Differential Forms

- Differential Forms
- Vector Spaces $\Lambda^p_P(M)$ $\quad 0 \leq p \leq n$
- Wedge Product
- Exterior Derivative d
- Levi-Civita Tensor ϵ
- Duality
- Hodge-$*$ operator/duality

- p- forms are important because:
    - they can be differentiated
    - they can be integrated

without introducing additional structure

$\Rightarrow$ related to topological properties of the Manifold

- Let ω be a $(0,p)$ type tensor

If σ is a permutation of $1, 2, \dots, p$:

$$\sigma\omega(V_1, V_2, \dots, V_p) \equiv \omega(V_{\sigma(1)}, V_{\sigma(2)}, \dots, V_{\sigma(p)})$$

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If $\{e_\mu\}$ a basis in $T_p\mathcal{M}$, then

$$\omega_{\mu_1 \dots \mu_p} = \omega(e_{\mu_1}, \dots, e_{\mu_p})$$

$$\sigma\omega_{\mu_1 \dots \mu_p} = \sigma\omega(e_{\mu_1}, \dots, e_{\mu_p}) = \omega(e_{\sigma(\mu_1)}, \dots, e_{\sigma(\mu_p)}) = \omega_{\sigma(\mu_1) \dots \sigma(\mu_p)}$$

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Antisymmetrizer of ω:

$$A\omega = \frac{1}{p!} \sum_{\sigma} \mathrm{sign}(\sigma)\, \sigma\omega \qquad \mathrm{sign}(\sigma) = \begin{cases} +1 & \sigma \text{ even} \\ -1 & \sigma \text{ odd} \end{cases}$$

$$(A\omega)_{\mu_1 \dots \mu_p} = \frac{1}{p!} \sum_{\sigma} \mathrm{sign}(\sigma)\, \omega_{\sigma(\mu_1) \dots \sigma(\mu_p)} = \omega_{[\mu_1 \dots \mu_p]}$$

- Let ω be a $(0,p)$ type tensor
  Then ω is a p-form if

$$\omega = A\,\omega$$

or $$\omega_{\mu_1 \cdots \mu_p} = \omega_{[\mu_1 \cdots \mu_p]}$$ totally antisymmetric under exchange of indices

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- <u>p-forms form a vector space Λ^p_p</u>

Wedge product: If we have p one-forms $\{e^{\mu_i}\}_{i=1,\dots,p}$

$$e^{\mu_1} \wedge e^{\mu_2} \wedge \dots \wedge e^{\mu_p} = \sum_{\sigma} \operatorname{sign}(\sigma)\, e^{\sigma(\mu_1)} \otimes e^{\sigma(\mu_2)} \otimes \dots \otimes e^{\sigma(\mu_p)}$$

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(express one as linear combination of others and substitute, then each term in sum has at least two same factors)

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* If $\sigma_{\mu\nu} = \sigma_{\nu\mu} \Rightarrow \sigma_{\mu\nu} e^\mu \wedge e^\nu = 0$.

  $\sigma_{\mu_1 \dots \mu_p}$ symmetric under exchange of any index $\Rightarrow \sigma_{\mu_1 \dots \mu_p} e^{\mu_1} \wedge \dots \wedge e^{\mu_p} = 0$

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  $\sigma_{\mu_1 \dots \mu_p}$ symmetric under exchange of any index $\Rightarrow \sigma_{\mu_1 \dots \mu_p}\, e^{\mu_1} \wedge \dots \wedge e^{\mu_p} = 0$

- $\omega_{\mu_1 \dots \mu_p}\, e^{\mu_1} \wedge \dots \wedge e^{\mu_p} = \omega_{[\mu_1 \dots \mu_p]}\, e^{\mu_1} \wedge \dots \wedge e^{\mu_p}$

  only antisymmetric part of (0,p) tensor contributes

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- **p-forms form a vector space Λ^p_p**

Any $(0,p)$ tensor: $\omega = \omega_{\mu_1 \cdots \mu_p} dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_p}$

$\hookrightarrow$ defines components in $\{dx^\mu\}$ basis

- **p-forms form a vector space Λ^p_p**

Any $(0,p)$ tensor: $\omega = \omega_{\mu_1 \cdots \mu_p} dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_p}$

If $\omega_{\mu_1 \cdots \mu_p} = \omega_{[\mu_1 \cdots \mu_p]}$, then all $p!$ terms

$$\omega_{[\mu_1 \cdots \mu_p]} \, \mathrm{sign}(\sigma) \, dx^{\sigma(\mu_1)} \otimes \cdots \otimes dx^{\sigma(\mu_p)}$$

are equal.

- ## p-forms form a vector space Λ^p_P

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If $\omega_{\mu_1 \cdots \mu_p} = \omega_{[\mu_1 \cdots \mu_p]}$, then all $p!$ terms

$$\omega_{[\mu_1 \cdots \mu_p]} \operatorname{sign}(\sigma)\, dx^{\sigma(\mu_1)} \otimes \cdots \otimes dx^{\sigma(\mu_p)}$$

are equal. Summing over all σ, we obtain $p!\,\omega$, therefore:

$$\omega = \omega_{[\mu_1 \cdots \mu_p]} \frac{1}{p!} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \quad \Rightarrow$$

- p-forms form a vector space Λ^p_P

$$\omega = \frac{1}{p!}\,\omega_{\mu_1\cdots\mu_p}\, dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}$$

* all p-forms are linear combinations of

$$\{dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\}$$

- <u>p-forms form a vector space Λ^p_p</u>

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* all p-forms are linear combinations of

$$\{ dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \}$$

* only $\binom{n}{p} = \dfrac{n!}{p!(n-p)!}$ are linearly independent

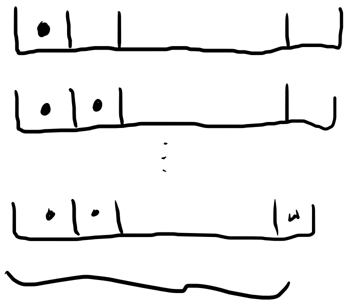
  the rest are obtained, up to a sign, by index permutation

- **p-forms form a vector space Λ^p_P**

$$\omega = \frac{1}{p!}\, \omega_{\mu_1 \cdots \mu_p}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$

Indeed:

* all indices must be different: choose p among n



n ways

$n(n-1)$ ways

⋮

$n(n-1)\cdots(n-p+1) = \dfrac{n!}{(n-p)!}$ ways

p-slots

- **p-forms form a vector space Λ^p_P**

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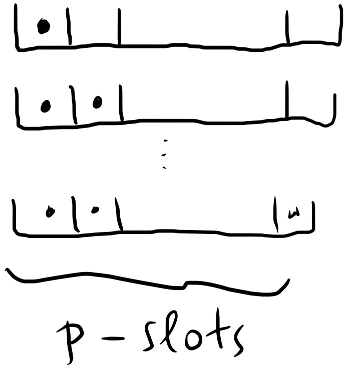


n ways

$n(n-1)$ ways

$\vdots$

$n(n-1)\cdots(n-p+1) = \dfrac{n!}{(n-p)!}$ ways (p-permutations of n)

* the $p!$ permutations give the same (up to sign) $dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}$

(order does not matter $\Rightarrow$ p-combinations of n)

- p-forms form a vector space Λ^p_p

$$\omega = \frac{1}{p!}\,\omega_{\mu_1\cdots\mu_p}\, dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}$$

* $\{dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\}$ has $\binom{n}{p} = \frac{n!}{p!(n-p)!}$ linearly independent p-forms

every p-form a linear combination of them

$\Rightarrow$ $\binom{n}{p}$-dim basis

* the $p!$ permutations give the same (up to sign) $dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}$

- **p-forms form a vector space $\Lambda^p_{\underline{p}}$**

$$\omega = \frac{1}{p!} \omega_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$$

* $\{dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}\}$ has $\binom{n}{p} = \frac{n!}{p!(n-p)!}$ linearly independent p-forms

every p-form a linear combination of them

$\Rightarrow$ $\binom{n}{p}$-dim basis

$\Rightarrow$ $\boxed{\dim \Lambda^p_{\underline{p}} \text{ a } \binom{n}{p}\text{-dim vector space}}$

$$p=0 \qquad \dim \Lambda^0_x = 1 \qquad \text{functions}$$

$$p=1 \qquad \dim \Lambda^1_x = n \qquad \text{1-forms}$$

$$\vdots$$

$$p=n \qquad \dim \Lambda^n_x = 1 \qquad \text{n-forms}$$

* there are no non-trivial $p>n$ forms (e.g. $\omega_{\mu_1 \cdots \mu_n \mu_{n+1}}$ has at least two same indices)

* $\dim \Lambda^p = \dim \Lambda^{n-p}$ and Λ^p, Λ^{n-p} isomorphic

$$\binom{n}{p} = \binom{n}{n-p} = \frac{n!}{p!(n-p)!}$$

* $\dim \Lambda^n = 1$ (e.g. all forms proportional to $dx^1 \wedge \cdots \wedge dx^n$)

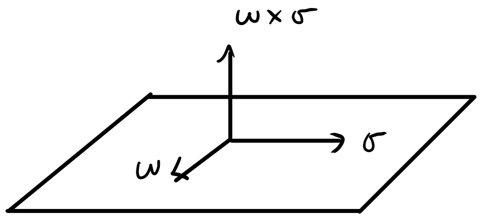
Example: $n=3$

$\dim \Lambda^0 = 1$
$\dim \Lambda^1 = 3$
$\dim \Lambda^2 = 3$
$\dim \Lambda^3 = 1$

$* \quad \Lambda_1 \cong \Lambda_2$ for $\omega, \sigma \in \Lambda^1$: $\omega \wedge \sigma \longleftrightarrow \omega \times \sigma$

