Stress-Energy Tensor

Conservation Laws

Hartle ch 22

Ferrari ch 5, 7.4.1

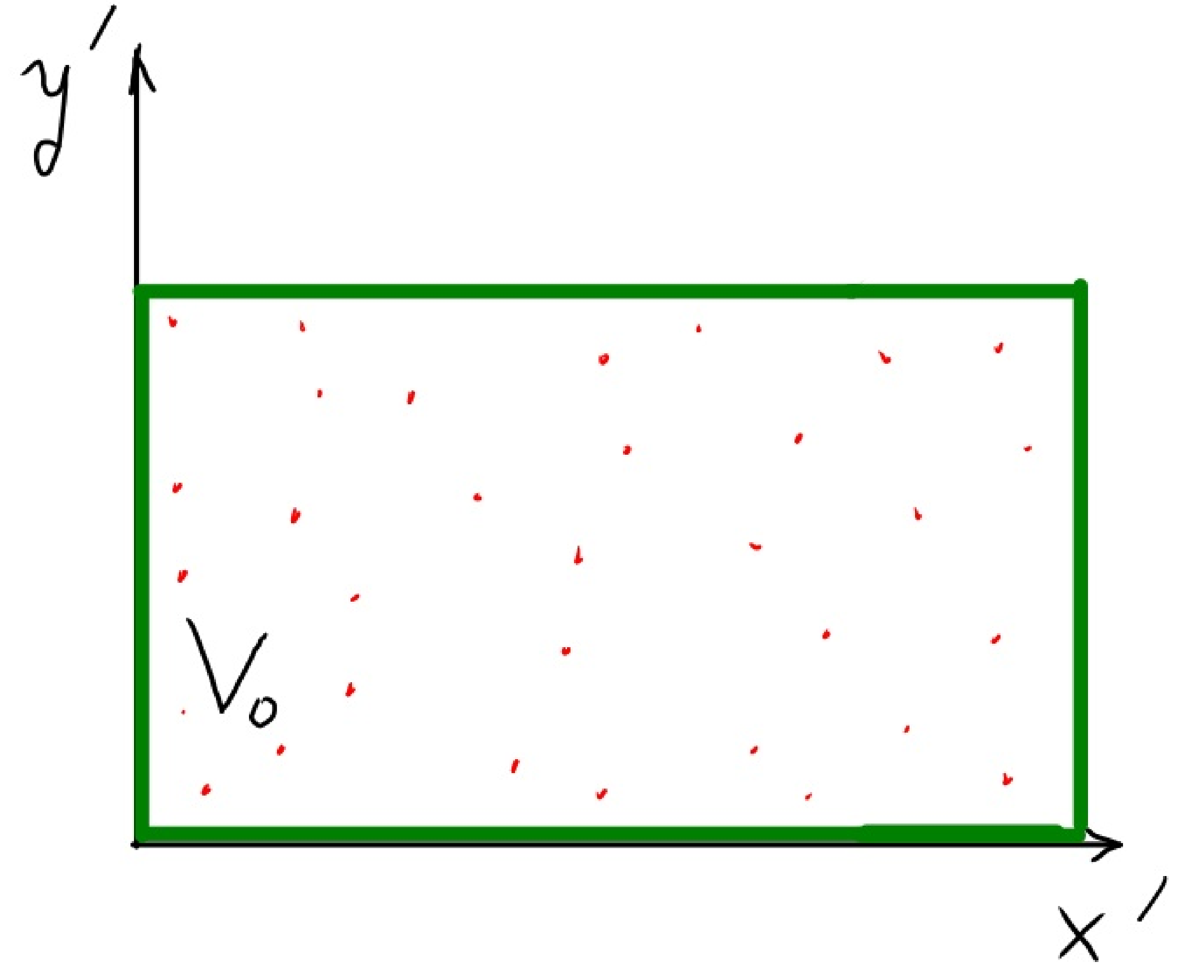
Misner et al p. 130 – 160

Stress-Energy Tensor in Special Relativity

Number Density:

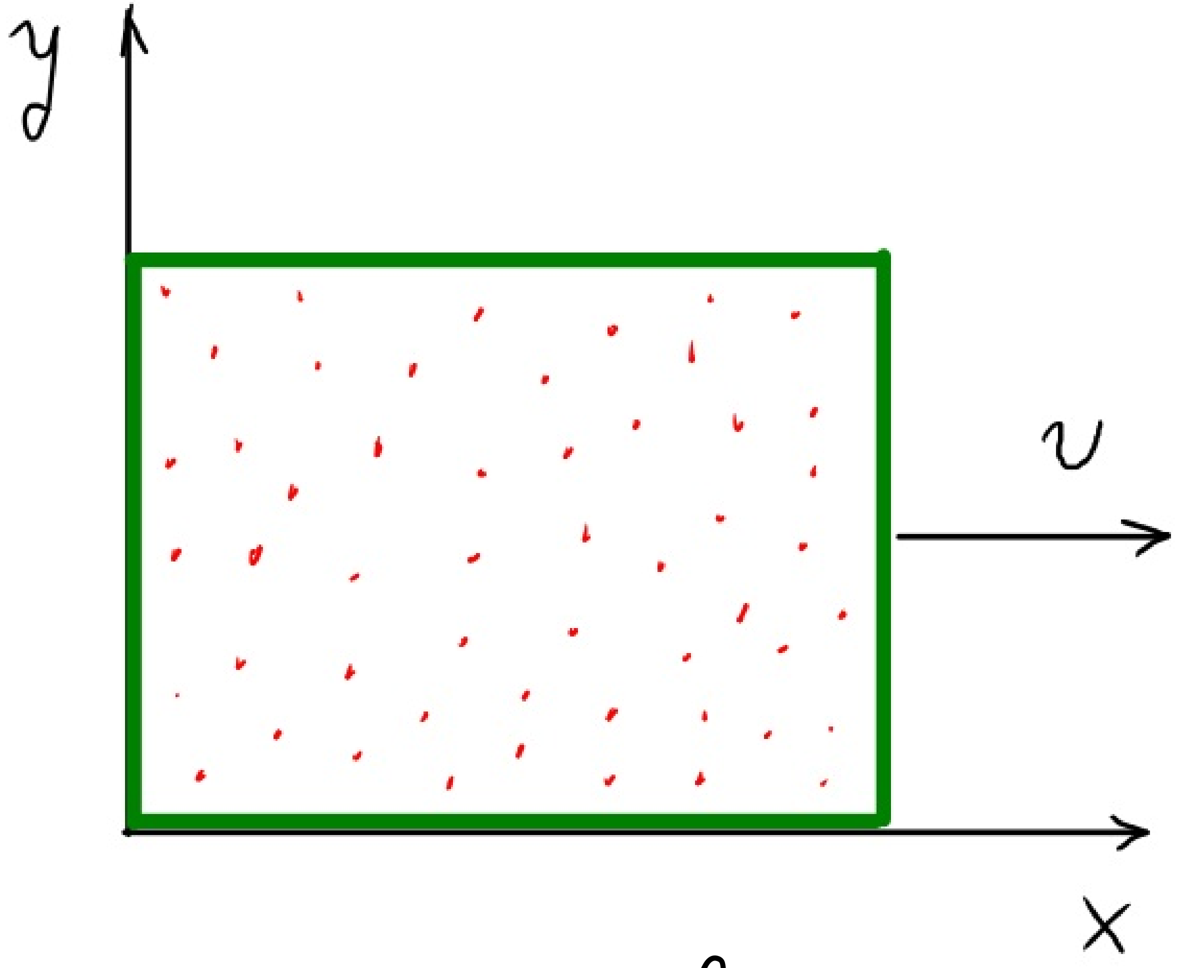
N particles

V_0 volume of box in their common rest frame



Stress-Energy Tensor in Special Relativity

Number Density:



N particles

V_0 volume of box in their common rest frame

v common velocity of particles in (x, y) inertial frame

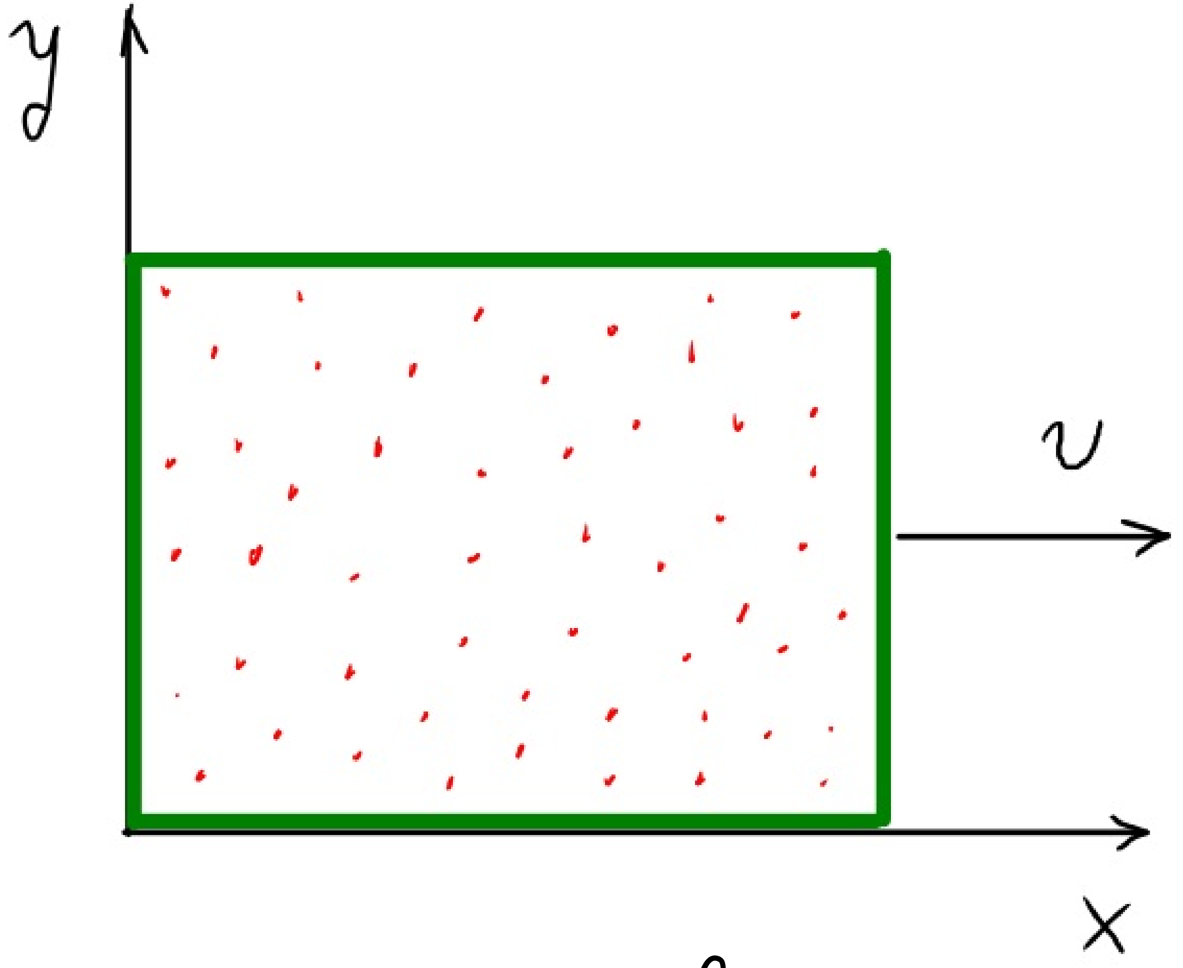
$V = \frac{V_0}{\gamma} = (1-v^2)^{1/2} V_0$ volume of box in (x,y) frame

Stress-Energy Tensor in Special Relativity

$n_0 = \frac{N}{V_0}$ number density in rest frame

same

$n = \frac{N}{V}$ " " " (x,y) "



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Stress-Energy Tensor in Special Relativity

$n_0 = \frac{N}{V_0}$ number density in rest frame

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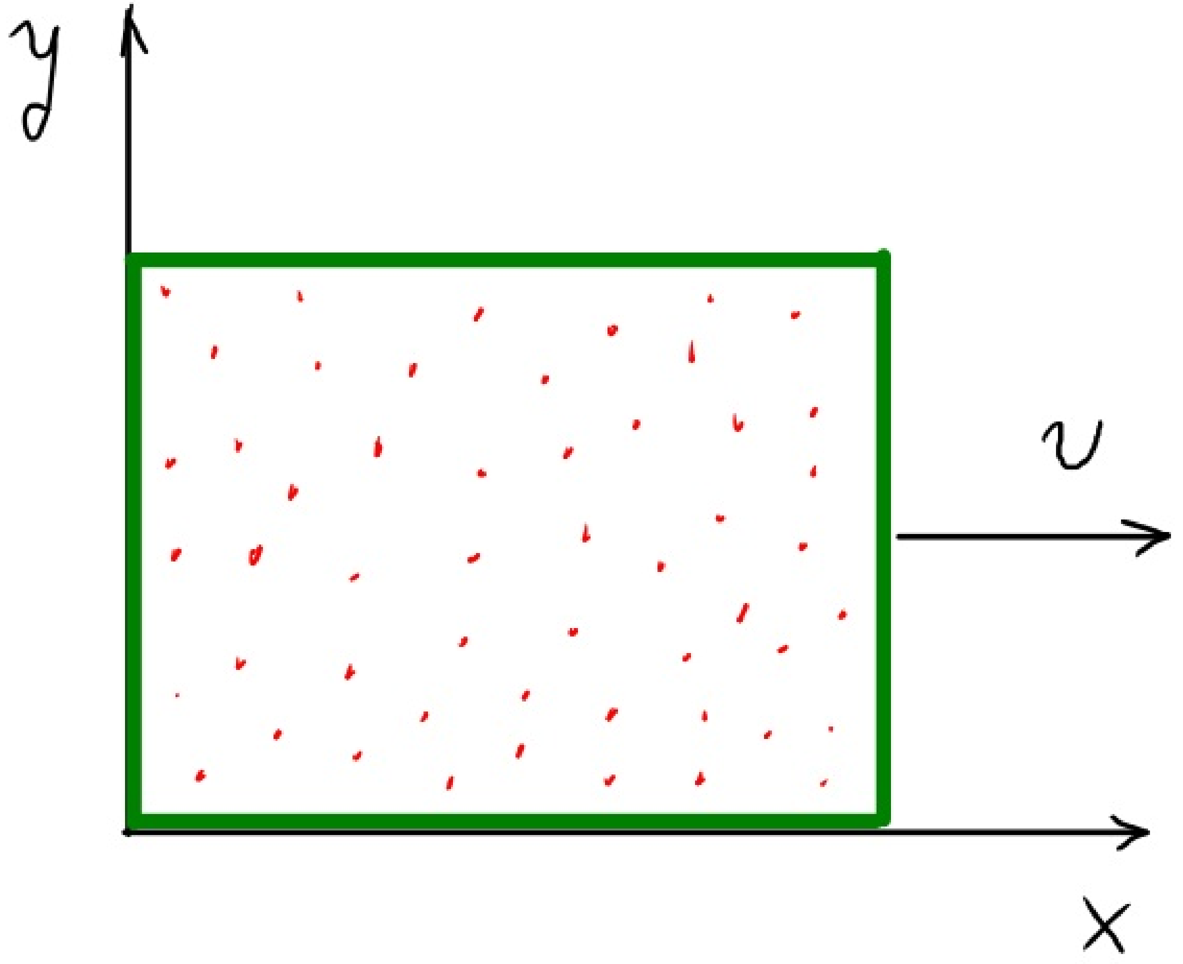
$n = \frac{N}{V}$ " " " (x,y) "

$n = \frac{N}{V_0/\gamma} = \gamma \frac{N}{V_0} = \gamma n_0$



Stress-Energy Tensor in Special Relativity

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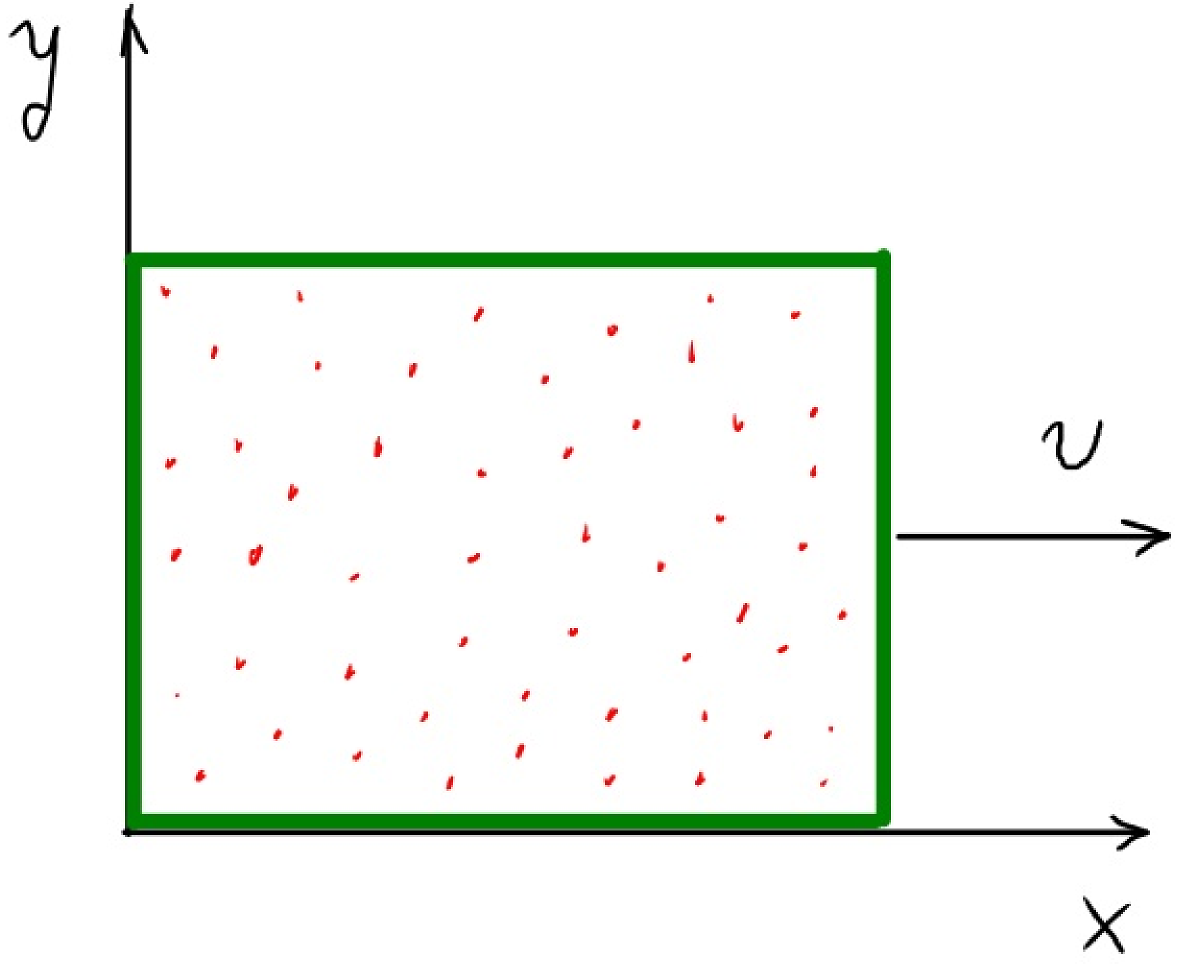

$$n = \frac{N}{V_0/\gamma} = \gamma \frac{N}{V_0} = \gamma n_0$$

But $u^0 = \gamma$, so $n = n_0 u^0$

$$u^\mu = (\gamma, \gamma \vec{v})$$

Stress-Energy Tensor in Special Relativity

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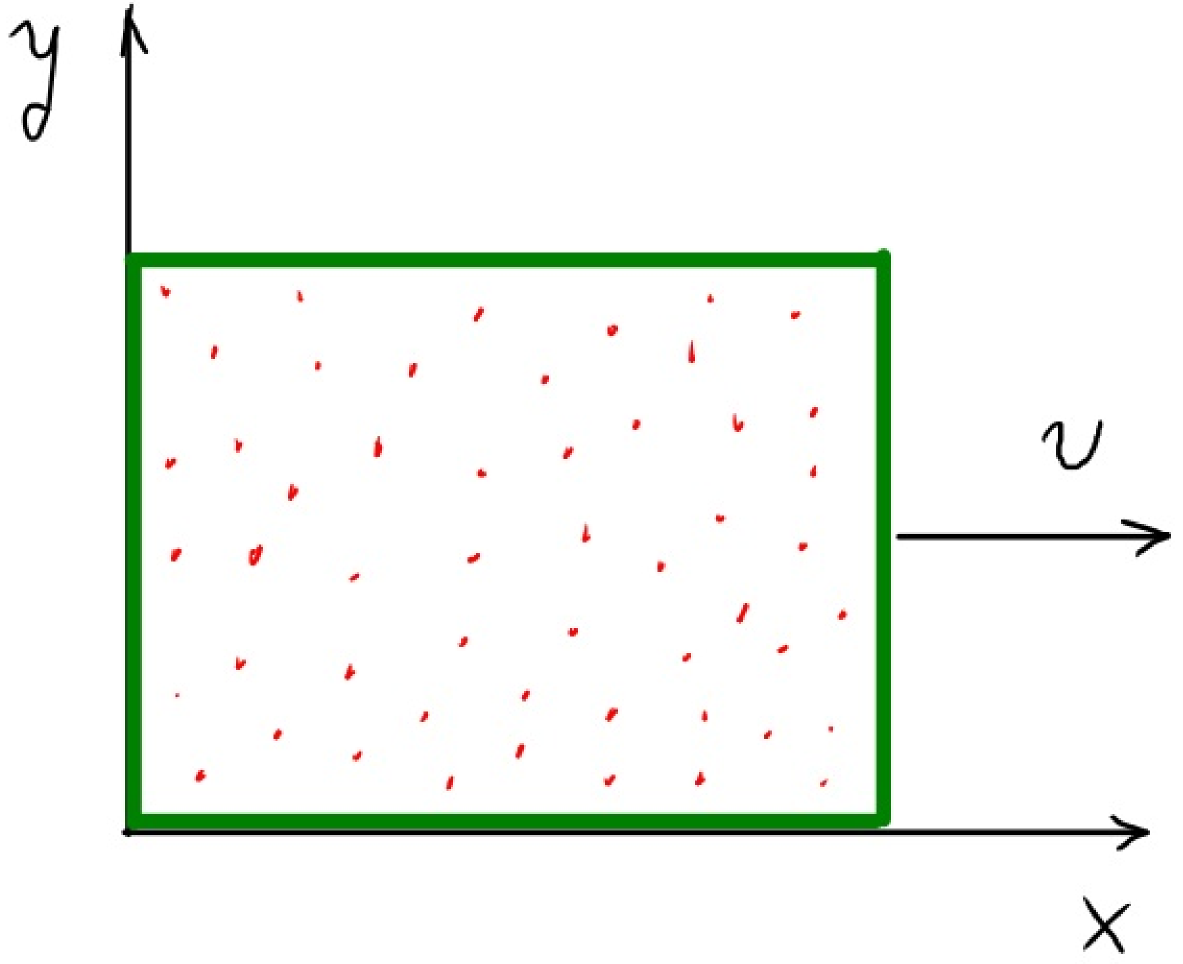

