The Schwarzschild Solution

Part I: Exterior Region $r > r_s$

The geometry outside a spherically symmetric star

- The metric in (t, r, θ, φ) coordinates:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \, d\Omega^2$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$$

- The metric in (t, r, θ, φ) coordinates:

$$ds^2 = -\left(1-\frac{2M}{r}\right) dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$$

- The unique solution to Einstein's equation in the vacuum which is:

  - static
  - spherically symmetric
  - asymptotically flat

- The metric in (t, r, θ, φ) coordinates:

$$ds^2 = -\left(1-\frac{2M}{r}\right) dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$$

- The unique solution to Einstein's equation in the vacuum which is:

$\hookrightarrow R_{\mu\nu} = 0$

  - static
  - spherically symmetric
  - asymptotically flat $\longrightarrow r \to \infty \Rightarrow \left(1-\frac{2M}{r}\right) \to 1 \Rightarrow g_{\mu\nu} \to \eta_{\mu\nu}$

- $(g_{\mu\nu}) = \mathrm{diag}\left(-\left(1-\frac{2M}{r}\right), \left(1-\frac{2M}{r}\right)^{-1}, r^2, r^2\sin^2\theta\right)$

$$\Rightarrow \begin{cases} \partial_t g_{\mu\nu} = 0 \\ \partial_\varphi g_{\mu\nu} = 0 \end{cases}$$

- $(g_{\mu\nu}) = \mathrm{diag}\left(-\left(1-\frac{2M}{r}\right), \left(1-\frac{2M}{r}\right)^{-1}, r^2, r^2\sin^2\theta\right)$

$$\Rightarrow \left\{\begin{array}{l} \partial_t g_{\mu\nu} = 0 \\ \partial_\varphi g_{\mu\nu} = 0 \end{array}\right\} \Rightarrow \begin{array}{l} \xi = \partial_t \\ \eta = \partial_\varphi \end{array} \quad \text{are Killing vector fields}$$

$$\bullet \ (g_{\mu\nu}) = \mathrm{diag}\left(-\left(1-\frac{2M}{r}\right), \left(1-\frac{2M}{r}\right)^{-1}, r^2, r^2\sin^2\theta\right)$$

$$\Rightarrow \left\{\begin{array}{l} \partial_t g_{\mu\nu} = 0 \\ \partial_\varphi g_{\mu\nu} = 0 \end{array}\right\} \Rightarrow \begin{array}{l} \xi = \partial_t \\ \eta = \partial_\varphi \end{array} \quad \text{are Killing vector fields}$$

$\Rightarrow$ ($\xi^\mu u_\mu$, $\eta^\mu u_\mu$ are conserved along geodesics w/ tangent vector u^μ)

- $(g_{\mu\nu}) = \mathrm{diag}\left(-\left(1-\frac{2M}{r}\right), \left(1-\frac{2M}{r}\right)^{-1}, r^2, r^2\sin^2\theta\right)$

$$\Rightarrow \left\{\begin{array}{l} \partial_t g_{\mu\nu} = 0 \\ \partial_\varphi g_{\mu\nu} = 0 \end{array}\right\} \Rightarrow \begin{array}{l} \xi = \partial_t \\ \eta = \partial_\varphi \end{array} \text{ are Killing vector fields}$$

- The geometry for $dt = dr = 0$ is that of a sphere of radius r:

$$d\Sigma^2 = r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$$

- $(g_{\mu\nu}) = \mathrm{diag}\left(-\left(1-\frac{2M}{r}\right), \left(1-\frac{2M}{r}\right)^{-1}, r^2, r^2\sin^2\theta\right)$

$$\Rightarrow \left\{\begin{array}{l} \partial_t g_{\mu\nu} = 0 \\ \partial_\varphi g_{\mu\nu} = 0 \end{array}\right\} \Rightarrow \begin{array}{l} \xi = \partial_t \\ \eta = \partial_\varphi \end{array} \quad \text{are Killing vector fields}$$

- The geometry for $dt = dr = 0$ is that of a sphere of radius r:

$$d\Sigma^2 = r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2$$

careful: not a sphere with center at $r=0$ with points lying at distance r from it

- $(g_{\mu\nu}) = \mathrm{diag}\left(-\left(1-\frac{2M}{r}\right), \left(1-\frac{2M}{r}\right)^{-1}, r^2, r^2\sin^2\theta\right)$

$$\Rightarrow \left\{\begin{array}{l} \partial_t g_{\mu\nu} = 0 \\ \partial_\varphi g_{\mu\nu} = 0 \end{array}\right\} \Rightarrow \begin{array}{l} \xi = \partial_t \\ \eta = \partial_\varphi \end{array} \quad \text{are Killing vector fields}$$

- The geometry for $dt = dr = 0$ is that of a sphere of radius r:

$$d\Sigma^2 = r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$$

careful: not a sphere with center at $r=0$ with points lying at distance r from it

but : area of sphere: $A = 4\pi r^2$ (defines "r")

- Mass M:

For $r \gg 2M$

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - 2\frac{M}{r}\right)^{-1} dr^2 + r^2 \, d\Omega^2 \qquad (1+x)^{-1} \approx 1 - x \quad x \ll 1$$

$$\approx -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 + 2\frac{M}{r}\right) dr^2 + r^2 \, d\Omega^2$$

- Mass M:

For $r \gg 2M$

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-2\frac{M}{r}\right)^{-1}dr^2 + r^2\,d\Omega^2$$

$(1+x)^{-1} \approx 1-x \quad x \ll 1$

$$\approx -\left(1-\underbrace{\frac{2M}{r}}_{2\Phi(r)}\right)dt^2 + \left(1+\underbrace{\frac{2M}{r}}_{2\Phi(r)}\right)dr^2 + r^2\,d\Omega^2$$

$$\Rightarrow \quad \Phi(r) = \frac{M}{r}$$

• Mass M:

For $r \gg 2M$

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - 2\frac{M}{r}\right)^{-1} dr^2 + r^2\, d\Omega^2 \qquad (1+x)^{-1} \approx 1 - x \quad x \ll 1$$

$$\approx -\left(1 - \underbrace{\frac{2M}{r}}_{2\Phi(r)}\right) dt^2 + \left(1 + \underbrace{\frac{2M}{r}}_{2\Phi(r)}\right) dr^2 + r^2\, d\Omega^2$$

$$\Rightarrow \quad \Phi(r) = \frac{M}{r}$$

Newtonian potential of spherical mass distribution of total mass M

- Mass M:

For $r \gg 2M$

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-2\frac{M}{r}\right)^{-1}dr^2 + r^2\, d\Omega^2 \qquad (1+x)^{-1} \approx 1-x \quad x \ll 1$$

$$\approx -\left(1-\underbrace{\frac{2M}{r}}_{2\Phi(r)}\right)dt^2 + \left(1+\underbrace{\frac{2M}{r}}_{2\Phi(r)}\right)dr^2 + r^2\, d\Omega^2$$

$\Rightarrow \Phi(r) = \dfrac{M}{r}$ Newtonian potential of spherical mass distribution of total mass M

* the parameter M is total mass of source of curvature
* there is no "particle" at $r=0$ of mass m. ($r=0$ not in spacetime)

- The same metric is obtained outside a spherical symmetric mass distribution

- The same metric is obtained outside a spherical symmetric mass distribution

- $r_s = 2M$ is the Schwarzschild radius

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_s}{r}} + r^2 d\Omega^2$$

$$r > r_s \Rightarrow \begin{cases} \partial_t & \text{timelike} \\ \partial_r & \text{spacelike} \end{cases}$$

$$\left(\partial_t \cdot \partial_t = g_{00} = -\left(1 - \frac{r_s}{r}\right) < 0 \right)$$

$$\left(\partial_r \cdot \partial_r = g_{11} = \left(1 - \frac{r_s}{r}\right)^{-1} > 0 \right)$$

- The same metric is obtained outside a spherical symmetric mass distribution

- $r_s = 2M$ is the Schwarzschild radius

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_s}{r}} + r^2 d\Omega^2$$

$$r > r_s \Rightarrow \begin{cases} \partial_t & \text{timelike} \\ \partial_r & \text{spacelike} \end{cases}$$

$$\left(\partial_t \cdot \partial_t = g_{00} = -\left(1 - \frac{r_s}{r}\right) < 0 \right)$$

$$\left(\partial_r \cdot \partial_r = g_{11} = \left(1 - \frac{r_s}{r}\right)^{-1} > 0 \right)$$

$$r < r_s \Rightarrow \begin{cases} \partial_t & \text{spacelike} \\ \partial_r & \text{timelike} \end{cases}$$

$\rightsquigarrow$ direction of "time"

- The same metric is obtained outside a spherical symmetric mass distribution

- $r_s = 2M$ is the Schwarzschild radius

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_s}{r}} + r^2 d\Omega^2$$

$$r > r_s \Rightarrow \begin{cases} \partial_t & \text{timelike} \\ \partial_r & \text{spacelike} \end{cases} \qquad r = r_s \quad \begin{cases} \partial_t & \text{null} \\ g_{\mu\nu} & \text{singular} \end{cases}$$

$$r < r_s \Rightarrow \begin{cases} \partial_t & \text{spacelike} \\ \partial_r & \text{timelike} \end{cases}$$

- The same metric is obtained outside a spherical symmetric mass distribution

- $r_s = 2M$ is the Schwarzschild radius

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_s}{r}} + r^2 d\Omega^2$$

- $r = r_s$ a pathology of the coordinate system, not of spacetime

- The same metric is obtained outside a spherical symmetric mass distribution

- $r_s = 2M$ is the Schwarzschild radius

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_s}{r}} + r^2 d\Omega^2$$

- $r = r_s$ a pathology of the coordinate system, not of spacetime
- defines a "horizon": everything that comes in, cannot come out

- The same metric is obtained outside a spherical symmetric mass distribution

- $r_s = 2M$ is the Schwarzschild radius

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_s}{r}} + r^2 d\Omega^2$$

- $r = r_s$ a pathology of the coordinate system, not of spacetime
- defines a "horizon": everything that comes in, cannot come out
- timelike geodesics (= trajectories of freely falling massive particles) behave regularly across the horizon, which they cross in finite proper time

- The same metric is obtained outside a spherical symmetric mass distribution

- $r_s = 2M$ is the Schwarzschild radius

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_s}{r}} + r^2 d\Omega^2$$

- $r = r_s$ a pathology of the coordinate system, not of spacetime
- defines a "horizon": everything that comes in, cannot come out
- timelike geodesics (= trajectories of freely falling massive particles) behave regularly across the horizon, which they cross in finite proper time
- curvature scalars have finite value on the horizon

- The same metric is obtained outside a spherical symmetric mass distribution

- $r_s = 2M$ is the Schwarzschild radius

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_s}{r}} + r^2 d\Omega^2$$

- r_s is irrelevant for ordinary stars, planets, objects in everyday life, where

$$R \gg r_s$$

↳ size of object

- The same metric is obtained outside a spherical symmetric mass distribution

- $r_s = 2M$ is the Schwarzschild radius

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_s}{r}} + r^2 d\Omega^2$$

  - r_s is irrelevant for ordinary stars, planets, objects in everyday life, where

$$R \gg r_s$$

$r_s =$ 8.8 mm for the Earth
2.95 km " " Sun
0.2 lyrs " " galaxy

- The same metric is obtained outside a spherical symmetric mass distribution

- $r_s = 2M$ is the Schwarzschild radius

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_s}{r}} + r^2 d\Omega^2$$

- $r = 0$ a true spacetime singularity:
  - $K = R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} = \frac{48 M^2}{r^6} \rightarrow \infty$ (but regular at $r = 2M$)

↳ Kretschmann scalar

- The same metric is obtained outside a spherical symmetric mass distribution

- $r_s = 2M$ is the Schwarzschild radius

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_s}{r}} + r^2 d\Omega^2$$

– $r = 0$ a true spacetime singularity:

- $K = R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} = \frac{48M^2}{r^6} \rightarrow \infty$

- timelike geodesics reach $r=0$ at finite proper time

→ they end there: the end of time...

Units

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)(c\,dt)^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2\, d\Omega^2$$

SI:

$$G = 6.7 \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$$

$$\kappa = \frac{8\pi G}{c^4} = 2.1 \times 10^{-43}\ N^{-1}$$

$$F = G\frac{Mm}{r^2} \qquad \underline{\Phi} = \frac{GM}{r}$$

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

Units

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)(c\,dt)^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$$G = 6.7 \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$$

$$\kappa = \frac{8\pi G}{c^4} = 2.1 \times 10^{-43}\ N^{-1}$$

$$F = G\frac{Mm}{r^2} \qquad \Phi = \frac{GM}{r}$$

$$G_{\mu\nu} = \kappa\, T_{\mu\nu}$$

- $c = 1 \Rightarrow$ s, t, r measured in length units

Units

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)(c\,dt)^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

SI

$$G = 6.7 \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2} \qquad \kappa = \frac{8\pi G}{c^4} = 2.1 \times 10^{-43}\ N^{-1}$$

$$F = G\frac{Mm}{r^2} \qquad \Phi = \frac{GM}{r} \qquad G_{\mu\nu} = \kappa T_{\mu\nu}$$

- $c = 1 \Rightarrow s, t, r$ measured in length units
- $G = 1 \Rightarrow M, m$ measured in length units, s.t.

$$M(\text{in cm}) = \frac{G}{c^2} M(\text{in gr}) = 0.74 \times 10^{-28}\ \frac{cm}{gr}\ M(\text{in gr})$$

Geometrized units

$$\Rightarrow \begin{cases} M_\odot = 1.5\ km \\ M_\oplus = 4.4\ mm \end{cases} \qquad (= 2 r_s)$$

Units

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)(c\,dt)^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$$G = 6.7 \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$$

$$\kappa = \frac{8\pi G}{c^4} = 2.1 \times 10^{-43}\ N^{-1}$$

$$F = G\frac{Mm}{r^2} \qquad \underline{\Phi} = \frac{GM}{r}$$

$$G_{\mu\nu} = \kappa\, T_{\mu\nu}$$

- $c = 1 \Rightarrow s, t, r$ measured in length units
- $G = 1 \Rightarrow M, m$ measured in length units, s.t.

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Units

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)(c\,dt)^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$$G = 6.7 \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$$

$$\kappa = \frac{8\pi G}{c^4} = 2.1 \times 10^{-43}\ N^{-1}$$

$$F = G\frac{Mm}{r^2} \qquad \Phi = \frac{GM}{r}$$

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

- $c = 1 \Rightarrow$ s, t, r measured in length units
- $G = 1 \Rightarrow$ M, m measured in length units, s.t.

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Restore metric units: $M \to \frac{GM}{c^2} \qquad t \to ct, \quad \tau \to c\tau$

Quantity	SI dimension	Geometric dimension	Multiplication factor
Length	L	L	1
Time	T	L	c
Mass	M	L	$G\,c^{-2}$
Velocity	$L\,T^{-1}$	1	c^{-1}
Angular velocity	T^{-1}	L^{-1}	c^{-1}
Acceleration	$L\,T^{-2}$	L^{-1}	c^{-2}
Energy	$M\,L^{2}\,T^{-2}$	L	$G\,c^{-4}$
Energy density	$M\,L^{-1}\,T^{-2}$	L^{-2}	$G\,c^{-4}$
Angular momentum	$M\,L^{2}\,T^{-1}$	L^{2}	$G\,c^{-3}$
Force	$M\,L\,T^{-2}$	1	$G\,c^{-4}$
Power	$M\,L^{2}\,T^{-3}$	1	$G\,c^{-5}$
Pressure	$M\,L^{-1}\,T^{-2}$	L^{-2}	$G\,c^{-4}$
Density	$M\,L^{-3}$	L^{-2}	$G\,c^{-2}$
Electric charge	T I	L	$G^{1/2}\,c^{-2}\,\varepsilon_0^{-1/2}$
Electric potential	$M\,L^{2}\,T^{-3}\,I^{-1}$	1	$G^{1/2}\,c^{-2}\,\varepsilon_0^{1/2}$
Electric field	$M\,L\,T^{-3}\,I^{-1}$	L^{-1}	$G^{1/2}\,c^{-2}\,\varepsilon_0^{1/2}$
Magnetic field	$M\,T^{-2}\,I^{-1}$	L^{-1}	$G^{1/2}\,c^{-1}\,\varepsilon_0^{1/2}$

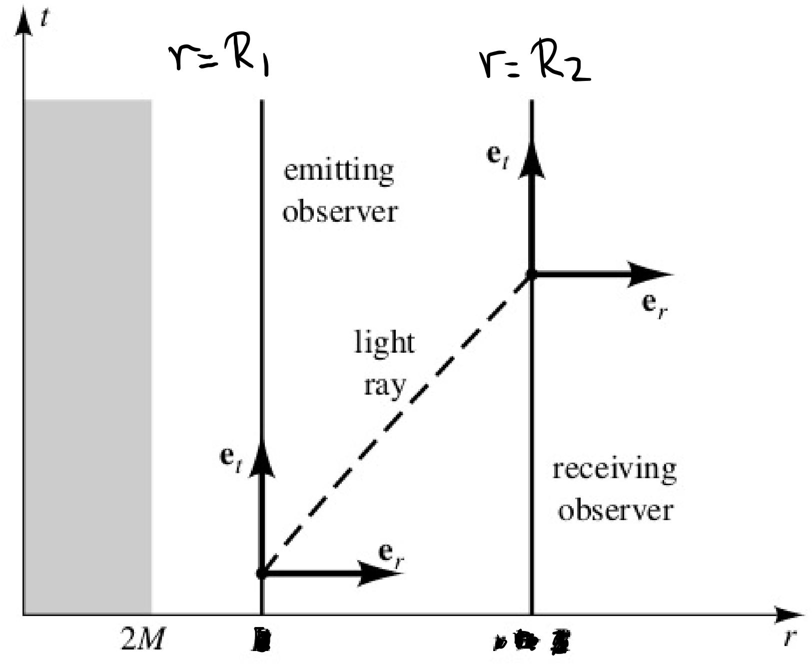
https://en.wikipedia.org/wiki/Geometrized_unit_system

Gravitational Redshift

- $\xi = \partial_t$ a Killing Vector field

  $p_\mu \xi^\mu$ conserved along geodesic w/ tangent vector p^μ

  $\xi^\mu = (1, 0, 0, 0)$



Hartle, Fig 9.1

Gravitational Redshift

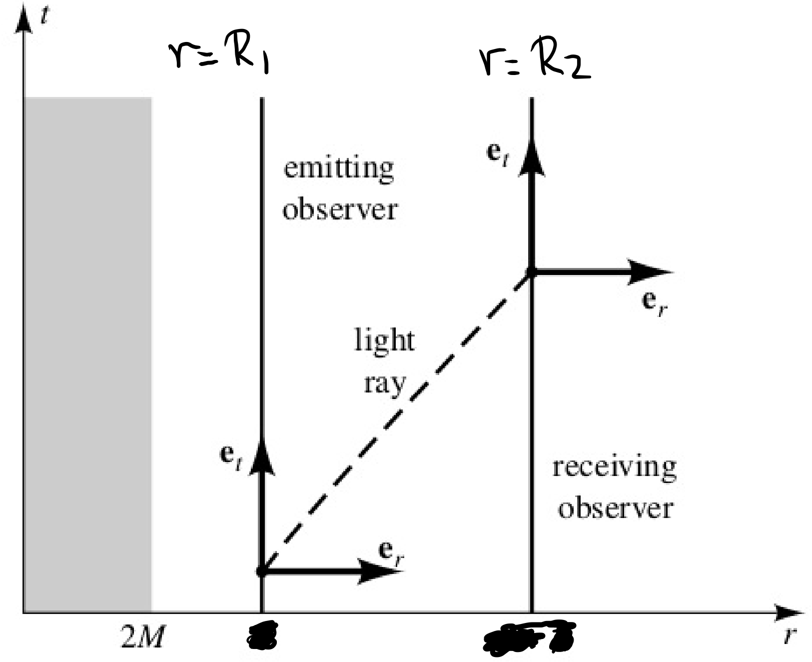
- $\xi = \partial_t$ a Killing Vector field