↳ cross product

- cross product is a vector only in 3 d



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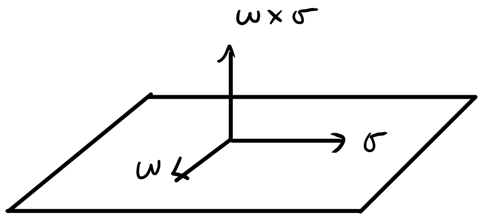

$$\left.\begin{aligned} \omega &= \omega_\mu dx^\mu \\ \sigma &= \sigma_\nu dx^\nu \end{aligned}\right\} \Rightarrow \omega \wedge \sigma = \omega_\mu \sigma_\nu \, dx^\mu \wedge dx^\nu = \omega_{[\mu} \sigma_{\nu]} \, dx^\mu \wedge dx^\nu$$

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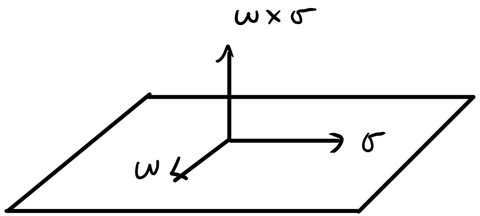

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$$= (\omega_1\sigma_2 - \omega_2\sigma_1)\, dx^1\wedge dx^2 + (\omega_2\sigma_3 - \omega_3\sigma_2)\, dx^2\wedge dx^3 + (\omega_3\sigma_1 - \omega_1\sigma_3)\, dx^3\wedge dx^1$$

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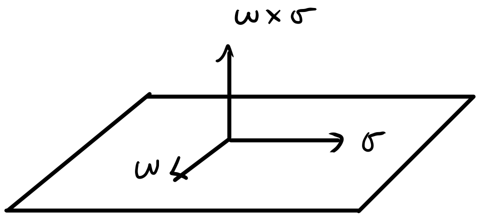

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$$= (\omega_1 \sigma_2 - \omega_2 \sigma_1) \, dx^1 \wedge dx^2 + (\omega_2 \sigma_3 - \omega_3 \sigma_2) \, dx^2 \wedge dx^3 + (\omega_3 \sigma_1 - \omega_1 \sigma_3) \, dx^3 \wedge dx^1$$

$$\omega \times \sigma = (\omega_2 \sigma_3 - \omega_3 \sigma_2) \, dx^1 + (\omega_3 \sigma_1 - \omega_1 \sigma_3) \, dx^2 + (\omega_1 \sigma_2 - \omega_2 \sigma_1) \, dx^3$$

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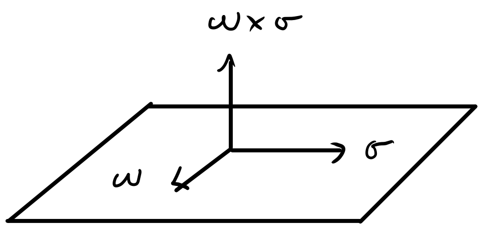

* $p=1$ $\qquad \omega_x\, dx + \omega_y\, dy + \omega_z\, dz$

$p=2$ $\qquad \omega_{xy}\, dx \wedge dy + \omega_{yz}\, dy \wedge dz + \omega_{zx}\, dz \wedge dx$

$p=3$ $\qquad \omega_{xyz}\, dx \wedge dy \wedge dz$ $\qquad$ 1-dim!

p-form fields

* a rule that assigns a p-form smoothly at each point P

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* n-form fields are special: differ only by a function

  - Let ϵ be a nowhere vanishing n-form field

  - $\forall$ n-form field ω: $\omega = f\,\epsilon$ for some function $f \in F(U)$

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ω a p-form

ξ a q-form

$\Rightarrow \omega \wedge \xi$ a $(p+q)$-form

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s.t.

$$(\omega \wedge \xi)(V_1, \dots, V_{p+q}) = \frac{1}{p!\,q!} \underbrace{\sum_{\sigma} \operatorname{sign}(\sigma)\, \omega\left(V_{\sigma(1)}, \dots, V_{\sigma(p)}\right) \xi\left(V_{\sigma(p+1)}, \dots, V_{\sigma(p+q)}\right)}_{(p+q)!\ \text{terms}}$$

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Components:

$$(\omega \wedge \xi)_{\mu_1 \cdots \mu_{p+q}} = (\omega \wedge \xi)(\partial_{\mu_1}, \dots, \partial_{\mu_{p+q}})$$

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$$= \frac{1}{p!\,q!} \sum_{\sigma} \operatorname{sign}(\sigma)\, \omega_{\sigma(\mu_1)\cdots\sigma(\mu_p)}\, \xi_{\sigma(\mu_{p+1})\cdots\sigma(\mu_{p+q})}$$

$$(\omega\wedge\xi)(V_1,\dots,V_{p+q}) = \frac{1}{p!\,q!} \sum_{\sigma} \operatorname{sign}(\sigma)\, \omega\big(V_{\sigma(1)},\dots,V_{\sigma(p)}\big)\, \xi\big(V_{\sigma(p+1)},\dots,V_{\sigma(p+q)}\big)$$

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$$= \frac{1}{p!\,q!} (p+q)!\, \omega_{[\mu_1\cdots\mu_p}\, \xi_{\mu_{p+1}\cdots\mu_{p+q}]}$$

(definition of $[\mu_1 \cdots \mu_{p+q}]$)

$$(\omega\wedge\xi)(V_1,\dots,V_{p+q}) = \frac{1}{p!\,q!} \sum_\sigma \operatorname{sign}(\sigma)\, \omega(V_{\sigma(1)},\dots,V_{\sigma(p)})\, \xi(V_{\sigma(p+1)},\dots,V_{\sigma(p+q)})$$

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$$= \frac{1}{p!\,q!} (p+q)!\; \omega_{[\mu_1\cdots\mu_p}\, \xi_{\mu_{p+1}\cdots\mu_{p+q}]}$$

$$\Rightarrow (\omega\wedge\xi)_{\mu_1\cdots\mu_{p+q}} = \frac{(p+q)!}{p!\,q!}\, \omega_{[\mu_1\cdots\mu_p}\, \xi_{\mu_{p+1}\cdots\mu_{p+q}]}$$

$$\boxed{(\omega\wedge\xi)(V_1,\dots,V_{p+q}) = \frac{1}{p!\,q!} \sum_\sigma \operatorname{sign}(\sigma)\, \omega\big(V_{\sigma(1)},\dots,V_{\sigma(p)}\big)\, \xi\big(V_{\sigma(p+1)},\dots,V_{\sigma(p+q)}\big)}$$

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$$= \frac{1}{p!\,q!} (p+q)!\ \omega_{[\mu_1\cdots\mu_p}\ \xi_{\mu_{p+1}\cdots\mu_{p+q}]}$$

$$\Rightarrow (\omega\wedge\xi)_{\mu_1\cdots\mu_{p+q}} = \frac{(p+q)!}{p!\ q!}\ \omega_{[\mu_1\cdots\mu_p}\ \xi_{\mu_{p+1}\cdots\mu_{p+q}]}$$

Since $\omega\wedge\xi$ a $(p+q)$-form: $\omega\wedge\xi = \frac{1}{(p+q)!} (\omega\wedge\xi)_{\mu_1\cdots\mu_{p+q}}\ dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{p+q}}$

$$= \frac{1}{p!\,q!} \sum_{\sigma} \operatorname{sign}(\sigma)\, \omega_{\sigma(\mu_1)\cdots\sigma(\mu_p)}\, \xi_{\sigma(\mu_{p+1})\cdots\sigma(\mu_{p+q})}$$

$$= \frac{1}{p!\,q!} (p+q)!\, \omega_{[\mu_1\cdots\mu_p}\, \xi_{\mu_{p+1}\cdots\mu_{p+q}]}$$

$$\Rightarrow (\omega\wedge\xi)_{\mu_1\cdots\mu_{p+q}} = \frac{(p+q)!}{p!\,q!}\, \omega_{[\mu_1\cdots\mu_p}\, \xi_{\mu_{p+1}\cdots\mu_{p+q}]}$$

Since $\omega\wedge\xi$ a $(p+q)$-form: $\omega\wedge\xi = \frac{1}{(p+q)!} (\omega\wedge\xi)_{\mu_1\cdots\mu_{p+q}}\, dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{p+q}}$, we obtain:

$$\omega\wedge\xi = \frac{1}{p!\,q!}\, \omega_{[\mu_1\cdots\mu_p}\, \xi_{\mu_{p+1}\cdots\mu_{p+q}]}\, dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{p+q}}$$

↳ brackets can be removed

$$= \frac{1}{p!\,q!} \sum_{\sigma} \mathrm{sign}(\sigma)\ \omega_{\sigma(\mu_1)\cdots\sigma(\mu_p)}\ \xi_{\sigma(\mu_{p+1})\cdots\sigma(\mu_{p+q})}$$

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Example:

ω is a 1-form

ξ 2-form

$\omega \wedge \xi$ 3-form

$$(\omega \wedge \xi)_{\mu\nu\rho} = \omega_\mu \xi_{\nu\rho} + \omega_\rho \xi_{\mu\nu} + \omega_\nu \xi_{\rho\mu}$$

$\mu \to \nu \to \rho \to \mu$

$$(\omega \wedge \xi)_{\mu\nu\rho} = \omega_\mu \xi_{\nu\rho} + \omega_\rho \xi_{\mu\nu} + \omega_\nu \xi_{\rho\mu}$$

$$\begin{tikzcd}
 & \mu \arrow[dr] & \\
\rho \arrow[ur] & & \nu \arrow[ll]
\end{tikzcd}$$