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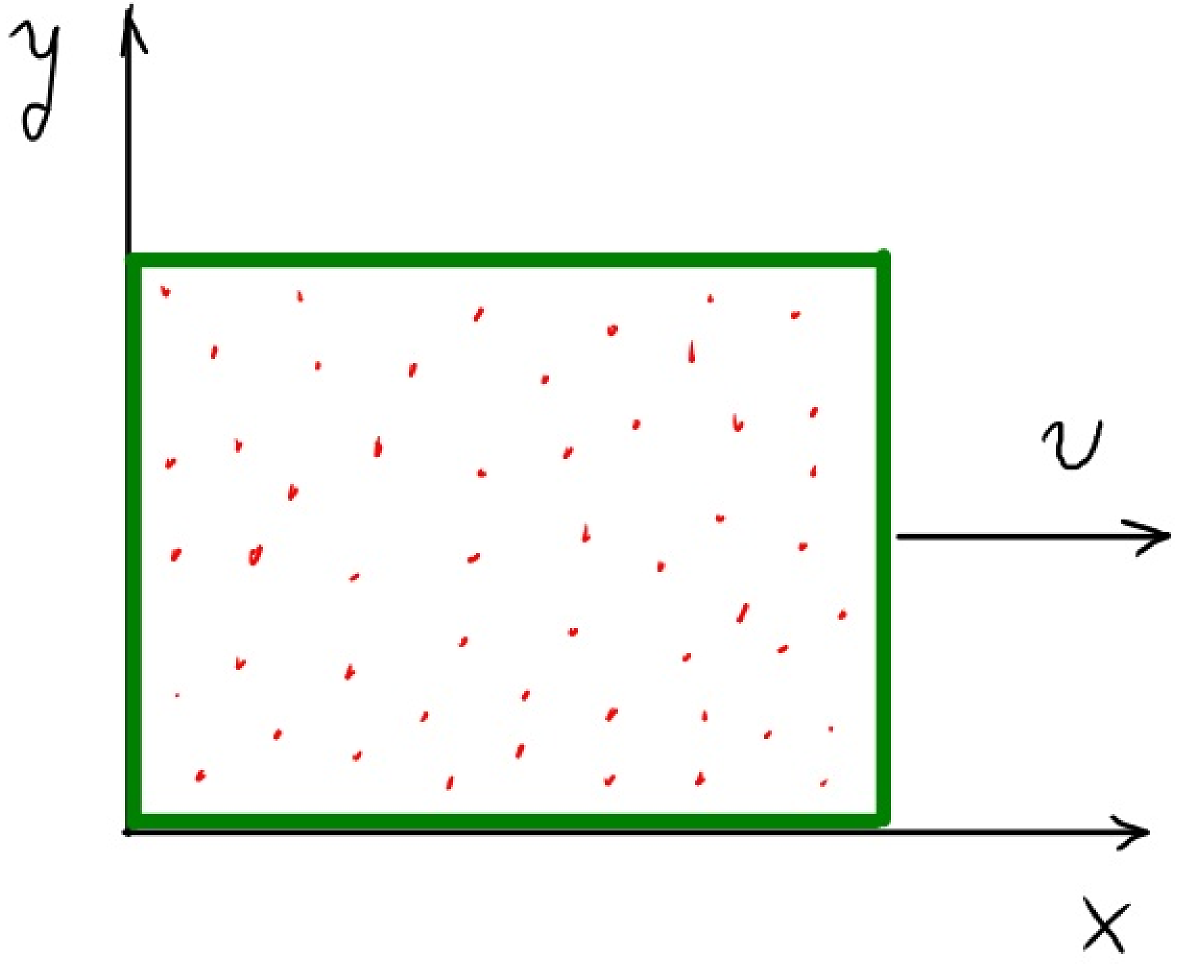
But $u^0 = \gamma$, so $n = n_0 u^0$

We define $j^\mu = n_0 u^\mu$, the number current density

Stress-Energy Tensor in Special Relativity

$$j^\mu = n_0(\gamma, \gamma\vec{v})$$



$$j^0 = n_0\gamma \qquad j^i = n_0\gamma v^i$$

We define $j^\mu = n_0 U^\mu$, the number current density

Stress-Energy Tensor in Special Relativity

$$j^\mu = n_0(\gamma, \gamma\vec{u})$$

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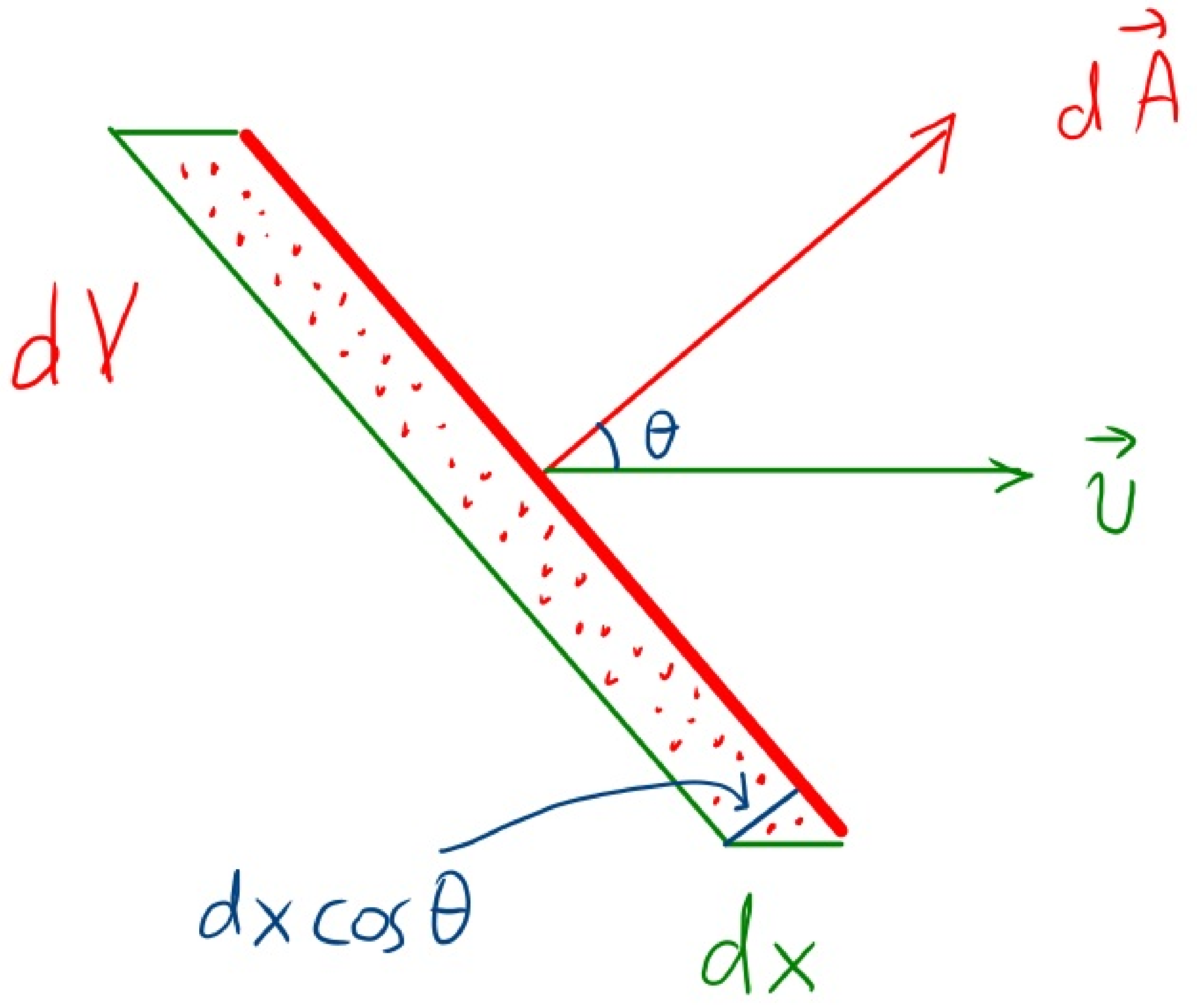

In time dt, all particles in dV cross surface $d\vec{A}$:

$$dN = n \cdot dV$$

$$dx = u\,dt$$

Stress-Energy Tensor in Special Relativity

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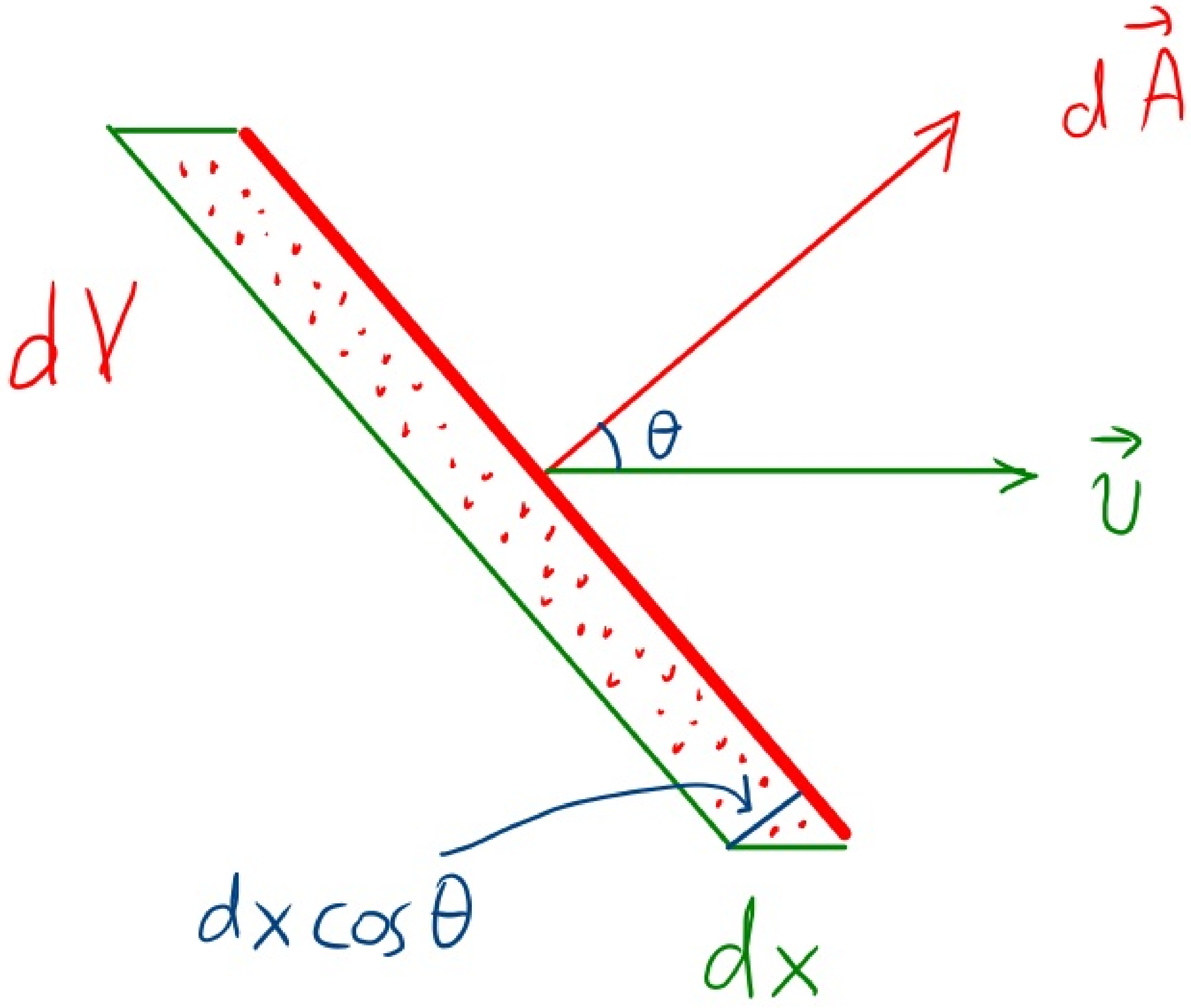
$$j^0 = n_0\gamma \qquad j^i = n_0\gamma u^i$$



In time dt, all particles in dV cross surface $d\vec{A}$:

$$dN = n \cdot dV = n(dx\cos\theta)\,dA$$

Stress-Energy Tensor in Special Relativity

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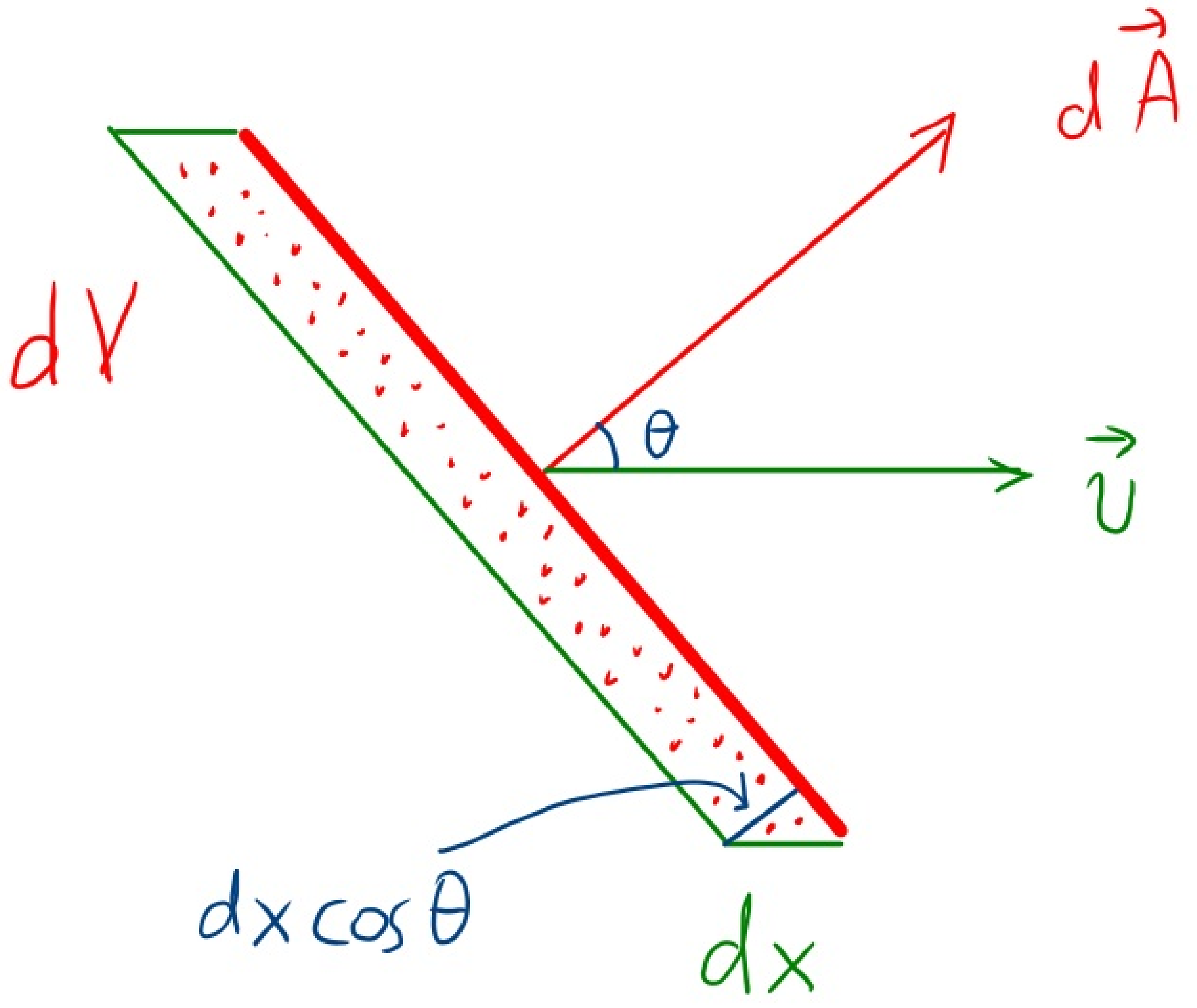


In time dt, all particles in dV cross surface $d\vec{A}$:

$$dN = n \cdot dV = n(dx\cos\theta)\,dA \Rightarrow$$

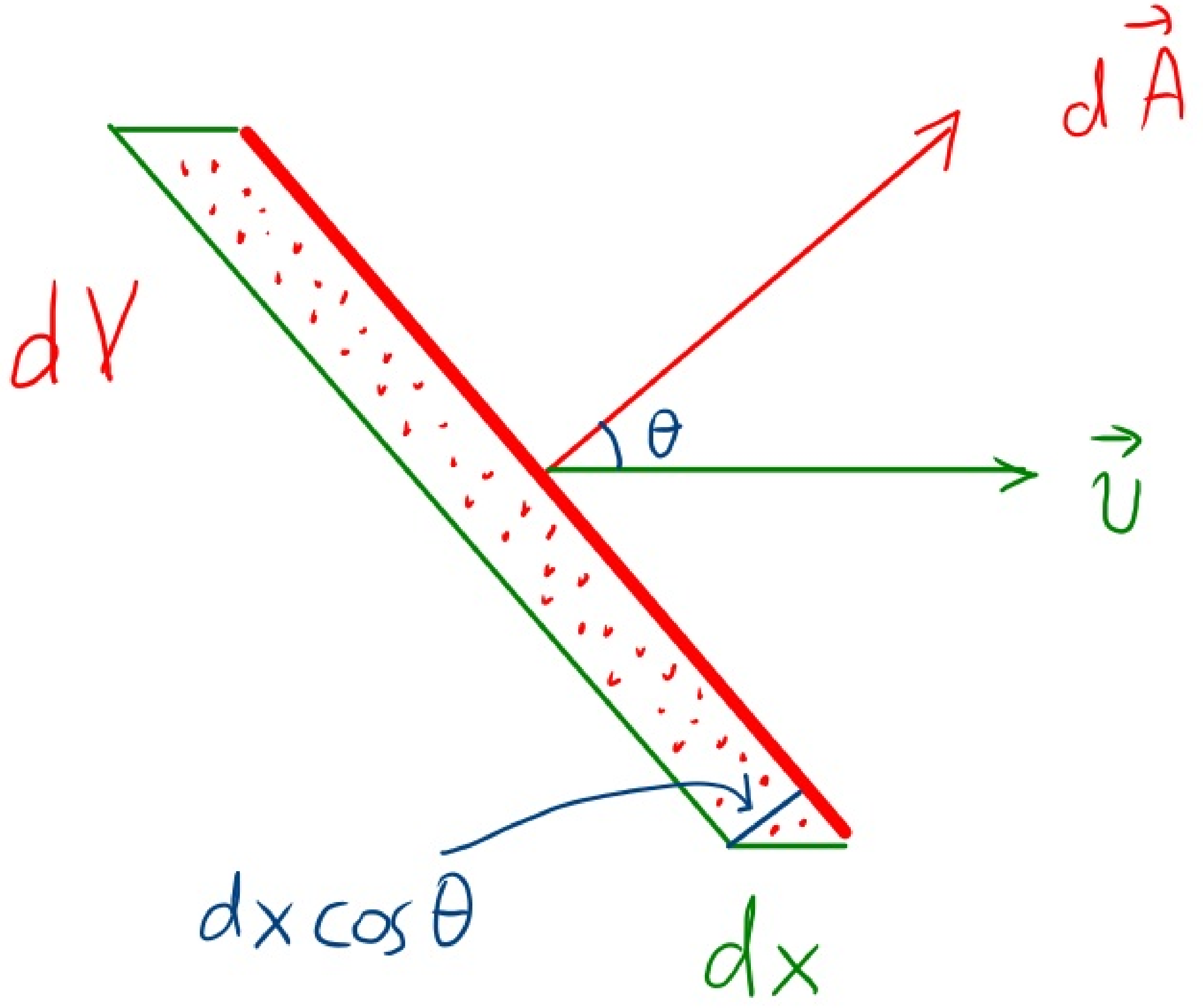
$$\frac{dN}{dt} = n\frac{dx}{dt}\,dA\cos\theta$$

$$u = \frac{dx}{dt}$$

Stress-Energy Tensor in Special Relativity

$$j^\mu = n_0(\gamma, \gamma\vec{v})$$

$$j^0 = n_0\gamma \qquad j^i = n_0\gamma v^i$$



In time dt, all particles in dV cross surface $d\vec{A}$:

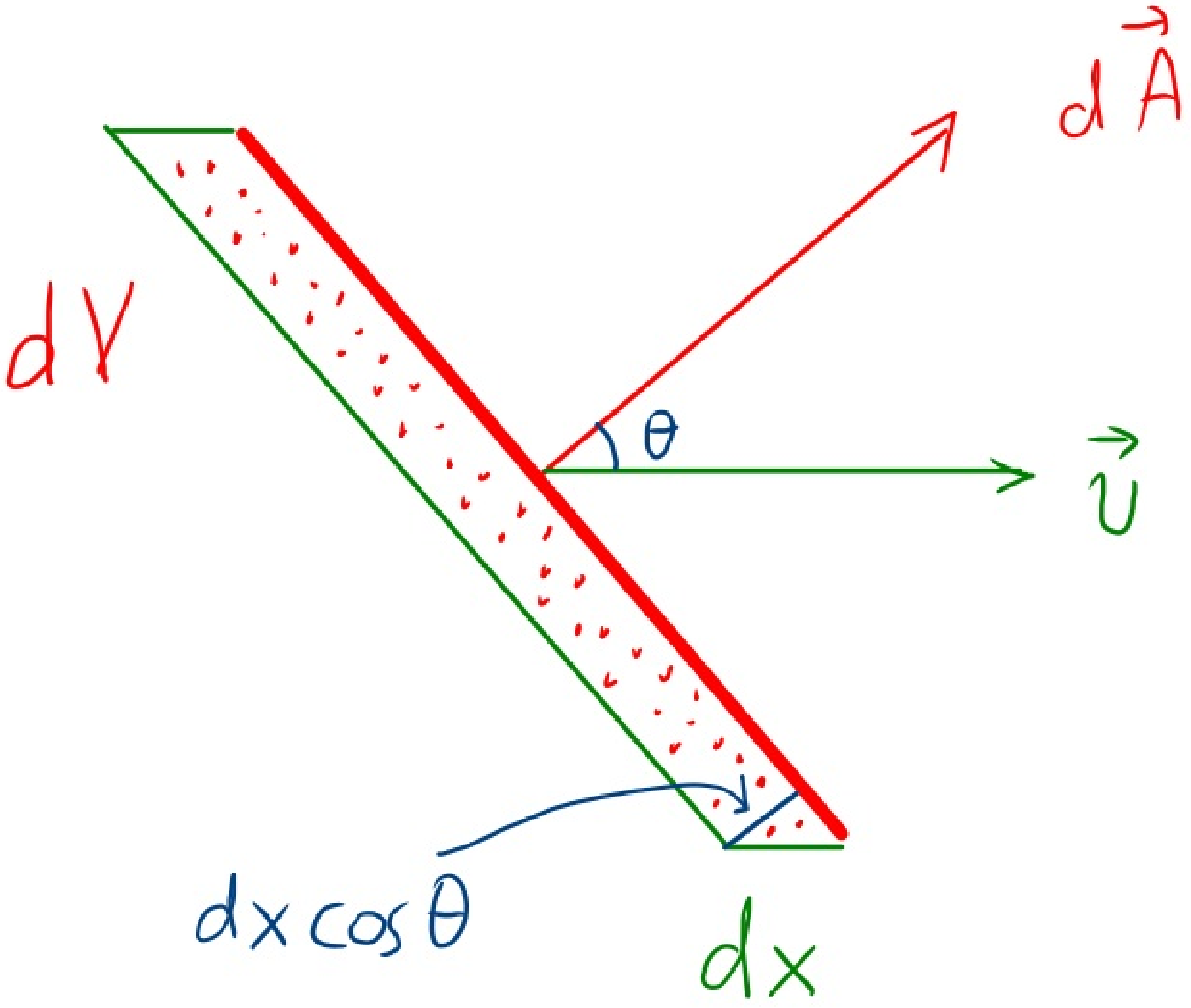
$$dN = n \cdot dV = n(dx\cos\theta)\,dA \Rightarrow$$

$$\frac{dN}{dt} = n\frac{dx}{dt}\,dA\cos\theta = n\,v\cdot dA\cos\theta = n\,\vec{v}\cdot d\vec{A}$$