  $p_\mu \xi^\mu$ conserved along geodesic w/ tangent vector p^μ

  $\xi^\mu = (1, 0, 0, 0)$

- photon emitted by observer at $r = R_1$

  " received " " " $r = R_2$



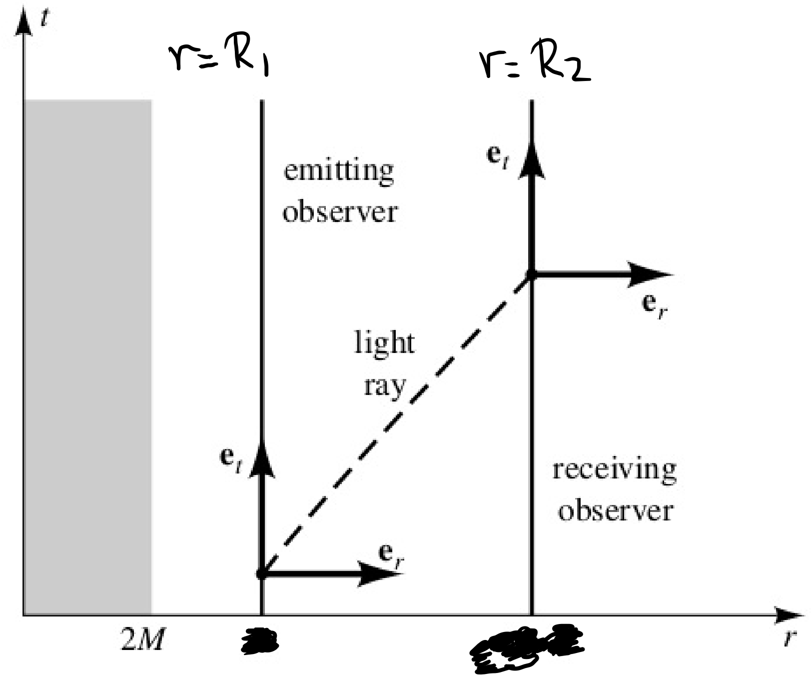
Hartle, Fig 9.1

Gravitational Redshift

- $\xi = \partial_t$ a Killing Vector field

  $p_\mu \xi^\mu$ conserved along geodesic w/ tangent vector p^μ

  $\xi^\mu = (1, 0, 0, 0)$



Hartle , Fig 9.1

- photon emitted by observer at $r = R_1$

  " received " " " $r = R_2$

- 4 velocity of observer at R_1: $u_1^\mu = (u_1^0, 0, 0, 0)$

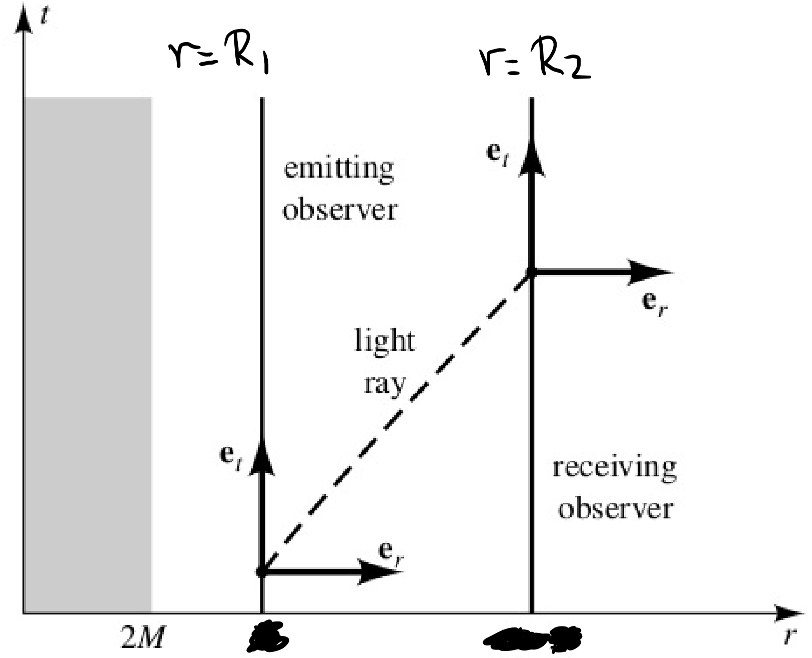
  $u_1^\mu u_{1,\mu} = -1 \Rightarrow g_{\mu\nu} u_1^\mu u_1^\nu = -1 \Rightarrow g_{00} u_1^0 u_1^0 = -1$

Gravitational Redshift

- $\xi = \partial_t$ a Killing Vector field

  $p_\mu \xi^\mu$ conserved along geodesic w/ tangent vector p^μ

  $\xi^\mu = (1, 0, 0, 0)$



Hartle, Fig 9.1

- photon emitted by observer at $r = R_1$

  " received " " " $r = R_2$

- 4 velocity of observer at R_1: $u_1^\mu = (u_1^0, 0, 0, 0)$

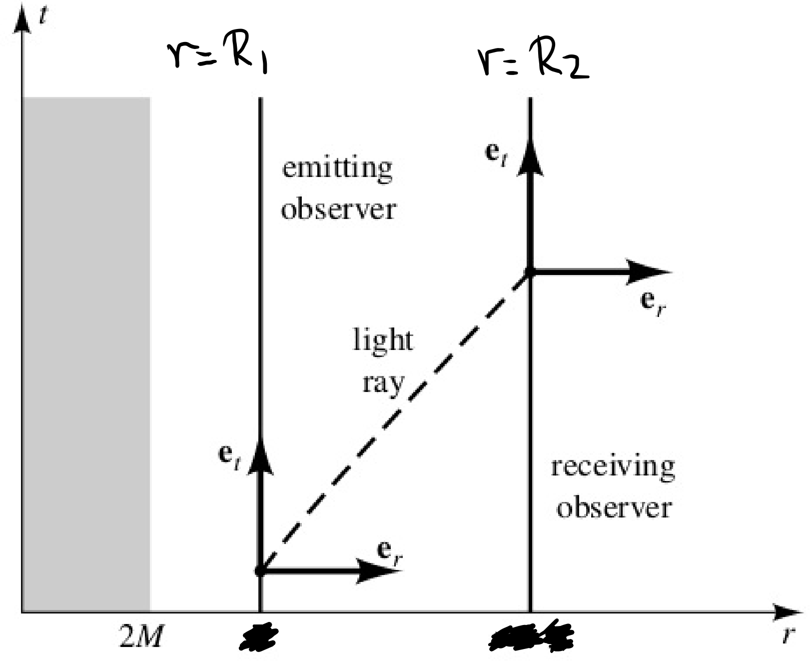
$$u_1^\mu u_{1,\mu} = -1 \Rightarrow g_{\mu\nu} u_1^\mu u_1^\nu = -1 \Rightarrow g_{00} u_1^0 u_1^0 = -1 \Rightarrow -\left(1 - \frac{2M}{R_1}\right)(u_1^0)^2 = -1$$

Gravitational Redshift

- $\xi = \partial_t$ a Killing Vector field

  $p_\mu \xi^\mu$ conserved along geodesic w/ tangent vector p^μ

  $\xi^\mu = (1, 0, 0, 0)$



Hartle, Fig 9.1

- photon emitted by observer at $r = R_1$

  " received " " " $r = R_2$

- 4 velocity of observer at R_1: $u_1^\mu = (u_1^0, 0, 0, 0)$

$$u_1^\mu u_{1\mu} = -1 \Rightarrow g_{\mu\nu} u_1^\mu u_1^\nu = -1 \Rightarrow g_{00} u_1^0 u_1^0 = -1 \Rightarrow -\left(1 - \frac{2M}{R_1}\right)(u_1^0)^2 = -1 \Rightarrow$$

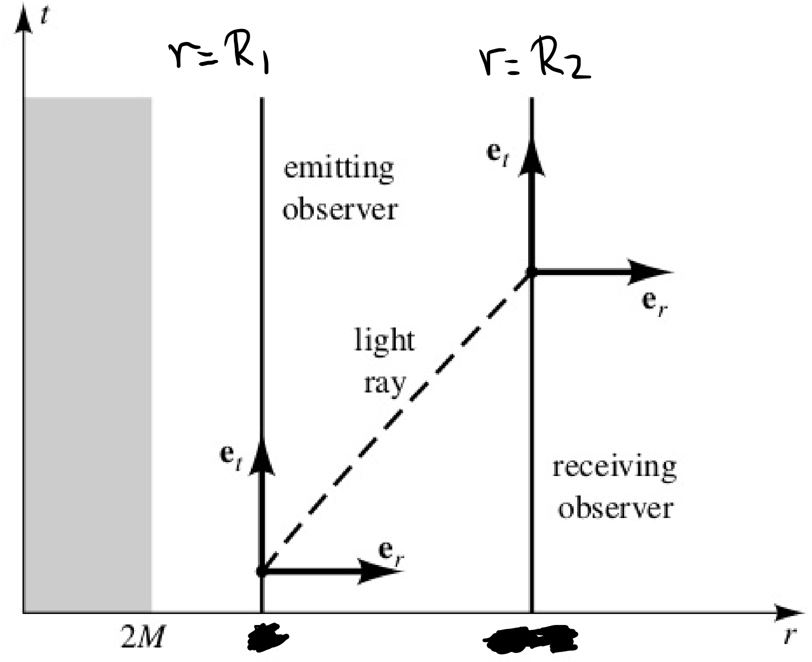
$$u_1^0 = \left(1 - \frac{2M}{r}\right)^{-1/2}$$

Gravitational Redshift

- $\xi = \partial_t$ a Killing Vector field

  $p_\mu \xi^\mu$ conserved along geodesic w/ tangent vector p^μ

  $\xi^\mu = (1, 0, 0, 0)$



Hartle, Fig 9.1

- photon emitted by observer at $r = R_1$

  " received " " " $r = R_2$

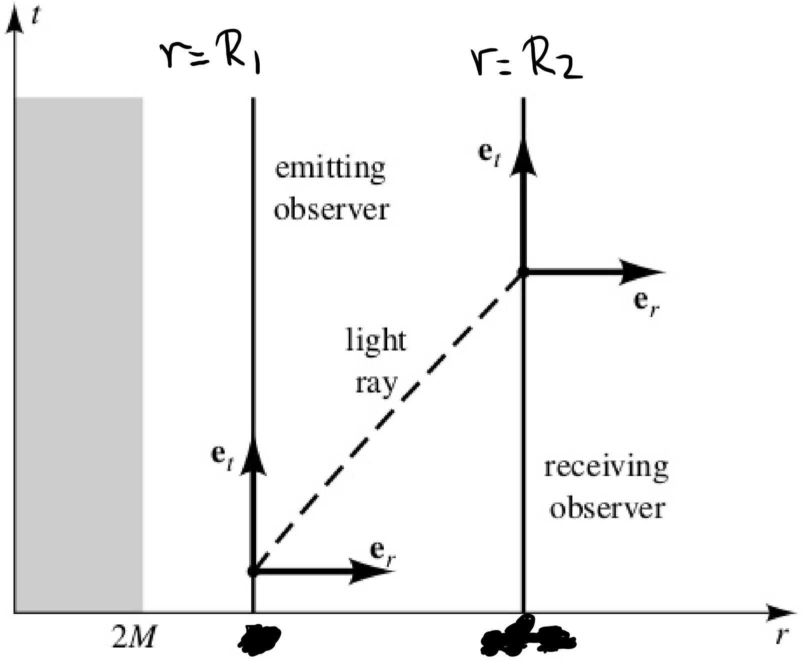
- 4 velocity of observer at R_1: $u_1^\mu = (u_1^0, 0, 0, 0) = \left(\left(1 - \frac{2M}{R_1}\right)^{-1/2}, 0, 0, 0\right)$

Gravitational Redshift

- $\xi = \partial_t$ a Killing Vector field

  $p_\mu \xi^\mu$ conserved along geodesic w/ tangent vector p^μ

  $\xi^\mu = (1, 0, 0, 0)$



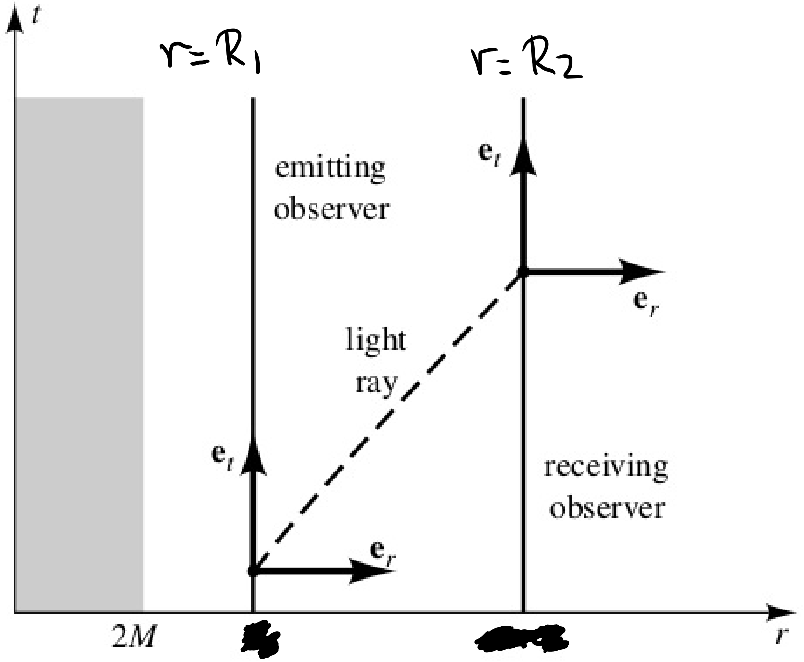
Hartle, Fig 9.1

- photon emitted by observer at $r = R_1$

  " received " " " $r = R_2$

- 4 velocity of observer at R_1: $u_1^\mu = (u_1^0, 0, 0, 0) = \left(\left(1 - \frac{2M}{R_1}\right)^{-1/2}, 0, 0, 0 \right)$
- " " R_2: $u_2^\mu = (u_2^0, 0, 0, 0) = \left(\left(1 - \frac{2M}{R_2}\right)^{-1/2}, 0, 0, 0 \right)$

$$\Rightarrow \quad u_1^{\mu} = \left(1 - \frac{2M}{R_1}\right)^{-1/2} \xi^{\mu} \qquad u_2^{\mu} = \left(1 - \frac{2M}{R_2}\right)^{-1/2} \xi^{\mu}$$



Hartle , Fig 9.1

$$\xi^{\mu} = (1, 0, 0, 0)$$

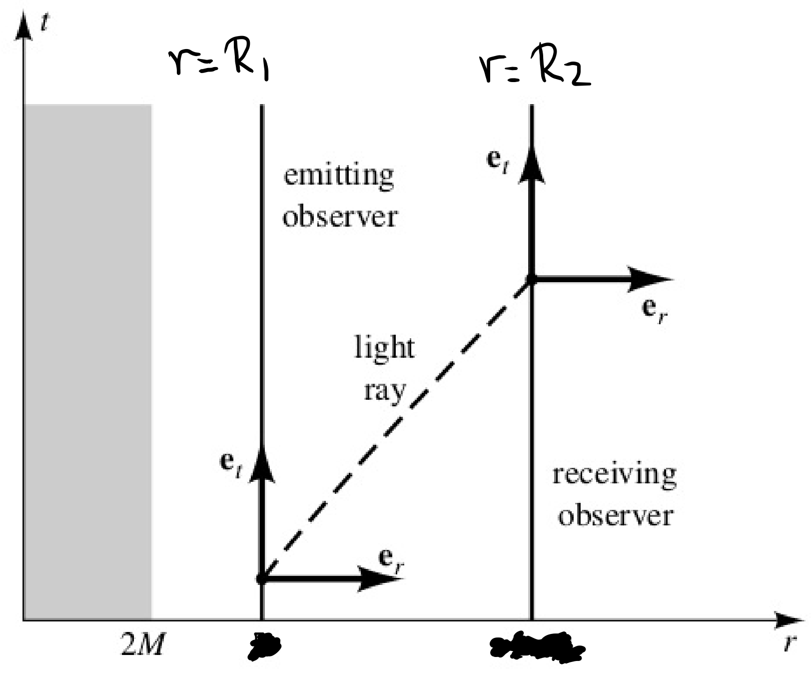
- photon emitted by observer at $r = R_1$
- " received " " " $r = R_2$

- 4 velocity of observer at R_1: $u_1^{\mu} = (u_1^0, 0, 0, 0) = \left(\left(1 - \frac{2M}{R_1}\right)^{-1/2}, 0, 0, 0\right)$
- " " R_2: $u_2^{\mu} = (u_2^0, 0, 0, 0) = \left(\left(1 - \frac{2M}{R_2}\right)^{-1/2}, 0, 0, 0\right)$

$$\Rightarrow \quad u_1^{\mu} = \left(1-\frac{2M}{R_1}\right)^{-1/2}\xi^{\mu} \qquad u_2^{\mu} = \left(1-\frac{2M}{R_2}\right)^{-1/2}\xi^{\mu}$$

Energies of photons, as measured by observers:

$$E_1 = -P_1^{\mu}\, u_{1\,\mu} = -\,P_1^{\mu}\,\xi_{\mu}\left(1-\frac{2M}{R_1}\right)^{-1/2}$$



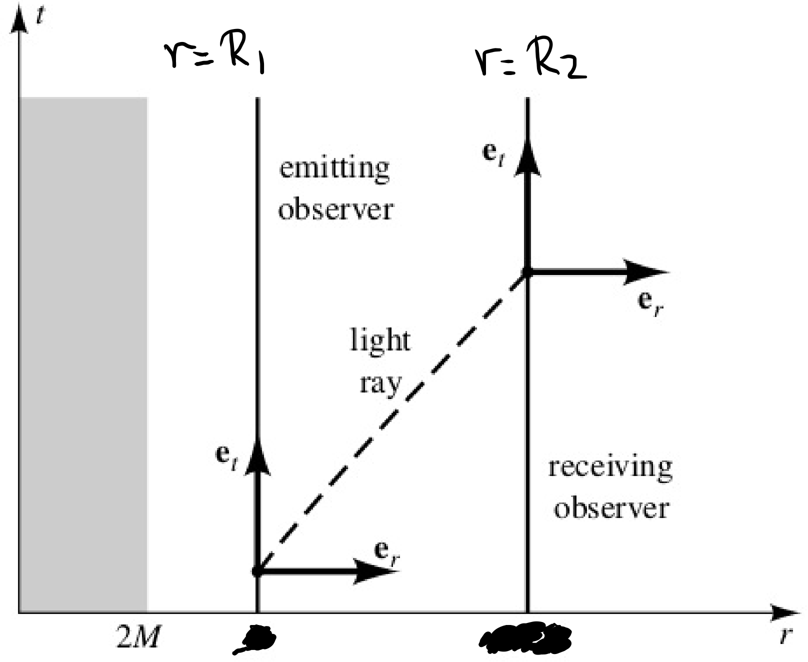
Hartle, Fig 9.1

$$\Rightarrow \quad u_1^{\mu} = \left(1 - \frac{2M}{R_1}\right)^{-1/2} \xi^{\mu} \qquad u_2^{\mu} = \left(1 - \frac{2M}{R_2}\right)^{-1/2} \xi^{\mu}$$

Energies of photons, as measured by observers:

$$E_1 = -P_1^{\mu}\, u_{1\,\mu} = -\, P_1^{\mu}\, \xi_{\mu} \left(1 - \frac{2M}{R_1}\right)^{-1/2}$$

$$E_2 = -P_2^{\mu}\, u_{2\,\mu} = -\, P_2^{\mu}\, \xi_{\mu} \left(1 - \frac{2M}{R_2}\right)^{-1/2}$$



Hartle, Fig 9.1

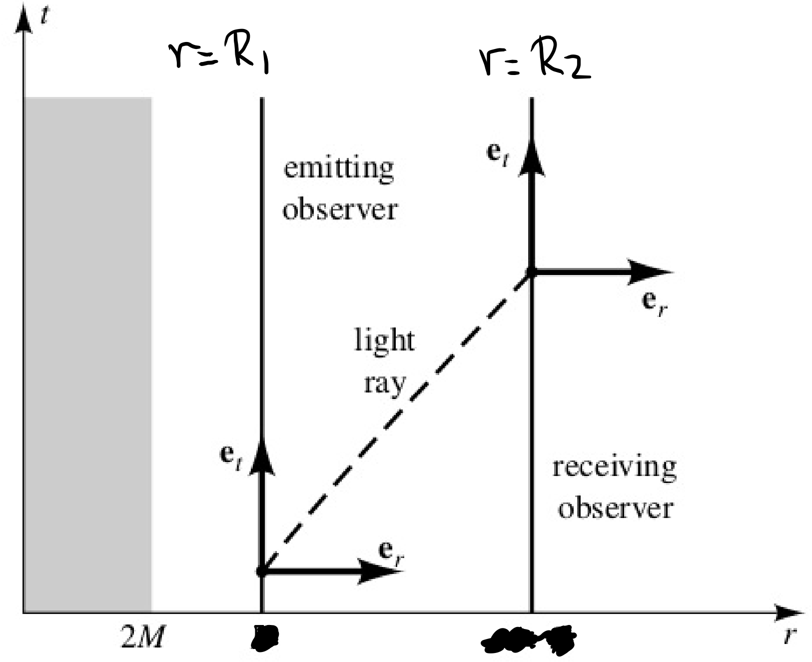
$$\Rightarrow \quad u_1^{\mu} = \left(1 - \frac{2M}{R_1}\right)^{-1/2} \xi^{\mu} \qquad u_2^{\mu} = \left(1 - \frac{2M}{R_2}\right)^{-1/2} \xi^{\mu}$$

Energies of photons, as measured by observers:

$$E_1 = -P_1^{\mu} u_{1\mu} = -P_1^{\mu} \xi_{\mu} \left(1 - \frac{2M}{R_1}\right)^{-1/2}$$