But:

$$(\omega \wedge \xi)_{\mu\nu\rho} = \frac{(2+1)!}{2!\,1!}\,\omega_{[\mu}\xi_{\nu\rho]} = 3\,\omega_{[\mu}\xi_{\nu\rho]}$$

$$3\,\omega_{[\mu}\xi_{\nu\rho]} = \frac{3}{3!}\Big(\omega_\mu \xi_{\underline{\nu\rho}} + \omega_\rho \xi_{\underline{\mu\nu}} + \omega_\nu \xi_{\underline{\rho\mu}}$$

$$- \omega_\mu \xi_{\underline{\rho\nu}} - \omega_\nu \xi_{\underline{\mu\rho}} - \omega_\rho \xi_{\underline{\nu\mu}} \Big)$$

$$= \frac{\cancel{3}}{\cancel{3}\cdot\cancel{2}}\,\cancel{2}\,\big(\omega_\mu \xi_{\nu\rho} + \omega_\rho \xi_{\mu\nu} + \omega_\nu \xi_{\rho\mu} \big)$$

$$(\omega \wedge \xi)_{\mu\nu\rho} = \omega_\mu \xi_{\nu\rho} + \omega_\rho \xi_{\mu\nu} + \omega_\nu \xi_{\rho\mu}$$

$$\begin{tikzcd} & \mu \arrow[dr] & \\ \rho \arrow[ur] & & \nu \arrow[ll] \end{tikzcd}$$

But:

$$(\omega\wedge\xi)_{\mu\nu\rho} = \frac{(2+1)!}{2!\;1!}\,\omega_{[\mu}\xi_{\nu\rho]} = 3\,\omega_{[\mu}\xi_{\nu\rho]}$$

- $\omega \wedge (\xi \wedge \eta) = (\omega \wedge \xi) \wedge \eta$ associative
- $\omega \wedge \xi = (-1)^{pq} \xi \wedge \omega$ graded anticommutative
- p odd $\Rightarrow$ $\omega \wedge \omega = 0$

$$\omega \wedge \omega = (-1)^{p \cdot p} \omega \wedge \omega = \begin{cases} \omega \wedge \omega & p \text{ even} \\ -\omega \wedge \omega & p \text{ odd} \end{cases}$$

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- p odd $\Rightarrow$ $\omega \wedge \omega = 0$
- $\wedge$ defines a product that makes the space

$$\Lambda^*_P = \Lambda^0_P \oplus \Lambda^1_P \oplus \cdots \oplus \Lambda^n_P$$

a graded algebra s.t. $\Lambda^p_P \wedge \Lambda^q_P \subset \Lambda^{p+q}_P$

Λ^*_P is called a Grassmann algebra

If V and W vector spaces, then

$$V \oplus W$$

is a vector space with element $(v,w) \in V \times W$ and

addition: $(v_1,w_1) + (v_2,w_2) = (v_1+v_2, w_1+w_2)$

multiplication: $\alpha(v,w) = (\alpha v, \alpha w)$

- $\wedge$ defines a product that makes the space

$$\Lambda^*_p = \Lambda^0_p \oplus \Lambda^1_p \oplus \cdots \oplus \Lambda^n_p \qquad (\omega_0, \omega_1, \omega_2, \ldots, \omega_n) \in \Lambda^*_p$$

a graded algebra s.t. $\Lambda^p_p \wedge \Lambda^q_p \subset \Lambda^{p+q}_p$

Λ^*_p is called a Grassmann algebra

Let's prove $\omega \wedge \xi = (-1)^{pq}\, \xi \wedge \omega$:

$$\omega = \frac{1}{p!}\, \omega_{\mu_1 \cdots \mu_p}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$

$$\xi = \frac{1}{q!}\, \xi_{\mu_{p+1} \cdots \mu_{p+q}}\, dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_{p+q}}$$

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$$\xi \wedge \omega = \frac{1}{p!\,q!} \xi_{\mu_{p+1} \cdots \mu_{p+q}} \omega_{\mu_1 \cdots \mu_p} dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_{p+q}} \wedge dx^{\mu_1} \cdots \wedge dx^{\mu_p}$$

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- move in total p one forms dx^μ over q one forms to go from

$$dx^{\mu_{p+1}}\wedge\cdots\wedge dx^{\mu_{p+q}}\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}$$

$$\downarrow$$

$$dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\wedge dx^{\mu_{p+1}}\wedge\cdots\wedge dx^{\mu_{p+q}}$$

change sign: $(-1)^{p\cdot q}$

Exterior Derivative

Consider smooth p-form fields $\omega \in \Lambda^p$

Exterior derivative $d : \Lambda^p \to \Lambda^{p+1}$

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If $\omega = \frac{1}{p!} \omega_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$, then

$$d\omega = \frac{1}{p!} \partial_\nu \omega_{\mu_1 \cdots \mu_p} dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$

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component of $(p+1)$-form

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Exterior Derivative

$$(d\omega)_{\nu\mu_1\cdots\mu_p} = (p+1)\,\partial_{[\nu}\,\omega_{\mu_1\cdots\mu_p]}$$

$$d\omega = \frac{1}{p!}\,\partial_\nu\,\omega_{\mu_1\cdots\mu_p}\;dx^\nu\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}$$

Exterior Derivative

Examples:

$$(df)_\mu = \partial_\mu f$$

$$(d\omega)_{\mu\nu} = (1+1)\,\partial_{[\mu}\omega_{\nu]} = 2\cdot\frac{1}{2!}\left(\partial_\mu\omega_\nu - \partial_\nu\omega_\mu\right) = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu$$

$$(d\sigma)_{\mu\nu\rho} = (2+1)\,\partial_{[\mu}\sigma_{\nu\rho]} = 3\cdot\frac{1}{3!}\left(\partial_\mu\sigma_{\nu\rho} + \partial_\rho\sigma_{\mu\nu} + \partial_\nu\sigma_{\rho\mu} - \partial_\mu\sigma_{\rho\nu} - \partial_\nu\sigma_{\mu\rho} - \partial_\rho\sigma_{\nu\mu}\right)$$

$$= \frac{1}{2}\cdot 2\left(\partial_\mu\sigma_{\nu\rho} + \partial_\rho\sigma_{\mu\nu} + \partial_\nu\sigma_{\rho\mu}\right)$$

$$= \partial_\mu\sigma_{\nu\rho} + \partial_\rho\sigma_{\mu\nu} + \partial_\nu\sigma_{\rho\mu}$$

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Properties

- $d(\alpha\omega + \beta\xi) = \alpha\, d\omega + \beta\, d\xi$ $\qquad \omega, \xi \in \Lambda^p \quad \alpha, \beta \in \mathbb{R}$

- $d(\overset{p\text{-form}}{\omega} \wedge \underset{q\text{-form}}{\xi}) = d\omega \wedge \xi + (-1)^p\, \omega \wedge d\xi$ $\qquad$ antiderivation

- $d(d\omega) = 0$ $\qquad$ nilpotent

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Theorem: the properties above, together with definition for df, determine d uniquely

Examples:

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d(\omega \wedge \xi) &= \partial_\rho(\omega_\mu \xi_\nu)\, dx^\rho \wedge dx^\mu \wedge dx^\nu \\
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&= \partial_\rho \omega_\mu \xi_\nu \, dx^\rho \wedge dx^\mu \wedge dx^\nu - \omega_\mu \partial_\rho \xi_\nu \, dx^\mu \wedge dx^\rho \wedge dx^\nu
\end{aligned}$$

- sign   swapped

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&= d\omega \wedge \xi - \omega \wedge d\xi
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$$= \underbrace{\partial_\rho \partial_\mu}_{\text{symmetric}} \omega_\nu \; \underbrace{dx^\rho \wedge dx^\mu}_{\text{antisymmetric}} \wedge dx^\nu = 0$$

Contraction or Interior Product

$V \in TM \qquad \omega \in \Lambda^p$

$V = V^\mu \partial_\mu \qquad \omega = \dfrac{1}{p!}\, \omega_{\mu_1 \cdots \mu_p}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$

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- $i_V : \Lambda^p \to \Lambda^{p-1}$
- $i_V^2 = 0$ nilpotent
- $\mathcal{L}_V \omega = (d\, i_V + i_V d)\,\omega$
- $[\mathcal{L}_V, i_V]\,\omega = 0$

De Rahm Cohomology

d induces the sequence: $0 \xrightarrow{i} \Lambda^0 \xrightarrow{d} \Lambda^1 \rightarrow \dots \xrightarrow{d} \Lambda^n \xrightarrow{d} 0$

($i : 0 \hookrightarrow \Lambda^0$ is the inclusion map)

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$dF = 0$	F is closed	$F \in Z^p(M)$
$F = d\omega$	F is exact	$F \in B^p(M)$

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Equivalence relation: $F_1 \sim F_2 \Leftrightarrow \exists \omega$ s.t. $F_1 - F_2 = d\omega$

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F_1 and F_2 are called cohomologous and they are the same point of the set:

$$H^p(M) = \frac{Z^p(M)}{B^p(M)}$$

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* $H^p(M)$ is the p^{th} de Rahm cohomology group of M

* $H^p(M)$ isomorphic to $H_p(M)$, the singular homology group of M

De Rahm Cohomology

F_1 and F_2 are called cohomologous and they are the same point of the set:

$$H^p(M) = \frac{Z^p(M)}{B^p(M)}$$

* $H^p(M)$ is the p^{th} de Rahm cohomology group of M

* $H^p(M)$ isomorphic to $H_p(M)$, the singular homology group of M

* $H^p(M)$ depends on the topology of M

Levi-Civita Symbols

$$\tilde{\epsilon}_{\mu_1 \dots \mu_n} = \tilde{\epsilon}^{\mu_1 \dots \mu_n} = \begin{cases} +1 & \text{if } \mu_1, \dots, \mu_p \text{ an even permutation of } 1, 2, \dots, n \\ -1 & \text{" odd "} \\ 0 & \text{otherwise} \end{cases}$$