Stress-Energy Tensor in Special Relativity

$$j^\mu = n_0(\gamma, \gamma\vec{v})$$

$$j^0 = n_0\gamma = n, \quad j^i = n_0 \gamma v^i = n v^i$$



In time dt, all particles in dV cross surface $d\vec{A}$:

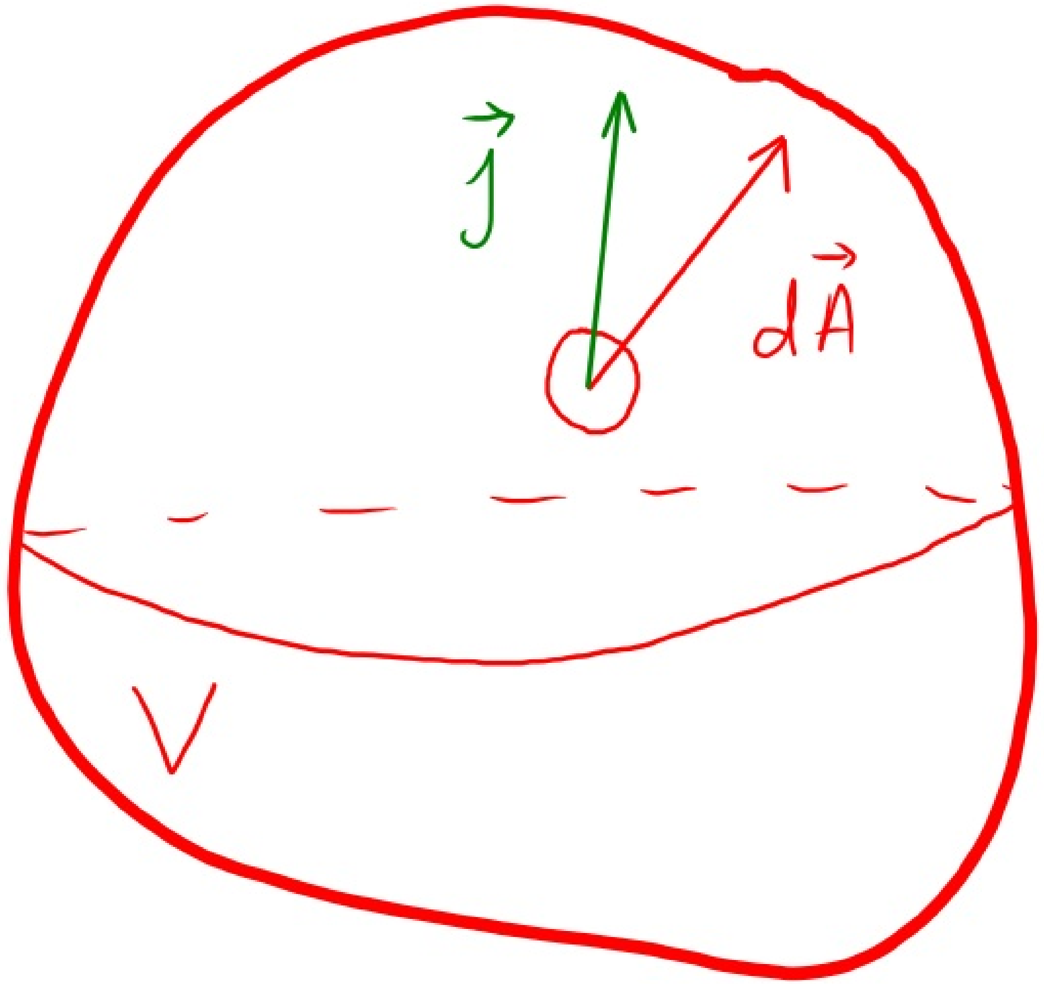
$$dN = n \cdot dV = n(dx\cos\theta)\,dA \Rightarrow$$

$$\frac{dN}{dt} = n\frac{dx}{dt}\,dA\cos\theta = n v \cdot dA\cos\theta = n\vec{v}\cdot d\vec{A}$$

$$= \vec{j}\cdot d\vec{A}$$

Stress-Energy Tensor in Special Relativity

Therefore if we have a closed surface $S = \partial V$, the rate of change of particles enclosed is:



$$\frac{dN}{dt} = -\int_{\partial V} \vec{j} \cdot d\vec{A}$$

((-) because $\vec{j} \cdot d\vec{A} > 0 \Rightarrow$ N decreases)

$$\frac{dN}{dt} = n \frac{dx}{dt} \, dA \cos\theta = n \, v \cdot dA \cos\theta = n \, \vec{v} \cdot d\vec{A}$$

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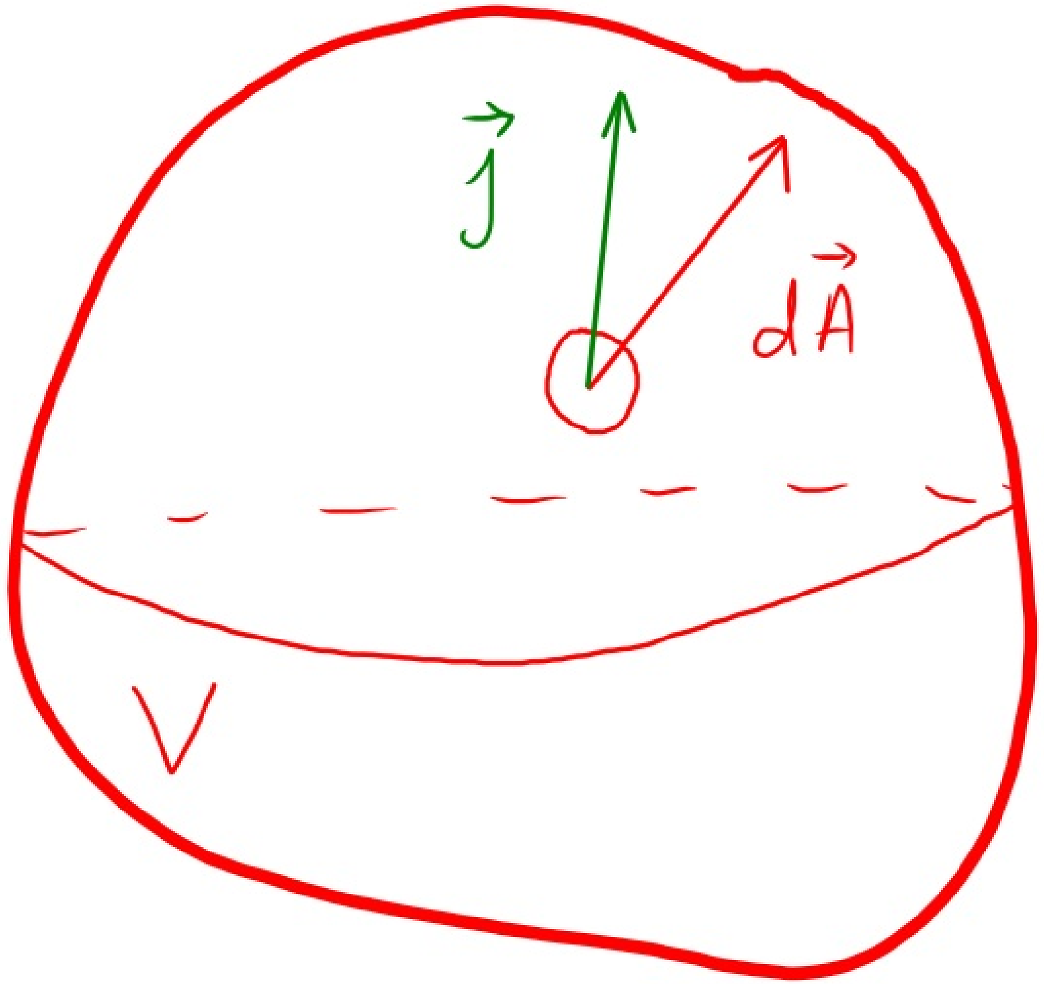

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$$\frac{dN}{dt} = \frac{d}{dt}\int_V n \cdot dV = \frac{d}{dt}\int_V j^0 \, dV = \int_V \frac{\partial j^0}{\partial t} dV$$

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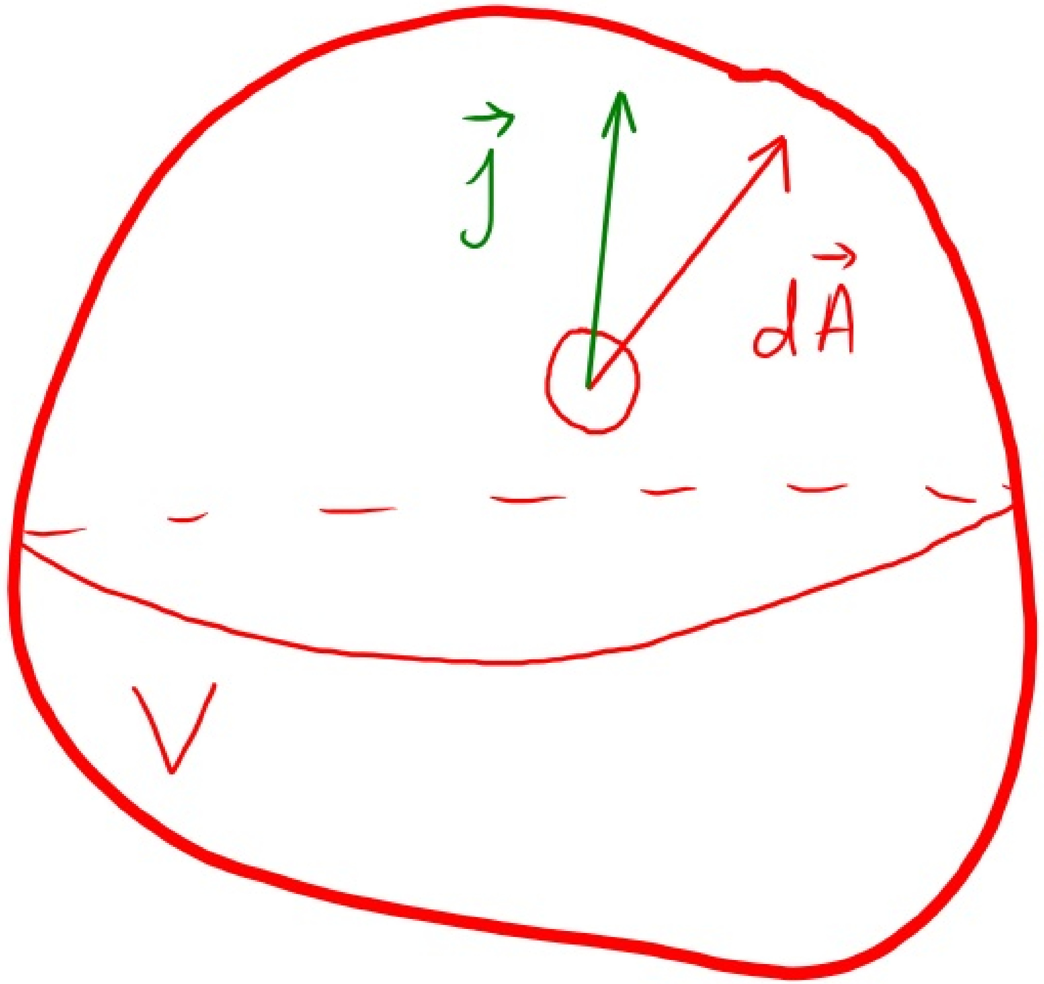

$$\frac{dN}{dt} = -\int_{\partial V} \vec{j} \cdot d\vec{A} = -\int_V \vec{\nabla} \cdot \vec{j} \; dV$$

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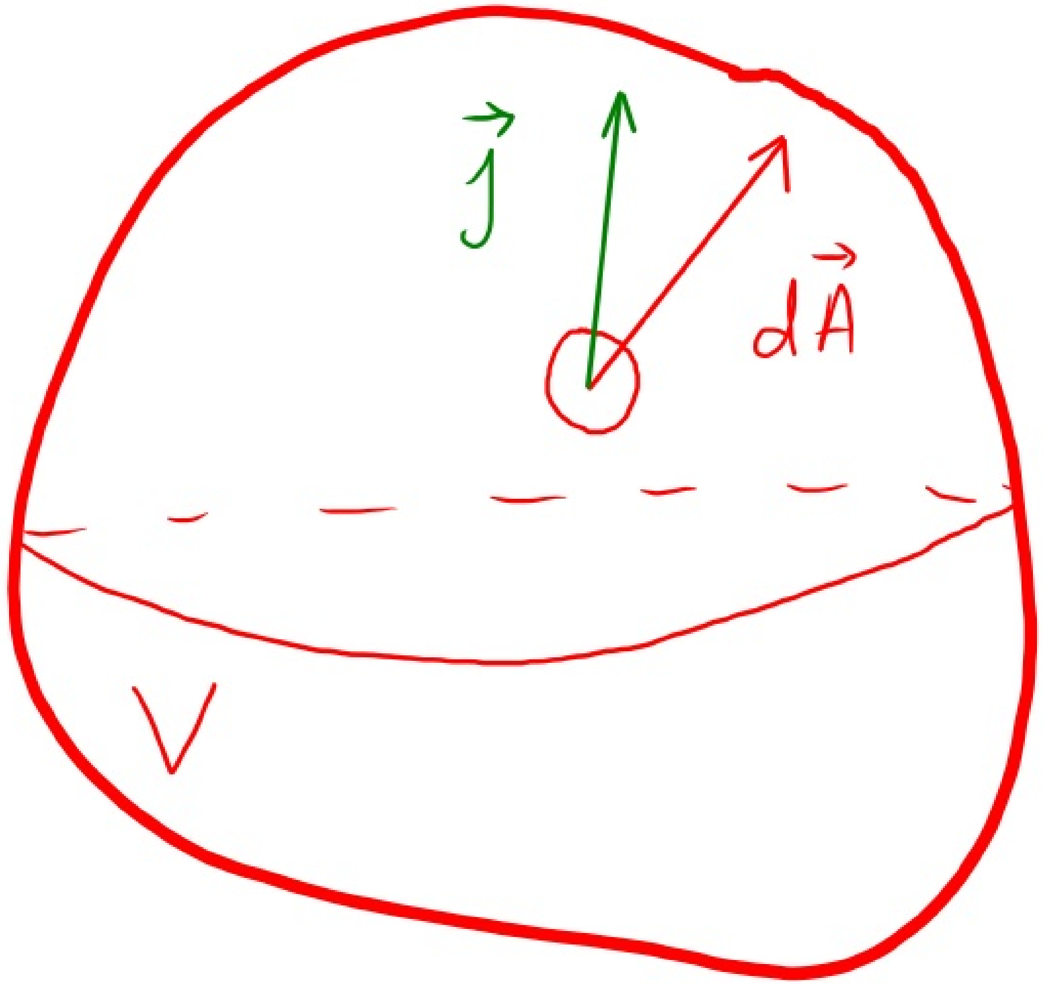

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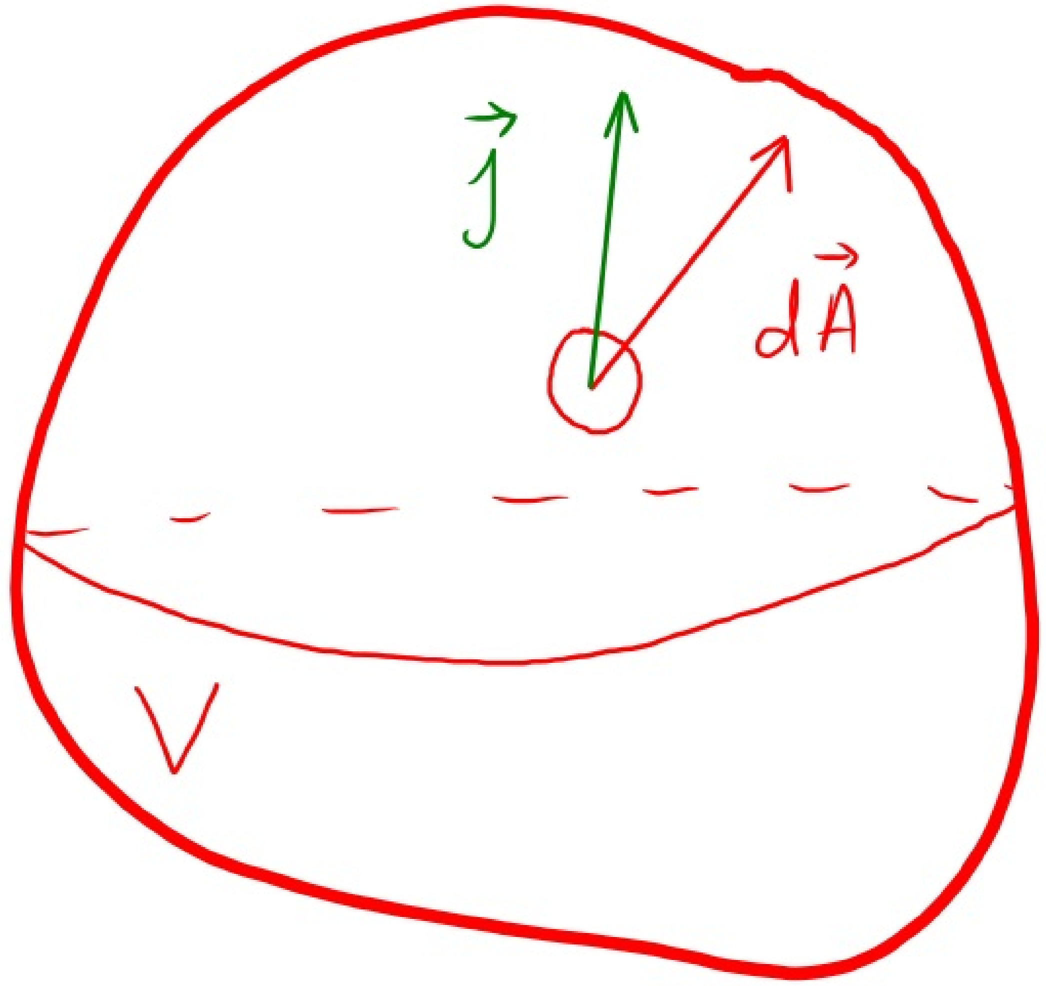

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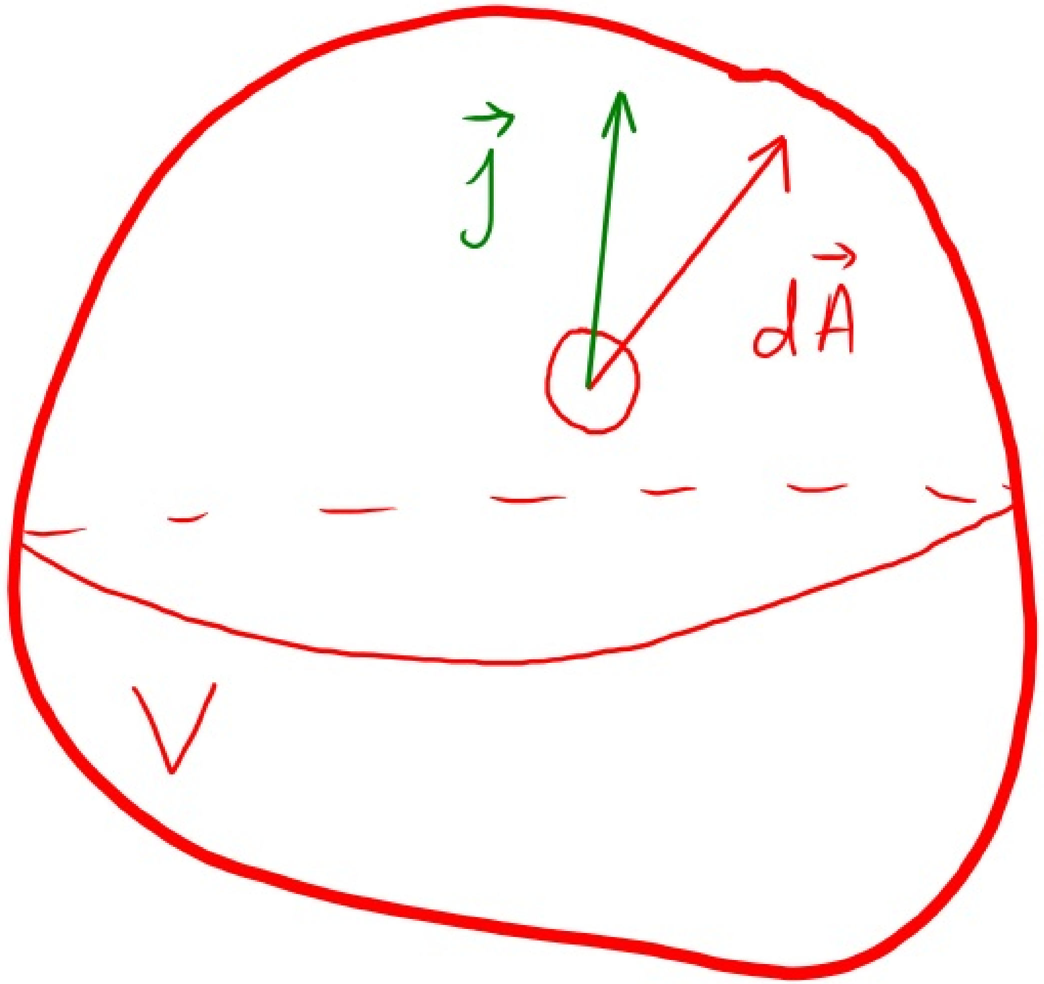
$$\Rightarrow \quad \frac{\partial j^0}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0 \Rightarrow \partial_0 j^0 + \partial_i j^i = 0$$

Stress-Energy Tensor in Special Relativity

Therefore if we have a closed surface $S = \partial V$, the rate of change of particles enclosed is:



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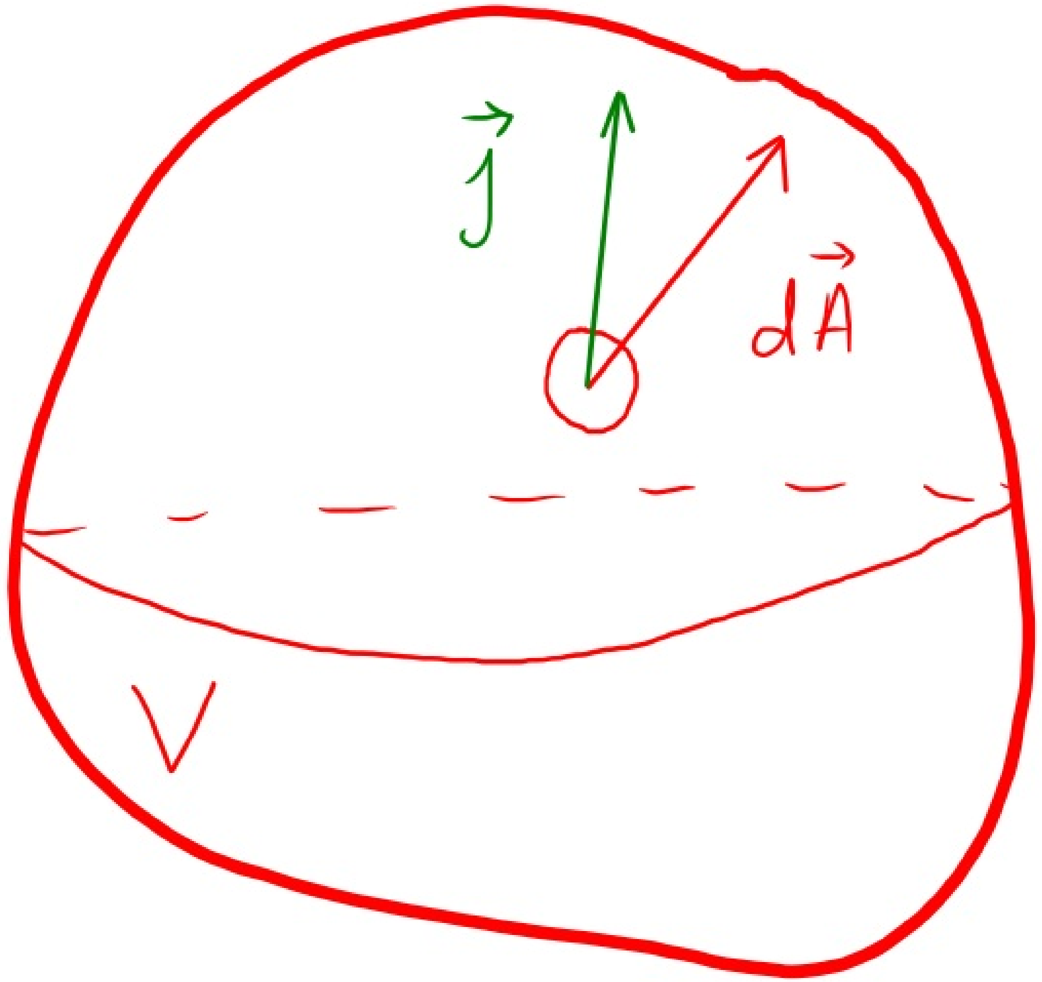
Stress - Energy Tensor in Special Relativity



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$$\Rightarrow \quad \underbrace{\frac{\partial j^0}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0}_{\text{continuity equation}} \quad \Rightarrow \quad \partial_0 j^0 + \partial_i j^i = 0 \quad \Rightarrow \quad \partial_\mu j^\mu = 0$$

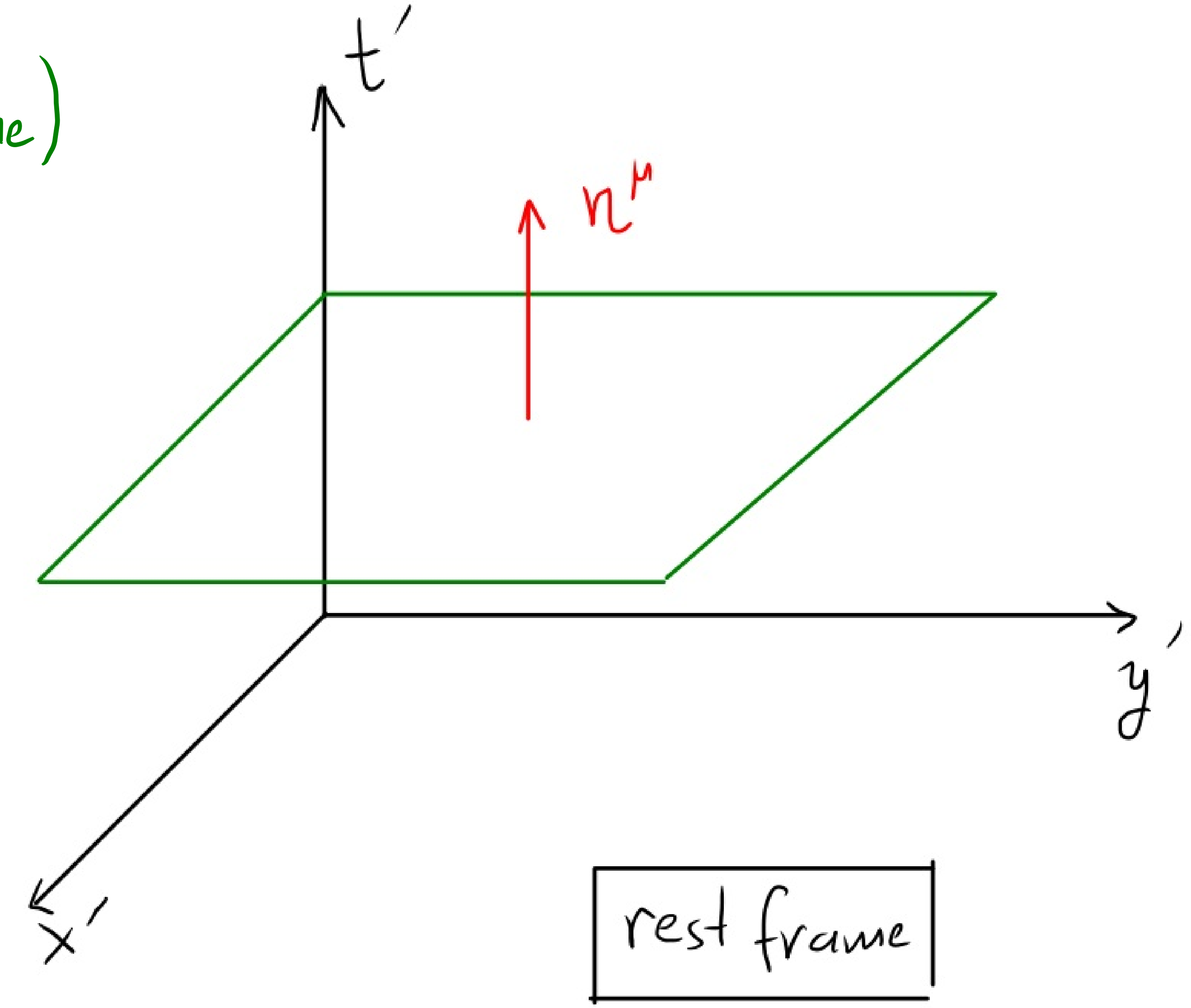
continuity equation $\longrightarrow$ we say that j^μ "conserved"

Stress-Energy Tensor in Special Relativity

Consider the spacelike 3-surface (3 volume)
$t' = \text{const.}$ in rest frame of particles:

$n^\mu = (1,0,0,0)$ normal to surface

$u^\mu = (1,0,0,0)$ 4-velocity of particles



Stress-Energy Tensor in Special Relativity

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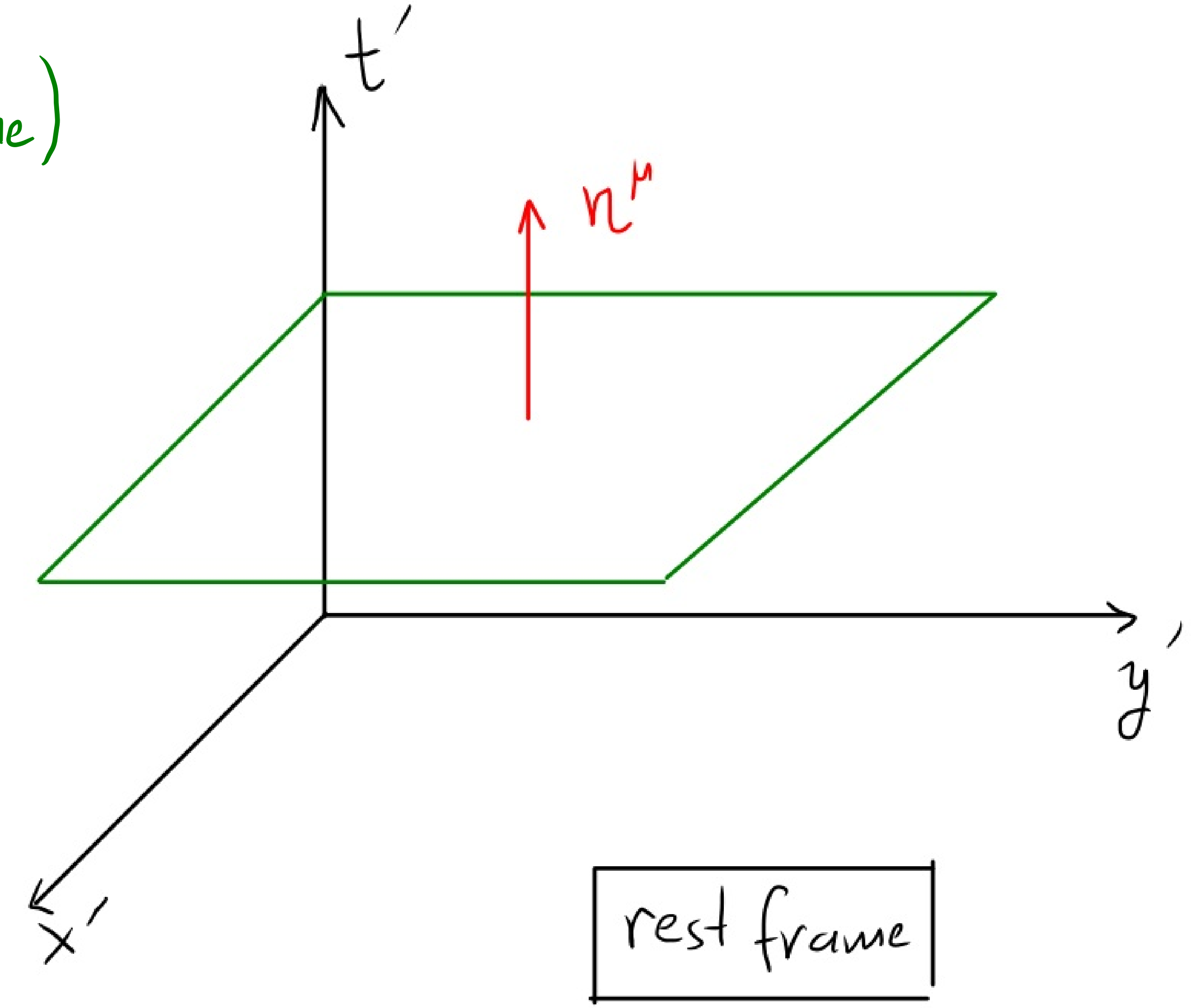

$n^\mu = (1,0,0,0)$ normal to surface

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timelike: $n_\mu n^\mu = \eta_{\mu\nu} n^\mu n^\nu = -1$

$u_\mu u^\mu = \eta_{\mu\nu} u^\mu u^\nu = -1$

Stress-Energy Tensor in Special Relativity

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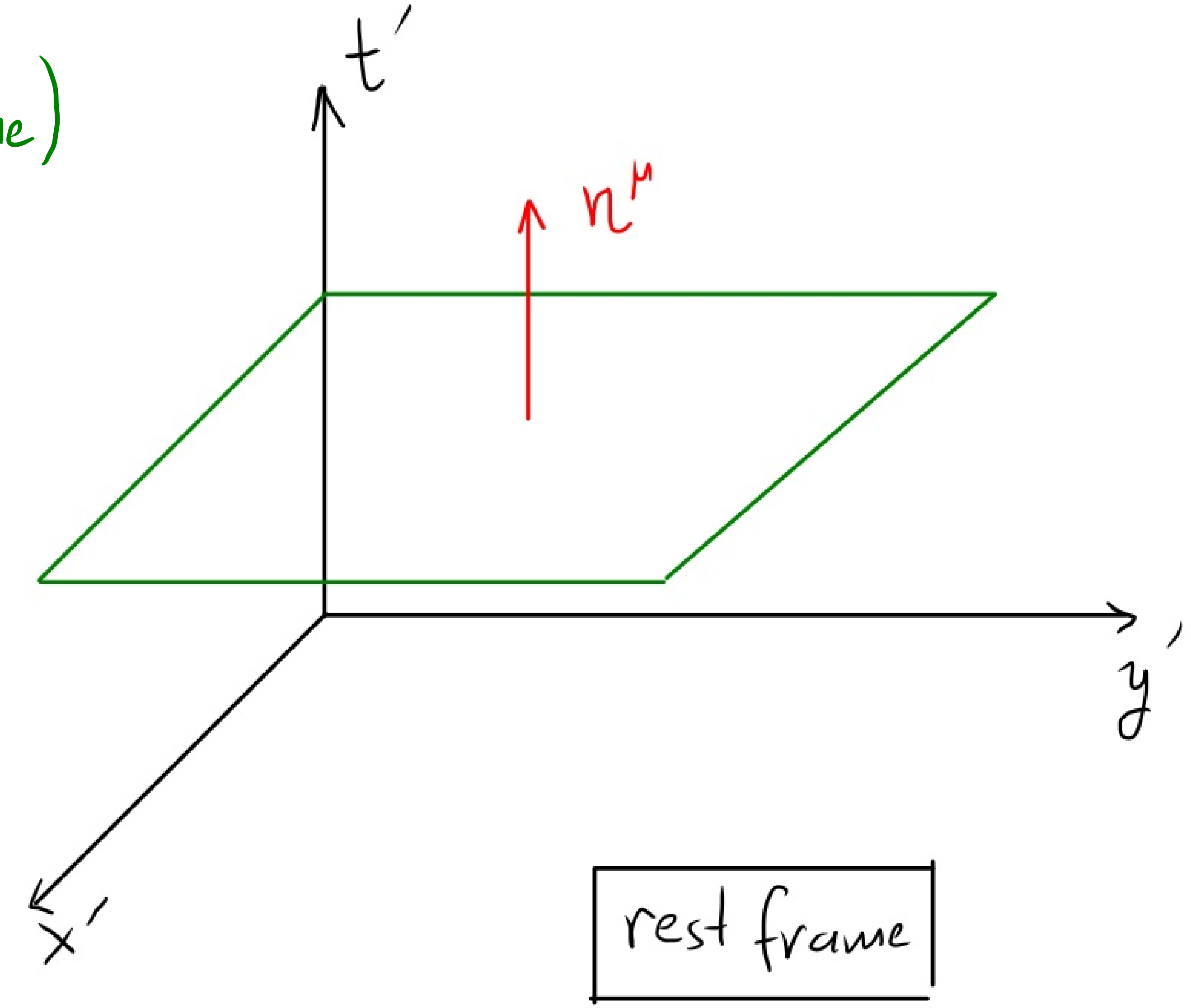

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then $j_\mu n^\mu = n_0 \eta_{\mu\nu} u^\mu n^\nu$

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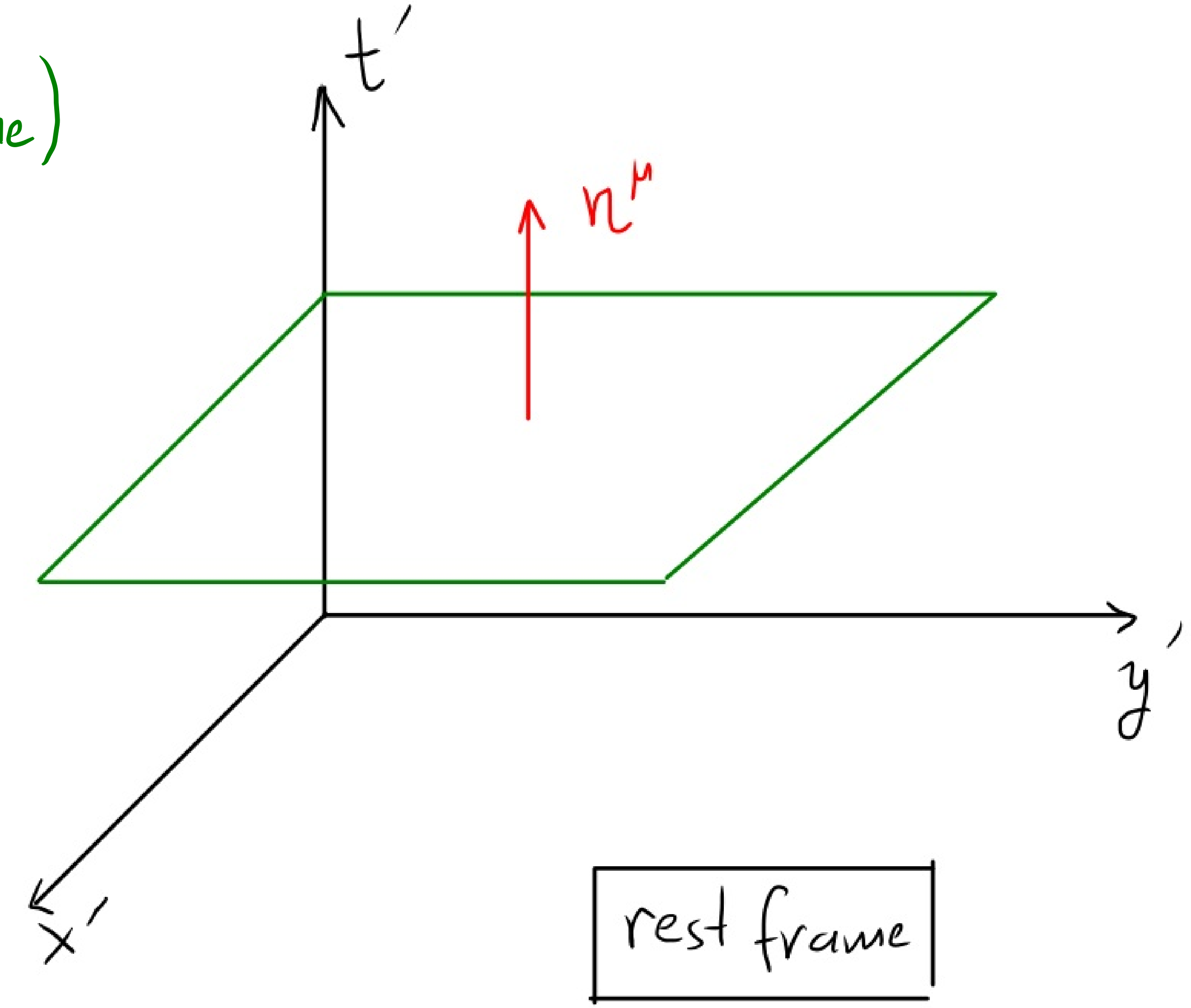

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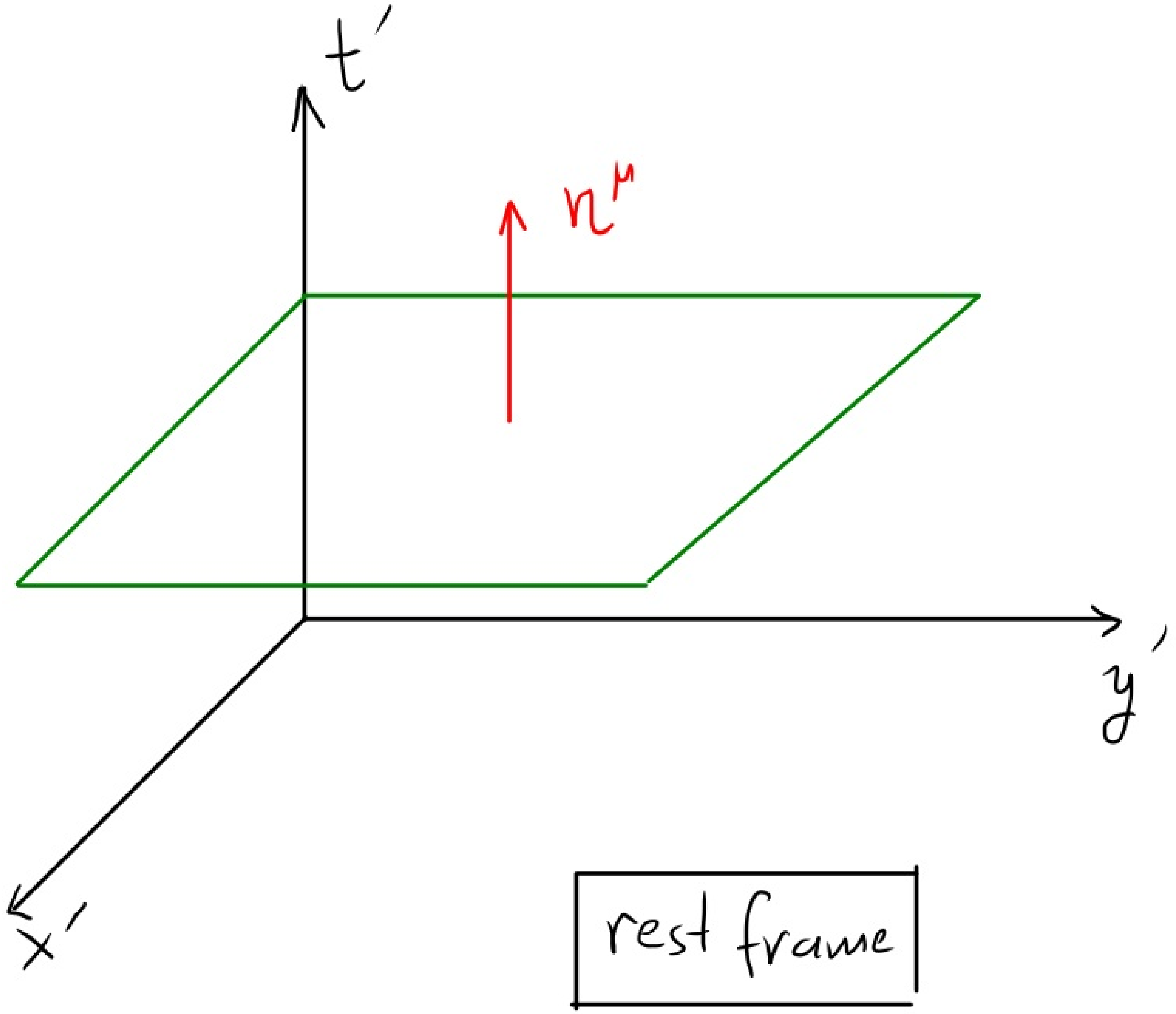
then $j_\mu n^\mu = n_0 \eta_{\mu\nu} u^\mu n^\nu = n_0 \eta_{00} u^0 n^0 = -n_0$