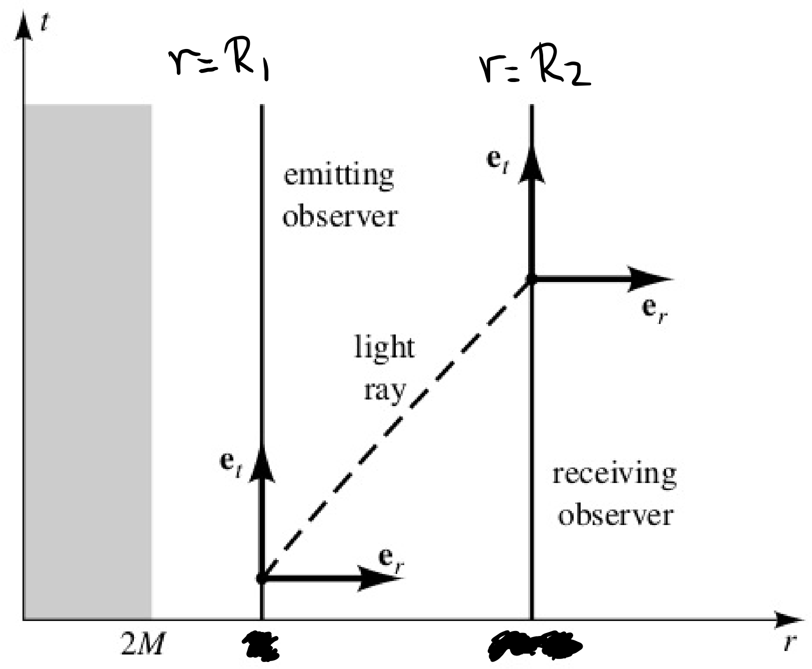
$$E_2 = -P_2^{\mu} u_{2\mu} = -P_2^{\mu} \xi_{\mu} \left(1 - \frac{2M}{R_2}\right)^{-1/2}$$

$$\Rightarrow \frac{E_1}{E_2} = \frac{-P_1^{\mu} \xi_{\mu} \left(1 - \frac{2M}{R_1}\right)^{-1/2}}{-P_2^{\mu} \xi_{\mu} \left(1 - \frac{2M}{R_1}\right)^{-1/2}}$$



Hartle, Fig 9.1

$$\Rightarrow \quad u_1^{\mu} = \left(1 - \frac{2M}{R_1}\right)^{-1/2} \xi^{\mu} \qquad u_2^{\mu} = \left(1 - \frac{2M}{R_2}\right)^{-1/2} \xi^{\mu}$$



Hartle, Fig 9.1

Energies of photons, as measured by observers:

$$E_1 = -P_1^{\mu} u_{1\,\mu} = -P_1^{\mu} \xi_{\mu} \left(1 - \frac{2M}{R_1}\right)^{-1/2}$$

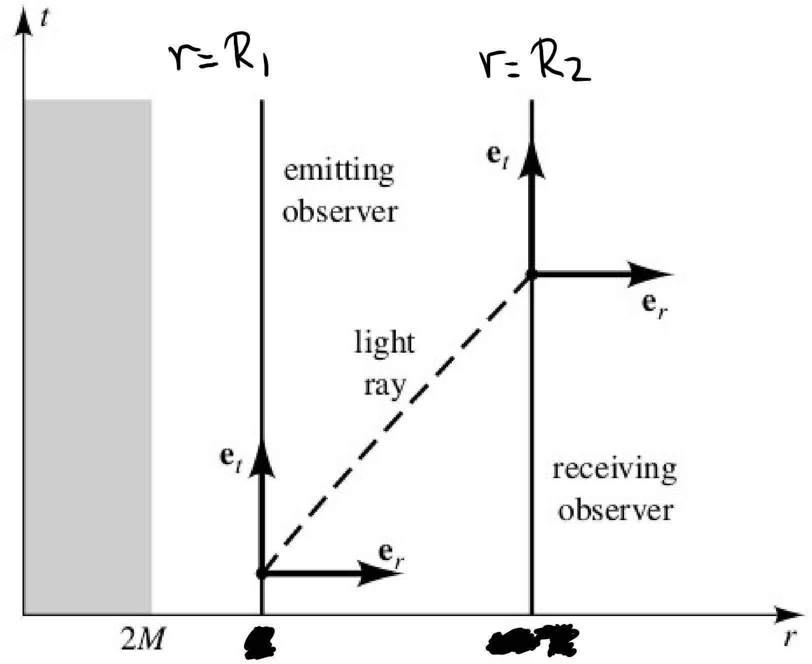
$$E_2 = -P_2^{\mu} u_{2\,\mu} = -P_2^{\mu} \xi_{\mu} \left(1 - \frac{2M}{R_2}\right)^{-1/2}$$

$$\Rightarrow \quad \frac{E_1}{E_2} = \frac{-\cancel{P_1^{\mu}} \xi_{\mu} \left(1 - \frac{2M}{R_1}\right)^{-1/2}}{-\cancel{P_2^{\mu}} \xi_{\mu} \left(1 - \frac{2M}{R_1}\right)^{-1/2}} \quad \Rightarrow$$

conserved!

$$E_2 = \frac{\left(1 - \frac{2M}{R_1}\right)^{1/2}}{\left(1 - \frac{2M}{R_2}\right)^{1/2}} E_1$$

$$\Rightarrow \quad u_1^{\mu} = \left(1-\frac{2M}{R_1}\right)^{-1/2}\xi^{\mu} \qquad u_2^{\mu} = \left(1-\frac{2M}{R_2}\right)^{-1/2}\xi^{\mu}$$



Hartle, Fig 9.1

Energies of photons, as measured by observers:

$$E_1 = -p_1^{\mu}\, u_{1\,\mu} = -\,p_1^{\mu}\,\xi_{\mu}\left(1-\frac{2M}{R_1}\right)^{-1/2}$$

$$E_2 = -p_2^{\mu}\, u_{2\,\mu} = -\,p_2^{\mu}\,\xi_{\mu}\left(1-\frac{2M}{R_2}\right)^{-1/2}$$

conserved!

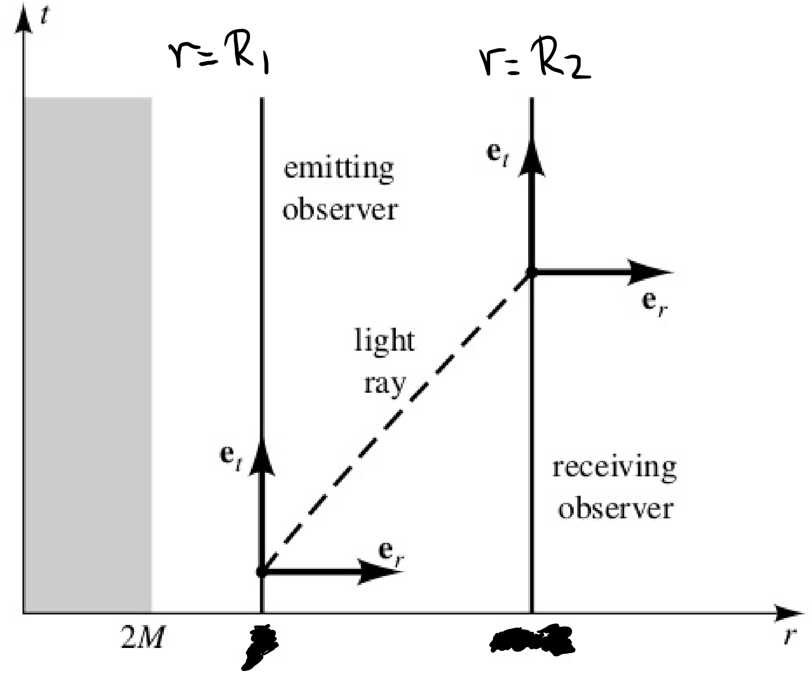
$$\Rightarrow \quad \frac{E_1}{E_2} = \frac{-\cancel{p_1^{\mu}\,\xi_{\mu}}\left(1-\frac{2M}{R_1}\right)^{-1/2}}{-\cancel{p_2^{\mu}\,\xi_{\mu}}\left(1-\frac{2M}{R_1}\right)^{-1/2}} \quad \Rightarrow$$

$$E_2 = \frac{\left(1-\frac{2M}{R_1}\right)^{1/2}}{\left(1-\frac{2M}{R_2}\right)^{1/2}}\, E_1$$

If $R_2 = \infty$

$$E_{\infty} = \left(1-\frac{2M}{R}\right)^{1/2} E(R)$$

$$\Rightarrow \quad u_1^{\mu} = \left(1-\frac{2M}{R_1}\right)^{-1/2}\xi^{\mu} \qquad u_2^{\mu} = \left(1-\frac{2M}{R_2}\right)^{-1/2}\xi^{\mu}$$



Hartle, Fig 9.1

Energies of photons, as measured by observers:

$$E_1 = -p_1^{\mu} u_{1\mu} = -p_1^{\mu}\xi_{\mu}\left(1-\frac{2M}{R_1}\right)^{-1/2}$$

$$E_2 = -p_2^{\mu} u_{2\mu} = -p_2^{\mu}\xi_{\mu}\left(1-\frac{2M}{R_2}\right)^{-1/2}$$

conserved!

$$\Rightarrow \quad \frac{E_1}{E_2} = \frac{-\cancel{p_1^{\mu}}\xi_{\mu}\left(1-\frac{2M}{R_1}\right)^{-1/2}}{-\cancel{p_2^{\mu}}\xi_{\mu}\left(1-\frac{2M}{R_1}\right)^{-1/2}} \Rightarrow$$

$$E_2 = \frac{\left(1-\frac{2M}{R_1}\right)^{1/2}}{\left(1-\frac{2M}{R_2}\right)^{1/2}} E_1$$

If $R_2 = \infty$

$E = \hbar\omega$

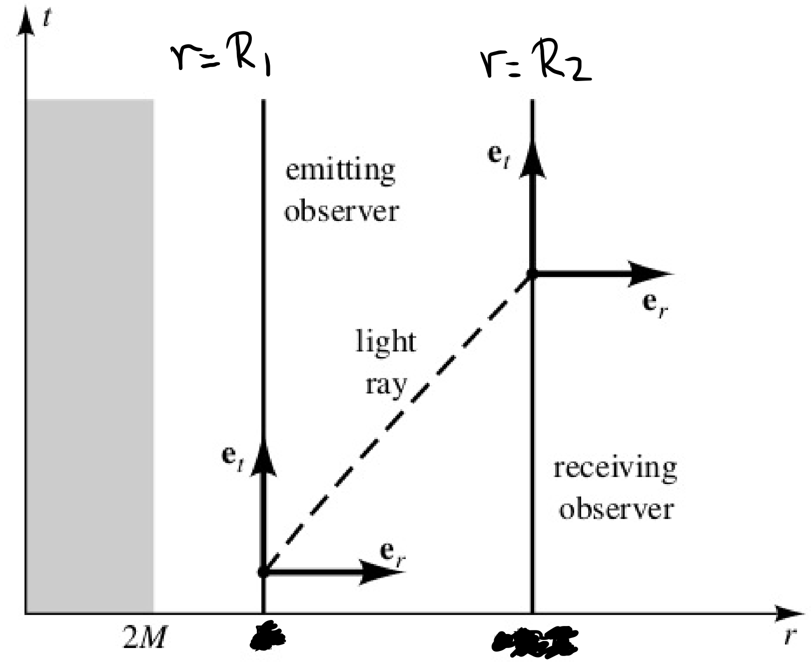
$$\omega_{\infty} = \left(1-\frac{2M}{R}\right)^{1/2}\omega(R)$$

For $R, R_1, R_2 \gg 2M$

$$\omega_\infty \approx \left(1 - \frac{M}{R}\right)\omega(R) = \left(1 + \Phi(R)\right)\omega(R)$$

$$\omega_2 \approx \left(1 - \frac{M}{R_1}\right)\left(1 + \frac{M}{R_2}\right)\omega_1$$

$$(1-x)^{-\frac{1}{2}} \approx 1 - \left(-\tfrac{1}{2}\right)x = 1 + \tfrac{1}{2}x$$



Hartle, Fig 9.1

$$\omega_2 = \frac{\left(1 - \frac{2M}{R_1}\right)^{1/2}}{\left(1 - \frac{2M}{R_2}\right)^{1/2}}\,\omega_1$$

If $R_2 = \infty$ $\qquad E = \hbar\omega$

$$\omega_\infty = \left(1 - \frac{2M}{R}\right)^{1/2}\omega(R)$$

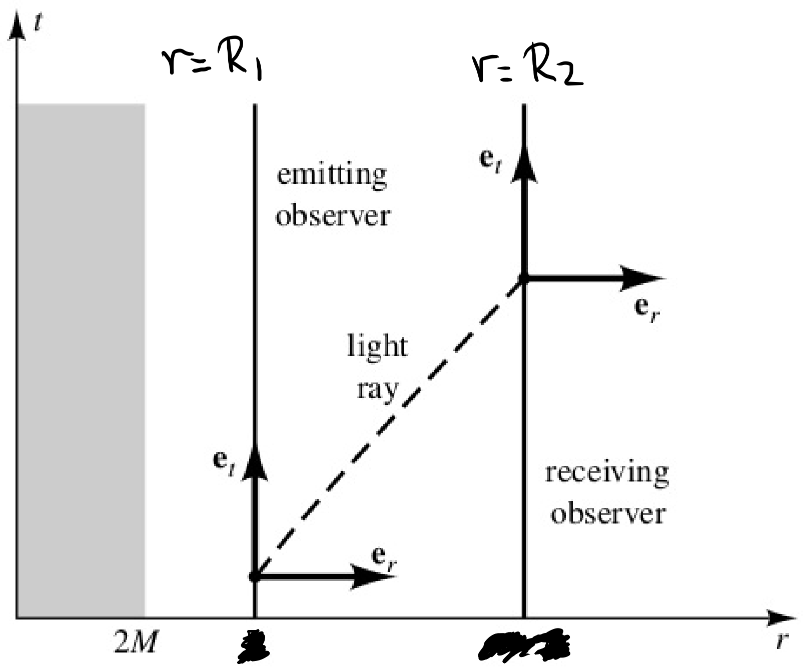
For $R, R_1, R_2 \gg 2M$

$$\omega_\infty \approx \left(1 - \frac{M}{R}\right)\omega(R) = \left(1 + \Phi(R)\right)\omega(R)$$

$$\omega_2 \approx \left(1 - \frac{M}{R_1}\right)\left(1 + \frac{M}{R_2}\right)\omega_1$$

$$\approx \left(1 - \frac{M}{R_1} + \frac{M}{R_2}\right)\omega_1$$

$$= \left(1 + \Phi(R_1) - \Phi(R_2)\right)\omega_1$$



Hartle, Fig 9.1

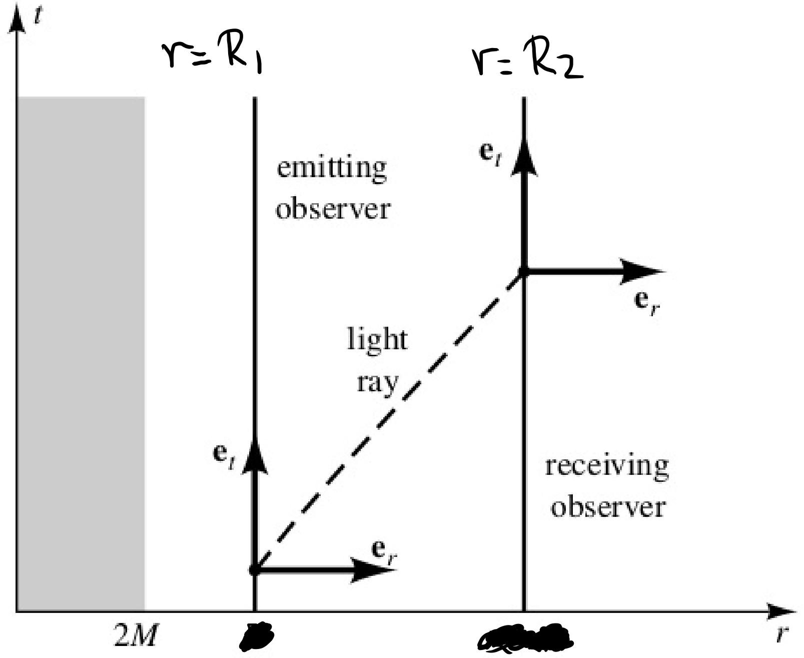
$$\omega_2 = \frac{\left(1 - \frac{2M}{R_1}\right)^{1/2}}{\left(1 - \frac{2M}{R_2}\right)^{1/2}}\omega_1$$

If $R_2 = \infty$

$E = \hbar\omega$

$$\omega_\infty = \left(1 - \frac{2M}{R}\right)^{1/2}\omega(R)$$

$$\left.\begin{array}{l} \text{If } R = 2M \\ \omega(R) < \infty \end{array}\right\} \Rightarrow \omega_\infty = 0$$



Hartle, Fig 9.1

$$\omega_2 = \frac{\left(1 - \frac{2M}{R_1}\right)^{1/2}}{\left(1 - \frac{2M}{R_2}\right)^{1/2}} \, \omega_1$$

If $R_2 = \infty$

$E = \hbar\omega$

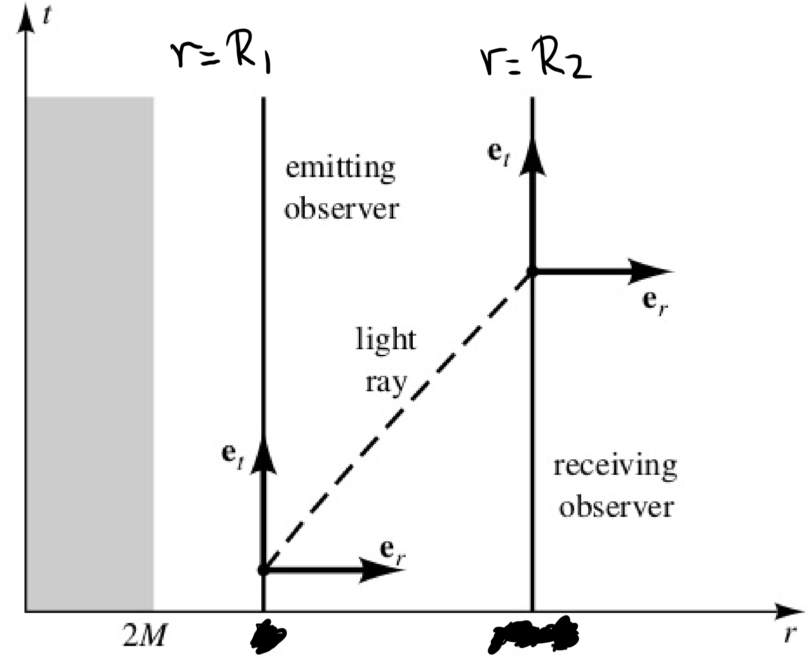
$$\omega_\infty = \left(1 - \frac{2M}{R}\right)^{1/2} \omega(R)$$

$$\left.\begin{array}{l} \text{If } R = 2M \\ \omega(R) < \infty \end{array}\right\} \Rightarrow \omega_\infty = 0$$

$\Rightarrow$ • infinite redshift $\dfrac{\omega - \omega_\infty}{\omega_\infty}$

• cannot "see" objects as they approach $R \to 2M$

their light signals are redshifted to zero



Hartle, Fig 9.1

$$\omega_2 = \frac{\left(1 - \frac{2M}{R_1}\right)^{1/2}}{\left(1 - \frac{2M}{R_2}\right)^{1/2}}\, \omega_1$$

If $R_2 = \infty$ $\quad E = \hbar\omega$

$$\omega_\infty = \left(1 - \frac{2M}{R}\right)^{1/2} \omega(R)$$

Free massive particle trajectories

- Particles move on timelike geodesics
- Will use conserved quantities: will not need geodesic equations!
- $\xi = \partial_t$ timelike Killing vector field: $e = -\xi^\mu u_\mu$ conserved

$$\xi^\mu = (1,0,0,0) \qquad u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau}\right)$$

4-velocity of freely falling particle

u^μ: tangent to its timelike geodesic world line

Free massive particle trajectories

- Particles move on timelike geodesics
- Will use conserved quantities: will not need geodesic equations!
- $\xi = \partial_t$ timelike Killing vector field: $e = -\xi^\mu u_\mu$ conserved

$$\xi^\mu = (1, 0, 0, 0) \qquad u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau} \right)$$

$$(g_{\mu\nu}) = \mathrm{diag}\left(-\left(1 - \frac{2M}{r}\right), \left(1 - \frac{2M}{r}\right)^{-1}, r^2, r^2 \sin^2\theta \right)$$

Free massive particle trajectories

- Particles move on timelike geodesics
- Will use conserved quantities: will not need geodesic equations!
- $\xi = \partial_t$ timelike Killing vector field: $e = -\xi^\mu u_\mu$ conserved

$$\xi^\mu = (1, 0, 0, 0) \qquad u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau} \right)$$

$$(g_{\mu\nu}) = \mathrm{diag}\left(-\left(1 - \frac{2M}{r}\right), \left(1 - \frac{2M}{r}\right)^{-1}, r^2, r^2 \sin^2\theta \right)$$

$$-\xi^\mu u_\mu = -g_{\mu\nu} \xi^\mu u^\nu = -g_{00} \xi^0 u^0$$

Free massive particle trajectories

- Particles move on timelike geodesics
- Will use conserved quantities: will not need geodesic equations!
- $\xi = \partial_t$ timelike Killing vector field: $e = -\xi^\mu u_\mu$ conserved

$$\xi^\mu = (1,0,0,0) \qquad u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau} \right)$$

$$(g_{\mu\nu}) = \mathrm{diag}\left(-\left(1 - \frac{2M}{r}\right), \left(1 - \frac{2M}{r}\right)^{-1}, r^2, r^2 \sin^2\theta \right)$$

$$-\xi^\mu u_\mu = -g_{\mu\nu}\xi^\mu u^\nu = -g_{00}\xi^0 u^0 = +\left(1 - \frac{2M}{r}\right) \cdot 1 \cdot \frac{dt}{d\tau}$$