$n = \dim M$

Levi-Civita Symbols

$$\tilde{\epsilon}_{\mu_1 \cdots \mu_n} = \tilde{\epsilon}^{\mu_1 \cdots \mu_n} = \begin{cases} +1 & \text{if } \mu_1, \ldots, \mu_p \text{ an even permutation of } 1, 2, \ldots, n \\ -1 & \text{" odd "} \\ 0 & \text{otherwise} \end{cases}$$

* Not tensors: just symbols

  - they do not transform under change of basis
  - upstairs / downstairs indices the same - placed for convenience

Levi-Civita Symbols

Contraction identities : useful in calculations
Kronecker's delta

fixed

e.g. $$\tilde{\epsilon}_{\mu\nu}\tilde{\epsilon}^{\lambda\rho} = \delta_\mu^{\ \lambda}\delta_\nu^{\ \rho} - \delta_\nu^{\ \lambda}\delta_\mu^{\ \rho}$$

exchange

Levi-Civita Symbols

Contraction identities: useful in calculations

Kronecker's delta

fixed

e.g. $\tilde{\epsilon}_{\mu\nu}\tilde{\epsilon}^{\lambda\rho} = \delta_\mu^{\ \lambda}\delta_\nu^{\ \rho} - \delta_\nu^{\ \lambda}\delta_\mu^{\ \rho}$

exchange

$$\Rightarrow \tilde{\epsilon}_{\mu\nu}\tilde{\epsilon}^{\lambda\nu} = \delta_\mu^{\ \lambda}\underbrace{\delta_\nu^{\ \nu}}_{= n=2} - \underbrace{\delta_\nu^{\ \lambda}\delta_\mu^{\ \nu}}_{\delta^\lambda_{\ \mu}} = (n-1)\,\delta_\mu^{\ \lambda}$$

contraction

Levi - Civita Symbols

Contraction identities : useful in calculations
Kronecker's delta

Generalization for $n > 2$

$$\tilde{\epsilon}_{\mu_1 \dots \mu_n} \tilde{\epsilon}^{\nu_1 \dots \nu_n} = n! \, \delta^{\nu_1}_{[\mu_1} \delta^{\nu_2}_{\mu_2} \cdots \delta^{\nu_n}_{\mu_n]}$$

Levi - Civita Symbols

Contraction identities : useful in calculations

Kronecker's delta

Generalization for $n>2$

$$\tilde{\epsilon}_{\mu_1 \cdots \mu_n} \tilde{\epsilon}^{\nu_1 \cdots \nu_n} = n! \, \delta^{\nu_1}_{[\mu_1} \delta^{\nu_2}_{\mu_2} \cdots \delta^{\nu_n}_{\mu_n]} \equiv \delta^{\nu_1 \cdots \nu_n}_{\mu_1 \cdots \mu_n}$$

Define p-delta symbol

$$\delta^{\nu_1 \cdots \nu_p}_{\mu_1 \cdots \mu_p} = p! \, \delta^{\nu_1}_{[\mu_1} \delta^{\nu_2}_{\mu_2} \cdots \delta^{\nu_n}_{\mu_p]}$$

Levi-Civita Symbols

Contractions

$$\delta^{\nu_1 \cdots \nu_p}_{\mu_1 \cdots \mu_p} \equiv p! \; \delta^{\nu_1}_{[\mu_1} \delta^{\nu_2}_{\mu_2} \cdots \delta^{\nu_p}_{\mu_p]}$$

$$\delta^{\lambda_1 \nu_2 \cdots \nu_p}_{\lambda_1 \underbrace{\mu_2 \quad \mu_p}_{p-1}} = \underbrace{(n-p+1)}_{= [n-(p-1)]} \delta^{\nu_2 \cdots \nu_p}_{\underbrace{\mu_2 \cdots \mu_p}_{p-1}}$$

$$\delta^{\lambda_1 \lambda_2 \nu_3 \cdots \nu_p}_{\lambda_1 \lambda_2 \underbrace{\mu_3 \cdots \mu_p}_{p-2}} = (n-p+1)(n-p+2)\, \delta^{\nu_3 \cdots \nu_p}_{\underbrace{\mu_3 \cdots \mu_p}_{p-2}}$$

$$\vdots$$

$$\delta^{\lambda_1 \lambda_2 \cdots \lambda_p}_{\lambda_1 \lambda_2 \cdots \lambda_p} = (n-p+1)(n-p+2)\cdots n = \frac{n!}{(n-p)!}$$

Levi-Civita Symbols

Contractions

$$\delta^{\nu_1\cdots\nu_p}_{\mu_1\cdots\mu_p} \equiv p!\,\delta^{\nu_1}_{[\mu_1}\delta^{\nu_2}_{\mu_2}\cdots\delta^{\nu_p}_{\mu_p]}$$

$$\delta^{\lambda_1\nu_2\cdots\nu_p}_{\lambda_1\underbrace{\mu_2\cdots\mu_p}_{p-1}} = \underbrace{(n-p+1)}_{=[n-(p-1)]}\,\delta^{\nu_2\cdots\nu_p}_{\underbrace{\mu_2\cdots\mu_p}_{p-1}}$$

$$\delta^{\lambda_1\lambda_2\nu_3\cdots\nu_p}_{\lambda_1\lambda_2\underbrace{\mu_3\cdots\mu_p}_{p-2}} = (n-p+1)(n-p+2)\,\delta^{\nu_3\cdots\nu_p}_{\underbrace{\mu_3\cdots\mu_p}_{p-2}}$$

$$\vdots$$

$$\delta^{\lambda_1\lambda_2\cdots\lambda_p}_{\lambda_1\lambda_2\cdots\lambda_p} = (n-p+1)(n-p+2)\cdots n = \frac{n!}{(n-p)!}$$

$p=n$

$$\delta^{\nu_1\cdots\nu_n}_{\mu_1\cdots\mu_n} = n!\,\delta^{\nu_1}_{[\mu_1}\delta^{\nu_2}_{\mu_2}\cdots\delta^{\nu_n}_{\mu_n]}$$

$$\delta^{\lambda_1\nu_2\cdots\nu_n}_{\lambda_1\mu_2\cdots\mu_n} = \delta^{\nu_2\cdots\nu_n}_{\mu_2\cdots\mu_n}$$

$$\delta^{\lambda_1\lambda_2\nu_3\cdots\nu_n}_{\lambda_1\lambda_2\mu_3\cdots\mu_n} = 1\cdot 2\,\delta^{\nu_3\cdots\nu_n}_{\mu_3\cdots\mu_n}$$

$$\vdots$$

$$\delta^{\lambda_1\lambda_2\cdots\lambda_n}_{\lambda_1\lambda_2\cdots\lambda_n} = 1\cdot 2\cdots n = n!$$

Levi-Civita Symbols

Contractions

$p = n$

$$\tilde{\epsilon}_{\mu_1 \dots \mu_n} \tilde{\epsilon}^{\nu_1 \dots \nu_n} =$$

$$\delta^{\nu_1 \dots \nu_n}_{\mu_1 \dots \mu_n} = n! \, \delta^{\nu_1}_{[\mu_1} \delta^{\nu_2}_{\mu_2} \dots \delta^{\nu_n}_{\mu_n]}$$

$$\tilde{\epsilon}_{\lambda_1 \mu_2 \dots \mu_n} \tilde{\epsilon}^{\lambda_1 \nu_2 \dots \nu_n} = (n-1)! \, \delta^{\nu_2}_{[\mu_2} \delta^{\nu_3}_{\mu_3} \dots \delta^{\nu_n}_{\mu_n]}$$

$$\delta^{\lambda_1 \nu_2 \dots \nu_n}_{\lambda_1 \mu_2 \dots \mu_n} = \delta^{\nu_2 \dots \nu_n}_{\mu_2 \dots \mu_n}$$

$$\tilde{\epsilon}_{\lambda_1 \lambda_2 \mu_3 \dots \mu_n} \tilde{\epsilon}^{\lambda_1 \lambda_2 \nu_3 \dots \nu_n} = 1 \cdot 2 \cdot (n-2)! \, \delta^{\nu_3}_{[\mu_3} \delta^{\nu_4}_{\mu_4} \dots \delta^{\nu_n}_{\mu_n]}$$

$$\delta^{\lambda_1 \lambda_2 \nu_3 \dots \nu_n}_{\lambda_1 \lambda_2 \mu_3 \dots \mu_n} = 1 \cdot 2 \, \delta^{\nu_3 \dots \nu_n}_{\mu_3 \dots \mu_n}$$

$$\vdots$$

$$\tilde{\epsilon}_{\lambda_1 \lambda_2 \dots \lambda_n} \tilde{\epsilon}^{\lambda_1 \dots \lambda_n} = n!$$

$$\delta^{\lambda_1 \lambda_2 \dots \lambda_n}_{\lambda_1 \lambda_2 \dots \lambda_n} = 1 \cdot 2 \cdots n = n!$$

Levi-Civita Symbols

Contractions

these simplify from [...] definition.

$$\tilde{\epsilon}_{\mu_1 \dots \mu_n} \tilde{\epsilon}^{\nu_1 \dots \nu_n} =$$

$$\tilde{\epsilon}_{\lambda_1 \mu_2 \dots \mu_n} \tilde{\epsilon}^{\lambda_1 \nu_2 \dots \nu_n} = (n-1)! \, \delta^{\nu_2}_{[\mu_2} \delta^{\nu_3}_{\mu_3} \dots \delta^{\nu_n}_{\mu_n]}$$

$$\tilde{\epsilon}_{\lambda_1 \lambda_2 \mu_3 \dots \mu_n} \tilde{\epsilon}^{\lambda_1 \lambda_2 \nu_3 \dots \nu_n} = 1 \cdot 2 \cdot (n-2)! \, \delta^{\nu_3}_{[\mu_3} \delta^{\nu_4}_{\mu_4} \dots \delta^{\nu_n}_{\mu_n]}$$

$$\vdots$$

$$\tilde{\epsilon}_{\lambda_1 \lambda_2 \dots \lambda_n} \tilde{\epsilon}^{\lambda_1 \dots \lambda_n} = n!$$

