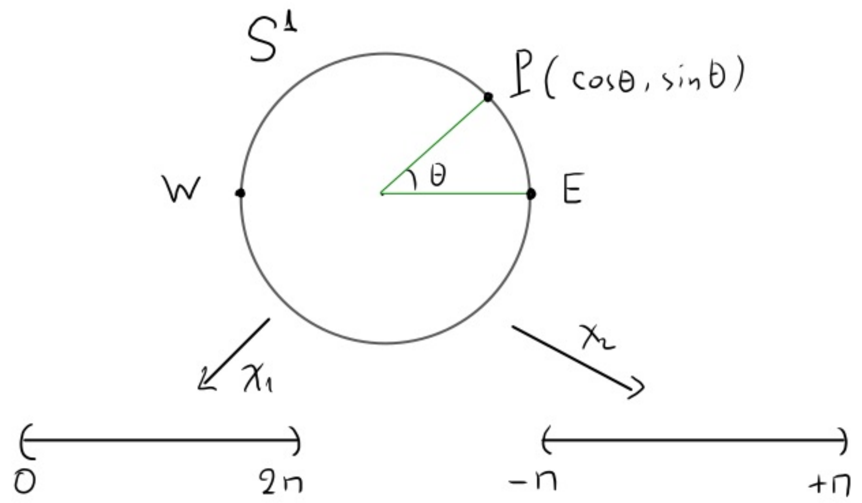

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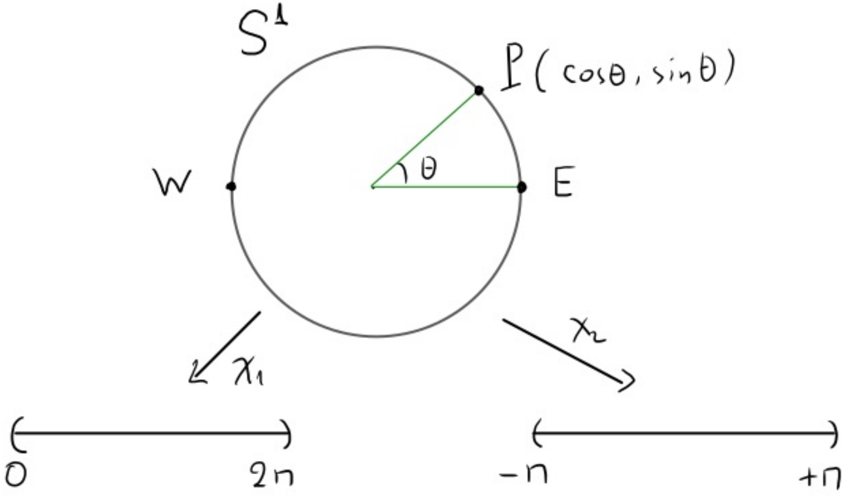

Integrals on $\mathcal{M}$ with a metric

If we choose a metric on $\mathcal{M}$, the volume element is:

$$\omega = \epsilon \qquad \text{the Levi-Civita tensor}$$

$$\epsilon = \frac{1}{n!}\sqrt{|g|}\;\tilde{\epsilon}_{\mu_1 \dots \mu_n}\, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}$$

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Indeed: If we diagonalize the metric, then: $\left(|\partial_\mu|^2 \equiv \partial_\mu \cdot \partial_\mu = g(\partial_\mu, \partial_\mu), \text{ can be } < 0 \right)$

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So: $\epsilon(\Delta x^0 \partial_0, \dots, \Delta x^{n-1}\partial_{n-1}) = (|\partial_0|\,\Delta x^0)\,(|\partial_1|\,\Delta x^1)\cdots(|\partial_{n-1}|\,\Delta x^{n-1}) = (-1)^s\,\Delta v \quad , \Delta v > 0$

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In the physics literature, many times we relaaaaax, and write:

$$\int_M f \epsilon = \int f(x) \sqrt{|g|} \, d^n x \qquad \left(\text{we imply that } d^n x \text{ is } dx^0 \wedge \cdots \wedge dx^{n-1} \ldots\right)$$