Free massive particle trajectories

- Particles move on timelike geodesics
- Will use conserved quantities: will not need geodesic equations!
- $\xi = \partial_t$ timelike Killing vector field: $e = -\xi^\mu u_\mu = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$

Free massive particle trajectories

- Particles move on timelike geodesics
- Will use conserved quantities: will not need geodesic equations!
- $\xi = \partial_t$ timelike Killing vector field: $e = -\xi^\mu u_\mu = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$
- $\eta = \partial_\varphi$ spacelike " " ": $l = \eta^\mu u_\mu$ conserved

Free massive particle trajectories

- Particles move on timelike geodesics
- Will use conserved quantities: will not need geodesic equations!
- $\xi = \partial_t$ timelike Killing vector field: $e = -\xi^\mu u_\mu = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$
- $\eta = \partial_\varphi$ spacelike " " ": $l = \eta^\mu u_\mu$ conserved

$$\eta^\mu = (0, 0, 0, 1) \qquad u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau}\right)$$

$$g_{\mu\nu} = \mathrm{diag}\left(-\left(1 - \frac{2M}{r}\right), \left(1 - \frac{2M}{r}\right)^{-1}, r^2, r^2 \sin^2\theta\right)$$

Free massive particle trajectories

- Particles move on timelike geodesics
- Will use conserved quantities: will not need geodesic equations!
- $\xi = \partial_t$ timelike killing vector field: $e = -\xi^\mu u_\mu = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$
- $\eta = \partial_\varphi$ spacelike " " ": $l = \eta^\mu u_\mu$ conserved

$$\eta^\mu = (0, 0, 0, 1) \qquad u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau}\right)$$

$$g_{\mu\nu} = \mathrm{diag}\left(-\left(1 - \frac{2M}{r}\right), \left(1 - \frac{2M}{r}\right)^{-1}, r^2, r^2 \sin^2\theta\right)$$

$$\eta^\mu u_\mu = g_{\mu\nu}\, \eta^\mu u^\nu = g_{\varphi\varphi}\, \eta^\varphi u^\varphi$$

Free massive particle trajectories

- Particles move on timelike geodesics
- Will use conserved quantities: will not need geodesic equations!
- $\xi = \partial_t$ timelike Killing vector field: $e = -\xi^\mu u_\mu = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$
- $\eta = \partial_\varphi$ spacelike " " ": $l = \eta^\mu u_\mu$ conserved

$$\eta^\mu = (0, 0, 0, 1) \qquad u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau}\right)$$

$$g_{\mu\nu} = \mathrm{diag}\left(-\left(1 - \frac{2M}{r}\right), \left(1 - \frac{2M}{r}\right)^{-1}, r^2, r^2 \sin^2\theta\right)$$

$$\eta^\mu u_\mu = g_{\mu\nu}\eta^\mu u^\nu = g_{\varphi\varphi}\eta^\varphi u^\varphi = r^2 \sin^2\theta \cdot 1 \cdot \frac{d\varphi}{d\tau}$$

Free massive particle trajectories

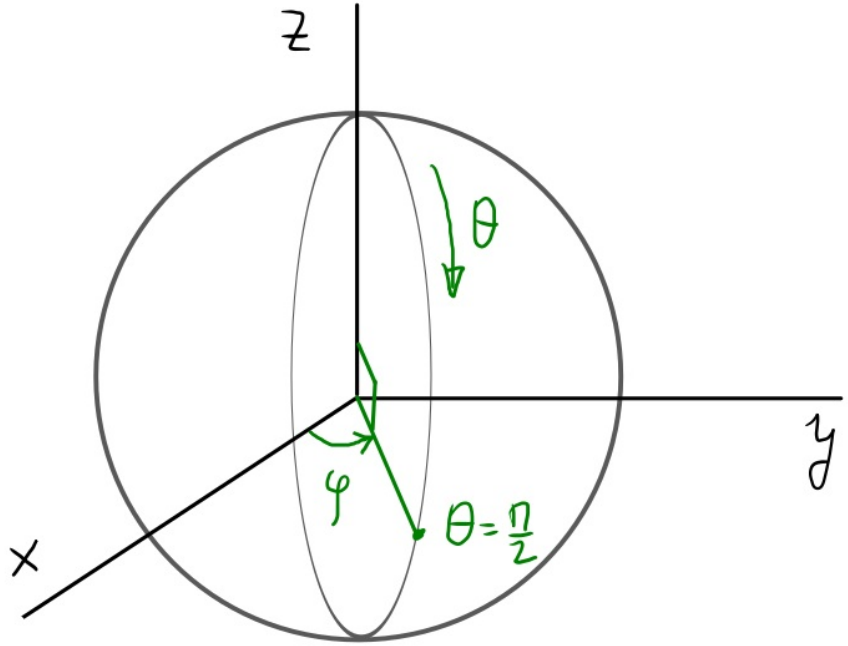
- Particles move on timelike geodesics
- Will use conserved quantities: will not need geodesic equations!
- $\xi = \partial_t$ timelike Killing vector field: $e = -\xi^\mu u_\mu = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$
- $\eta = \partial_\varphi$ spacelike " " " : $l = \eta^\mu u_\mu = r^2 \sin^2\theta \frac{d\varphi}{d\tau}$

Free massive particle trajectories

- Particles move on timelike geodesics
- Will use conserved quantities: will not need geodesic equations!
- $\xi = \partial_t$ timelike Killing vector field: $e = -\xi^\mu u_\mu = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$
- $\eta = \partial_\varphi$ spacelike " " ": $\ell = \eta^\mu u_\mu = r^2 \sin^2\theta \, \frac{d\varphi}{d\tau}$

ℓ conserved $\Rightarrow$ motion on a plane!

$l = r^2 \sin^2\theta \dfrac{d\varphi}{d\tau}$ conserved $\Rightarrow$ motion on a plane

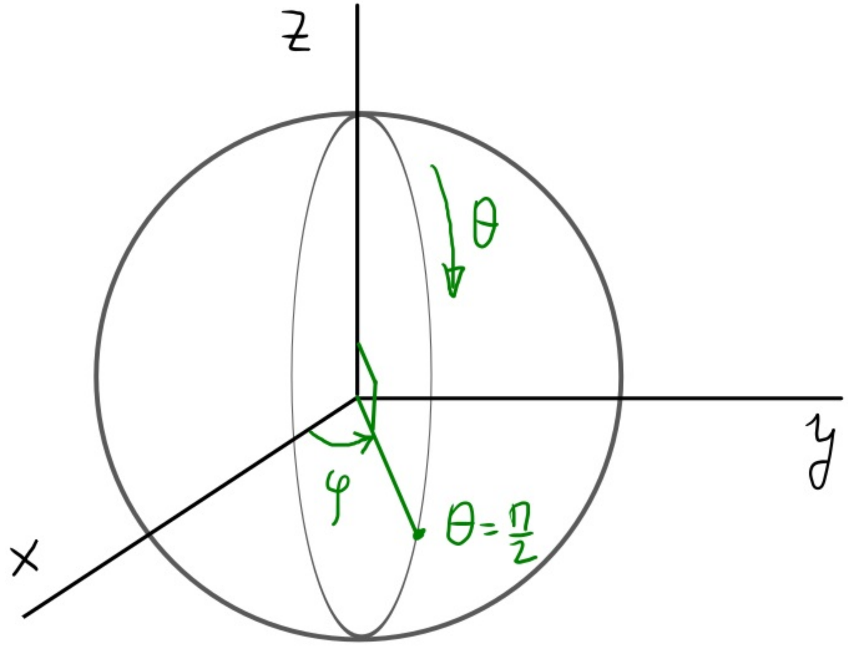


- consider $u^\mu = (u^0, \vec{u})$

- orient coordinate system so that $u^\phi = \dfrac{d\phi}{d\tau} = 0$ at some instant of time τ_0

$$\Rightarrow l = 0 \qquad \text{at } \tau_0$$

$\ell = r^2 \sin^2\theta \dfrac{d\varphi}{d\tau}$ conserved $\Rightarrow$ motion on a plane

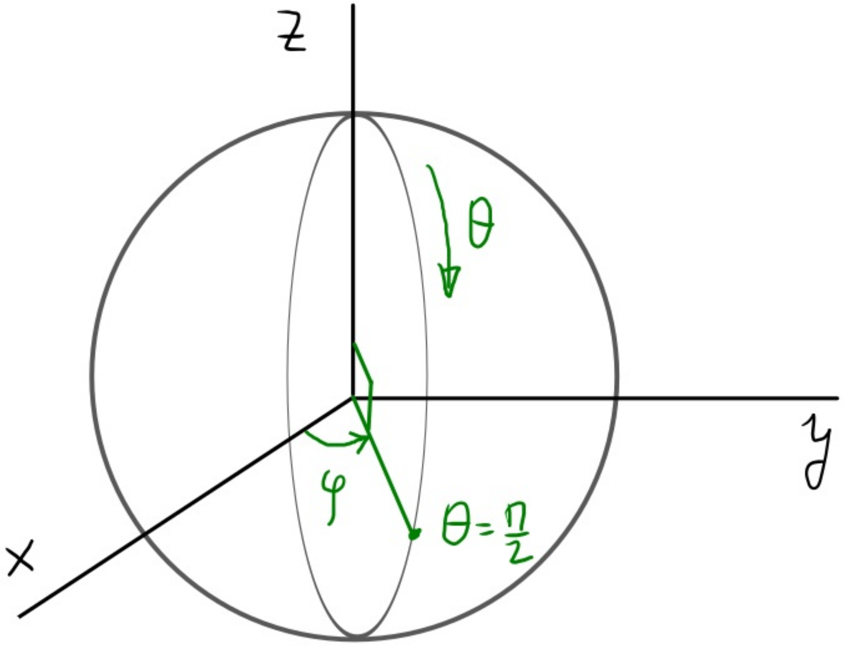


- consider $u^\mu = (u^0, \vec{u})$
- orient coordinate system so that $u^\phi = \dfrac{d\phi}{d\tau} = 0$ at some instant of time τ_0

$\Rightarrow \ell = 0$ at τ_0

$\Rightarrow \ell = 0 \quad \forall \ \tau$

$\ell = r^2 \sin^2\theta \frac{d\varphi}{d\tau}$ conserved $\Rightarrow$ motion on a plane



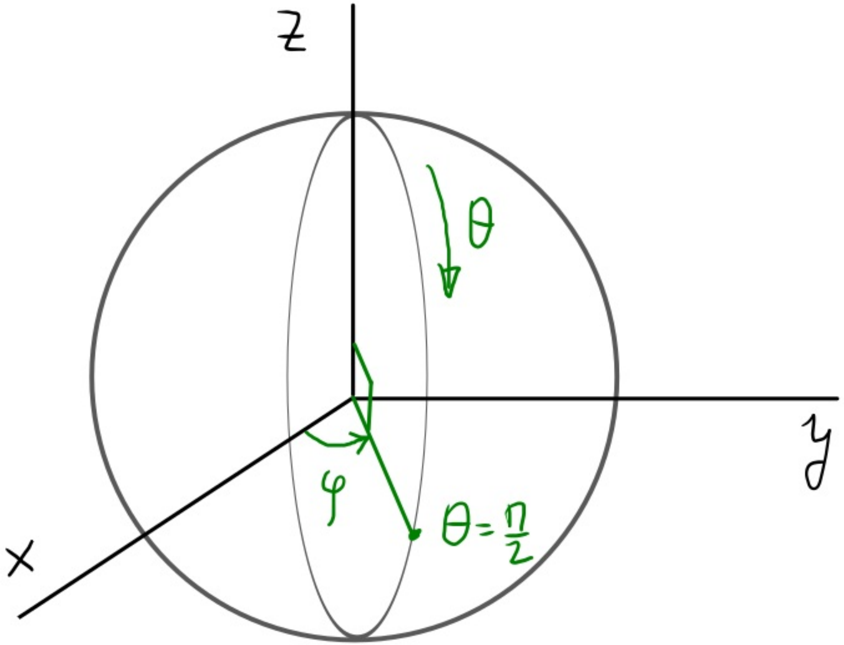
- consider $u^\mu = (u^0, \vec{u})$
- orient coordinate system so that $u^\phi = \frac{d\phi}{d\tau} = 0$ at some instant of time τ_0

$\Rightarrow \ell = 0$ at τ_0

$\Rightarrow \ell = 0 \quad \forall \tau$

$\Rightarrow \frac{d\phi}{d\tau} = 0 \Rightarrow \phi =$ const $\Rightarrow$ stays on $\phi =$ const plane

$\ell = r^2 \sin^2\theta \, \frac{d\varphi}{d\tau}$ conserved $\Rightarrow$ motion on a plane



- consider $u^\mu = (u^0, \vec{u})$
- orient coordinate system so that $u^\phi = \frac{d\phi}{d\tau} = 0$ at some instant of time τ_0

$\Rightarrow \ell = 0$ at τ_0

$\Rightarrow \ell = 0 \quad \forall \tau$

$\Rightarrow \frac{d\phi}{d\tau} = 0 \Rightarrow \phi = \text{const} \Rightarrow$ stays on $\phi = \text{const}$ plane

- now reorient coordinates, so that the plane of motion is $\theta = \frac{\pi}{2}$

$\Rightarrow u^\theta = \frac{d\theta}{d\tau} = 0$

Free massive particle trajectories

- Particles move on timelike geodesics
- Will use conserved quantities: will not need geodesic equations!
- $\xi = \partial_t$ timelike Killing vector field: $e = -\xi^\mu u_\mu = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$
- $\eta = \partial_\varphi$ spacelike " " ": $l = \eta^\mu u_\mu = r^2 \sin^2\theta \frac{d\varphi}{d\tau}$

l conserved $\Rightarrow$ motion on $\theta = \frac{\pi}{2}$ plane $\Rightarrow \begin{cases} \sin\theta = 1 \\ u^\theta = \frac{d\theta}{d\tau} = 0 \end{cases}$

Free massive particle trajectories

- Particles move on timelike geodesics
- Will use conserved quantities: will not need geodesic equations!
- $\xi = \partial_t$ timelike Killing vector field: $e = -\xi^\mu u_\mu = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$
- $\eta = \partial_\varphi$ spacelike " " " : $\ell = \eta^\mu u_\mu = r^2 \underbrace{\cancel{\sin^2\theta}}_{=1} \frac{d\varphi}{d\tau}$

ℓ conserved $\Rightarrow$ motion on $\theta = \frac{\pi}{2}$ plane $\Rightarrow \begin{cases} \sin\theta = 1 \\ u^\theta = \frac{d\theta}{d\tau} = 0 \end{cases}$

$\Rightarrow u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, \frac{d\phi}{d\tau}\right)$ $\quad (g_{\mu\nu}) = \mathrm{diag}\left(-\left(1-\frac{2M}{r}\right), \left(1-\frac{2M}{r}\right)^{-1}, r^2, r^2\right)$

$\hookrightarrow \sin\theta = 1$

$$u^\mu u_\mu = -1 \Rightarrow g_{\mu\nu} u^\mu u^\nu = -1$$

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} = \text{const} \qquad l = r^2 \frac{d\varphi}{d\tau} = \text{const}$$

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, \frac{d\phi}{d\tau}\right) \qquad (g_{\mu\nu}) = \text{diag}\left(-\left(1-\frac{2M}{r}\right), \left(1-\frac{2M}{r}\right)^{-1}, r^2, r^2\right)$$

$$u^\mu u_\mu = -1 \Rightarrow g_{\mu\nu} u^\mu u^\nu = -1 \Rightarrow$$

$$-\left(1-\frac{2M}{r}\right)(u^0)^2 + \left(1-\frac{2M}{r}\right)^{-1}(u^1)^2 + r^2 (u^3)^2 = -1$$

$$e = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} = \text{const} \qquad l = r^2 \frac{d\varphi}{d\tau} = \text{const}$$

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, \frac{d\phi}{d\tau}\right) \qquad (g_{\mu\nu}) = \mathrm{diag}\left(-\left(1-\frac{2M}{r}\right), \left(1-\frac{2M}{r}\right)^{-1}, r^2, r^2\right)$$

$$u^\mu u_\mu = -1 \Rightarrow g_{\mu\nu} u^\mu u^\nu = -1 \Rightarrow$$

$$-\left(1-\frac{2M}{r}\right)(u^0)^2 + \left(1-\frac{2M}{r}\right)^{-1}(u^1)^2 + r^2 (u^3)^2 = -1 \Rightarrow$$

$$-\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\varphi}{d\tau}\right)^2 = -1$$

$$e = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} = \text{const} \qquad l = r^2 \frac{d\varphi}{d\tau} = \text{const}$$

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, \frac{d\phi}{d\tau}\right) \qquad (g_{\mu\nu}) = \mathrm{diag}\left(-\left(1-\frac{2M}{r}\right), \left(1-\frac{2M}{r}\right)^{-1}, r^2, r^2\right)$$

$$u^\mu u_\mu = -1 \Rightarrow g_{\mu\nu} u^\mu u^\nu = -1 \Rightarrow$$

$$-\left(1-\frac{2M}{r}\right)(u^0)^2 + \left(1-\frac{2M}{r}\right)^{-1}(u^1)^2 + r^2 (u^3)^2 = -1 \Rightarrow$$

$$-\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\varphi}{d\tau}\right)^2 = -1 \Rightarrow$$

$$-\left(1-\frac{2M}{r}\right)^{-1} e^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2 \frac{\ell^2}{r^4} = -1$$

$$e = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} \Rightarrow \frac{dt}{d\tau} = \left(1-\frac{2M}{r}\right)^{-1} e, \quad \ell = r^2 \frac{d\varphi}{d\tau} \Rightarrow \frac{d\varphi}{d\tau} = \frac{\ell}{r^2}$$

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, \frac{d\phi}{d\tau}\right) \quad (g_{\mu\nu}) = \mathrm{diag}\left(-\left(1-\frac{2M}{r}\right), \left(1-\frac{2M}{r}\right)^{-1}, r^2, r^2\right)$$

$$u^\mu u_\mu = -1 \Rightarrow g_{\mu\nu} u^\mu u^\nu = -1 \Rightarrow$$

$$-\left(1-\frac{2M}{r}\right)(u^0)^2 + \left(1-\frac{2M}{r}\right)^{-1}(u^1)^2 + r^2 (u^3)^2 = -1 \Rightarrow$$

$$-\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\varphi}{d\tau}\right)^2 = -1 \Rightarrow$$

$$-\left(1-\frac{2M}{r}\right)^{-1} e^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2 \frac{\ell^2}{r^4} = -1 \Rightarrow$$

$$- e^2 + \left(\frac{dr}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)\frac{\ell^2}{r^2} = -\left(1-\frac{2M}{r}\right)$$

$$e = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} \Rightarrow \frac{dt}{d\tau} = \left(1-\frac{2M}{r}\right)^{-1} e, \quad \ell = r^2 \frac{d\varphi}{d\tau} \Rightarrow \frac{d\varphi}{d\tau} = \frac{\ell}{r^2}$$