$p = n$

$$\delta^{\nu_1 \dots \nu_n}_{\mu_1 \dots \mu_n} = n! \, \delta^{\nu_1}_{[\mu_1} \delta^{\nu_2}_{\mu_2} \dots \delta^{\nu_n}_{\mu_n]}$$

$$\delta^{\lambda_1 \nu_2 \dots \nu_n}_{\lambda_1 \mu_2 \dots \mu_n} = \delta^{\nu_2 \dots \nu_n}_{\mu_2 \dots \mu_n}$$

$$\delta^{\lambda_1 \lambda_2 \nu_3 \dots \nu_n}_{\lambda_1 \lambda_2 \mu_3 \dots \mu_n} = 1 \cdot 2 \, \delta^{\nu_3 \dots \nu_n}_{\mu_3 \dots \mu_n}$$

$$\vdots$$

$$\delta^{\lambda_1 \lambda_2 \dots \lambda_n}_{\lambda_1 \lambda_2 \dots \lambda_n} = 1 \cdot 2 \cdots n = n!$$

Levi-Civita Symbols

Determinants: If $A^{\mu}{}_{\nu}$ a $n \times n$ matrix, then:

row column

Levi-Civita Symbols

Determinants: If $A^{\mu}{}_{\nu}$ a $n \times n$ matrix, then:

$$\tilde{\epsilon}_{\mu_1' \dots \mu_n'} \det A = \tilde{\epsilon}_{\mu_1 \dots \mu_n} A^{\mu_1}{}_{\mu_1'} A^{\mu_2}{}_{\mu_2'} \cdots A^{\mu_n}{}_{\mu_n'}$$

Levi-Civita Symbols

Determinants: If $A^\mu{}_\nu$ a $n \times n$ matrix, then:

$$\tilde{\epsilon}_{\mu_1' \cdots \mu_n'} \det A = \tilde{\epsilon}_{\mu_1 \cdots \mu_n} A^{\mu_1}{}_{\mu_1'} A^{\mu_2}{}_{\mu_2'} \cdots A^{\mu_n}{}_{\mu_n'} \Rightarrow$$

$$\det A = \frac{1}{n!} \tilde{\epsilon}_{\mu_1 \cdots \mu_n} \tilde{\epsilon}^{\mu_1' \cdots \mu_n'} A^{\mu_1}{}_{\mu_1'} A^{\mu_2}{}_{\mu_2'} \cdots A^{\mu_n}{}_{\mu_n'}$$

indices up/down the same: $\tilde{\epsilon}^{\mu_1' \cdots \mu_n'} = \tilde{\epsilon}_{\mu_1' \cdots \mu_n'}$

symbol - not tensor

Levi-Civita Symbols

Determinants: If $A^\mu{}_\nu$ a $n\times n$ matrix, then:

$$\tilde{\epsilon}_{\mu_1'\cdots\mu_n'} \det A = \tilde{\epsilon}_{\mu_1\cdots\mu_n} A^{\mu_1}{}_{\mu_1'} A^{\mu_2}{}_{\mu_2'} \cdots A^{\mu_n}{}_{\mu_n'} \Rightarrow$$

$$\det A = \frac{1}{n!} \tilde{\epsilon}_{\mu_1\cdots\mu_n} \tilde{\epsilon}^{\mu_1'\cdots\mu_n'} A^{\mu_1}{}_{\mu_1'} A^{\mu_2}{}_{\mu_2'} \cdots A^{\mu_n}{}_{\mu_n'}$$

Consider the matrix: $A^\mu{}_{\mu'} = \dfrac{\partial x^\mu}{\partial x^{\mu'}}$

Levi-Civita Symbols

Determinants: If $A^{\mu}{}_{\nu}$ a $n\times n$ matrix, then:

$$\tilde{\epsilon}_{\mu_1' \dots \mu_n'} \det A = \tilde{\epsilon}_{\mu_1 \dots \mu_n} A^{\mu_1}{}_{\mu_1'} A^{\mu_2}{}_{\mu_2'} \dots A^{\mu_n}{}_{\mu_n'} \quad \Rightarrow$$

$$\det A = \frac{1}{n!} \tilde{\epsilon}_{\mu_1 \dots \mu_n} \tilde{\epsilon}^{\mu_1' \dots \mu_n'} A^{\mu_1}{}_{\mu_1'} A^{\mu_2}{}_{\mu_2'} \dots A^{\mu_n}{}_{\mu_n'}$$

Consider the matrix: $A^{\mu}{}_{\mu'} = \dfrac{\partial x^{\mu}}{\partial x^{\mu'}}$

$$\left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right| = \left| \left(\frac{\partial x^{\mu}}{\partial x^{\mu'}} \right)^{-1} \right| = \left| \frac{\partial x^{\mu}}{\partial x^{\mu'}} \right|^{-1}$$

$$\tilde{\epsilon}_{\mu_1' \dots \mu_n'} \left| \frac{\partial x^{\mu}}{\partial x^{\mu'}} \right| = \tilde{\epsilon}_{\mu_1 \dots \mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \dots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}} \quad \Leftrightarrow \quad \tilde{\epsilon}_{\mu_1' \dots \mu_n'} = \left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right| \tilde{\epsilon}_{\mu_1 \dots \mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \dots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}}$$

Levi-Civita Symbols

Determinants: If $A^{\mu}{}_{\nu}$ a $n \times n$ matrix, then:

$$\tilde{\epsilon}_{\mu_1' \cdots \mu_n'} \det A = \tilde{\epsilon}_{\mu_1 \cdots \mu_n} A^{\mu_1}{}_{\mu_1'} A^{\mu_2}{}_{\mu_2'} \cdots A^{\mu_n}{}_{\mu_n'} \Rightarrow$$

$$\det A = \frac{1}{n!} \tilde{\epsilon}_{\mu_1 \cdots \mu_n} \tilde{\epsilon}^{\mu_1' \cdots \mu_n'} A^{\mu_1}{}_{\mu_1'} A^{\mu_2}{}_{\mu_2'} \cdots A^{\mu_n}{}_{\mu_n'}$$

Consider the matrix: $A^{\mu}{}_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}$

$$\tilde{\epsilon}_{\mu_1' \cdots \mu_n'} \left| \frac{\partial x^{\mu}}{\partial x^{\mu'}} \right| = \tilde{\epsilon}_{\mu_1 \cdots \mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}} \Leftrightarrow \tilde{\epsilon}_{\mu_1' \cdots \mu_n'} = \left| \frac{\partial x^{\mu'}}{\partial x^{\mu}} \right| \tilde{\epsilon}_{\mu_1 \cdots \mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}}$$

$$\Rightarrow \left| \frac{\partial x^{\mu}}{\partial x^{\mu'}} \right| = \frac{1}{n!} \tilde{\epsilon}_{\mu_1 \cdots \mu_n} \tilde{\epsilon}^{\mu_1' \cdots \mu_n'} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \frac{\partial x^{\mu_2}}{\partial x^{\mu_2'}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}}$$

Levi - Civita tensor

* We pick a metric g on M:
  - g a $(0,2)$ tensor
  - $g(v,w) = g(w,v)$ symmetric
  - $|\det g| \equiv |g| > 0 \Rightarrow \exists\, g^{-1} : g^{\mu\nu} g_{\nu\lambda} = \delta^{\mu}{}_{\lambda}$

    $g \equiv \det g$

  - s = signature of g = number of negative eigenvalues

$$(-1)^s = \frac{g}{|g|}$$

Levi - Civita tensor

* coordinate xfm:

$$g_{\mu'\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} g_{\mu\nu} = \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} g_{\mu\nu} \qquad , \quad \Lambda^{\mu}{}_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}$$

Levi-Civita tensor

* coordinate xfm:

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu} = \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} g_{\mu\nu} \qquad , \quad \Lambda^\mu{}_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}$$

$$= \Lambda^\mu{}_{\mu'} g_{\mu\nu} \Lambda^\nu{}_{\nu'}$$

Levi - Civita tensor

* coordinate xfm:

$$g_{\mu'\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} g_{\mu\nu} = \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} g_{\mu\nu} \qquad , \Lambda^{\mu}{}_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}$$

$$= \Lambda^{\mu}{}_{\mu'} g_{\mu\nu} \Lambda^{\nu}{}_{\nu'}$$

$$= (\Lambda^{T})_{\mu'}{}^{\mu} g_{\mu\nu} \Lambda^{\nu}{}_{\nu'}$$

Levi - Civita tensor

* coordinate xfm:

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$$= \Lambda^{\mu}{}_{\mu'} g_{\mu\nu} \Lambda^{\nu}{}_{\nu'}$$

$$= (\Lambda^{T})_{\mu'}{}^{\mu} g_{\mu\nu} \Lambda^{\nu}{}_{\nu'}$$

$$= (\Lambda^{T} \cdot g \cdot \Lambda)_{\mu'\nu'}$$

Levi - Civita tensor

* coordinate xfm:

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu} = \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} g_{\mu\nu} \quad , \quad \Lambda^\mu{}_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}$$

$$= \Lambda^\mu{}_{\mu'} g_{\mu\nu} \Lambda^\nu{}_{\nu'}$$

$$= (\Lambda^T)_{\mu'}{}^\mu g_{\mu\nu} \Lambda^\nu{}_{\nu'}$$

$$= (\Lambda^T \cdot g \cdot \Lambda)_{\mu'\nu'}$$

$$\Rightarrow \det g' = \det \Lambda^T \det g \det \Lambda = \det g \, (\det \Lambda)^2$$

Levi - Civita tensor

* coordinate xfm:

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu} = \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} g_{\mu\nu} \quad , \quad \Lambda^\mu{}_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}$$

$$= \Lambda^\mu{}_{\mu'} g_{\mu\nu} \Lambda^\nu{}_{\nu'}$$

$$= (\Lambda^T)_{\mu'}{}^{\mu} g_{\mu\nu} \Lambda^\nu{}_{\nu'}$$

$$= (\Lambda^T \cdot g \cdot \Lambda)_{\mu'\nu'}$$

$$\Rightarrow \det g' = \det \Lambda^T \det g \det \Lambda = \det g \, (\det \Lambda)^2$$

$$\Rightarrow g' = g \left|\frac{\partial x^\mu}{\partial x^{\mu'}}\right|^2 \quad \Rightarrow \sqrt{|g'|} = \sqrt{g} \, \left|\left|\frac{\partial x^\mu}{\partial x^{\mu'}}\right|\right|$$

Levi - Civita tensor

Levi - Civita tensor: $\epsilon_{\mu_1 \cdots \mu_n} = \sqrt{|g|}\, \tilde{\epsilon}_{\mu_1 \cdots \mu_n}$

Assume a change of coordinates s.t. $\left|\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right| > 0$ (orientation preserving):

Levi - Civita tensor

Levi - Civita tensor: $\epsilon_{\mu_1 \cdots \mu_n} = \sqrt{|g|}\,\tilde{\epsilon}_{\mu_1 \cdots \mu_n}$

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$$\epsilon_{\mu_1 \cdots \mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}} \equiv \sqrt{|g|}\,\tilde{\epsilon}_{\mu_1 \cdots \mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}}$$

Levi - Civita tensor

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$$= \sqrt{|g|} \left| \frac{\partial x^{\mu}}{\partial x^{\mu'}} \right| \tilde{\epsilon}_{\mu_1' \cdots \mu_n'}$$

Levi - Civita tensor

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$$= \sqrt{|g|} \left| \frac{\partial x^{\mu}}{\partial x^{\mu'}} \right| \tilde{\epsilon}_{\mu_1' \cdots \mu_n'}$$

$$= \sqrt{|g'|}\, \tilde{\epsilon}_{\mu_1' \cdots \mu_n'}$$

(used $\left| \left| \frac{\partial x^{\mu}}{\partial x^{\mu'}} \right| \right| = \left| \frac{\partial x^{\mu}}{\partial x^{\mu'}} \right|$)

Levi - Civita tensor

Levi-Civita tensor: $\epsilon_{\mu_1 \cdots \mu_n} = \sqrt{|g|}\, \tilde{\epsilon}_{\mu_1 \cdots \mu_n}$

Assume a change of coordinates s.t. $\left|\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right| > 0$ (orientation preserving):

$$\epsilon_{\mu_1 \cdots \mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}} \equiv \sqrt{|g|}\, \tilde{\epsilon}_{\mu_1 \cdots \mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu_1'}}$$

$$= \sqrt{|g|} \left|\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right| \tilde{\epsilon}_{\mu_1' \cdots \mu_n'}$$

$$= \sqrt{|g'|}\, \tilde{\epsilon}_{\mu_1' \cdots \mu_n'}$$

$$= \epsilon_{\mu_1' \cdots \mu_n'}$$

transforms as a $(0,n)$ tensor!

Levi - Civita tensor

Levi-Civita tensor: $\epsilon_{\mu_1 \cdots \mu_n} = \sqrt{|g|}\, \tilde{\epsilon}_{\mu_1 \cdots \mu_n}$

$\epsilon = \frac{1}{n!} \sqrt{|g|}\, \tilde{\epsilon}_{\mu_1 \cdots \mu_n}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}$ a <u>nowhere vanishing</u> n-form

Levi - Civita tensor

Levi-Civita tensor: $\epsilon_{\mu_1 \cdots \mu_n} = \sqrt{|g|}\, \tilde{\epsilon}_{\mu_1 \cdots \mu_n}$

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---

Define the n-vector:

$$\underbrace{\epsilon^{\mu_1 \cdots \mu_n}}_{\text{fully antisymmetric}} = g^{\mu_1 \nu_1} \cdots g^{\mu_n \nu_n}\, \epsilon_{\nu_1 \cdots \nu_n}$$

Levi - Civita tensor

Levi-Civita tensor: $\epsilon_{\mu_1 \dots \mu_n} = \sqrt{|g|}\, \tilde{\epsilon}_{\mu_1 \dots \mu_n}$

$\epsilon = \frac{1}{n!} \sqrt{|g|}\, \tilde{\epsilon}_{\mu_1 \dots \mu_n} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n}$ a <u>nowhere vanishing</u> n-form

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Define the n-vector:

$$\epsilon^{\mu_1 \dots \mu_n} = g^{\mu_1 \nu_1} \dots g^{\mu_n \nu_n} \epsilon_{\nu_1 \dots \nu_n} = \underbrace{g^{\mu_1 \nu_1} \dots g^{\mu_n \nu_n} \sqrt{|g|}\, \tilde{\epsilon}_{\nu_1 \dots \nu_n}}_{\tilde{\epsilon} \det g^{-1}}$$

Levi-Civita tensor

Levi-Civita tensor: $\epsilon_{\mu_1 \cdots \mu_n} = \sqrt{|g|}\, \tilde{\epsilon}_{\mu_1 \cdots \mu_n}$

$\epsilon = \frac{1}{n!} \sqrt{|g|}\, \tilde{\epsilon}_{\mu_1 \cdots \mu_n}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}$ a <u>nowhere vanishing</u> n-form

---

Define the n-vector:

$$\epsilon^{\mu_1 \cdots \mu_n} = g^{\mu_1 \nu_1} \cdots g^{\mu_n \nu_n}\, \epsilon_{\nu_1 \cdots \nu_n} = g^{\mu_1 \nu_1} \cdots g^{\mu_n \nu_n} \sqrt{|g|}\, \tilde{\epsilon}_{\nu_1 \cdots \nu_n}$$

$$= \frac{1}{g}\, \tilde{\epsilon}^{\mu_1 \cdots \mu_n} \sqrt{|g|}$$

Levi - Civita tensor

Levi - Civita tensor: $\epsilon_{\mu_1 \cdots \mu_n} = \sqrt{|g|}\, \tilde{\epsilon}_{\mu_1 \cdots \mu_n}$

$\epsilon = \frac{1}{n!} \sqrt{|g|}\, \tilde{\epsilon}_{\mu_1 \cdots \mu_n}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n}$ a <u>nowhere vanishing</u> n-form

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Define the n-vector:

$$\epsilon^{\mu_1 \cdots \mu_n} = g^{\mu_1 \nu_1} \cdots g^{\mu_n \nu_n}\, \epsilon_{\nu_1 \cdots \nu_n} = g^{\mu_1 \nu_1} \cdots g^{\mu_n \nu_n} \sqrt{|g|}\, \tilde{\epsilon}_{\nu_1 \cdots \nu_n}$$

$$= \frac{1}{g}\, \tilde{\epsilon}^{\mu_1 \cdots \mu_n} \sqrt{|g|} = \frac{(-1)^s}{\sqrt{|g|}}\, \tilde{\epsilon}^{\mu_1 \cdots \mu_n}$$

Levi - Civita tensor

Contraction Identities computed from:

$$\epsilon_{\mu_1 \dots \mu_n} \epsilon^{\nu_1 \dots \nu_n} = \sqrt{|g|}\, \tilde{\epsilon}_{\mu_1 \dots \mu_n} \frac{(-1)^s}{\sqrt{|g|}} \tilde{\epsilon}^{\nu_1 \dots \nu_n} = (-1)^s \, \delta^{\nu_1 \dots \nu_n}_{\mu_1 \dots \mu_n}$$

Levi - Civita tensor

Contraction Identities computed from:

$$\epsilon_{\mu_1 \cdots \mu_n} \epsilon^{\nu_1 \cdots \nu_n} = \sqrt{|g|}\, \tilde{\epsilon}_{\mu_1 \cdots \mu_n} \frac{(-1)^s}{\sqrt{|g|}} \tilde{\epsilon}^{\nu_1 \cdots \nu_n} = (-1)^s \, \delta^{\nu_1 \cdots \nu_n}_{\mu_1 \cdots \mu_n} \Rightarrow$$

$$\epsilon_{\lambda_1 \cdots \lambda_p \mu_1 \cdots \mu_{n-p}} \epsilon^{\lambda_1 \cdots \lambda_p \nu_1 \cdots \nu_{n-p}} = (-1)^s (n-p)! \, \delta^{\nu_1 \cdots \nu_{n-p}}_{\mu_1 \cdots \mu_{n-p}}$$

$$\vdots$$

$$\epsilon_{\lambda_1 \cdots \lambda_{n-1} \nu} \epsilon^{\lambda_1 \cdots \lambda_{n-1} \mu} = (-1)^s (n-1)! \, \delta_\nu{}^\mu$$

$$\epsilon_{\lambda_1 \cdots \lambda_n} \epsilon^{\lambda_1 \cdots \lambda_n} = (-1)^s \, n!$$

Dual Forms

Let ω be a nowhere zero n-form

$$T^{\mu_1 \cdots \mu_p} = T^{[\mu_1 \cdots \mu_p]} \quad \text{a } p\text{-vector}$$

Dual Forms

Let ω be a nowhere zero n-form

$T^{\mu_1 \cdots \mu_p} = T^{[\mu_1 \cdots \mu_p]}$ a p-vector, then

$$ {}^{*}T_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!}\, \omega_{\underline{\nu_1 \cdots \nu_p}\, \mu_1 \cdots \mu_{n-p}} T^{\underline{\nu_1 \cdots \nu_p}} $$

$$ {}^{*}T = \omega(T) \equiv \omega(T, \underbrace{\cdots}_{n-p \text{ slots}}) $$

is the dual $(n-p)$-form of T with respect to ω

Dual Forms

$$*T_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!}\, \omega_{\nu_1 \cdots \nu_p \mu_1 \cdots \mu_{n-p}}\, T^{\nu_1 \cdots \nu_p}$$