Stress-Energy Tensor in Special Relativity

Then $dN_0 = n_0 \, dV_0 = -j_\mu \eta^\mu \, dV_0$



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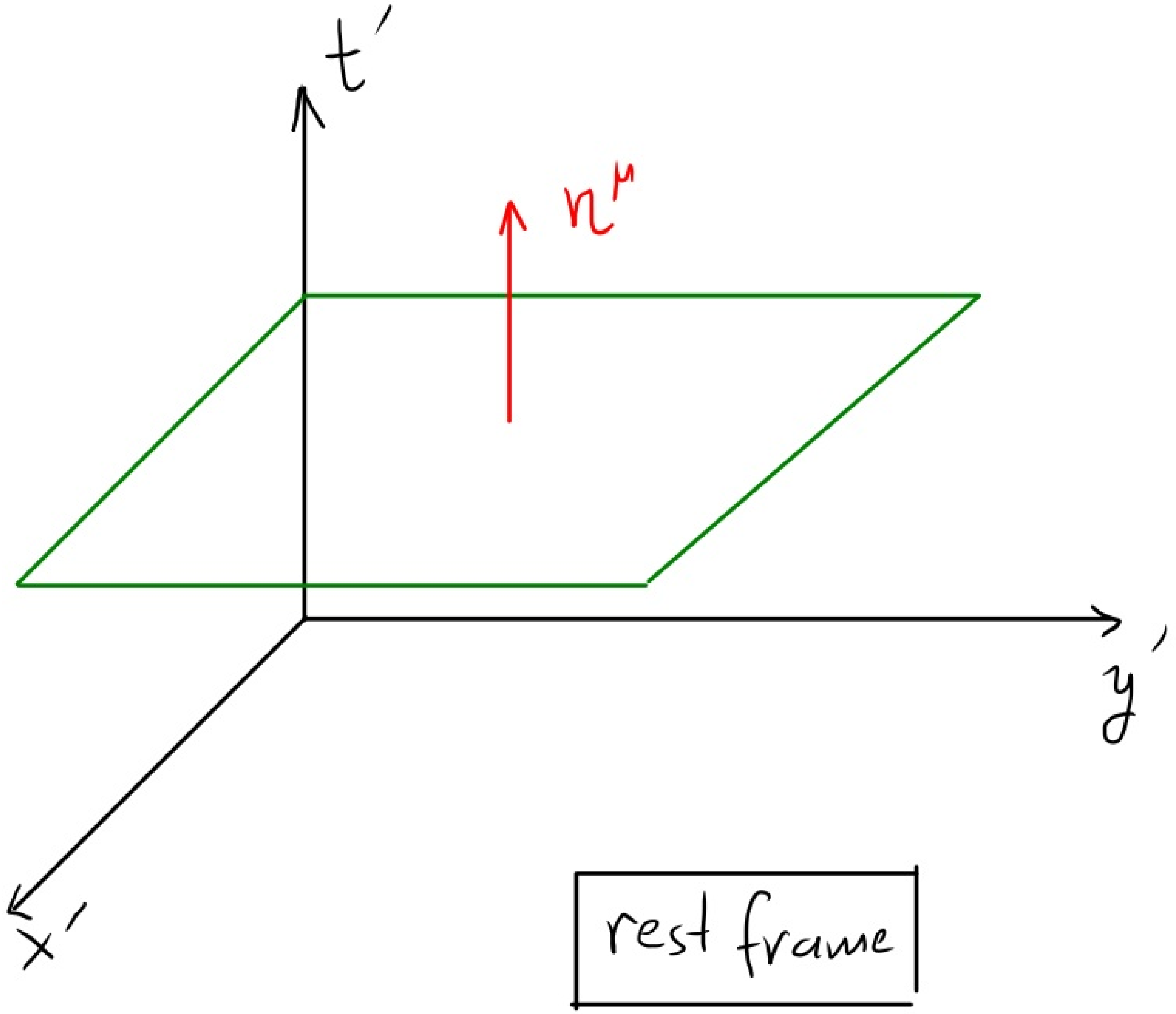
then $j_\mu \eta^\mu = n_0 \, \eta_{\mu\nu} u^\mu \eta^\nu = n_0 \, \eta_{00} u^0 \eta^0 = -n_0$

Stress-Energy Tensor in Special Relativity

Then $dN_0 = n_0\, dV_0 = -j_\mu \underbrace{n^\mu\, dV_0}_{\text{"area" element of 3-surface}}$



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Stress-Energy Tensor in Special Relativity

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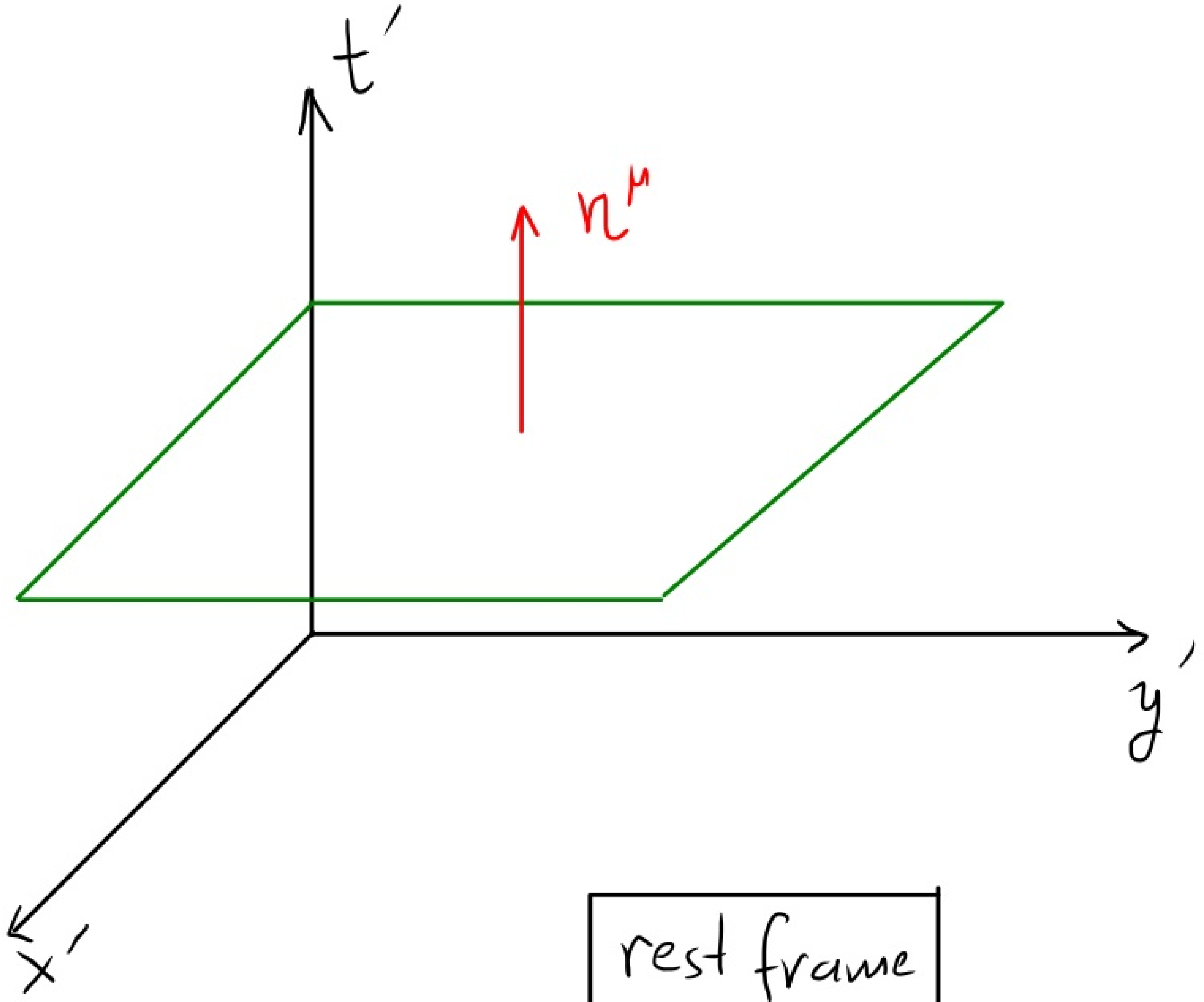
flux of number current through spacelike surface



rest frame

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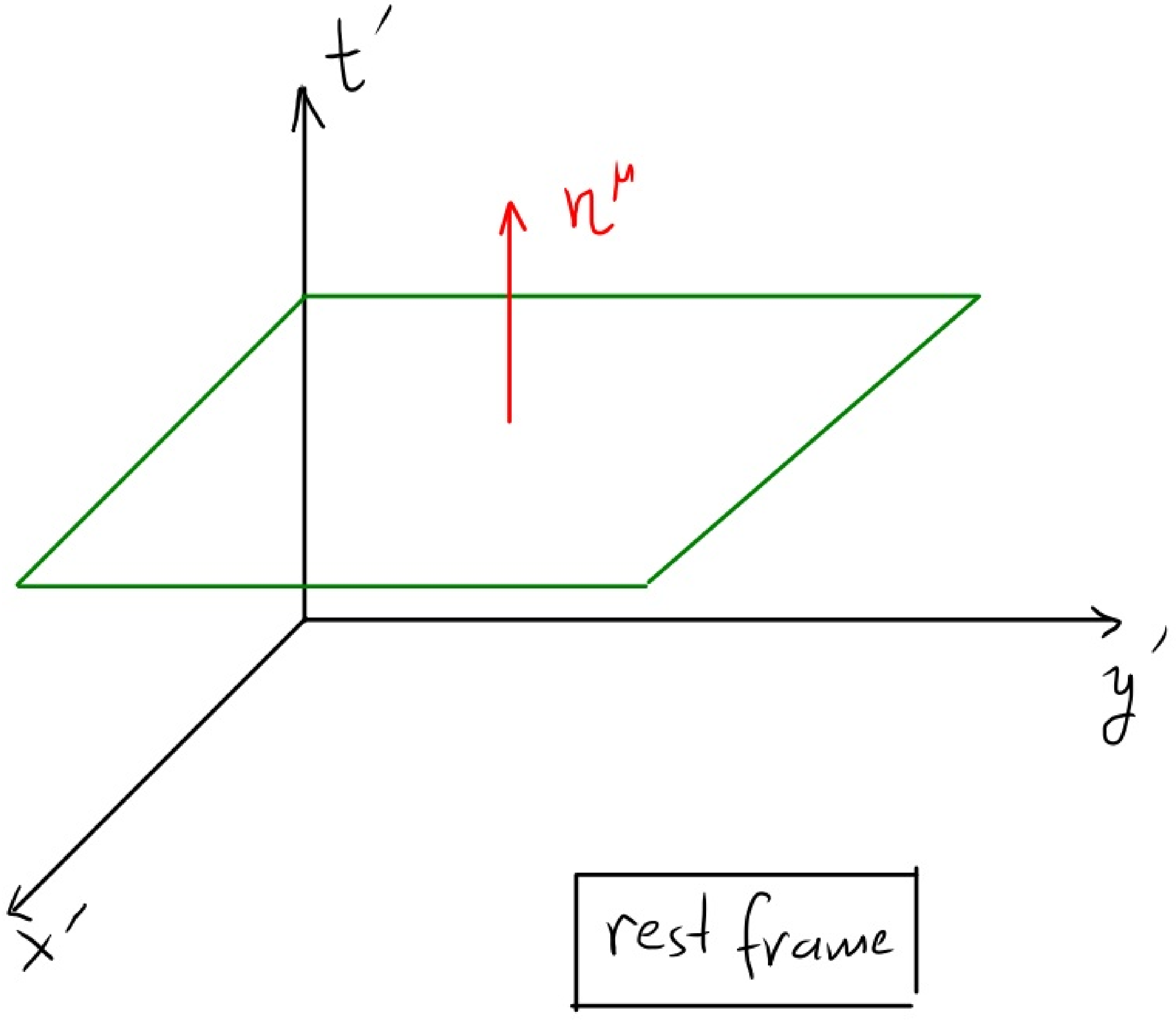
then $j_\mu n^\mu = n_0\, \eta_{\mu\nu} u^\mu n^\nu = n_0\, \eta_{00} u^0 n^0 = -n_0$

Stress-Energy Tensor in Special Relativity

Then $dN_0 = n_0\, dV_0 = -j_\mu \underbrace{\eta^\mu\, dV_0}_{\text{"area" element of 3-surface}}$

flux of number current through spacelike surface



- density n_0 is a flux of number current through spacelike surfaces

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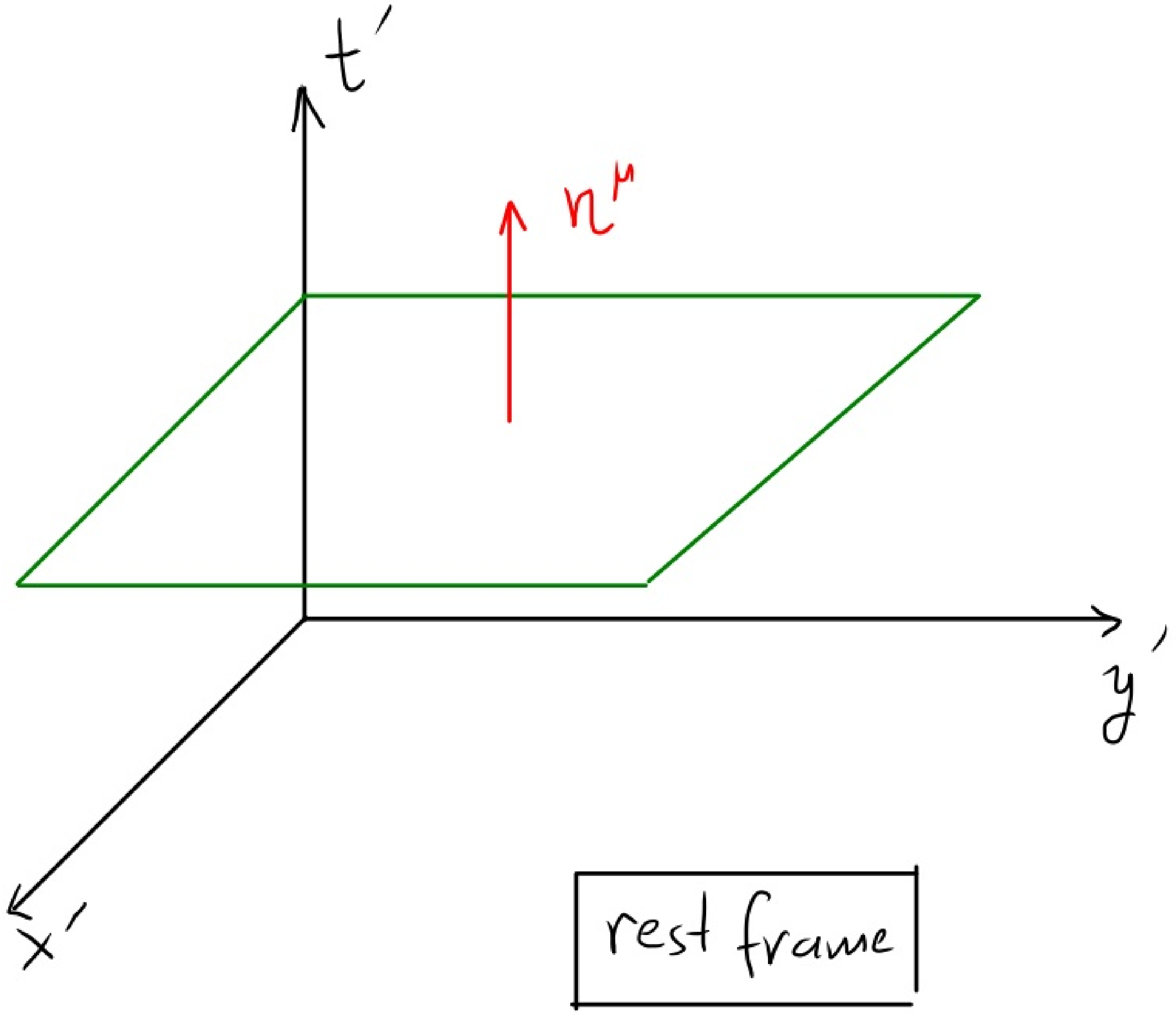
Stress-Energy Tensor in Special Relativity

Then $dN_0 = n_0 \, dV_0 = -j_\mu \underbrace{\eta^\mu \, dV_0}$

flux of number current through spacelike surface

"area" element of 3-surface



- density n_0 is a flux of number current through spacelike surfaces

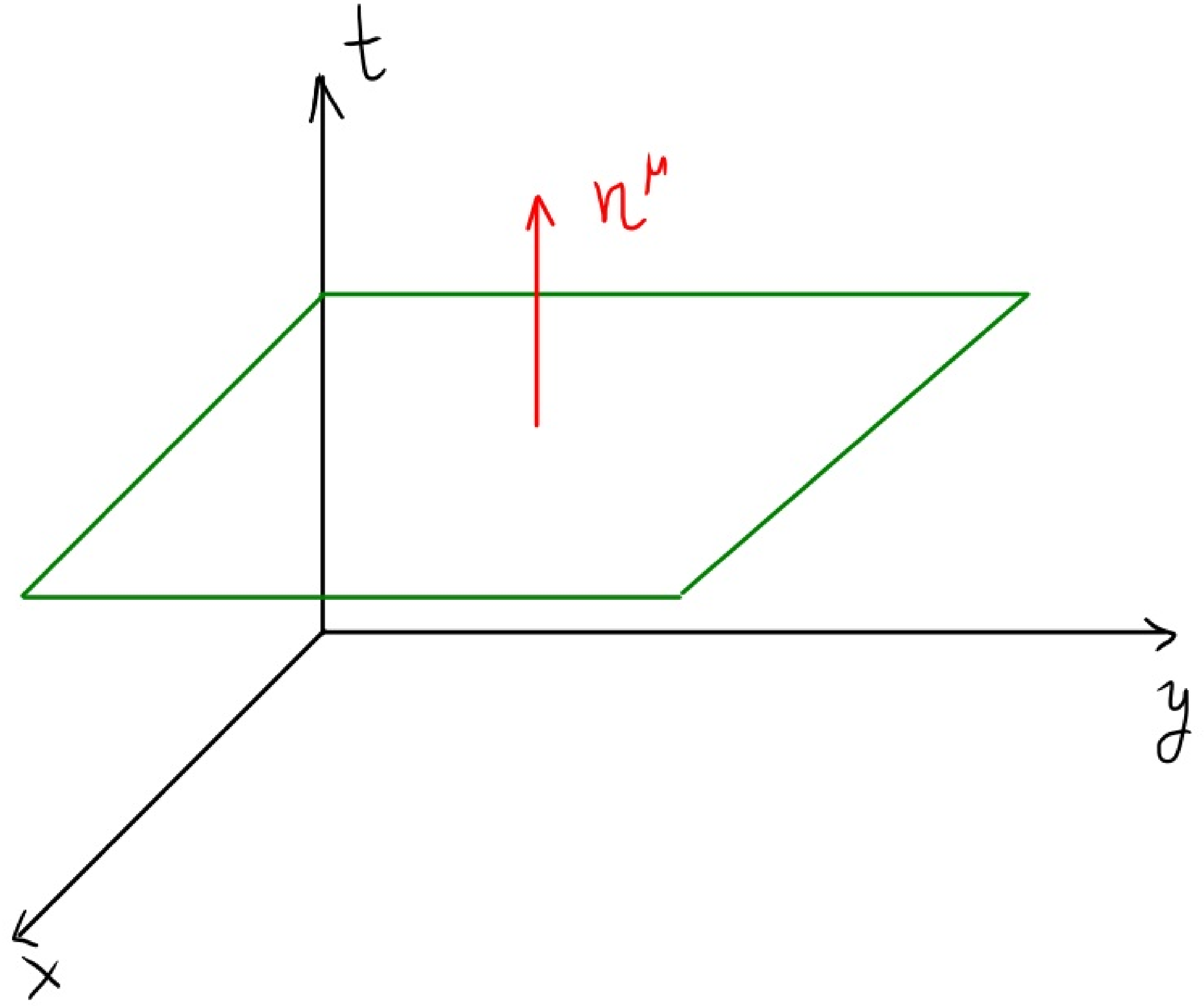
- likewise, currents $\vec{j}$ are fluxes in spacelike directions through timelike surfaces (spacelike normals)

Stress-Energy Tensor in Special Relativity

Similarly in (t,x,y,z) frame:

$j^\mu = (j^0, \vec{j})$ $\qquad$ $n^\mu = (1,0,0,0)$

$\hookrightarrow$ constant t surface

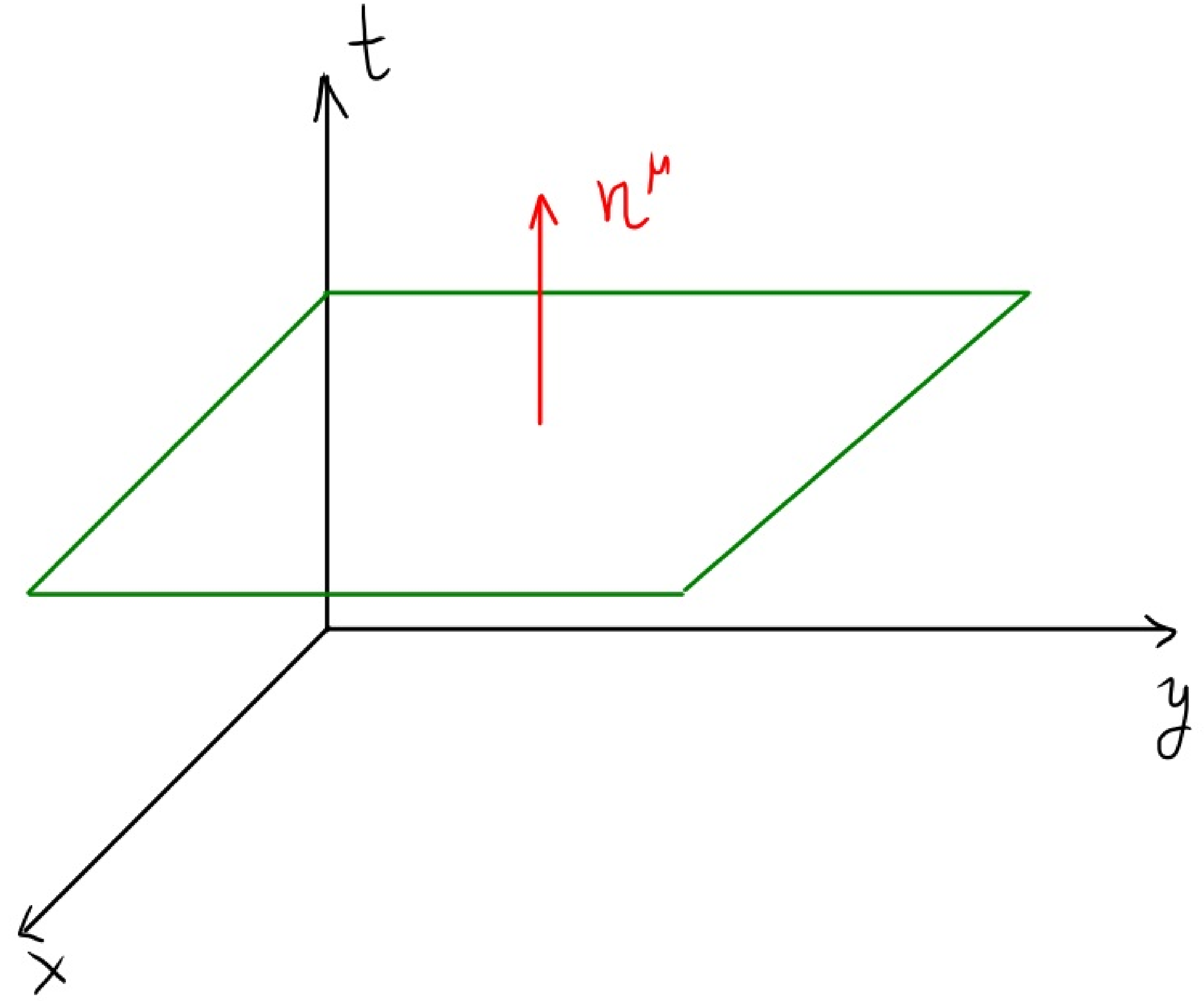


Stress-Energy Tensor in Special Relativity

Similarly in (t, x, y, z) frame:

$j^\mu = (j^0, \vec{j})$ $\qquad n^\mu = (1, 0, 0, 0)$

↳ constant t surface

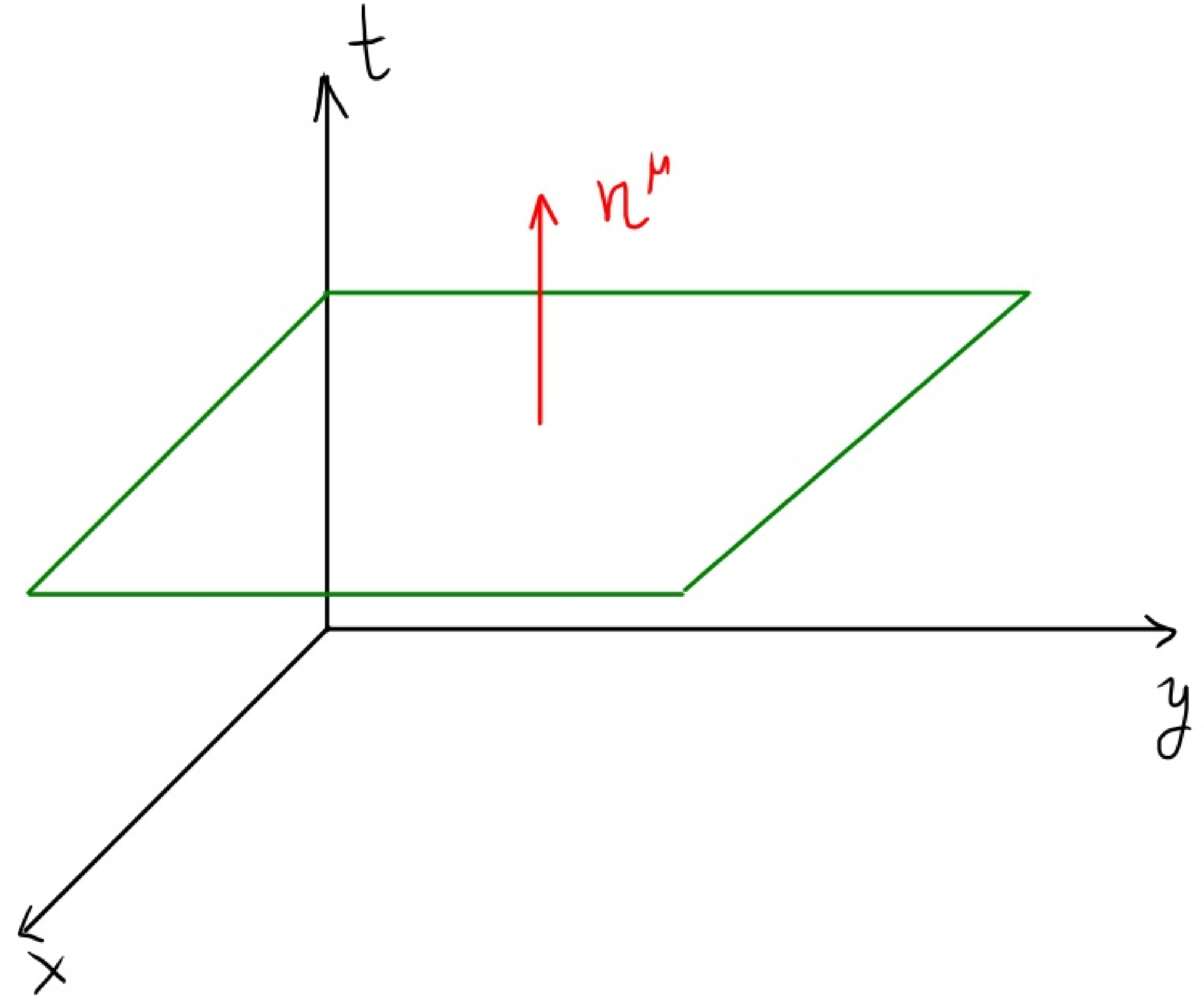


$$dN = n\, dV = j^0\, dV = -j^\mu n_\mu\, dV$$

Stress-Energy Tensor in Special Relativity

Consider 4-momentum in 3-volume ΔV

$$\Delta p^\mu = -\underbrace{T^{\mu\nu}}_{\text{"current"}} n_\nu \, \Delta V$$



compare with:

$$\Delta N = - j^\mu n_\mu \, \Delta V$$

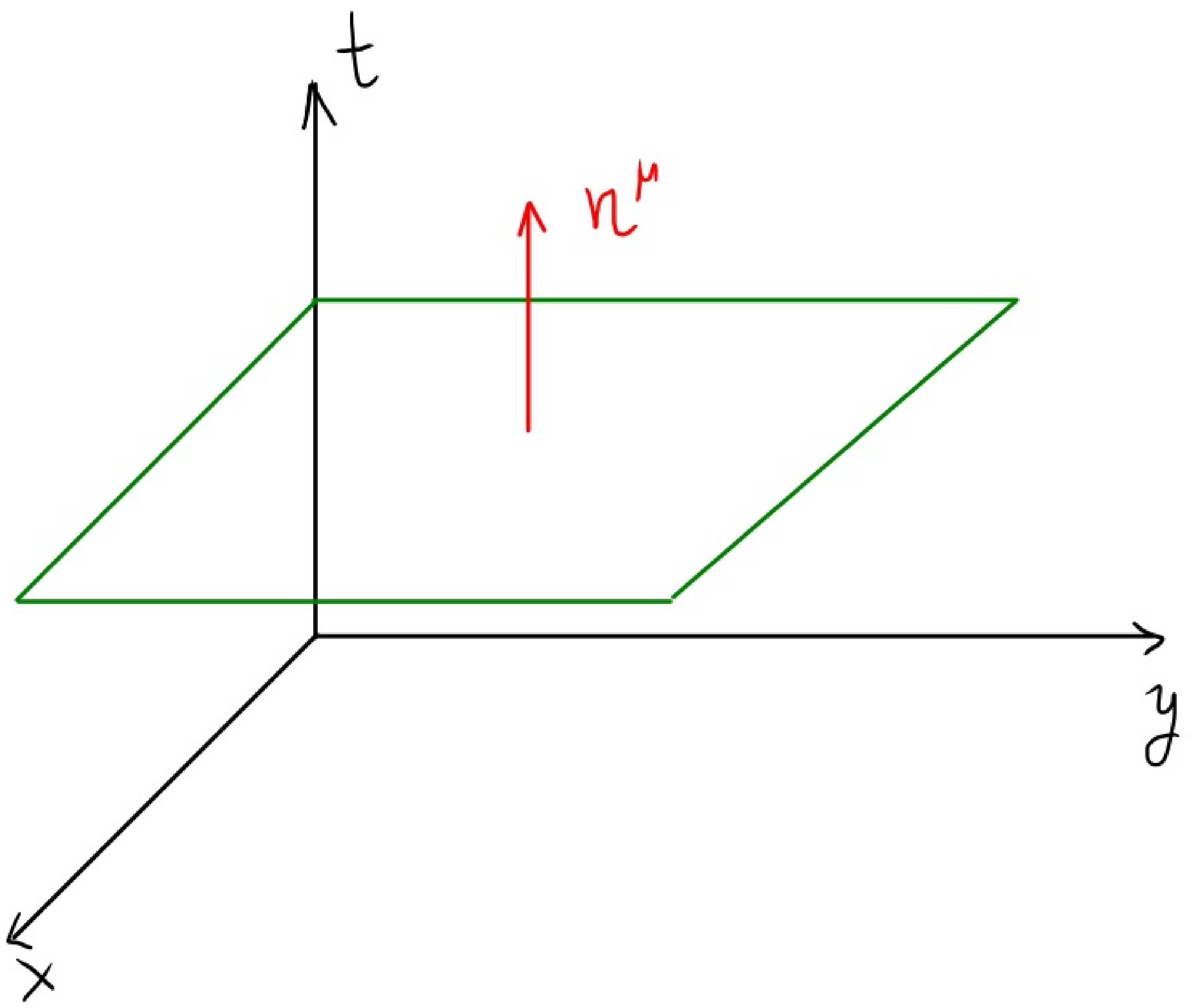
Stress-Energy Tensor in Special Relativity

Consider 4-momentum in 3-volume ΔV

$$\Delta p^\mu = - T^{\mu\nu} \, n_\nu \, \underbrace{\Delta V}_{\text{spacelike 3-surface "area"}}$$

need extra index

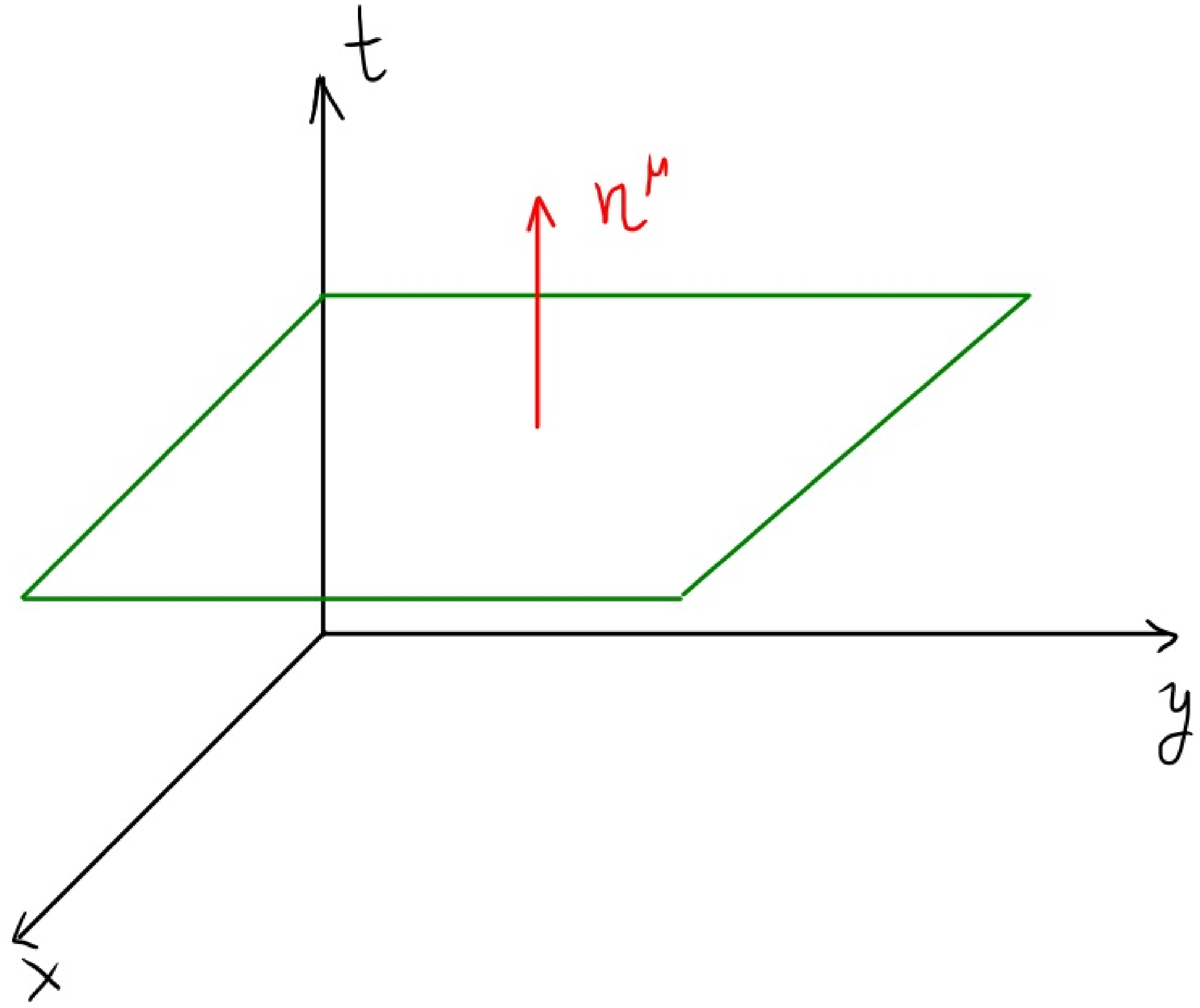
timelike



Stress-Energy Tensor in Special Relativity

Consider 4-momentum in 3-volume ΔV

$$\Delta p^\mu = -\underbrace{T^{\mu\nu}n_\nu}_{\text{4-momentum density}}\,\Delta V$$

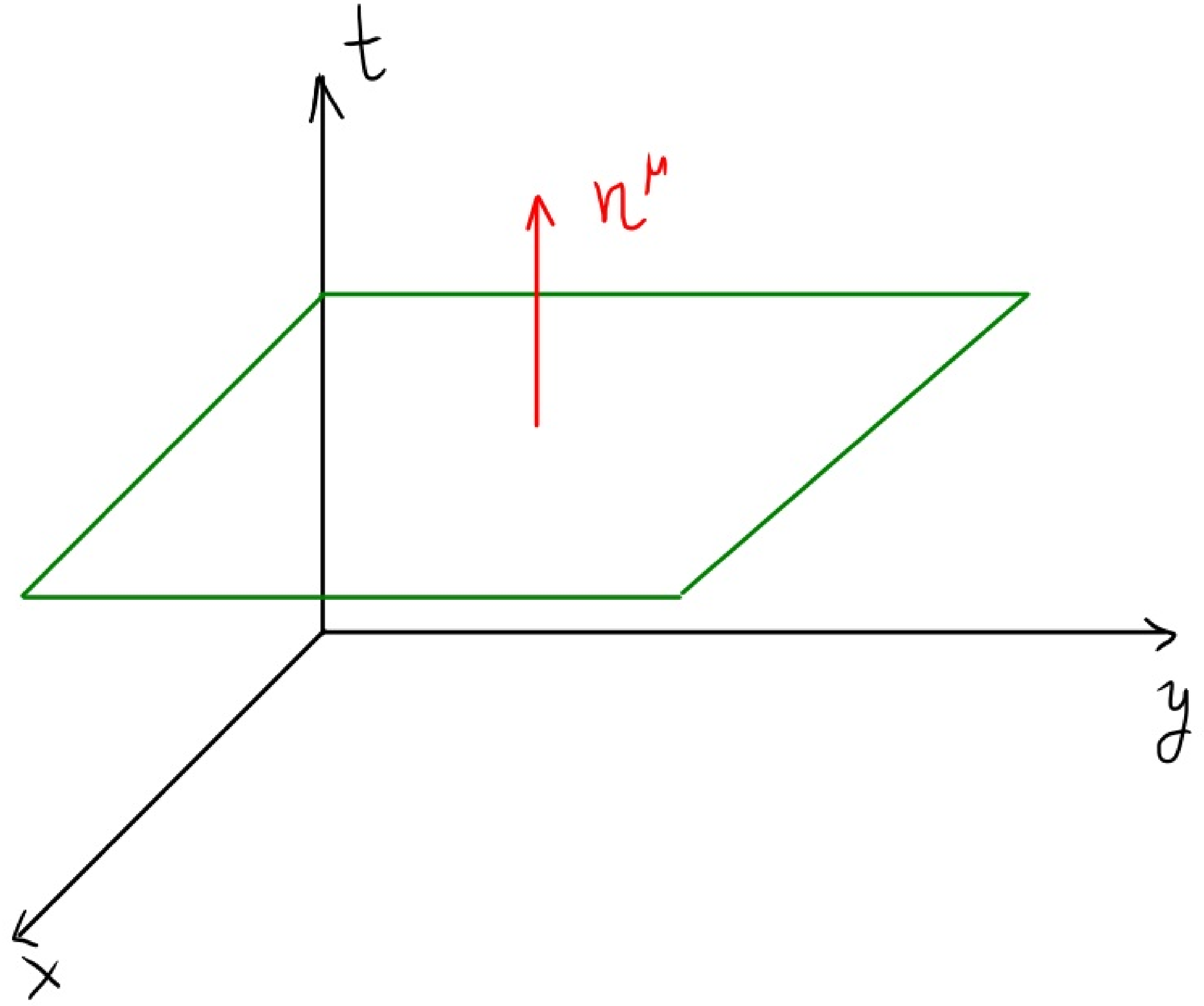


Stress-Energy Tensor in Special Relativity

Consider 4-momentum in 3-volume ΔV

$$\Delta p^\mu = - T^{\mu\nu} n_\nu \Delta V$$

If ΔV at rest, $n^\mu = (1, 0, 0, 0)$



Stress-Energy Tensor in Special Relativity

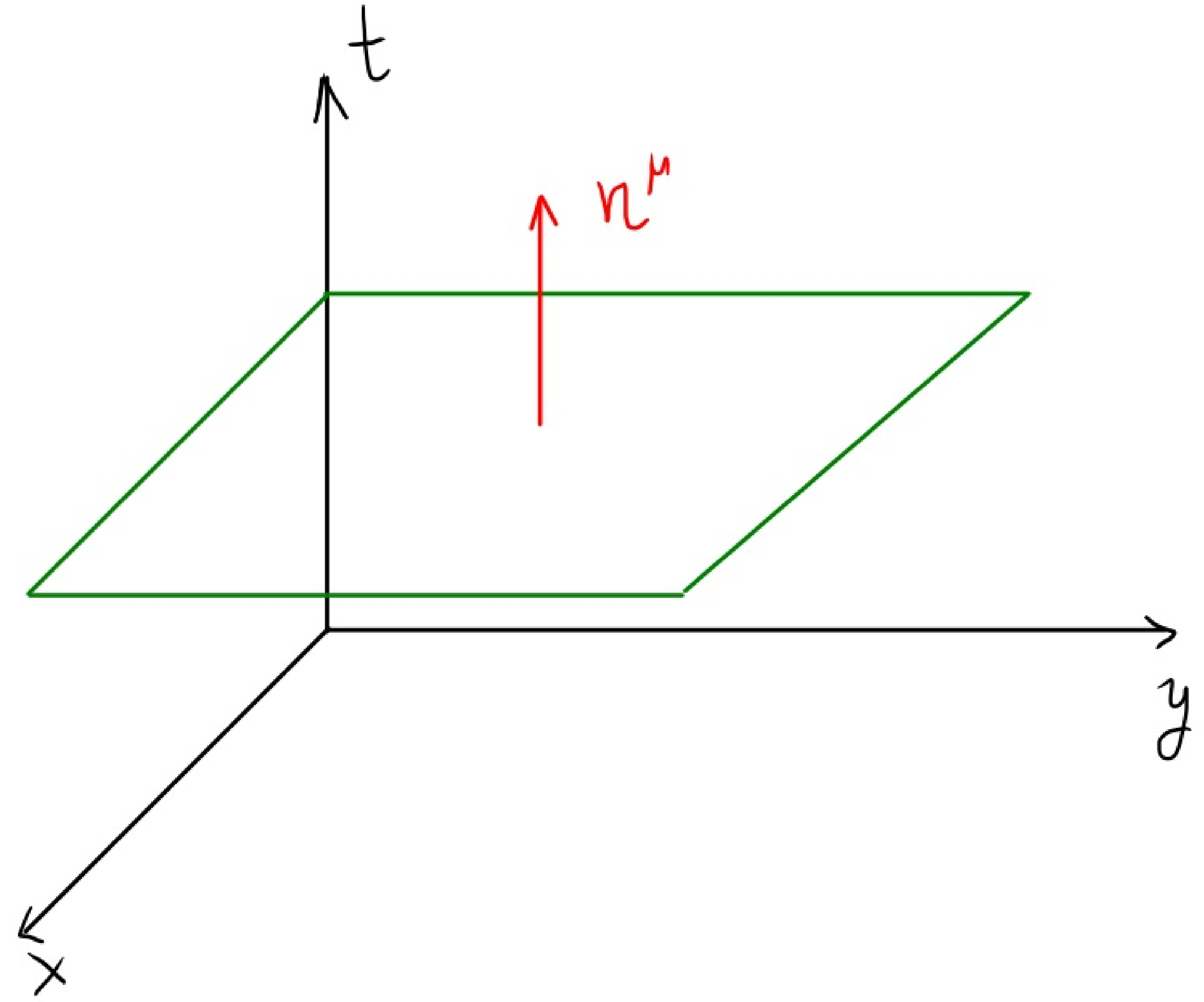
Consider 4-momentum in 3-volume ΔV

$$\Delta p^\mu = - T^{\mu\nu} n_\nu \Delta V$$

If ΔV at rest, $n^\mu = (1, 0, 0, 0)$

$$\Rightarrow \Delta p^\mu = + T^{\mu 0} \Delta V$$

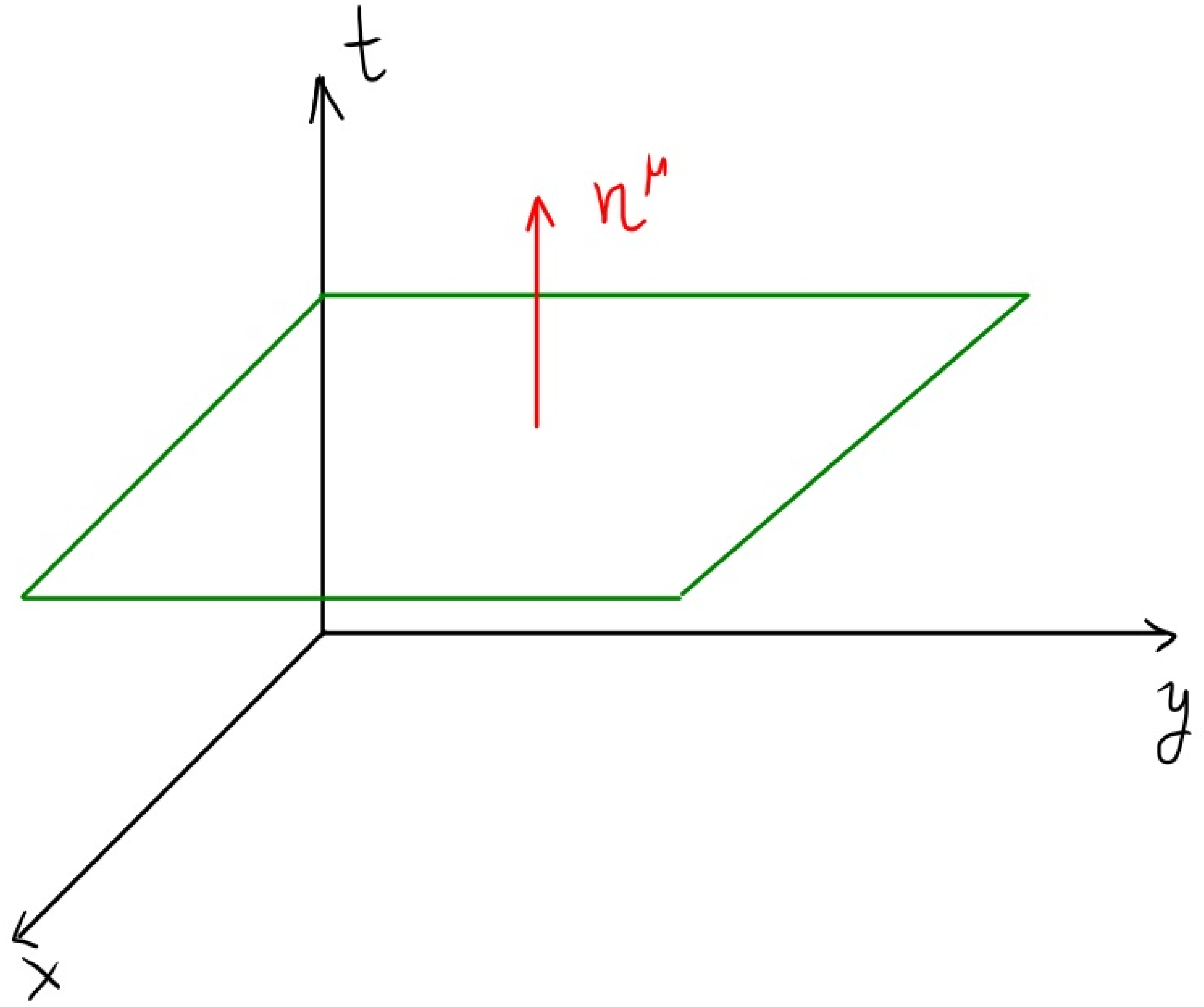
$n_\mu = (-1, 0, 0, 0)$



Stress-Energy Tensor in Special Relativity

Consider 4-momentum in 3-volume ΔV

$$\Delta p^\mu = - T^{\mu\nu} n_\nu \, \Delta V$$



If ΔV at rest, $n^\mu = (1, 0, 0, 0)$

$$\Rightarrow \Delta p^\mu = + T^{\mu 0} \, \Delta V \qquad n_\mu = (-1, 0, 0, 0)$$