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, \frac{d\phi}{d\tau}\right) \quad (g_{\mu\nu}) = \mathrm{diag}\left(-\left(1-\frac{2M}{r}\right), \left(1-\frac{2M}{r}\right)^{-1}, r^2, r^2\right)$$

$$u^\mu u_\mu = -1 \Rightarrow g_{\mu\nu} u^\mu u^\nu = -1 \Rightarrow$$

$$-\left(1-\frac{2M}{r}\right)(u^0)^2 + \left(1-\frac{2M}{r}\right)^{-1}(u^1)^2 + r^2 (u^3)^2 = -1 \Rightarrow$$

$$-\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\varphi}{d\tau}\right)^2 = -1 \Rightarrow$$

$$-\left(1-\frac{2M}{r}\right)^{-1} e^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2 \frac{\ell^2}{r^4} = -1 \Rightarrow$$

$$- e^2 + \left(\frac{dr}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)\frac{\ell^2}{r^2} = -\left(1-\frac{2M}{r}\right) \Rightarrow$$

$$- e^2 + \left(\frac{dr}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right) = 0$$

$$U^\mu U_\mu = -1 \Rightarrow g_{\mu\nu} U^\mu U^\nu = -1 \Rightarrow$$

$$-\left(1-\frac{2M}{r}\right)(U^0)^2 + \left(1-\frac{2M}{r}\right)^{-1}(U^1)^2 + r^2 (U^3)^2 = -1 \Rightarrow$$

$$-\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\varphi}{d\tau}\right)^2 = -1 \Rightarrow$$

$$-\left(1-\frac{2M}{r}\right)^{-1} e^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2 \frac{\ell^2}{r^4} = -1 \Rightarrow$$

$$- e^2 + \left(\frac{dr}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)\frac{\ell^2}{r^2} = -\left(1-\frac{2M}{r}\right) \Rightarrow$$

$$-e^2 + \left(\frac{dr}{d\tau}\right)^2 + \left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right) = 0 \Rightarrow$$

$$\frac{e^2-1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left[\left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right)-1\right]$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r)$$

$$\underbrace{\frac{e^2-1}{2}}_{\mathcal{E}} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \underbrace{\frac{1}{2}\left[\left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right)-1\right]}_{V_{eff}(r)} = 0 \quad \Rightarrow$$

$$\underbrace{\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r)}$$

analyze radial motion as we do in Newtonian theory

$$\underset{\mathcal{E}}{\underset{\shortparallel}{\frac{e^2-1}{2}}} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \underbrace{\frac{1}{2}\left[\left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right)-1\right]}_{V_{eff}(r)} = 0 \quad \Rightarrow$$

$$\underbrace{\mathcal{E}}_{\text{"energy" (conserved)}} = \underbrace{\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2}_{\text{"kinetic energy"}} + \underbrace{V_{eff}(r)}_{\text{"effective potential energy"}}$$

$$\underbrace{\frac{e^2-1}{2}}_{\mathcal{E}} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \underbrace{\frac{1}{2}\left[\left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right)-1\right]}_{V_{eff}(r)} = 0 \Rightarrow$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) \qquad V_{eff}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}$$

attractive Newtonian potential energy

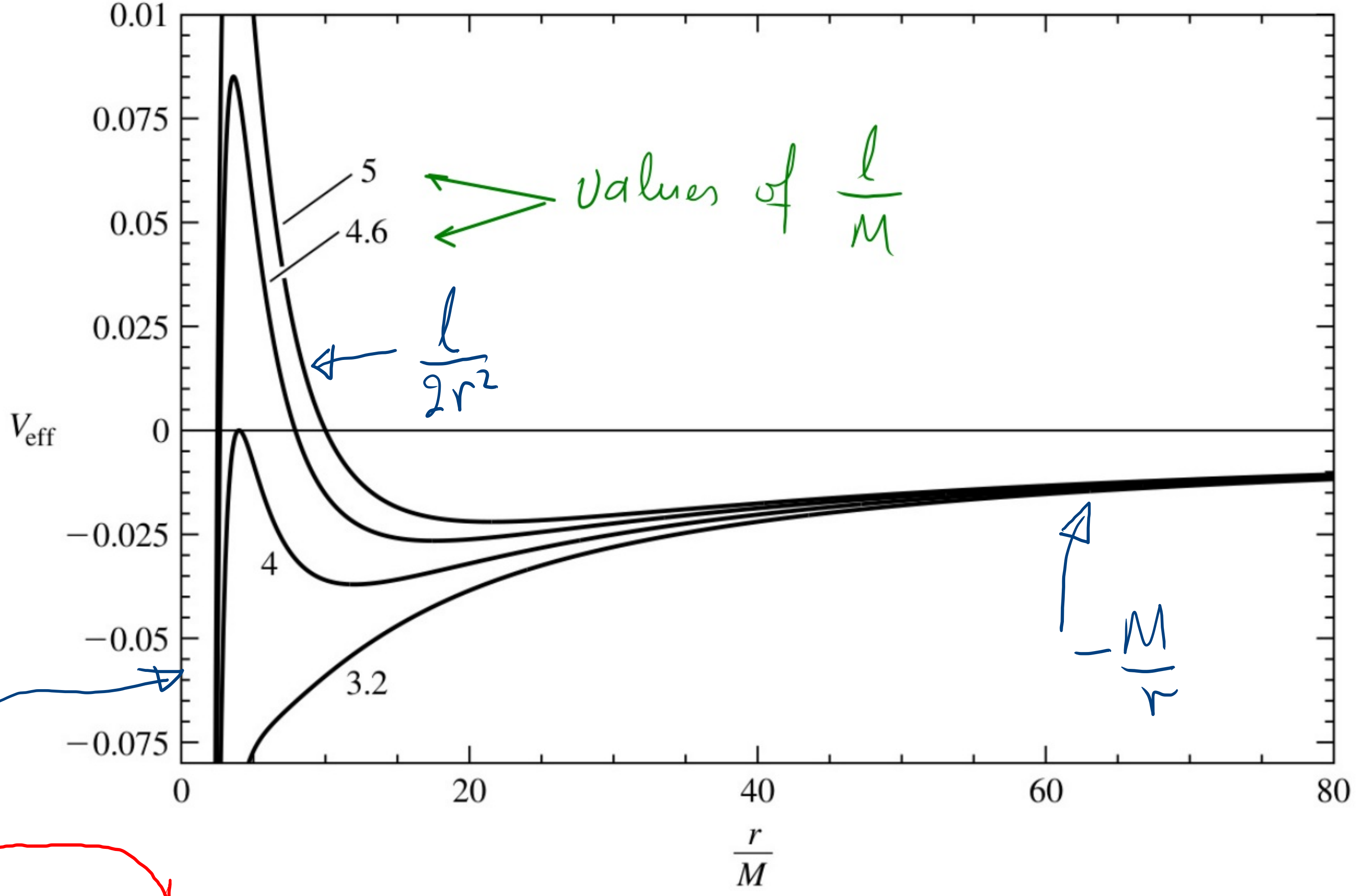
angular momentum repulsion

general relativity term (attractive, dominant for small r)

$$\underset{\mathcal{E}}{\underset{\parallel}{\frac{e^2-1}{2}}} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \underbrace{\frac{1}{2}\left[\left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right)-1\right]}_{V_{eff}(r)} = 0 \quad \Rightarrow$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r)$$

$$V_{eff}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}$$



values of $\frac{\ell}{M}$

$\frac{\ell}{2r^2}$

$-\frac{M}{r}$

$-\frac{M\ell^2}{r^3}$

$V(\infty) = 0$

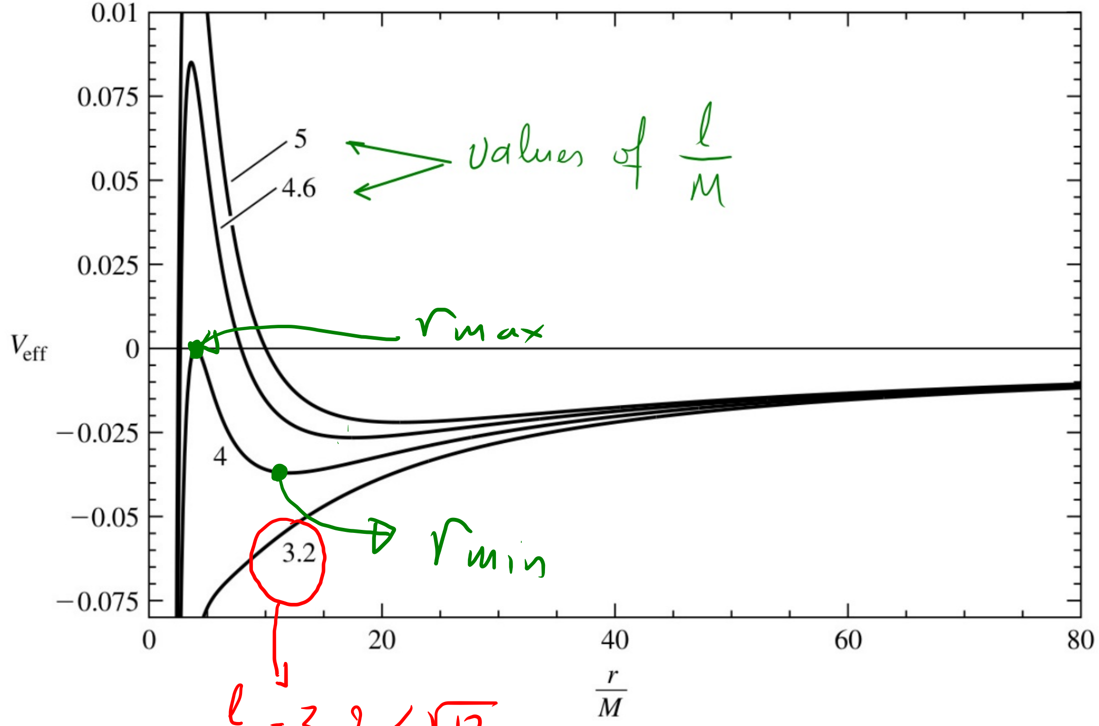
$V(2M) = -\frac{1}{2}$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) \qquad V_{eff}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}$$

$$\frac{dV_{eff}(r)}{dr} = 0 \Rightarrow$$

$$+\frac{M}{r^2} - \frac{\ell^2}{r^3} + \frac{3\ell^2 M}{r^4} = 0 \Rightarrow$$

$$r_{min,max} = \frac{\ell^2}{2M}\Big[1 \underbrace{\overbrace{\pm}^{r_{min}}}_{r_{max}} \sqrt{1 - 12\left(\frac{M}{\ell}\right)^2}\,\Big]$$



$$\frac{\ell}{M} = 3.2 < \sqrt{12}$$

$$\text{for } 1 - 12\left(\frac{M}{\ell}\right)^2 \geq 0 \Rightarrow \ell \geq \sqrt{12}\,M \Rightarrow \frac{\ell}{M} \geq \sqrt{12} \approx 3.464$$

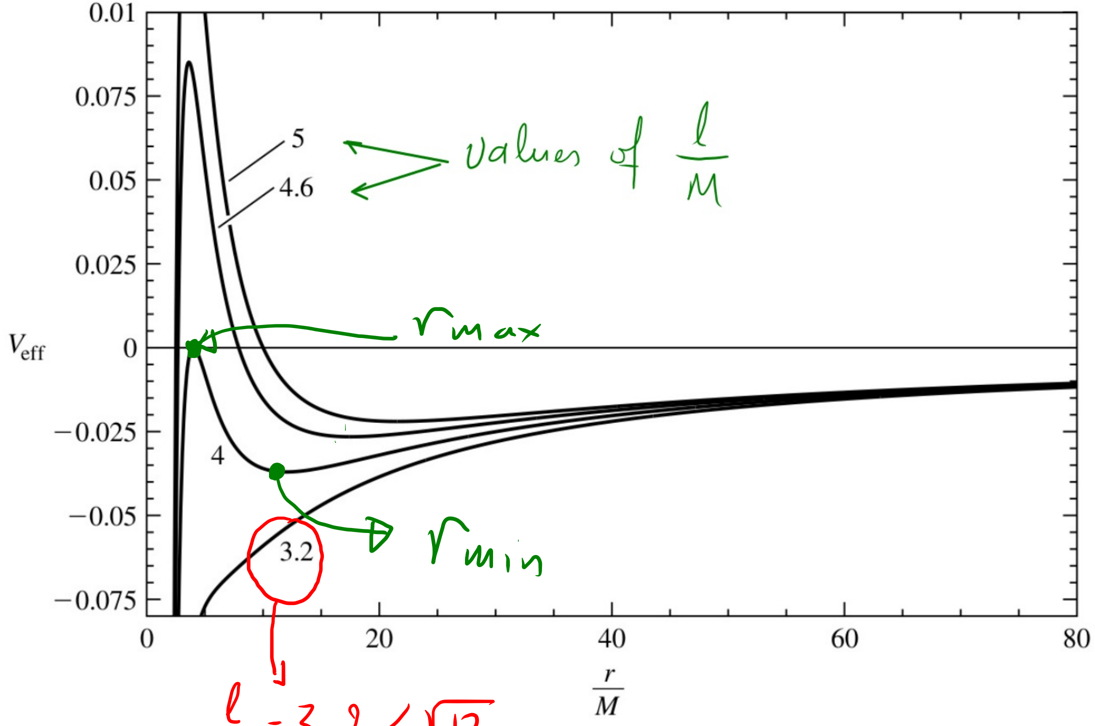
$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r)$$

$$V_{eff}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}$$

$$\frac{dV_{eff}(r)}{dr} = 0 \Rightarrow$$

$$+\frac{M}{r^2} - \frac{\ell^2}{r^3} + \frac{3\ell^2 M}{r^4} = 0 \Rightarrow$$

$$r_{min,max} = \frac{\ell^2}{2M}\left[1 \underset{r_{max}}{\overset{r_{min}}{\pm}} \sqrt{1-12\left(\frac{M}{\ell}\right)^2}\right]$$



$\frac{\ell}{M} = 3.2 < \sqrt{12}$

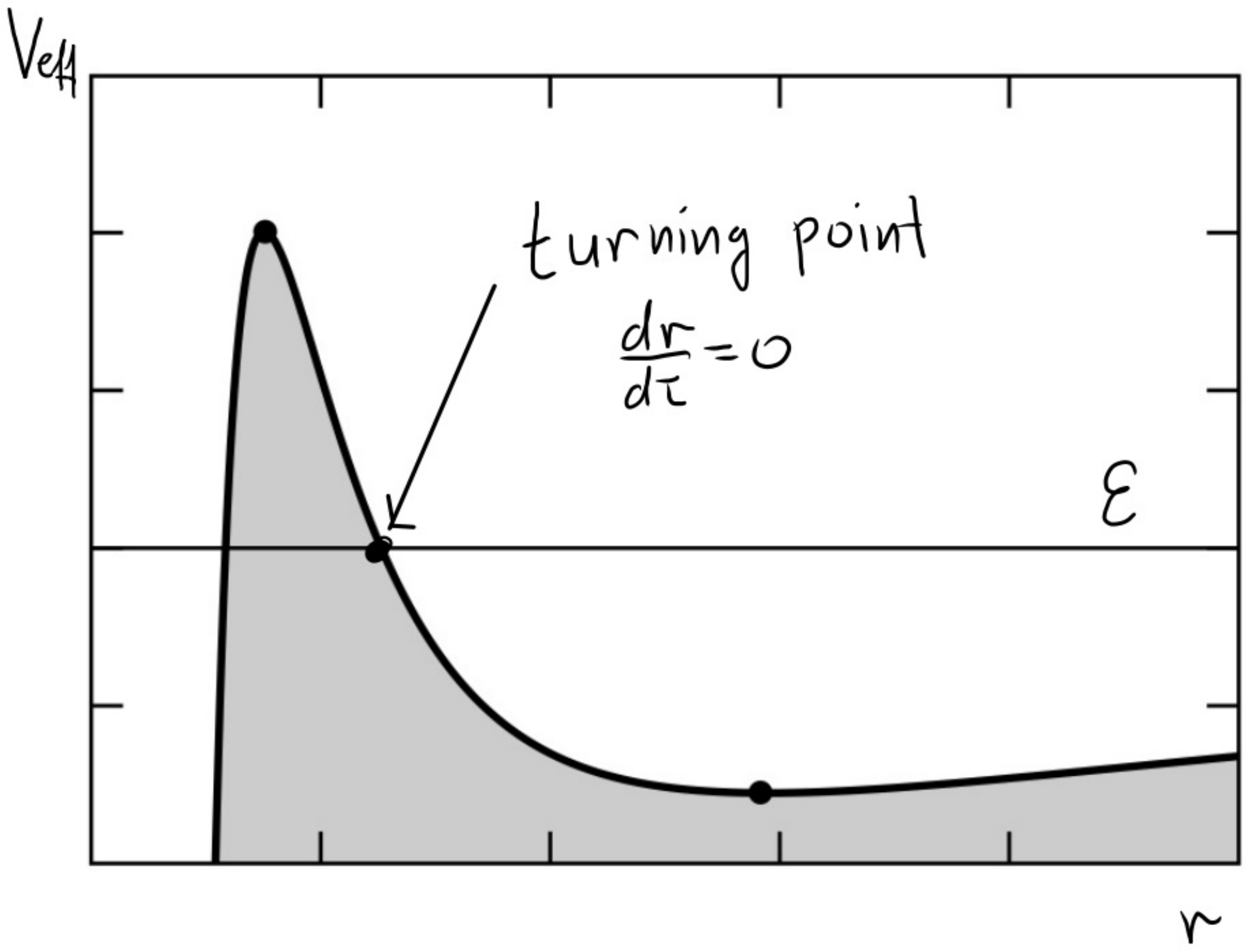
for $1 - 12\left(\frac{M}{\ell}\right)^2 \geq 0 \Rightarrow \ell \geq \sqrt{12}\, M \Rightarrow \frac{\ell}{M} \geq \sqrt{12} \approx 3.464$

$$V_{eff}(r_{max}) = \frac{2\left(8 - \left(\frac{\ell}{m}\right)^2 + \left(\frac{\ell}{m}\right)\sqrt{\left(\frac{\ell}{m}\right)^2 - 12}\right)}{\frac{\ell}{m}\left(\frac{\ell}{m} - \sqrt{\left(\frac{\ell}{m}\right)^2 - 12}\right)}$$

Radial motion:

Fix $\mathcal{E}$, determine turning points $\frac{dr}{d\tau} = 0$

$\Leftrightarrow \mathcal{E} = V_{eff}(r)$



Hartle, Fig 9.4

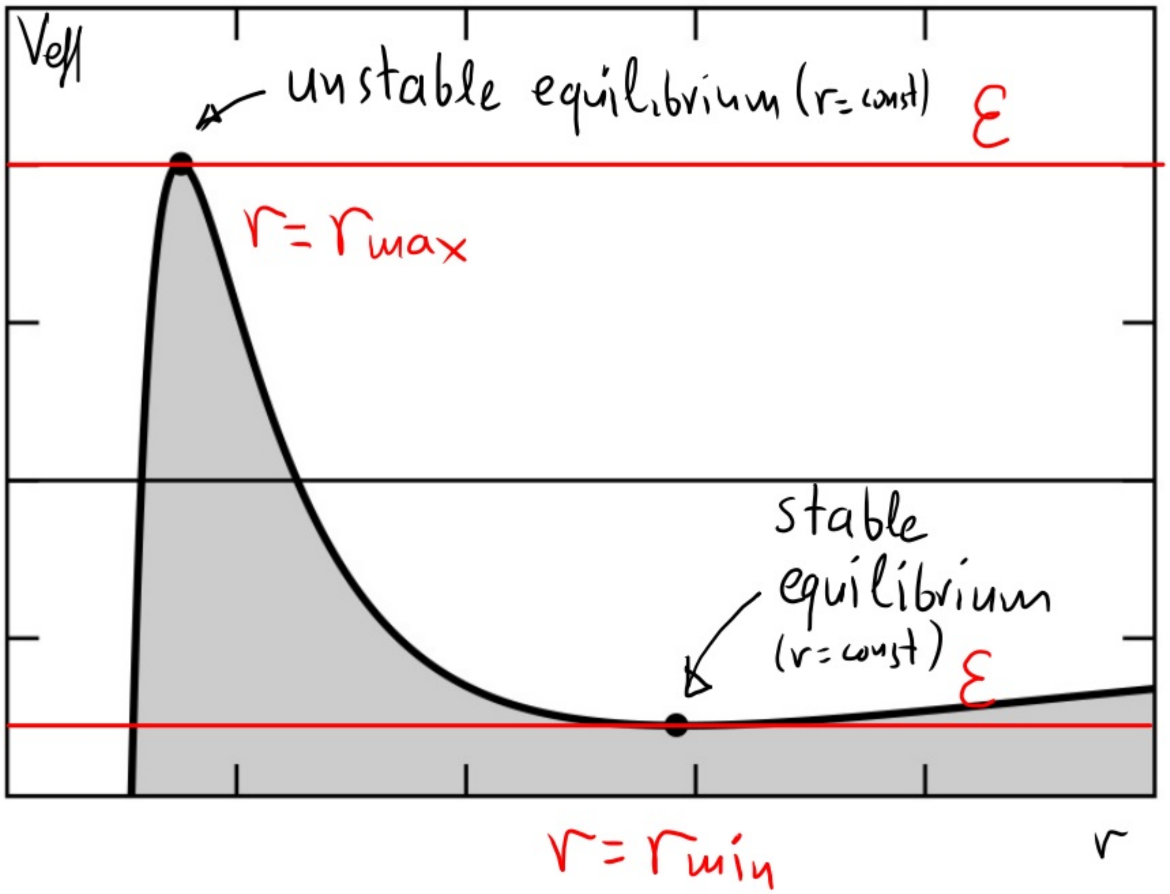
Radial motion:

Fix $\mathcal{E}$, determine turning points $\frac{dr}{d\tau} = 0$

$\Leftrightarrow \quad \mathcal{E} = V_{eff}(r)$

"Equilibrium" points:

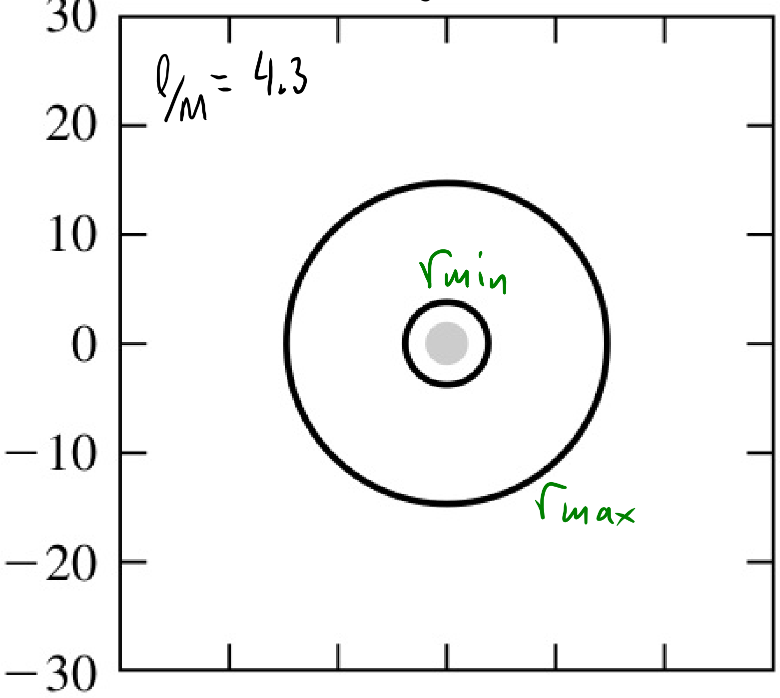
$r = \text{const} \Rightarrow$ circular orbits