Define the inverse of ω:

$$\underbrace{\omega^{\mu_1 \cdots \mu_n}\, \omega_{\mu_1 \cdots \mu_n}}_{n!\ \text{equal terms}} = n! \quad \Longleftrightarrow \quad \omega^{123 \cdots n} = \frac{1}{\omega_{123 \cdots n}}$$

Dual Forms

$$*T_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \omega_{\nu_1 \cdots \nu_p \mu_1 \cdots \mu_{n-p}} T^{\nu_1 \cdots \nu_p}$$

Define the inverse of ω:

$$\omega^{\mu_1 \cdots \mu_n} \omega_{\mu_1 \cdots \mu_n} = n! \iff \omega^{123 \cdots n} = \frac{1}{\omega_{123 \cdots n}}$$

If $\xi_{\mu_1 \cdots \mu_p}$ is a p-form, its dual $(n-p)$-vector is

$$*\xi^{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \omega^{\underline{\nu_1 \cdots \nu_p} \mu_1 \cdots \mu_{n-p}} \xi_{\underline{\nu_1 \cdots \nu_p}}$$

Dual Forms

$$*T_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!}\, \omega_{\nu_1 \cdots \nu_p \mu_1 \cdots \mu_{n-p}}\, T^{\nu_1 \cdots \nu_p}$$

Then

$$**T^{\mu_1 \cdots \mu_p} = \frac{1}{(n-p)!}\, \omega^{\underline{\nu_1 \cdots \nu_{n-p}}\, \mu_1 \cdots \mu_p}\, (*T)_{\underline{\nu_1 \cdots \nu_{n-p}}}$$

If $\eta_{\mu_1 \cdots \mu_p}$ is a p-form, its dual $(n-p)$-vector is

$$*\eta^{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!}\, \omega^{\nu_1 \cdots \nu_p \mu_1 \cdots \mu_{n-p}}\, \eta_{\nu_1 \cdots \nu_p}$$

Dual Forms

$$*T_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!}\, \omega_{\nu_1 \cdots \nu_p \mu_1 \cdots \mu_{n-p}}\, T^{\nu_1 \cdots \nu_p}$$

$$*\xi^{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!}\, \omega^{\nu_1 \cdots \nu_p \mu_1 \cdots \mu_{n-p}}\, \xi_{\nu_1 \cdots \nu_p}$$

Then

$$**T^{\mu_1 \cdots \mu_p} = \frac{1}{(n-p)!}\, \omega^{\underline{\nu_1 \cdots \nu_{n-p}}\, \mu_1 \cdots \mu_p}\, (*T)_{\underline{\nu_1 \cdots \nu_{n-p}}}$$

$$= \frac{1}{(n-p)!}\, \omega^{\underline{\nu_1 \cdots \nu_{n-p}}\, \mu_1 \cdots \mu_p}\, \frac{1}{p!}\, \omega_{\underline{\lambda_1 \cdots \lambda_p}\, \underline{\nu_1 \cdots \nu_{n-p}}}\, T^{\underline{\lambda_1 \cdots \lambda_p}}$$

Dual Forms

$$*T_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \omega_{\nu_1 \cdots \nu_p \mu_1 \cdots \mu_{n-p}} T^{\nu_1 \cdots \nu_p} \qquad * \eta^{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \omega^{\nu_1 \cdots \nu_p \mu_1 \cdots \mu_{n-p}} \eta_{\nu_1 \cdots \nu_p}$$

Then

$$**T^{\mu_1 \cdots \mu_p} = \frac{1}{(n-p)!} \omega^{\nu_1 \cdots \nu_{n-p} \mu_1 \cdots \mu_p} (*T)_{\nu_1 \cdots \nu_{n-p}}$$

$$= \frac{1}{(n-p)!} \omega^{\nu_1 \cdots \nu_{n-p} \mu_1 \cdots \mu_p} \frac{1}{p!} \omega_{\lambda_1 \cdots \lambda_p \underbrace{\nu_1 \cdots \nu_{n-p}}} T^{\lambda_1 \cdots \lambda_p}$$

each one must jump over p λ-indices giving $(-1)^p$

$$= \frac{1}{(n-p)!\,p!} \omega^{\nu_1 \cdots \nu_{n-p} \mu_1 \cdots \mu_p} \omega_{\underbrace{\nu_1 \cdots \nu_{n-p}} \lambda_1 \cdots \lambda_p} (-1)^{p\underbrace{(n-p)}} T^{\lambda_1 \cdots \lambda_p}$$

$(n-p)$ ν-indices contributed $(-1)^p$ each

Dual Forms

$$*T_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \omega_{\nu_1 \cdots \nu_p \mu_1 \cdots \mu_{n-p}} T^{\nu_1 \cdots \nu_p} \qquad * \xi^{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \omega^{\nu_1 \cdots \nu_p \mu_1 \cdots \mu_{n-p}} \xi_{\nu_1 \cdots \nu_p}$$

Then

$$**T^{\mu_1 \cdots \mu_p} = \frac{1}{(n-p)!} \omega^{\nu_1 \cdots \nu_{n-p} \mu_1 \cdots \mu_p} (*T)_{\nu_1 \cdots \nu_{n-p}}$$

$$= \frac{1}{(n-p)!} \omega^{\nu_1 \cdots \nu_{n-p} \mu_1 \cdots \mu_p} \frac{1}{p!} \omega_{\lambda_1 \cdots \lambda_p \nu_1 \cdots \nu_{n-p}} T^{\lambda_1 \cdots \lambda_p}$$

$$= \frac{1}{(n-p)! \cdot p!} \omega^{\underline{\nu_1 \cdots \nu_{n-p}} \, \underline{\mu_1 \cdots \mu_p}} \omega_{\underline{\nu_1 \cdots \nu_{n-p}} \, \underline{\lambda_1 \cdots \lambda_p}} (-1)^{p(n-p)} T^{\lambda_1 \cdots \lambda_p}$$

$$= \frac{1}{(n-p)! \cdot p!} (n-p)! \, \delta^{\underline{\mu_1 \cdots \mu_p}}_{\underline{\lambda_1 \cdots \lambda_p}} (-1)^{p(n-p)} T^{\lambda_1 \cdots \lambda_p}$$

Dual Forms

$$*T_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \, \omega_{\nu_1 \cdots \nu_p \mu_1 \cdots \mu_{n-p}} \, T^{\nu_1 \cdots \nu_p}$$

$$*\xi^{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \, \omega^{\nu_1 \cdots \nu_p \mu_1 \cdots \mu_{n-p}} \, \xi_{\nu_1 \cdots \nu_p}$$

Then

$$**T^{\mu_1 \cdots \mu_p} = \frac{1}{(n-p)!} \, \omega^{\nu_1 \cdots \nu_{n-p} \mu_1 \cdots \mu_p} \, (*T)_{\nu_1 \cdots \nu_{n-p}}$$

$$= \frac{1}{(n-p)!} \, \omega^{\nu_1 \cdots \nu_{n-p} \mu_1 \cdots \mu_p} \, \frac{1}{p!} \omega_{\lambda_1 \cdots \lambda_p \nu_1 \cdots \nu_{n-p}} \, T^{\lambda_1 \cdots \lambda_p}$$

$$= \frac{1}{(n-p)! \cdot p!} \omega^{\nu_1 \cdots \nu_{n-p} \mu_1 \cdots \mu_p} \, \omega_{\nu_1 \cdots \nu_{n-p} \lambda_1 \cdots \lambda_p} \, (-1)^{p(n-p)} \, T^{\lambda_1 \cdots \lambda_p}$$

$$= \frac{1}{\cancel{(n-p)!} \cdot p!} \, \cancel{(n-p)!} \, \delta^{\mu_1 \cdots \mu_p}_{\lambda_1 \cdots \lambda_p} \, (-1)^{p(n-p)} \, T^{\lambda_1 \cdots \lambda_p}$$

$$= \frac{1}{p!} \, p! \, \delta^{\mu_1}_{[\lambda_1} \delta^{\mu_2}_{\lambda_2} \cdots \delta^{\mu_p}_{\lambda_p]} \, (-1)^{p(n-p)} \, T^{\lambda_1 \cdots \lambda_p}$$

Dual Forms

$$*T_{\mu_1 \dots \mu_{n-p}} = \frac{1}{p!} \omega_{\nu_1 \dots \nu_p \mu_1 \dots \mu_{n-p}} T^{\nu_1 \dots \nu_p} \qquad * \eta^{\mu_1 \dots \mu_{n-p}} = \frac{1}{p!} \omega^{\nu_1 \dots \nu_p \mu_1 \dots \mu_{n-p}} \eta_{\nu_1 \dots \nu_p}$$

Then

$$* * T^{\mu_1 \dots \mu_p} = \frac{1}{(n-p)!} \omega^{\nu_1 \dots \nu_{n-p} \mu_1 \dots \mu_p} (*T)_{\nu_1 \dots \nu_{n-p}}$$

$$= \frac{1}{(n-p)!} \omega^{\nu_1 \dots \nu_{n-p} \mu_1 \dots \mu_p} \frac{1}{p!} \omega_{\lambda_1 \dots \lambda_p \nu_1 \dots \nu_{n-p}} T^{\lambda_1 \dots \lambda_p}$$

$$= \frac{1}{(n-p)! \cdot p!} \omega^{\nu_1 \dots \nu_{n-p} \mu_1 \dots \mu_p} \omega_{\nu_1 \dots \nu_{n-p} \lambda_1 \dots \lambda_p} (-1)^{p(n-p)} T^{\lambda_1 \dots \lambda_p}$$

$$= \frac{1}{\cancel{(n-p)!} \cdot p!} \cancel{(n-p)!} \, \delta^{\mu_1 \dots \mu_p}_{\lambda_1 \dots \lambda_p} (-1)^{p(n-p)} T^{\lambda_1 \dots \lambda_p}$$

$$= \frac{1}{p!} p! \, \delta^{\mu_1}_{[\lambda_1} \delta^{\mu_2}_{\lambda_2} \cdots \delta^{\mu_p}_{\lambda_p]} (-1)^{p(n-p)} T^{\lambda_1 \dots \lambda_p} = (-1)^{p(n-p)} \delta^{\mu_1}_{\lambda_1} \cdots \delta^{\mu_p}_{\lambda_p} T^{\lambda_1 \dots \lambda_p}$$