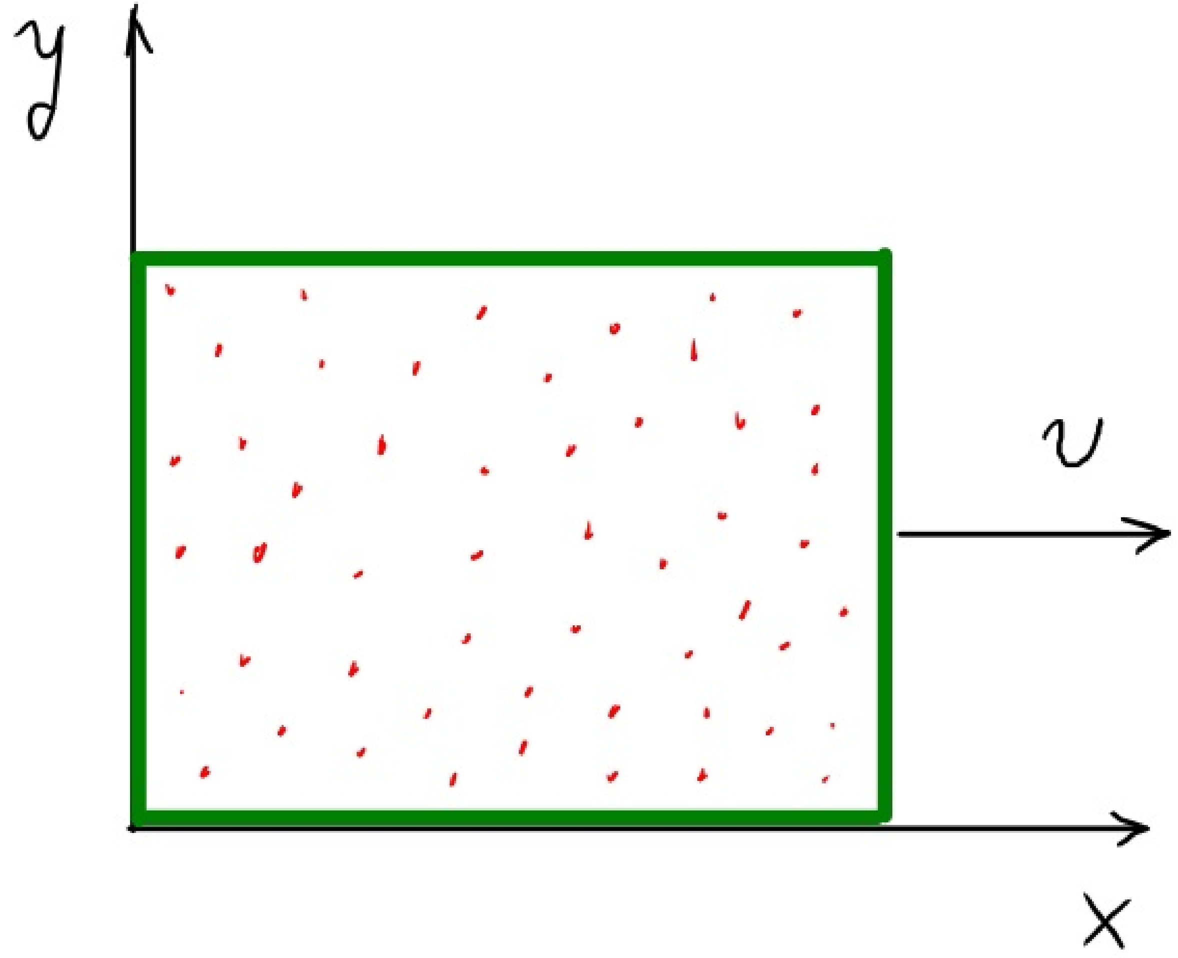
$$\rho = \frac{\Delta p^0}{\Delta V} = T^{00} \quad \text{energy density}$$

$$\pi^i = \frac{\Delta p^i}{\Delta V} = T^{i0} \quad \text{momentum density}$$

Stress-Energy Tensor in Special Relativity

Example:

$$\rho = \frac{E}{V} = \frac{N m \gamma}{V} = \left(\frac{N}{V}\right) m \gamma$$



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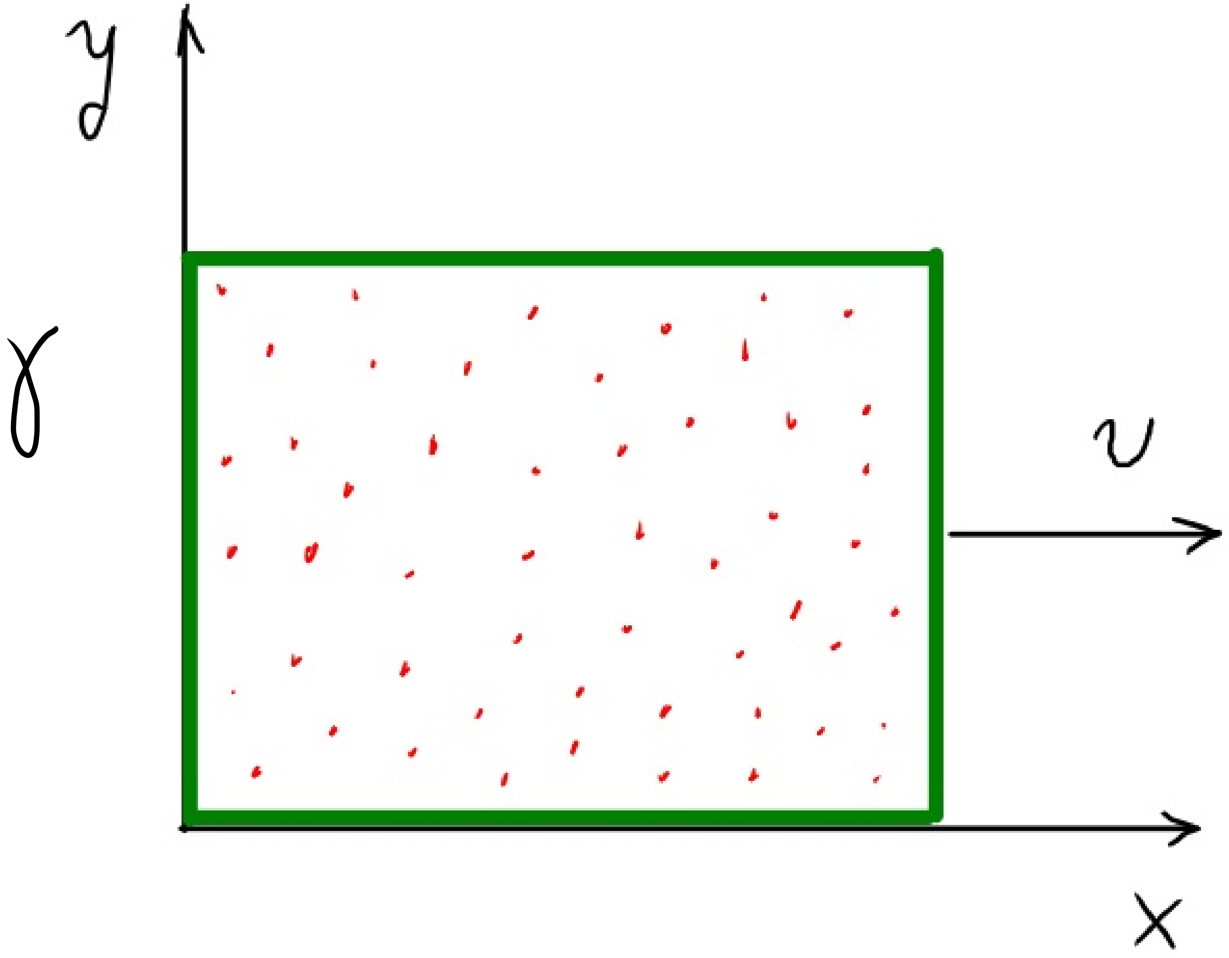

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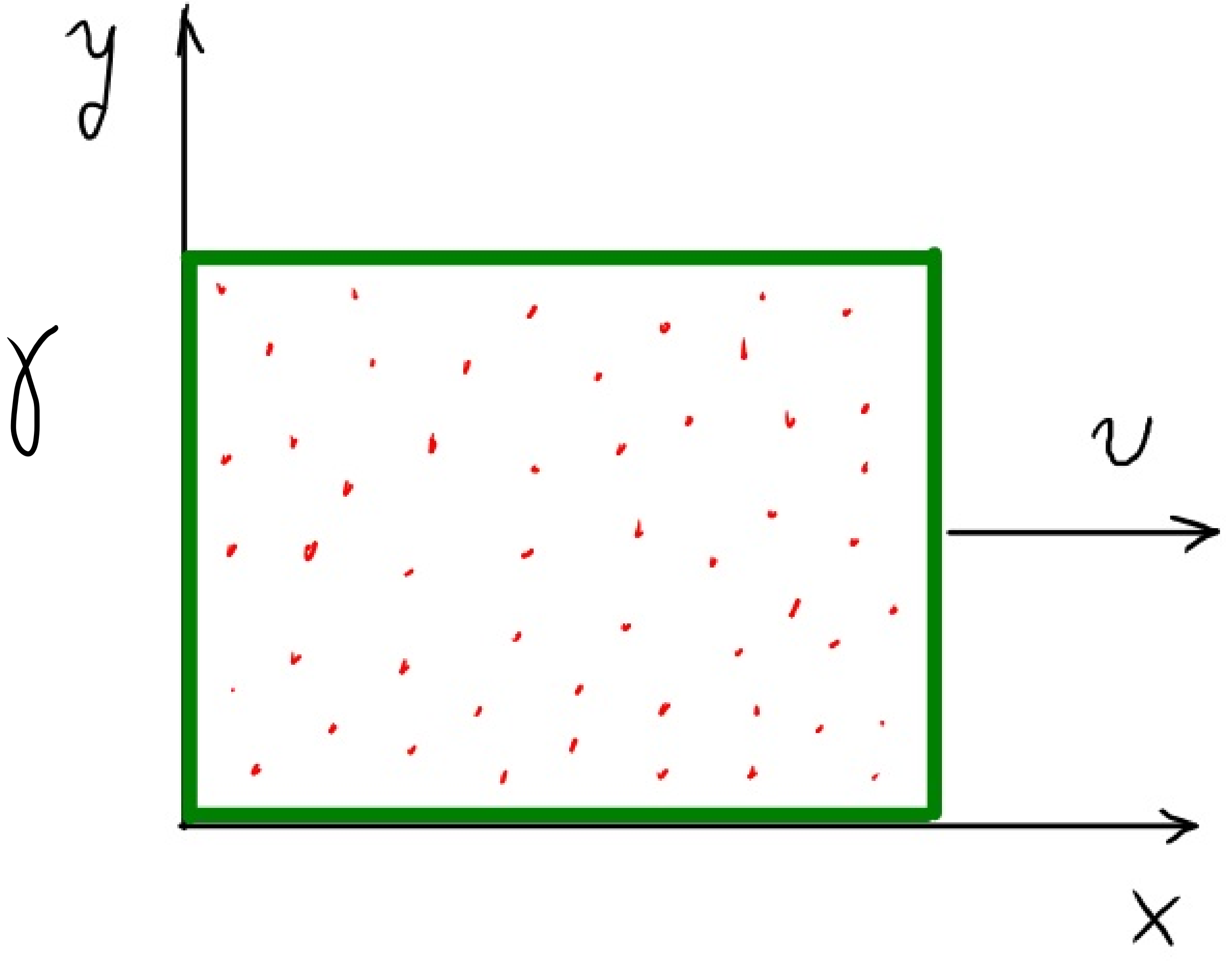

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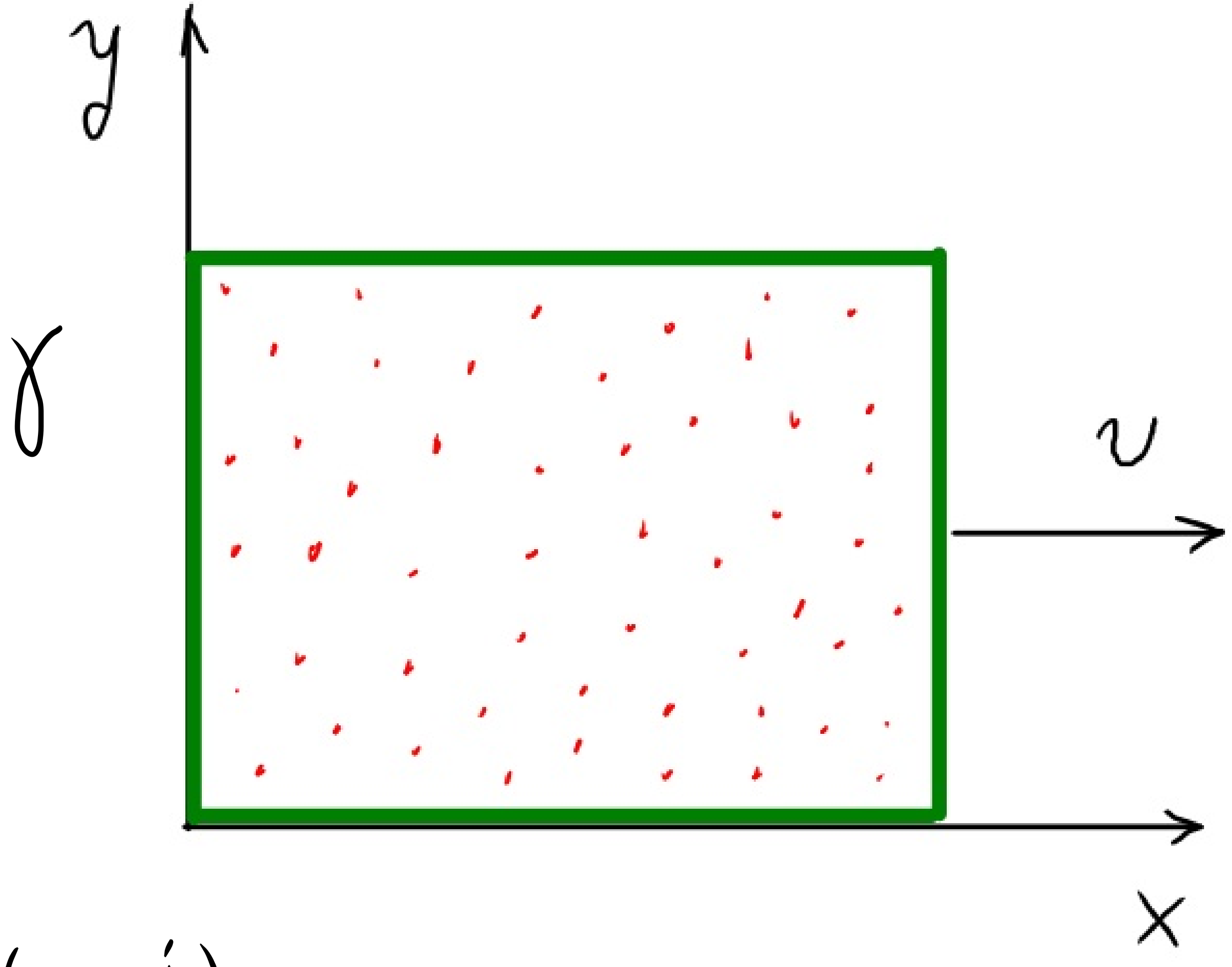

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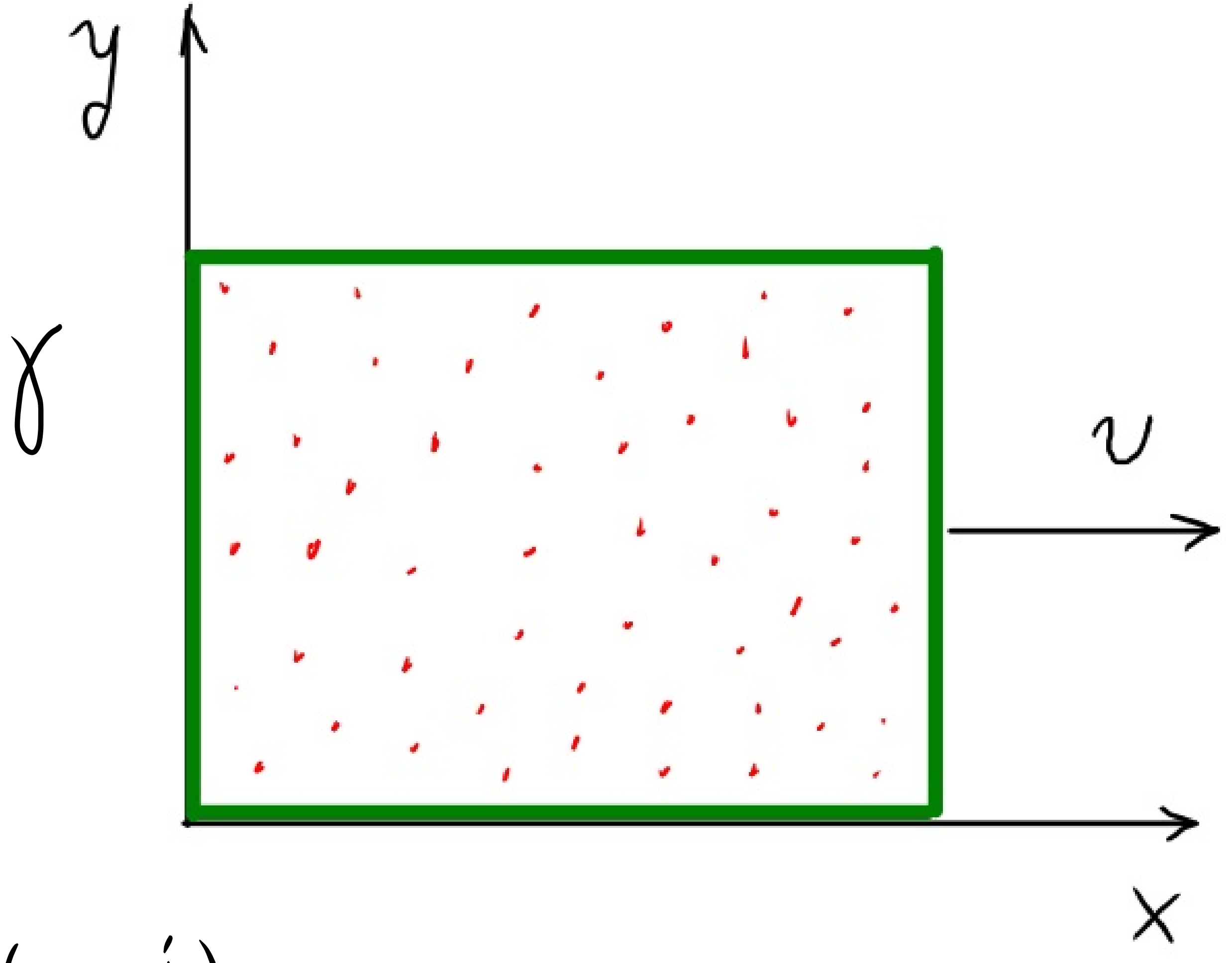

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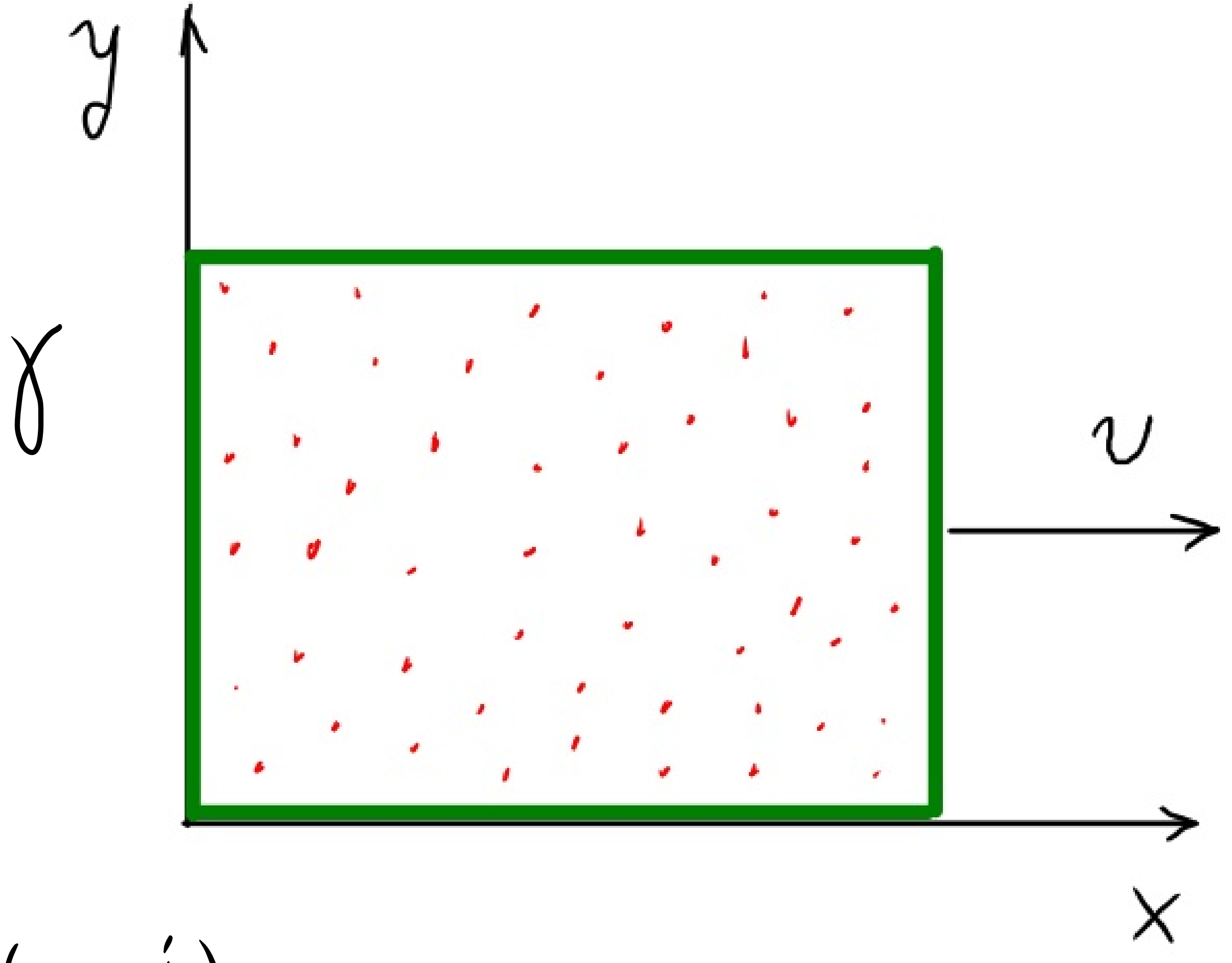