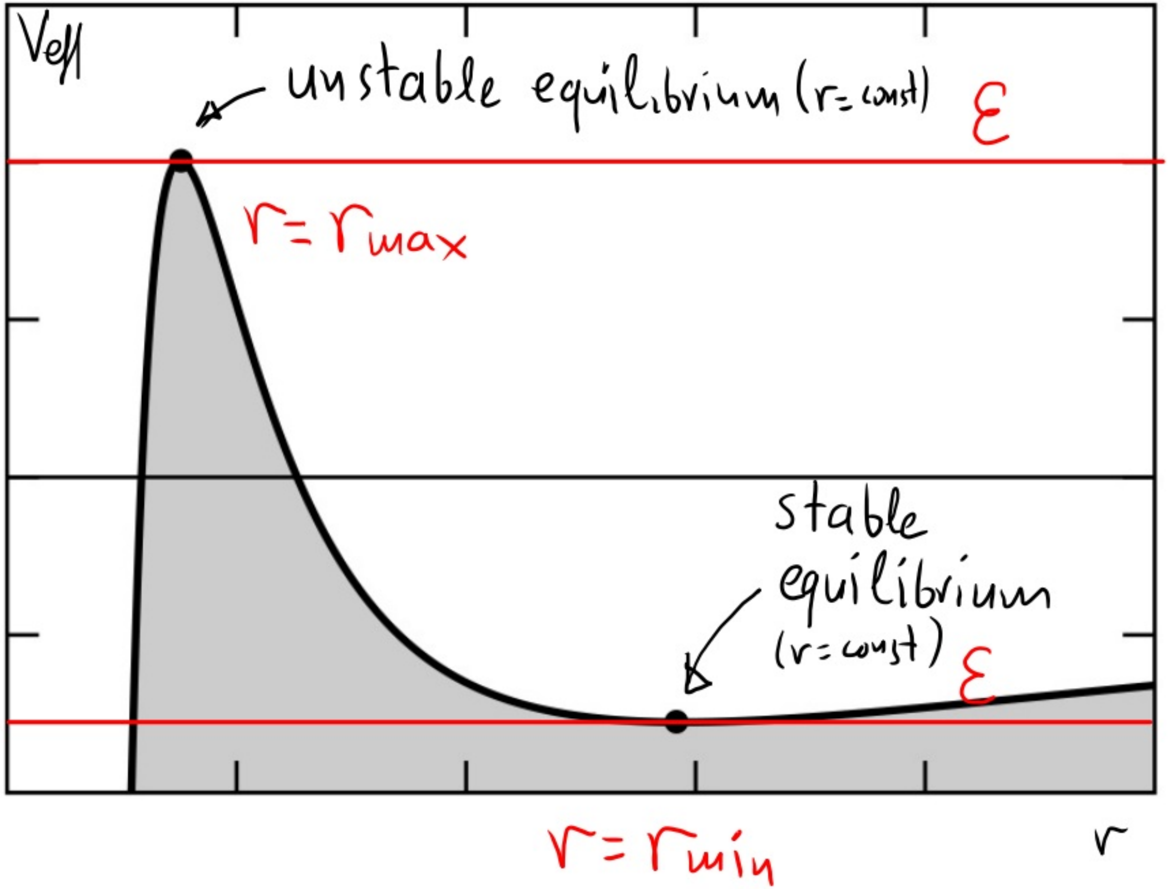


when $\mathcal{E} = V(r_{max})$ – unstable circular orbits

$\mathcal{E} = V(r_{min})$ – stable circular orbits

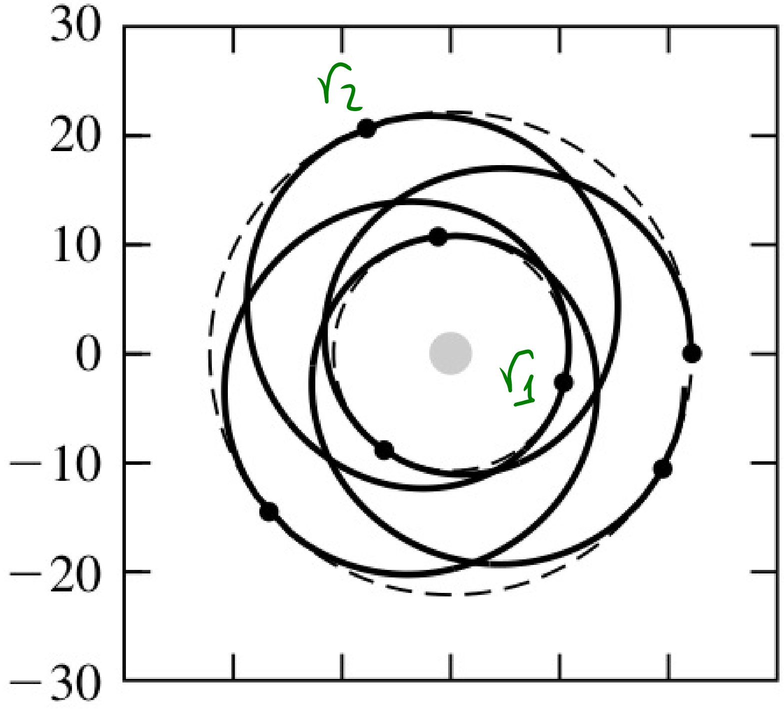
Hartle, Fig 9.4



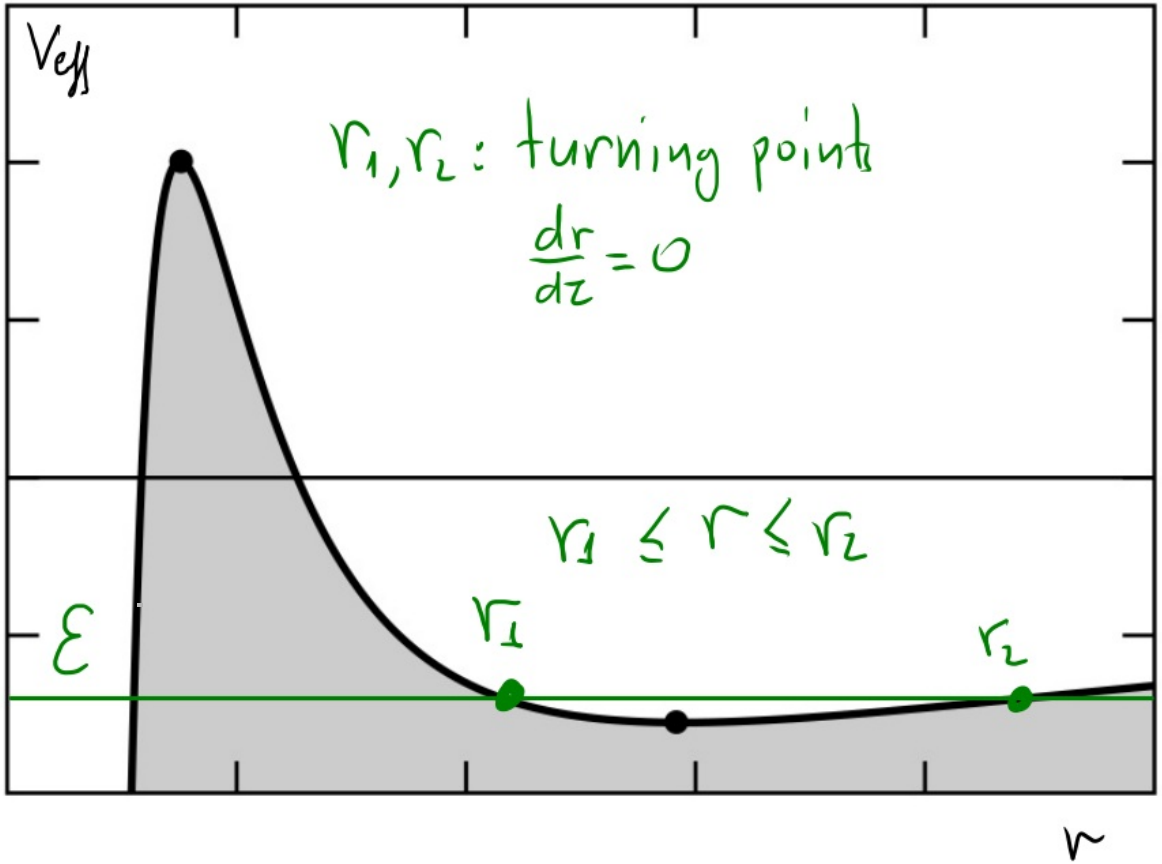


when $\mathcal{E} = V(r_{max})$ – unstable circular orbits

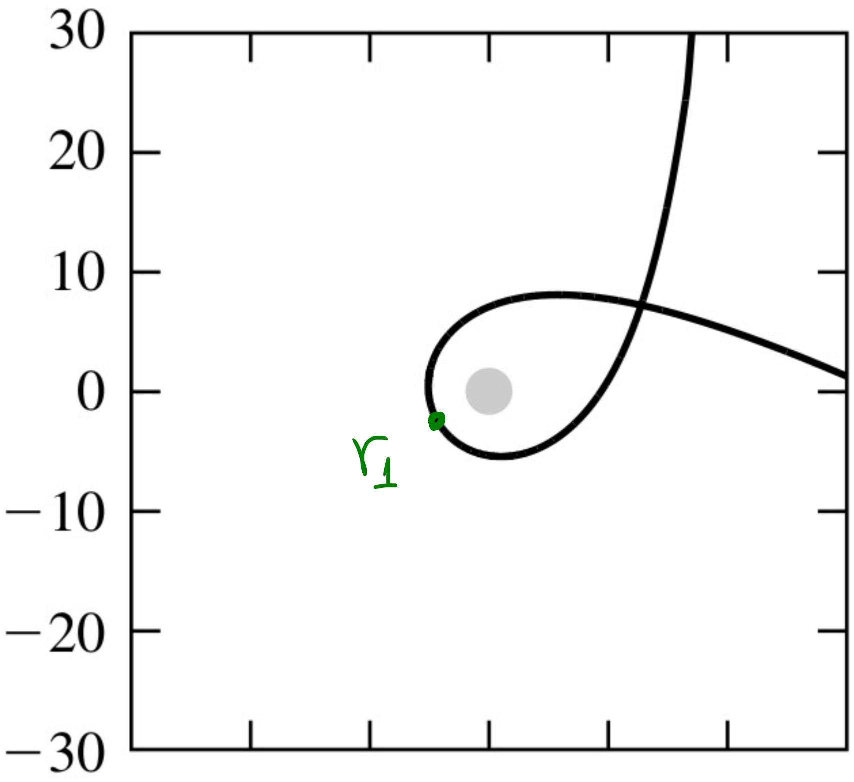
$\mathcal{E} = V(r_{min})$ – stable circular orbits



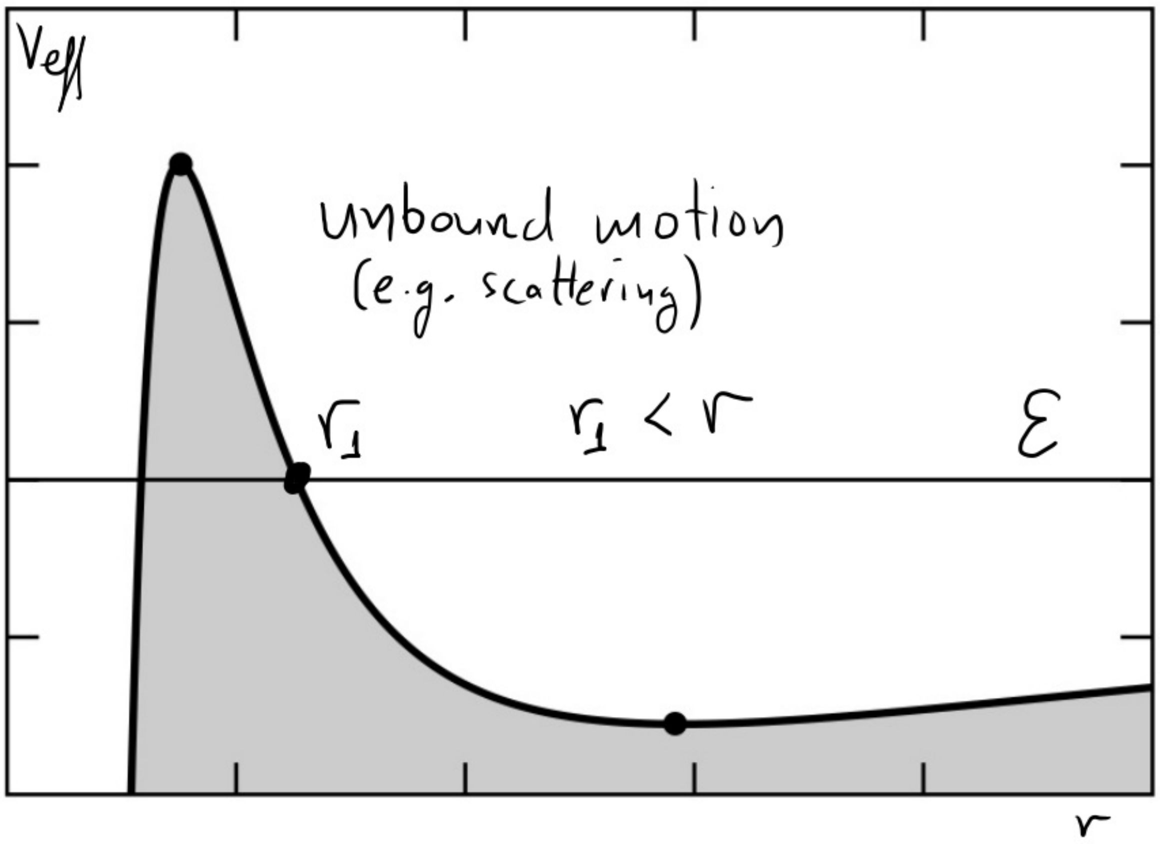
non-periodic ...



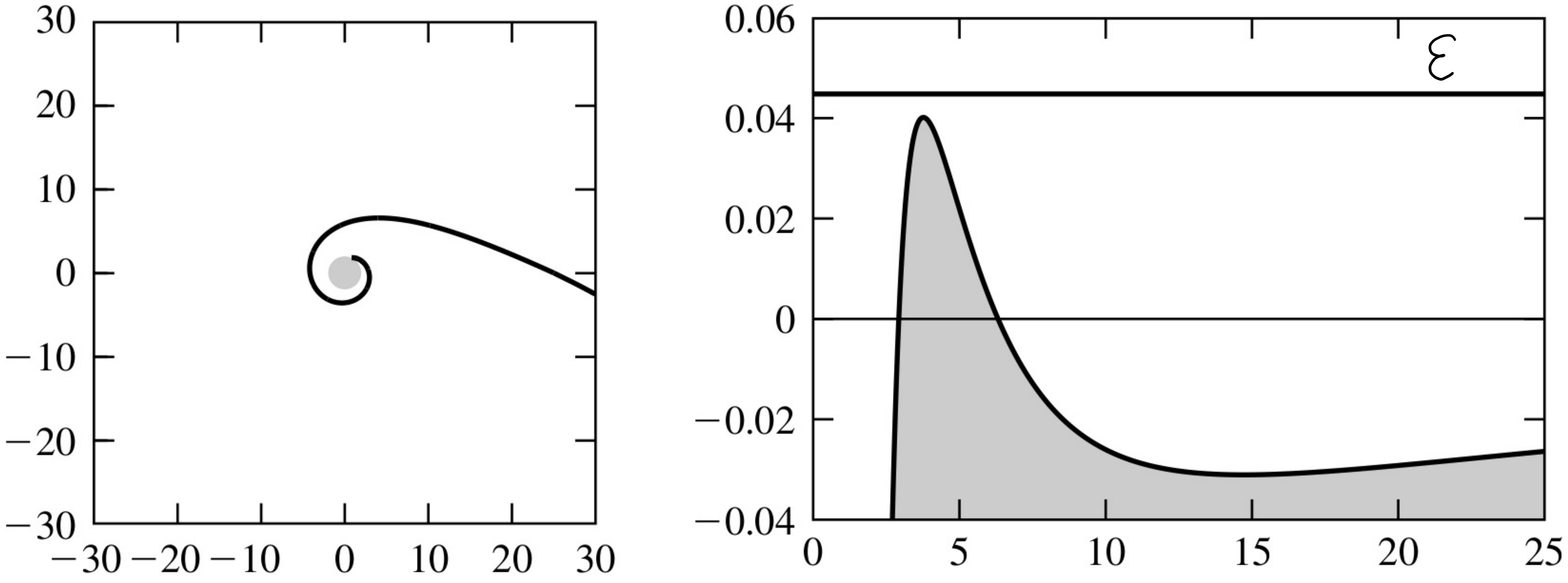
Bound motion



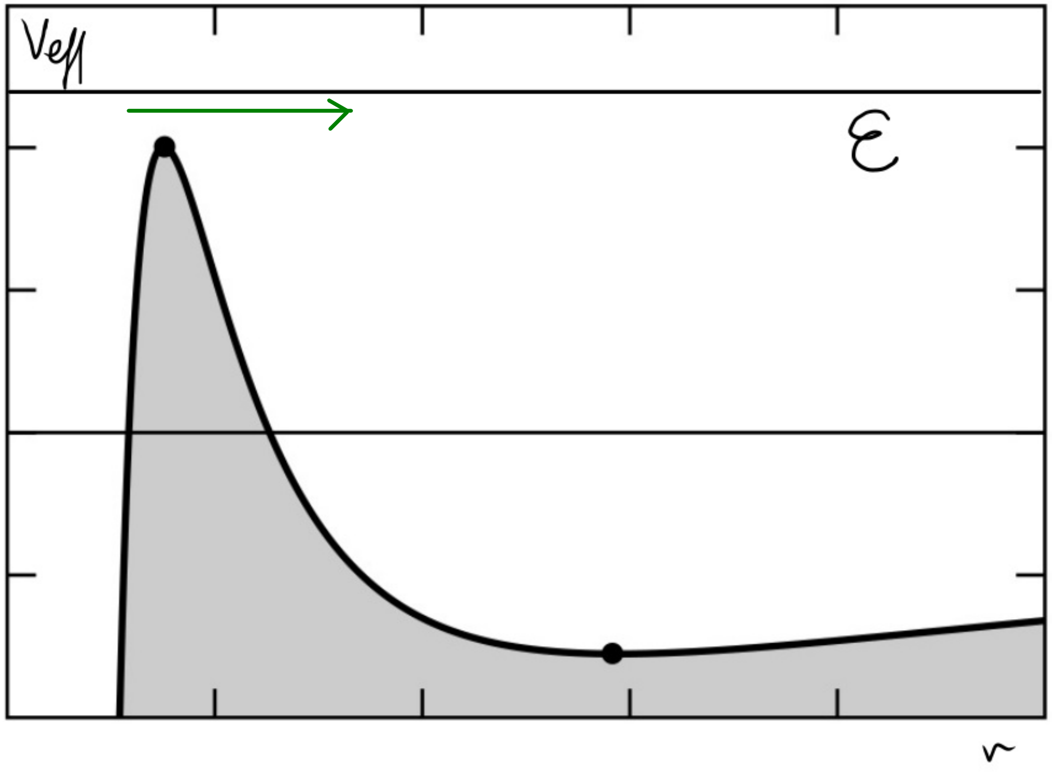
30
20
10
0
−10
−20
−30
r_1

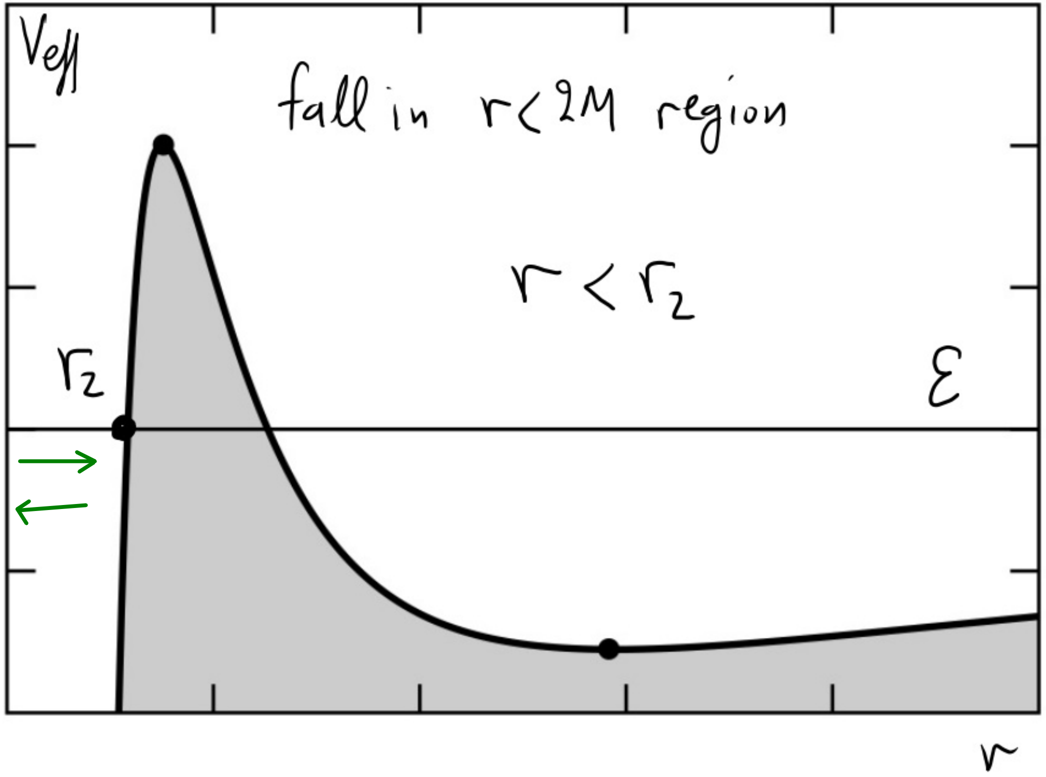


V_{eff}
unbound motion
(e.g. scattering)
r_1
$r_1 < r$
$\mathcal{E}$
r



Fall into the black hole!