Dual Forms

$$*T_{\mu_1\cdots\mu_{n-p}} = \frac{1}{p!}\,\omega_{\nu_1\cdots\nu_p\,\mu_1\cdots\mu_{n-p}}\,T^{\nu_1\cdots\nu_p} \qquad\qquad *\eta^{\mu_1\cdots\mu_{n-p}} = \frac{1}{p!}\,\omega^{\nu_1\cdots\nu_p\,\mu_1\cdots\mu_{n-p}}\,\eta_{\nu_1\cdots\nu_p}$$

Then

$$**T^{\mu_1\cdots\mu_p} = \frac{1}{(n-p)!}\,\omega^{\nu_1\cdots\nu_{n-p}\,\mu_1\cdots\mu_p}\,(*T)_{\nu_1\cdots\nu_{n-p}}$$

$$= \frac{1}{(n-p)!}\,\omega^{\nu_1\cdots\nu_{n-p}\,\mu_1\cdots\mu_p}\,\frac{1}{p!}\,\omega_{\lambda_1\cdots\lambda_p\,\nu_1\cdots\nu_{n-p}}\,T^{\lambda_1\cdots\lambda_p}$$

$$= \frac{1}{(n-p)!\,p!}\,\omega^{\nu_1\cdots\nu_{n-p}\,\mu_1\cdots\mu_p}\,\omega_{\nu_1\cdots\nu_{n-p}\,\lambda_1\cdots\lambda_p}\,(-1)^{p(n-p)}\,T^{\lambda_1\cdots\lambda_p}$$

$$= \frac{1}{\cancel{(n-p)!}\,p!}\,\cancel{(n-p)!}\,\delta^{\mu_1\cdots\mu_p}_{\lambda_1\cdots\lambda_p}\,(-1)^{p(n-p)}\,T^{\lambda_1\cdots\lambda_p}$$

$$= \frac{1}{p!}\,p!\,\delta^{\mu_1}_{[\lambda_1}\delta^{\mu_2}_{\lambda_2}\cdots\delta^{\mu_p}_{\lambda_p]}\,(-1)^{p(n-p)}\,T^{\lambda_1\cdots\lambda_p} = (-1)^{p(n-p)}\,\delta^{\mu_1}_{\lambda_1}\cdots\delta^{\mu_p}_{\lambda_p}\,T^{\lambda_1\cdots\lambda_p}$$

antisymmetry removed $[\cdots]$

Dual Forms

$$*T_{\mu_1\cdots\mu_{n-p}} = \frac{1}{p!}\,\omega_{\nu_1\cdots\nu_p\,\mu_1\cdots\mu_{n-p}}\,T^{\nu_1\cdots\nu_p}$$

$$*\xi^{\mu_1\cdots\mu_{n-p}} = \frac{1}{p!}\,\omega^{\nu_1\cdots\nu_p\,\mu_1\cdots\mu_{n-p}}\,\xi_{\nu_1\cdots\nu_p}$$

Then

$$**T^{\mu_1\cdots\mu_p} = (-1)^{p(n-p)}\,T^{\mu_1\cdots\mu_p}$$

antisymmetry removed

$[\cdots]$

$$= \frac{1}{p!}\,p!\,\delta^{\mu_1}_{[\lambda_1}\delta^{\mu_2}_{\lambda_2}\cdots\delta^{\mu_p}_{\lambda_p]}\,(-1)^{p(n-p)}\,T^{\lambda_1\cdots\lambda_p} = (-1)^{p(n-p)}\,\delta^{\mu_1}_{\lambda_1}\cdots\delta^{\mu_p}_{\lambda_p}\,T^{\lambda_1\cdots\lambda_p}$$

Dual Forms

$$*T_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \omega_{\nu_1 \cdots \nu_p \mu_1 \cdots \mu_{n-p}} T^{\nu_1 \cdots \nu_p}$$

$$*\xi^{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \omega^{\nu_1 \cdots \nu_p \mu_1 \cdots \mu_{n-p}} \xi_{\nu_1 \cdots \nu_p}$$

Then

$$**T^{\mu_1 \cdots \mu_p} = (-1)^{p(n-p)} T^{\mu_1 \cdots \mu_p}$$

$$**T = (-1)^{p(n-p)} T$$

similarly: $$**\xi = (-1)^{p(n-p)} \xi$$

Hodge Dual

Given Levi-Civita tensor $\epsilon_{\mu_1 \cdots \mu_n} = \sqrt{|g|}\ \tilde{\epsilon}_{\mu_1 \cdots \mu_n}$

and $\epsilon^{\mu_1 \cdots \mu_n} = \dfrac{(-1)^s}{\sqrt{|g|}}\ \tilde{\epsilon}^{\mu_1 \cdots \mu_n}$

$\Rightarrow \epsilon_{\mu_1 \cdots \mu_n} \epsilon^{\mu_1 \cdots \mu_n} = (-1)^s\, n!$

we can use $\omega_{\mu_1 \cdots \mu_n} \rightarrow \epsilon_{\mu_1 \cdots \mu_n}$

$\omega^{\mu_1 \cdots \mu_n} \rightarrow (-1)^s \epsilon^{\mu_1 \cdots \mu_n}$

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Given Levi-Civita tensor $\epsilon_{\mu_1 \cdots \mu_n} = \sqrt{|g|}\, \tilde{\epsilon}_{\mu_1 \cdots \mu_n}$

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$\Rightarrow \epsilon_{\mu_1 \cdots \mu_n} \epsilon^{\mu_1 \cdots \mu_n} = (-1)^s\, n!$

we can use $\omega_{\mu_1 \cdots \mu_n} \rightarrow \epsilon_{\mu_1 \cdots \mu_n}$

$$\omega^{\mu_1 \cdots \mu_n} \rightarrow (-1)^s \epsilon^{\mu_1 \cdots \mu_n}$$

usually $\epsilon^{\mu_1 \cdots \mu_n}$ is used instead

Hodge Dual

$$ {}^{*}T_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \epsilon_{\nu_1 \cdots \nu_p \mu_1 \cdots \mu_{n-p}} T^{\nu_1 \cdots \nu_p} $$

Hodge Dual

$$*\zeta_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \epsilon_{\nu_1 \cdots \nu_p \mu_1 \cdots \mu_{n-p}} \zeta^{\nu_1 \cdots \nu_p}$$

But the metric defines $\zeta_{\nu_1 \cdots \nu_p} \leftrightarrow \zeta^{\nu_1 \cdots \nu_p}$ as:

$$\zeta^{\nu_1 \cdots \nu_p} = g^{\nu_1 \lambda_1} \cdots g^{\nu_p \lambda_p} \zeta_{\lambda_1 \cdots \lambda_p}$$

Hodge Dual

$$*\zeta_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \epsilon_{\nu_1 \cdots \nu_p \mu_1 \cdots \mu_{n-p}} \zeta^{\nu_1 \cdots \nu_p}$$

But the metric defines $\zeta_{\nu_1 \cdots \nu_p} \leftrightarrow \zeta^{\nu_1 \cdots \nu_p}$ as:

$$\zeta^{\nu_1 \cdots \nu_p} = g^{\nu_1 \lambda_1} \cdots g^{\nu_p \lambda_p} \zeta_{\lambda_1 \cdots \lambda_p} \quad , \quad \text{then}$$

$$*\zeta_{\mu_1 \cdots \mu_{n-p}} = \frac{1}{p!} \epsilon_{\nu_1 \cdots \nu_p \mu_1 \cdots \mu_{n-p}} g^{\nu_1 \lambda_1} \cdots g^{\nu_p \lambda_p} \zeta_{\lambda_1 \cdots \lambda_p}$$

$$\equiv \frac{1}{p!} \epsilon^{\lambda_1 \cdots \lambda_p}{}_{\mu_1 \cdots \mu_{n-p}} \zeta_{\lambda_1 \cdots \lambda_p} \qquad \text{where} \quad \epsilon^{\lambda_1 \cdots \lambda_p}{}_{\mu_1 \cdots \mu_{n-p}} = g^{\lambda_1 \nu_1} \cdots g^{\lambda_p \nu_p} \epsilon_{\nu_1 \cdots \nu_p \mu_1 \cdots \mu_{n-p}}$$

Hodge Dual

$$*\zeta_{\mu_1 \dots \mu_{n-p}} = \frac{1}{p!} \epsilon^{\nu_1 \dots \nu_p}{}_{\mu_1 \dots \mu_{n-p}} \zeta_{\nu_1 \dots \nu_p}$$

- ζ in the $(n-p)$-form dual to the p-form ζ
- $*$ the Hodge-star operator
- $**\zeta = (-1)^{s+p(n-p)} \zeta$

Hodge Dual

$$*\xi_{\mu_1 \dots \mu_{n-p}} = \frac{1}{p!} \epsilon^{\nu_1 \dots \nu_p}{}_{\mu_1 \dots \mu_{n-p}} \xi_{\nu_1 \dots \nu_p}$$

- ξ in the $(n-p)$-form dual to the p-form ξ
- $*$ the Hodge-star operator
- $**\xi = (-1)^{s+p(n-p)} \xi$ Comes from the definition of $\epsilon^{\mu_1 \dots \mu_n}$, where $\epsilon^{\dots}\epsilon_{\dots} = (-1)^s n!$

  - if n is odd $p(n-p)$ is even $\forall p$

    " even " has parity of p