Escape from the black hole

Fall into the black hole, cannot escape!

Radial plunge into BH

Start at rest @ infinity

$$1 = \frac{dt}{d\tau} = e\left(1 - \frac{2M}{r}\right)^{-1} \underset{r \gg 2M}{\approx} e \Rightarrow e = 1$$

Radial plunge into BH

Start at rest @ infinity

$$1 = \frac{dt}{d\tau} = e\left(1 - \frac{2M}{r}\right)^{-1} \underset{r \gg 2M}{\approx} e \Rightarrow e = 1 \Rightarrow \mathcal{E} = \frac{e^2 - 1}{2} = 0$$

Radial plunge into BH

Start at rest @ infinity

$$1 = \frac{dt}{d\tau} = e\left(1 - \frac{2M}{r}\right)^{-1} \underset{r \gg 2M}{\approx} e \Rightarrow e = 1 \Rightarrow \mathcal{E} = \frac{e^2 - 1}{2} = 0$$

$$l = 0 < \sqrt{12}\,M \Rightarrow \frac{d\varphi}{d\tau} = 0$$

Radial plunge into BH

Start at rest @ infinity

$$1 = \frac{dt}{d\tau} = e\left(1 - \frac{2M}{r}\right)^{-1} \underset{r \gg 2M}{\approx} e \Rightarrow e = 1 \Rightarrow \mathcal{E} = \frac{e^2 - 1}{2} = 0$$

$$\ell = 0 < \sqrt{12}\, M \Rightarrow \frac{d\varphi}{d\tau} = 0$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} = 0$$

$\hookrightarrow$ $\ell = 0$, so $V_{eff}(r) = -\frac{M}{r}$

Radial plunge into BH

Start at rest @ infinity

$$1 = \frac{dt}{d\tau} = e\left(1-\frac{2M}{r}\right)^{-1} \underset{r \gg 2M}{\approx} e \Rightarrow e = 1 \Rightarrow \mathcal{E} = \frac{e^2-1}{2} = 0$$

$$l = 0 < \sqrt{12}\,M \Rightarrow \frac{d\varphi}{d\tau} = 0$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} = 0 \Rightarrow \frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2}$$

inward motion

Radial plunge into BH

Start at rest @ infinity

$$1 = \frac{dt}{d\tau} = e\left(1-\frac{2M}{r}\right)^{-1} \underset{r \gg 2M}{\approx} e \Rightarrow e = 1 \Rightarrow \mathcal{E} = \frac{e^2-1}{2} = 0$$

$$l = 0 < \sqrt{12}\, M \Rightarrow \frac{d\varphi}{d\tau} = 0$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} = 0 \Rightarrow \frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2}$$

$$e = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} = 1 \Rightarrow \frac{dt}{d\tau} = \left(1-\frac{2M}{r}\right)^{-1}$$

Radial plunge into BH

Start at rest @ infinity

$$1 = \frac{dt}{d\tau} = e\left(1-\frac{2M}{r}\right)^{-1} \underset{r\gg 2M}{\approx} e \Rightarrow e = 1 \Rightarrow \mathcal{E} = \frac{e^2-1}{2} = 0$$

$$l = 0 < \sqrt{12}M \Rightarrow \frac{d\varphi}{d\tau} = 0$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} = 0 \Rightarrow \frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2}$$

$$e = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} = 1 \Rightarrow \frac{dt}{d\tau} = \left(1-\frac{2M}{r}\right)^{-1}$$

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau}\right) = \left(\left(1-\frac{2M}{r}\right)^{-1}, -\left(\frac{2M}{r}\right)^{1/2}, 0, 0\right)$$

$$\frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2} \Rightarrow r^{1/2}\,dr = -(2M)^{1/2}\,d\tau$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} = 0 \Rightarrow \frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2}$$

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} = 1 \qquad \Rightarrow \frac{dt}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1}$$

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau}\right) = \left(\left(1 - \frac{2M}{r}\right)^{-1}, -\left(\frac{2M}{r}\right)^{1/2}, 0, 0\right)$$

$$\frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2} \Rightarrow r^{1/2}\,dr = -(2M)^{1/2}\,d\tau$$

$$\Rightarrow \int_0^r r'^{1/2}\,dr' = -\int_{\tau_*}^{\tau} (2M)^{1/2}\,d\tau' \quad , \quad r(\tau_*) = 0$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} = 0 \Rightarrow \frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2}$$

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} = 1 \qquad \Rightarrow \frac{dt}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1}$$

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau}\right) = \left(\left(1-\frac{2M}{r}\right)^{-1}, -\left(\frac{2M}{r}\right)^{1/2}, 0, 0\right)$$

$$\frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2} \Rightarrow r^{1/2}\, dr = -(2M)^{1/2}\, d\tau$$

$$\Rightarrow \int_0^r r'^{1/2}\, dr' = -\int_{\tau_*}^{\tau} (2M)^{1/2}\, d\tau' \quad , \quad r(\tau_*) = 0$$

$$\Rightarrow \frac{r^{3/2}}{\frac{3}{2}} = -(2M)^{1/2}(\tau - \tau_*)$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} = 0 \Rightarrow \frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2}$$

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} = 1 \qquad \Rightarrow \frac{dt}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1}$$

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau}\right) = \left(\left(1 - \frac{2M}{r}\right)^{-1}, -\left(\frac{2M}{r}\right)^{1/2}, 0, 0\right)$$

$$\frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2} \Rightarrow r^{1/2}\, dr = -(2M)^{1/2}\, d\tau$$

$$\Rightarrow \int_0^r r'^{1/2}\, dr' = -\int_{\tau_*}^{\tau} (2M)^{1/2}\, d\tau' \quad , \quad r(\tau_*) = 0$$

$$\Rightarrow \frac{r^{3/2}}{\frac{3}{2}} = -(2M)^{1/2} (\tau - \tau_*)$$

$$\Rightarrow r(\tau) = (3/2)^{2/3} (2M)^{1/3} (\tau_* - \tau)^{2/3}$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} = 0 \Rightarrow \frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2}$$

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} = 1 \qquad \Rightarrow \frac{dt}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1}$$

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau}\right) = \left(\left(1 - \frac{2M}{r}\right)^{-1}, -\left(\frac{2M}{r}\right)^{1/2}, 0, 0\right)$$

$$\frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2} \Rightarrow \quad r(\tau) = \left(3/2\right)^{2/3} (2M)^{1/3} (\tau_* - \tau)^{2/3}$$

$$\frac{(2)}{(1)} \Rightarrow \frac{dt/d\tau}{dr/d\tau} = \frac{dt}{dr} = -\left(\frac{2M}{r}\right)^{-1/2} \left(1 - \frac{2M}{r}\right)^{-1}$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} = 0 \Rightarrow \frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2} \qquad (1)$$

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} = 1 \qquad \Rightarrow \frac{dt}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1} \qquad (2)$$

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau}\right) = \left(\left(1 - \frac{2M}{r}\right)^{-1}, -\left(\frac{2M}{r}\right)^{1/2}, 0, 0\right)$$

$$\frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2} \Rightarrow r(\tau) = \left(3/2\right)^{2/3} (2M)^{1/3} (\tau_* - \tau)^{2/3}$$

$$\frac{(2)}{(1)} \Rightarrow \frac{dt/d\tau}{dr/d\tau} = \frac{dt}{dr} = -\left(\frac{2M}{r}\right)^{-1/2} \left(1 - \frac{2M}{r}\right)^{-1} \Rightarrow$$

$$dt = -\left(\frac{2M}{r}\right)^{-1/2} \left(1 - \frac{2M}{r}\right)^{-1} dr$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} = 0 \Rightarrow \frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2} \quad (1)$$

$$e = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} = 1 \Rightarrow \frac{dt}{d\tau} = \left(1 - \frac{2M}{r}\right)^{-1} \quad (2)$$

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau}\right) = \left(\left(1 - \frac{2M}{r}\right)^{-1}, -\left(\frac{2M}{r}\right)^{1/2}, 0, 0\right)$$

$$\frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2} \Rightarrow r(\tau) = \left(3/2\right)^{2/3}(2M)^{1/3}(\tau_* - \tau)^{2/3}$$

$$\frac{(2)}{(1)} \Rightarrow \frac{dt/d\tau}{dr/d\tau} = \frac{dt}{dr} = -\left(\frac{2M}{r}\right)^{-1/2}\left(1-\frac{2M}{r}\right)^{-1} \Rightarrow$$

$$\int dt = -\int \left(\frac{2M}{r}\right)^{-1/2}\left(1-\frac{2M}{r}\right)^{-1} dr \Rightarrow$$

$$t = t_* + 2M\left[-\frac{2}{3}\left(\frac{r}{2M}\right)^{3/2} - 2\left(\frac{r}{2M}\right)^{1/2} + \ln\left|\frac{\left(\frac{r}{2M}\right)^{1/2}+1}{\left(\frac{r}{2M}\right)^{1/2}-1}\right|\right]$$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} = 0 \Rightarrow \frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2} \qquad (1)$$

$$e = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau} = 1 \Rightarrow \frac{dt}{d\tau} = \left(1-\frac{2M}{r}\right)^{-1} \qquad (2)$$

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\varphi}{d\tau}\right) = \left(\left(1-\frac{2M}{r}\right)^{-1}, -\left(\frac{2M}{r}\right)^{1/2}, 0, 0\right)$$

$$\frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2} \Rightarrow r(\tau) = \left(3/2\right)^{2/3}(2M)^{1/3}(\tau_* - \tau)^{2/3} \qquad (1)$$

$$t = t_* + 2M\left[-\frac{2}{3}\left(\frac{r}{2M}\right)^{\frac{3}{2}} - 2\left(\frac{r}{2M}\right)^{1/2} + \ln\left|\frac{\left(\frac{r}{2M}\right)^{1/2} + 1}{\left(\frac{r}{2M}\right)^{1/2} - 1}\right|\right] \qquad (2)$$

- to find $t(\tau)$ substitute $r(\tau)$ in (2)

$$\frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2} \Rightarrow r(\tau) = (3/2)^{2/3}(2M)^{1/3}(\tau_* - \tau)^{2/3} \qquad (1)$$

$$t = t_* + 2M\left[-\frac{2}{3}\left(\frac{r}{2M}\right)^{\frac{3}{2}} - 2\left(\frac{r}{2M}\right)^{1/2} + \ln\left|\frac{\left(\frac{r}{2M}\right)^{1/2}+1}{\left(\frac{r}{2M}\right)^{1/2}-1}\right|\right] \qquad (2)$$

- to find $t(\tau)$ substitute $r(\tau)$ in (2)
- As $r \to 2M$, $t \to +\infty$ $\Rightarrow$

it takes infinite time t to cross $r = 2M$

(proper time of $r \gg 2M$, asymptotically flat observer)

$$\frac{dr}{d\tau} = -\left(\frac{2M}{r}\right)^{1/2} \Rightarrow \quad r(\tau) = (3/2)^{2/3} (2M)^{1/3} (\tau_* - \tau)^{2/3} \qquad (1)$$

$$t = t_* + 2M\left[-\frac{2}{3}\left(\frac{r}{2M}\right)^{\frac{3}{2}} - 2\left(\frac{r}{2M}\right)^{1/2} + \ln\left|\frac{\left(\frac{r}{2M}\right)^{1/2} + 1}{\left(\frac{r}{2M}\right)^{1/2} - 1}\right|\right] \qquad (2)$$

- to find $t(\tau)$ substitute $r(\tau)$ in (2)
- As $r \to 2M$, $t \to +\infty$ $\Rightarrow$

  it takes infinite time t to cross $r = 2M$

  (proper time of $r \gg 2M$, asymptotically flat observer)
- From $r(\tau)$ $\Rightarrow$ $\begin{cases} \text{takes finite } \tau \text{ to cross } r = 2M \\ \text{" \quad " \quad " to fall on } r = 0 \end{cases}$

Shape of bound orbits

We want $r = r(\phi)$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) \Rightarrow \left(\frac{dr}{d\tau}\right) = \pm\left[2(\mathcal{E} - V_{eff})\right]^{1/2}$$

Shape of bound orbits

We want $r = r(\phi)$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) \Rightarrow \left(\frac{dr}{d\tau}\right) = \pm\left[2\left(\mathcal{E} - V_{eff}\right)\right]^{1/2}$$

$$\ell = r^2 \frac{d\phi}{d\tau} \Rightarrow \frac{d\phi}{d\tau} = \frac{\ell}{r^2}$$

Shape of bound orbits

We want $r = r(\phi)$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) \Rightarrow \left(\frac{dr}{d\tau}\right) = \pm\left[2(\mathcal{E} - V_{eff})\right]^{1/2} \qquad (1)$$

$$\ell = r^2 \frac{d\phi}{d\tau} \Rightarrow \frac{d\phi}{d\tau} = \frac{\ell}{r^2} \qquad (2)$$

$$\frac{(2)}{(1)} \Rightarrow \frac{d\phi}{dr} = \pm \frac{\ell}{r^2}\left[2(\mathcal{E} - V_{eff}(r))\right]^{-1/2} = \pm \frac{\ell}{r^2}\left[e^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right)\right]^{-1/2}$$

Shape of bound orbits

We want $r = r(\phi)$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) \Rightarrow \left(\frac{dr}{d\tau}\right) = \pm\left[2(\mathcal{E} - V_{eff})\right]^{1/2} \qquad (1)$$

$$\ell = r^2 \frac{d\phi}{d\tau} \Rightarrow \frac{d\phi}{d\tau} = \frac{\ell}{r^2} \qquad (2)$$

$$\frac{(2)}{(1)} \Rightarrow \frac{d\phi}{dr} = \pm \frac{\ell}{r^2}\left[2(\mathcal{E} - V_{eff}(r))\right]^{-1/2} = \pm\frac{\ell}{r^2}\left[e^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right)\right]^{-1/2}$$

- angle between successive turning points is $\Delta\phi$
  if $\Delta\phi - 2\pi \neq 0$, orbit does not close

Shape of bound orbits

We want $r = r(\phi)$

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) \Rightarrow \left(\frac{dr}{d\tau}\right) = \pm\left[2\left(\mathcal{E} - V_{eff}\right)\right]^{1/2} \qquad (1)$$

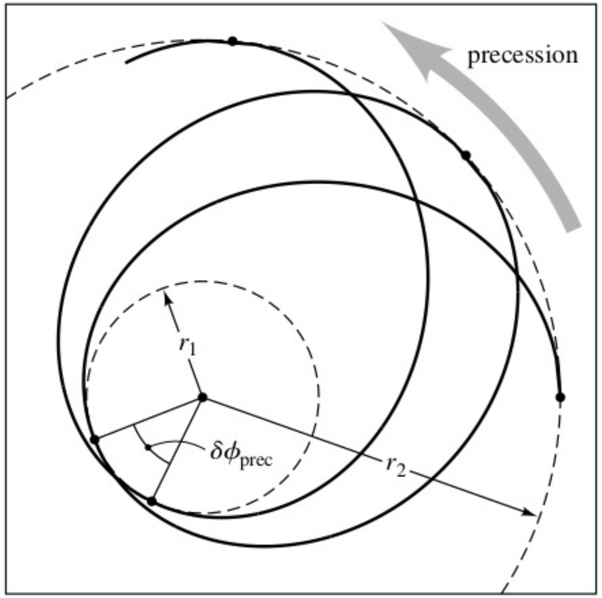
$$\ell = r^2 \frac{d\phi}{d\tau} \Rightarrow \frac{d\phi}{d\tau} = \frac{\ell}{r^2} \qquad (2)$$

$$\frac{(2)}{(1)} \Rightarrow \frac{d\phi}{dr} = \pm \frac{\ell}{r^2}\left[2\left(\mathcal{E} - V_{eff}(r)\right)\right]^{-1/2} = \pm \frac{\ell}{r^2}\left[e^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right)\right]^{-1/2}$$

- angle between successive turning points is $\Delta\phi$
  if $\Delta\phi - 2\pi \neq 0$, orbit does not close
- precession angle $\delta\phi = \Delta\phi - 2\pi$

$$\Delta\phi = 2\left[\int_{r_1}^{r_2} dr\, \frac{\ell}{r^2}\left[e^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right)\right]^{-1/2}\right]$$

$$\left.\frac{dr}{d\tau}\right|_{r_1} = \left.\frac{dr}{d\tau}\right|_{r_2} = 0$$



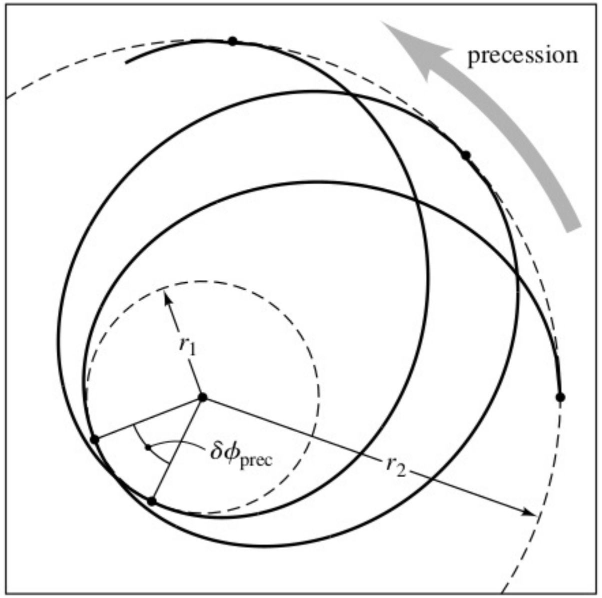
$\frac{(2)}{(1)} \Rightarrow$ $$\frac{d\phi}{dr} = \pm\frac{\ell}{r^2}\left[2\left(\mathcal{E}-V_{eff}(r)\right)\right]^{-1/2} = \pm\frac{\ell}{r^2}\left[e^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right)\right]^{-1/2}$$

- angle between successive turning points is $\Delta\phi$
  if $\Delta\phi - 2\pi \neq 0$, orbit does not close
- precession angle $\delta\phi = \Delta\phi - 2\pi$

$$\Delta\phi = 2\left[\int_{r_1}^{r_2} dr\, \frac{\ell}{r^2}\left[e^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right)\right]^{-1/2}\right]$$

$$\left.\frac{dr}{d\tau}\right|_{r_1} = \left.\frac{dr}{d\tau}\right|_{r_2} = 0$$

when $\frac{2 \cdot M \ell^2}{r^3}$ term neglected $\Rightarrow \Delta\phi = 2\pi$ (no precession)



$\frac{(2)}{(1)} \Rightarrow$ $$\frac{d\phi}{dr} = \pm\frac{\ell}{r^2}\left[2\left(\mathcal{E}-V_{eff}(r)\right)\right]^{-1/2} = \pm\frac{\ell}{r^2}\left[e^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right)\right]^{-1/2}$$

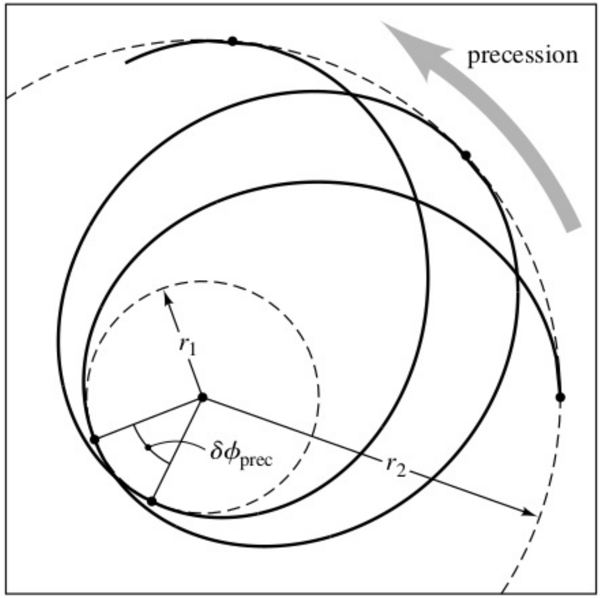
- angle between successive turning points is $\Delta\phi$
  if $\Delta\phi - 2\pi \neq 0$, orbit does not close
- precession angle $\delta\phi = \Delta\phi - 2\pi$

$$\Delta\phi = 2\left[\int_{r_1}^{r_2} dr\, \frac{\ell}{r^2}\left[e^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right)\right]^{-1/2}\right]$$

$$\left.\frac{dr}{d\tau}\right|_{r_1} = \left.\frac{dr}{d\tau}\right|_{r_2} = 0$$

when $\frac{2 \cdot M \ell^2}{r^3}$ term is small, then

$$\delta\phi \approx 6\pi\left(\frac{M}{\ell}\right)^2$$



For Mercury $\sim 43''$/century (detectable)

Stable Circular Orbits

$$r = \frac{\ell^2}{2M}\left[1 + \left(1 - 12\left(\frac{M}{\ell}\right)^2\right)^{1/2}\right]$$

Solution of $\frac{dV_{eff}}{dr} = 0 \rightsquigarrow r = \frac{\ell^2}{2M}\left[1 \pm \left(1 - 12\left(\frac{M}{\ell}\right)^2\right)^{1/2}\right]$

stable

unstable

Stable Circular Orbits

$$r = \frac{\ell^2}{2M}\left[1 + \left(1 - 12\left(\frac{M}{\ell}\right)^2\right)^{1/2}\right]$$

smallest when $\frac{\ell}{M} = \sqrt{12}$ $\Rightarrow$ $r_{ISCO} = \frac{(\sqrt{12}\,M)^2}{2M} = 6M$

$\hookrightarrow$ Innermost Stable Circular Orbit

Stable Circular Orbits

$$r = \frac{\ell^2}{2M}\left[1 + \left(1 - 12\left(\frac{M}{\ell}\right)^2\right)^{1/2}\right]$$

$$\Omega = \frac{d\varphi}{dt} = \frac{d\varphi/d\tau}{dt/d\tau}$$

Stable Circular Orbits

$$r = \frac{\ell^2}{2M}\left[1 + \left(1 - 12\left(\frac{M}{\ell}\right)^2\right)^{1/2}\right]$$

$$\Omega = \frac{d\varphi}{dt} = \frac{d\varphi/d\tau}{dt/d\tau} = \frac{\frac{\ell}{r^2}}{\frac{e}{1-\frac{2M}{r}}} = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\left(\frac{\ell}{e}\right)$$

Stable Circular Orbits

$$r = \frac{\ell^2}{2M}\left[1 + \left(1 - 12\left(\frac{M}{\ell}\right)^2\right)^{1/2}\right]$$

$$\Omega = \frac{d\varphi}{dt} = \frac{d\varphi/d\tau}{dt/d\tau} = \frac{\frac{\ell}{r^2}}{\frac{e}{1-\frac{2M}{r}}} = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\left(\frac{\ell}{e}\right)$$

Compute $\frac{\ell}{e}$ as function of M, r:

$$\frac{dV_{eff}}{dr} = 0 \Rightarrow \frac{M}{r} - \frac{\ell^2}{r^2} + \frac{3M\ell^2}{r^3} = 0$$

Stable Circular Orbits

$$r = \frac{\ell^2}{2M}\left[1 + \left(1 - 12\left(\frac{M}{\ell}\right)^2\right)^{1/2}\right]$$

$$\Omega = \frac{d\varphi}{dt} = \frac{d\varphi/d\tau}{dt/d\tau} = \frac{\frac{\ell}{r^2}}{\frac{e}{1-\frac{2M}{r}}} = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\left(\frac{\ell}{e}\right)$$

Compute $\frac{\ell}{e}$ as function of M, r:

$$\frac{dV_{\text{eff}}}{dr} = 0 \Rightarrow \frac{M}{r} - \frac{\ell^2}{r^2} + \frac{3M\ell^2}{r^3} = 0 \Rightarrow \frac{M}{r} - \frac{\ell^2}{r^2}\left(1 - \frac{3M}{r}\right) = 0 \Rightarrow \frac{\ell^2}{r^2} = \frac{1}{\frac{r}{M} - 3}$$

Stable Circular Orbits

$$r = \frac{\ell^2}{2M}\left[1 + \left(1 - 12\left(\frac{M}{\ell}\right)^2\right)^{1/2}\right]$$

$$\Omega = \frac{d\varphi}{dt} = \frac{d\varphi/d\tau}{dt/d\tau} = \frac{\frac{\ell}{r^2}}{\frac{e}{1-\frac{2M}{r}}} = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\left(\frac{\ell}{e}\right)$$

Compute $\frac{\ell}{e}$ as function of M, r:

$$\frac{dV_{eff}}{dr} = 0 \Rightarrow \frac{M}{r} - \frac{\ell^2}{r^2} + \frac{3M\ell^2}{r^3} = 0 \Rightarrow \frac{M}{r} - \frac{\ell^2}{r^2}\left(1 - \frac{3M}{r}\right) = 0 \Rightarrow \frac{\ell^2}{r^2} = \frac{1}{\frac{r}{M} - 3}$$

$$\frac{dr}{d\tau} = 0 \Rightarrow \mathcal{E} = V_{eff}(r) \Rightarrow e^2 = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right)$$

Stable Circular Orbits

$$r = \frac{\ell^2}{2M}\left[1 + \left(1 - 12\left(\frac{M}{\ell}\right)^2\right)^{1/2}\right]$$

$$\Omega = \frac{d\varphi}{dt} = \frac{d\varphi/d\tau}{dt/d\tau} = \frac{\frac{\ell}{r^2}}{\frac{e}{1-\frac{2M}{r}}} = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\left(\frac{\ell}{e}\right)$$

Compute $\frac{\ell}{e}$ as function of M, r:

$$\frac{dV_{eff}}{dr} = 0 \Rightarrow \frac{M}{r} - \frac{\ell^2}{r^2} + \frac{3M\ell^2}{r^3} = 0 \Rightarrow \frac{M}{r} - \frac{\ell^2}{r^2}\left(1 - \frac{3M}{r}\right) = 0 \Rightarrow \frac{\ell^2}{r^2} = \frac{1}{\frac{r}{M} - 3}$$

$$\frac{dr}{d\tau} = 0 \Rightarrow \mathcal{E} = V_{eff}(r) \Rightarrow e^2 = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right) = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{1}{\frac{r}{M} - 3}\right)$$

$$\Rightarrow e^2 = \left(1 - \frac{2M}{r}\right) \frac{\frac{r}{M} - 3 + 1}{\frac{r}{M} - 3} = \left(1 - \frac{2M}{r}\right) \frac{\frac{r}{M} - 2}{\frac{r}{M} - 3} = \left(1 - \frac{2M}{r}\right) \frac{\frac{r}{M}\left(1 - \frac{2M}{r}\right)}{\frac{r}{M} - 3}$$

$$\frac{dV_{eff}}{dr} = 0 \Rightarrow \frac{M}{r} - \frac{\ell^2}{r^2} + \frac{3M\ell^2}{r^3} = 0 \Rightarrow \frac{M}{r} - \frac{\ell^2}{r^2}\left(1 - \frac{3M}{r}\right) = 0 \Rightarrow \frac{\ell^2}{r^2} = \frac{1}{\frac{r}{M} - 3}$$

$$\frac{dr}{d\tau} = 0 \Rightarrow \mathcal{E} = V_{eff}(r) \Rightarrow e^2 = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right) = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{1}{\frac{r}{M} - 3}\right)$$

$$\Rightarrow e^2 = \left(1-\frac{2M}{r}\right)\frac{\frac{r}{M}-3+1}{\frac{r}{M}-3} = \left(1-\frac{2M}{r}\right)\frac{\frac{r}{M}-2}{\frac{r}{M}-3} = \left(1-\frac{2M}{r}\right)\frac{\frac{r}{M}\left(1-\frac{2M}{r}\right)}{\frac{r}{M}-3}$$

$$= \left(1-\frac{2M}{r}\right)^2 \frac{\frac{r}{M}}{\frac{r^2}{\ell^2}}$$

$$\frac{r}{M}-3 = \frac{r^2}{\ell^2}$$

$$\Uparrow$$

$$\frac{dV_{eff}}{dr} = 0 \Rightarrow \frac{M}{r} - \frac{\ell^2}{r^2} + \frac{3M\ell^2}{r^3} = 0 \Rightarrow \frac{M}{r} - \frac{\ell^2}{r^2}\left(1-\frac{3M}{r}\right) = 0 \Rightarrow \frac{\ell^2}{r^2} = \frac{1}{\frac{r}{M}-3}$$

$$\frac{dr}{d\tau} = 0 \Rightarrow \mathcal{E} = V_{eff}(r) \Rightarrow e^2 = \left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right) = \left(1-\frac{2M}{r}\right)\left(1+\frac{1}{\frac{r}{M}-3}\right)$$

$$\Rightarrow e^2 = \left(1-\frac{2M}{r}\right)\frac{\frac{r}{M}-3+1}{\frac{r}{M}-3} = \left(1-\frac{2M}{r}\right)\frac{\frac{r}{M}-2}{\frac{r}{M}-3} = \left(1-\frac{2M}{r}\right)\frac{\frac{r}{M}\left(1-\frac{2M}{r}\right)}{\frac{r}{M}-3}$$

$$= \left(1-\frac{2M}{r}\right)^2 \frac{\frac{r}{M}}{\frac{r^2}{\ell^2}} = \left(1-\frac{2M}{r}\right)^2 \frac{\ell^2}{Mr}$$

$$\frac{r}{M}-3 = \frac{r^2}{\ell^2}$$

$$\Uparrow$$

$$\frac{dV_{eff}}{dr} = 0 \Rightarrow \frac{M}{r} - \frac{\ell^2}{r^2} + \frac{3M\ell^2}{r^3} = 0 \Rightarrow \frac{M}{r} - \frac{\ell^2}{r^2}\left(1-\frac{3M}{r}\right) = 0 \Rightarrow \frac{\ell^2}{r^2} = \frac{1}{\frac{r}{M}-3}$$

$$\frac{dr}{d\tau} = 0 \Rightarrow \mathcal{E} = V_{eff}(r) \Rightarrow e^2 = \left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right) = \left(1-\frac{2M}{r}\right)\left(1+\frac{1}{\frac{r}{M}-3}\right)$$

$$\Rightarrow e^2 = \left(1-\frac{2M}{r}\right)\frac{\frac{r}{M}-3+1}{\frac{r}{M}-3} = \left(1-\frac{2M}{r}\right)\frac{\frac{r}{M}-2}{\frac{r}{M}-3} = \left(1-\frac{2M}{r}\right)\frac{\frac{r}{M}\left(1-\frac{2M}{r}\right)}{\frac{r}{M}-3}$$

$$= \left(1-\frac{2M}{r}\right)^2 \frac{\frac{r}{M}}{\frac{r^2}{\ell^2}} = \left(1-\frac{2M}{r}\right)^2 \frac{\ell^2}{Mr}$$

$$\Rightarrow \frac{\ell^2}{e^2} = Mr\left(1-\frac{2M}{r}\right)^{-2} \Rightarrow \frac{\ell}{e} = \sqrt{Mr}\left(1-\frac{2M}{r}\right)^{-1}$$

$$\frac{dV_{eff}}{dr} = 0 \Rightarrow \frac{M}{r} - \frac{\ell^2}{r^2} + \frac{3M\ell^2}{r^3} = 0 \Rightarrow \frac{M}{r} - \frac{\ell^2}{r^2}\left(1-\frac{3M}{r}\right) = 0 \Rightarrow \frac{\ell^2}{r^2} = \frac{1}{\frac{r}{M}-3}$$

$$\frac{dr}{d\tau} = 0 \Rightarrow \mathcal{E} = V_{eff}(r) \Rightarrow e^2 = \left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right) = \left(1-\frac{2M}{r}\right)\left(1+\frac{1}{\frac{r}{M}-3}\right)$$

$$\Rightarrow e^2 = \left(1-\frac{2M}{r}\right)\frac{\frac{r}{M}-3+1}{\frac{r}{M}-3} = \left(1-\frac{2M}{r}\right)\frac{\frac{r}{M}-2}{\frac{r}{M}-3} = \left(1-\frac{2M}{r}\right)\frac{\frac{r}{M}\left(1-\frac{2M}{r}\right)}{\frac{r}{M}-3}$$

$$= \left(1-\frac{2M}{r}\right)^2 \frac{\frac{r}{M}}{\frac{r^2}{\ell^2}} = \left(1-\frac{2M}{r}\right)^2 \frac{\ell^2}{Mr}$$

$$\Rightarrow \frac{\ell^2}{e^2} = Mr\left(1-\frac{2M}{r}\right)^{-2} \Rightarrow \frac{\ell}{e} = \sqrt{Mr}\left(1-\frac{2M}{r}\right)^{-1}$$

Then $\Omega = \frac{1}{r^2}\left(1-\frac{2M}{r}\right)\frac{\ell}{e} = \frac{1}{r^2}\left(1-\frac{2M}{r}\right)\sqrt{Mr}\left(1-\frac{2M}{r}\right)^{-1} = \frac{M^{1/2}}{r^{3/2}}$

$$\Rightarrow e^2 = \left(1-\frac{2M}{r}\right)\frac{\frac{r}{M}-3+1}{\frac{r}{M}-3} = \left(1-\frac{2M}{r}\right)\frac{\frac{r}{M}-2}{\frac{r}{M}-3} = \left(1-\frac{2M}{r}\right)\frac{\frac{r}{M}\left(1-\frac{2M}{r}\right)}{\frac{r}{M}-3}$$

$$= \left(1-\frac{2M}{r}\right)^2 \frac{\frac{r}{M}}{\frac{r^2}{\ell^2}} = \left(1-\frac{2M}{r}\right)^2 \frac{\ell^2}{Mr}$$

$$\Rightarrow \frac{\ell^2}{e^2} = Mr\left(1-\frac{2M}{r}\right)^{-2} \Rightarrow \frac{\ell}{e} = \sqrt{Mr}\left(1-\frac{2M}{r}\right)^{-1}$$

Then $\Omega = \frac{1}{r^2}\left(1-\frac{2M}{r}\right)\frac{\ell}{e} = \frac{1}{r^2}\left(1-\frac{2M}{r}\right)\sqrt{Mr}\left(1-\frac{2M}{r}\right)^{-1} = \frac{M^{1/2}}{r^{3/2}}$

$\Rightarrow \Omega^2 = \frac{M}{r^3}$ Kepler's law in GR!

Four velocity $u^\mu = \left(\frac{dt}{d\tau}, \cancel{\frac{dr}{d\tau}}, \cancel{\frac{d\theta}{d\tau}}, \frac{d\varphi}{d\tau} \right)$

$$= \left(\frac{dt}{d\tau}, 0, 0, \frac{d\varphi}{dt}\frac{dt}{d\tau} \right)$$

Four velocity $u^\mu = \left(\frac{dt}{d\tau}, \cancel{\frac{dr}{d\tau}}, \cancel{\frac{d\theta}{d\tau}}, \frac{d\varphi}{d\tau} \right)$

$$= \left(\frac{dt}{d\tau}, 0, 0, \frac{d\varphi}{dt}\frac{dt}{d\tau} \right)$$

$$= \frac{dt}{d\tau} \left(1, 0, 0, \Omega \right)$$

Four velocity $u^\mu = \left(\frac{dt}{d\tau}, \cancel{\frac{dr}{d\tau}}, \cancel{\frac{d\theta}{d\tau}}, \frac{d\varphi}{d\tau} \right)$

$$= \left(\frac{dt}{d\tau}, 0, 0, \frac{d\varphi}{dt}\frac{dt}{d\tau} \right)$$

$$= \frac{dt}{d\tau}\left(1, 0, 0, \Omega \right)$$

$$u^\mu u_\mu = g_{\mu\nu} u^\mu u^\nu = -1 \Rightarrow -\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + r^2 \Omega^2 \left(\frac{dt}{d\tau}\right)^2 = -1$$

Four velocity $u^\mu = \left(\frac{dt}{d\tau}, \cancel{\frac{dr}{d\tau}}, \cancel{\frac{d\theta}{d\tau}}, \frac{d\varphi}{d\tau} \right)$

$$= \left(\frac{dt}{d\tau}, 0, 0, \frac{d\varphi}{dt}\frac{dt}{d\tau} \right)$$

$$= \frac{dt}{d\tau}\left(1, 0, 0, \Omega \right)$$

$$u^\mu u_\mu = g_{\mu\nu} u^\mu u^\nu = -1 \Rightarrow -\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + r^2 \Omega^2 \left(\frac{dt}{d\tau}\right)^2 = -1 \Rightarrow$$

$$\left(\frac{dt}{d\tau}\right)^2 \left(1 - \frac{2M}{r} - r^2 \Omega^2 \right) = +1$$

Four velocity $u^\mu = \left(\frac{dt}{d\tau}, \cancel{\frac{dr}{d\tau}}, \cancel{\frac{d\theta}{d\tau}}, \frac{d\varphi}{d\tau} \right)$

$$= \left(\frac{dt}{d\tau}, 0, 0, \frac{d\varphi}{dt}\frac{dt}{d\tau} \right)$$

$$= \frac{dt}{d\tau} \left(1, 0, 0, \Omega \right)$$

$$u^\mu u_\mu = g_{\mu\nu} u^\mu u^\nu = -1 \Rightarrow -\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + r^2 \Omega^2 \left(\frac{dt}{d\tau}\right)^2 = -1 \Rightarrow$$

$$\left(\frac{dt}{d\tau}\right)^2 \left(1 - \frac{2M}{r} - r^2 \Omega^2 \right) = +1 \Rightarrow$$

$$\frac{dt}{d\tau} = \left(1 - \frac{2M}{r} - r^2 \Omega^2\right)^{-1/2}$$

Four velocity $u^\mu = \left(\frac{dt}{d\tau}, \cancel{\frac{dr}{d\tau}}, \cancel{\frac{d\theta}{d\tau}}, \frac{d\varphi}{d\tau} \right)$

$$= \left(\frac{dt}{d\tau}, 0, 0, \frac{d\varphi}{dt}\frac{dt}{d\tau} \right)$$

$$= \frac{dt}{d\tau} \left(1, 0, 0, \Omega \right)$$

$$u^\mu u_\mu = g_{\mu\nu} u^\mu u^\nu = -1 \Rightarrow -\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + r^2 \Omega^2 \left(\frac{dt}{d\tau}\right)^2 = -1 \quad \Rightarrow$$

$$\left(\frac{dt}{d\tau}\right)^2 \left(1 - \frac{2M}{r} - r^2 \Omega^2 \right) = +1 \quad \Rightarrow$$

$$\frac{dt}{d\tau} = \left(1 - \frac{2M}{r} - r^2 \Omega^2\right)^{-1/2} = \left(1 - \frac{2M}{r} - r^2 \frac{M}{r^3}\right)^{-1/2}$$

Four velocity $u^\mu = \left(\frac{dt}{d\tau}, \cancel{\frac{dr}{d\tau}}, \cancel{\frac{d\theta}{d\tau}}, \frac{d\varphi}{d\tau} \right)$

$$= \left(\frac{dt}{d\tau}, 0, 0, \frac{d\varphi}{dt}\frac{dt}{d\tau} \right)$$

$$= \frac{dt}{d\tau} (1, 0, 0, \Omega)$$

$$u^\mu u_\mu = g_{\mu\nu} u^\mu u^\nu = -1 \Rightarrow -\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + r^2 \Omega^2 \left(\frac{dt}{d\tau}\right)^2 = -1 \Rightarrow$$

$$\left(\frac{dt}{d\tau}\right)^2 \left(1 - \frac{2M}{r} - r^2\Omega^2 \right) = +1 \Rightarrow$$

$$\frac{dt}{d\tau} = \left(1 - \frac{2M}{r} - r^2\Omega^2\right)^{-1/2} = \left(1 - \frac{2M}{r} - r^2 \frac{M}{r^3}\right)^{-1/2}$$

$$= \left(1 - \frac{3M}{r}\right)^{-1/2}$$

Four velocity $u^\mu = \left(\frac{dt}{d\tau}, \cancel{\frac{dr}{d\tau}}, \cancel{\frac{d\theta}{d\tau}}, \frac{d\varphi}{d\tau} \right)$

$$= \left(\frac{dt}{d\tau}, 0, 0, \frac{d\varphi}{dt}\frac{dt}{d\tau} \right)$$

$$= \frac{dt}{d\tau}\left(1, 0, 0, \Omega \right)$$

$$\Rightarrow \quad u^\mu = \left(\left(1 - \frac{3M}{r}\right)^{-1/2}, 0, 0, \left(1 - \frac{3M}{r}\right)^{-1/2} \frac{M}{r^3} \right)$$

$$\frac{dt}{d\tau} = \left(1 - \frac{2M}{r} - r^2\Omega^2\right)^{-1/2} = \left(1 - \frac{2M}{r} - r^2\frac{M}{r^3}\right)^{-1/2}$$

$$= \left(1 - \frac{3M}{r}\right)^{-1/2}$$