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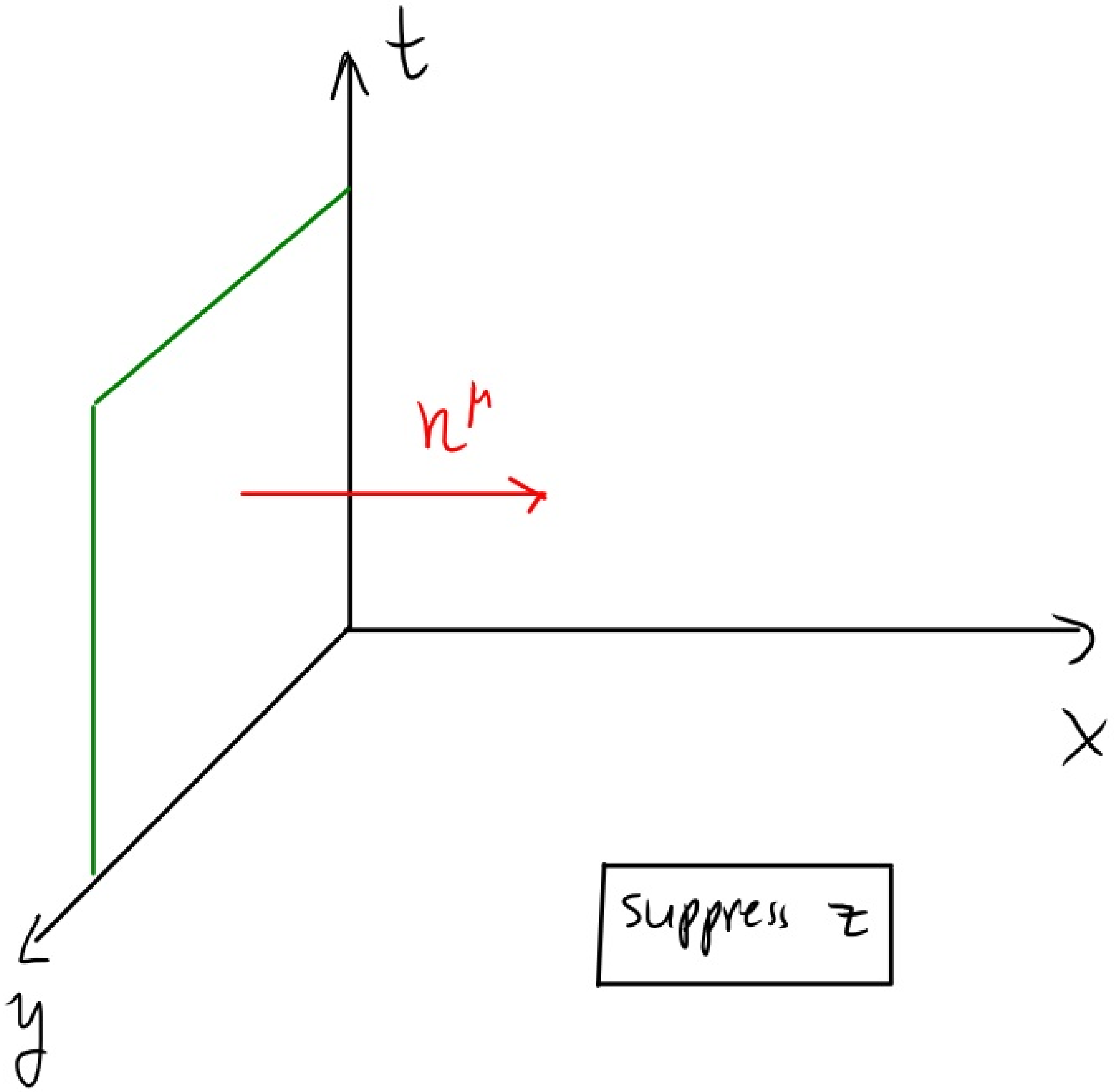
Now consider $T^{\mu j}$, e.g. $T^{\mu 1}$



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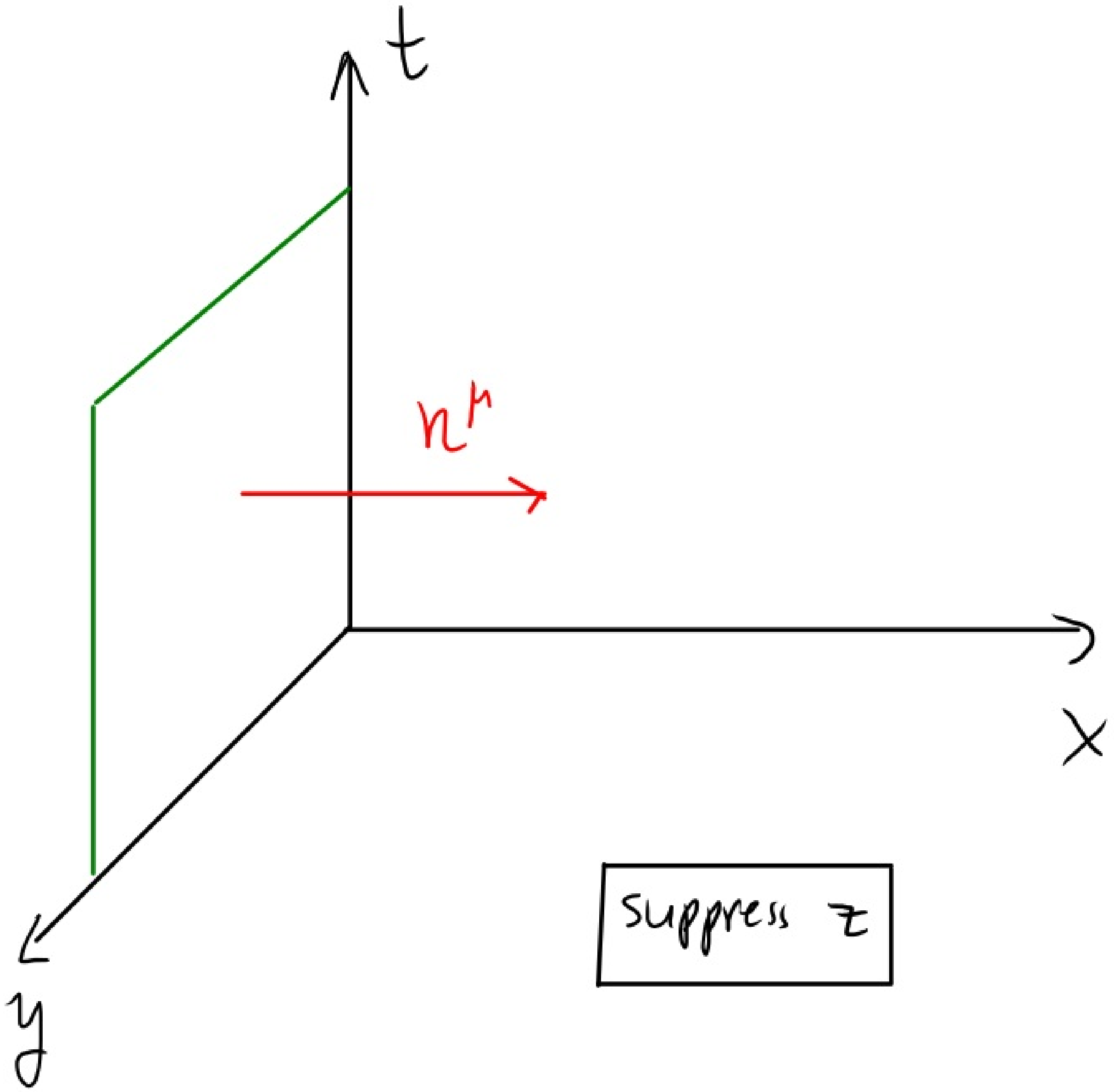
$n^\mu = (0, 1, 0, 0)$



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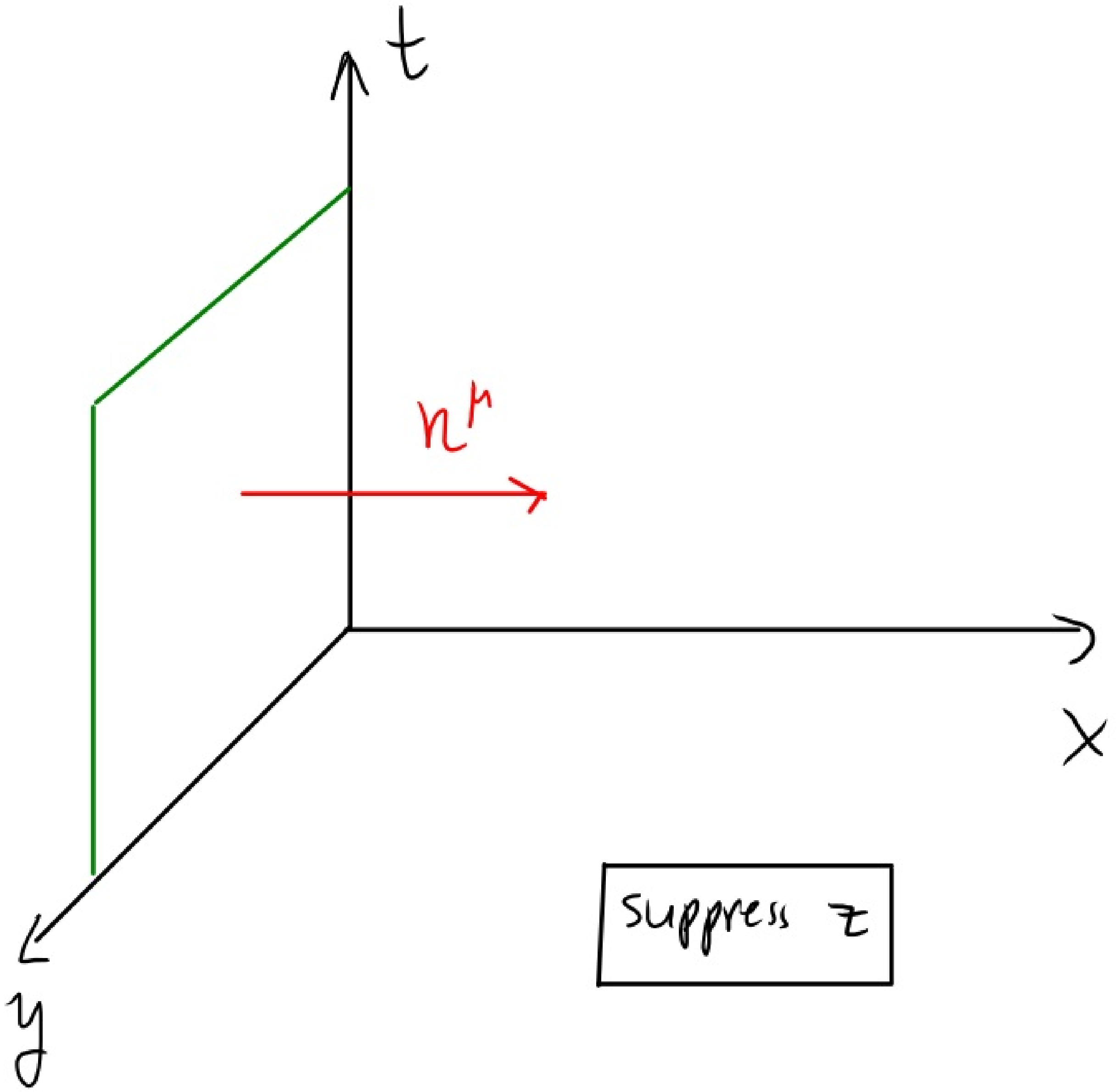

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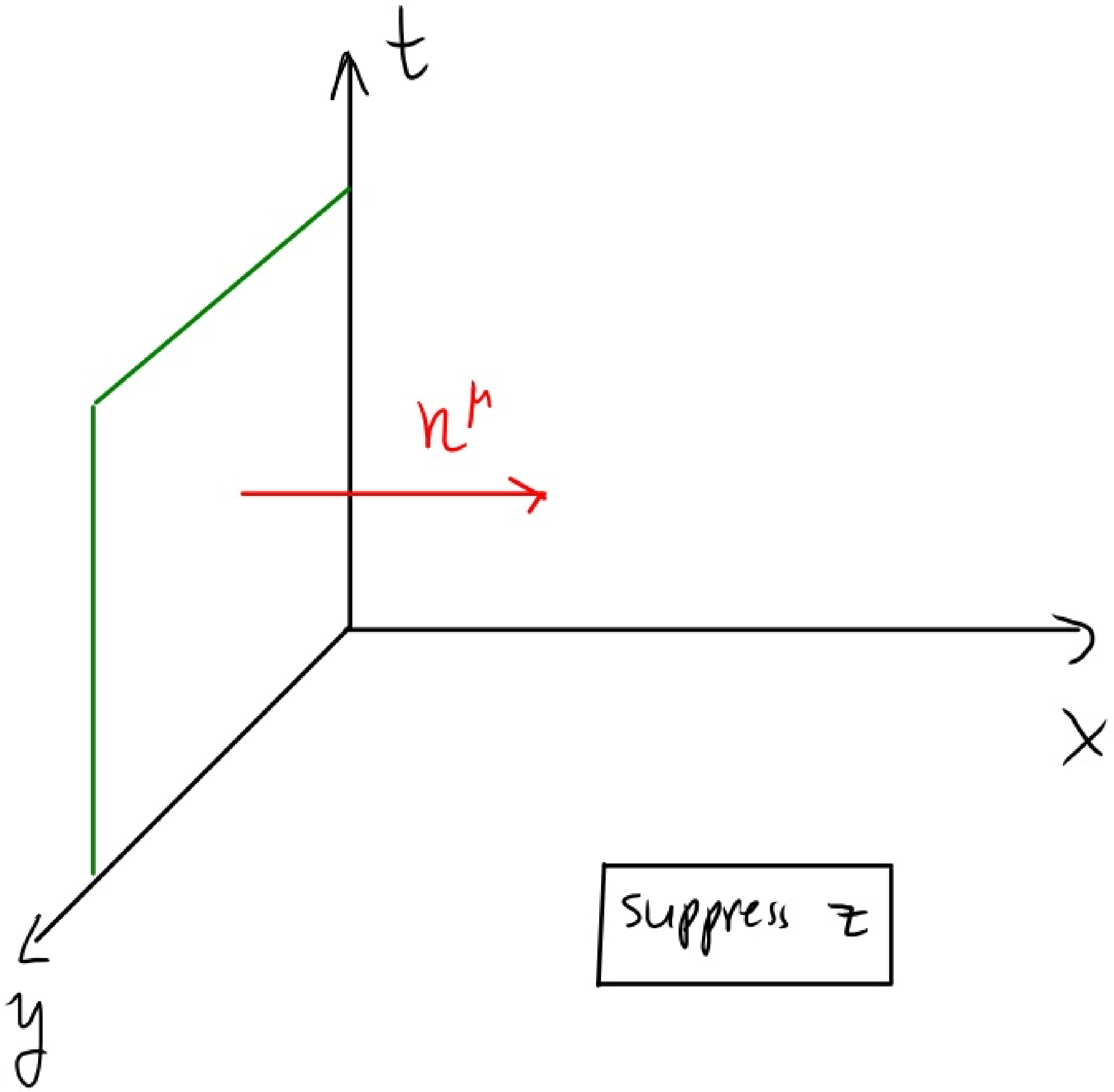

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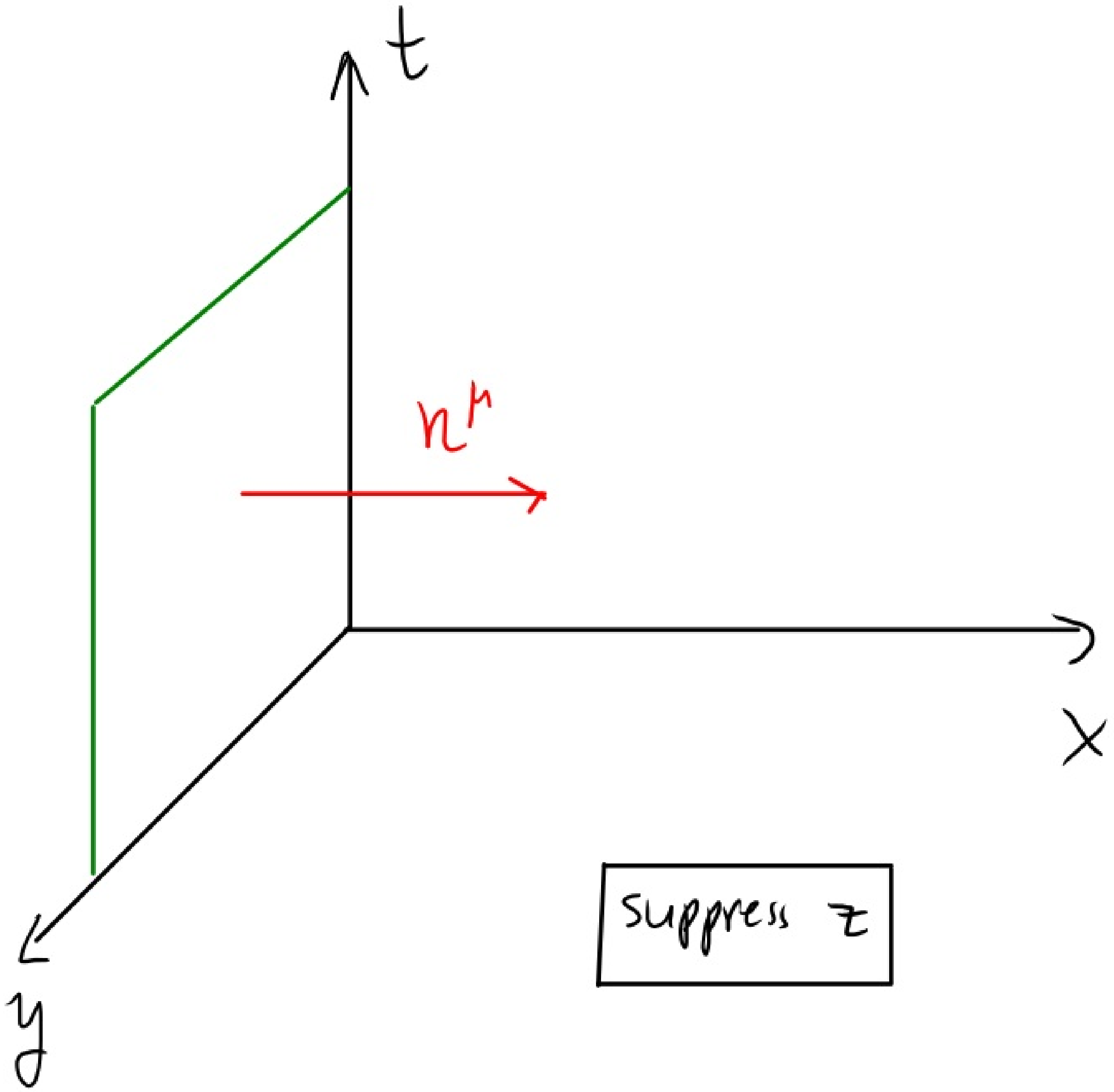

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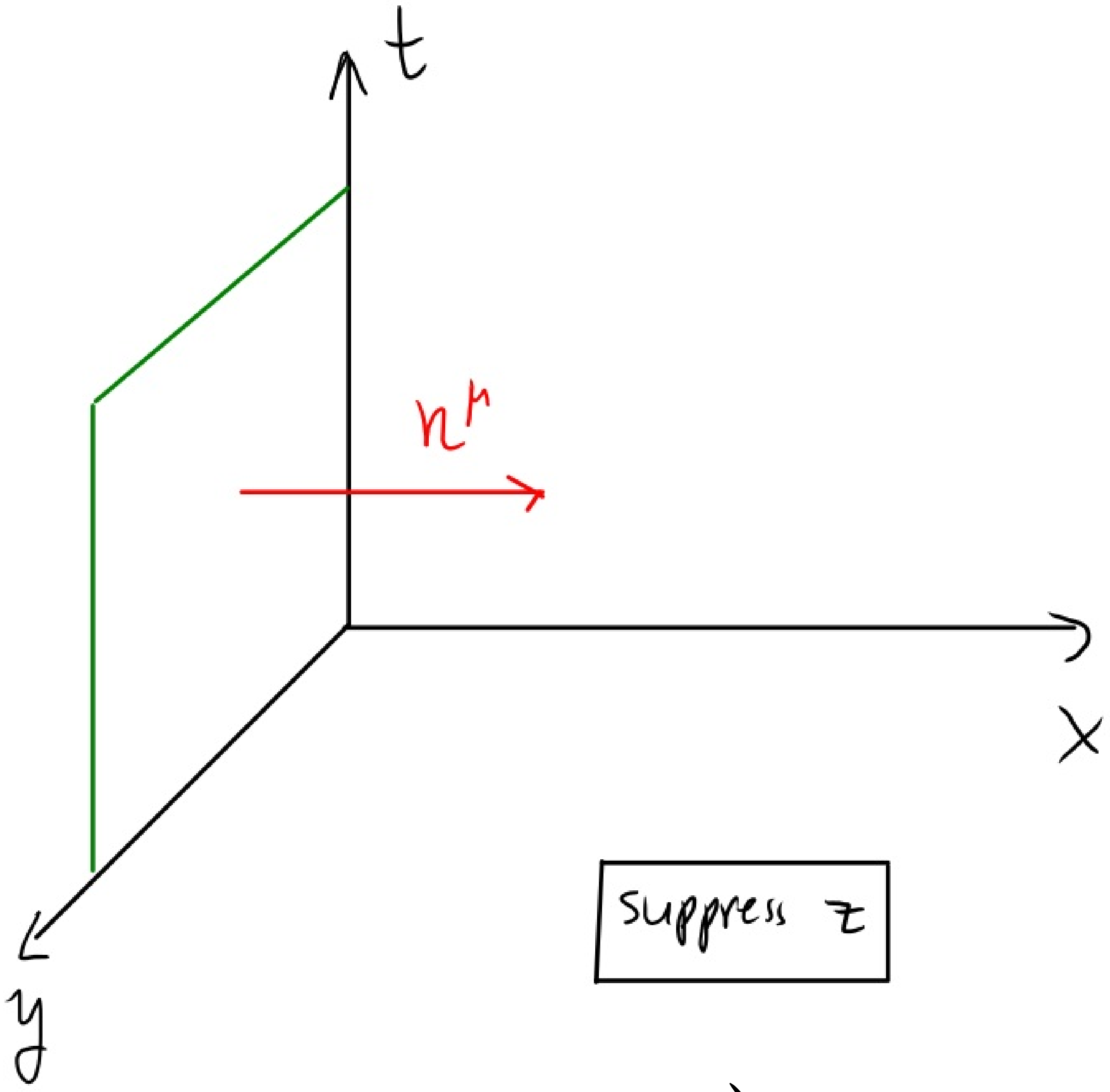

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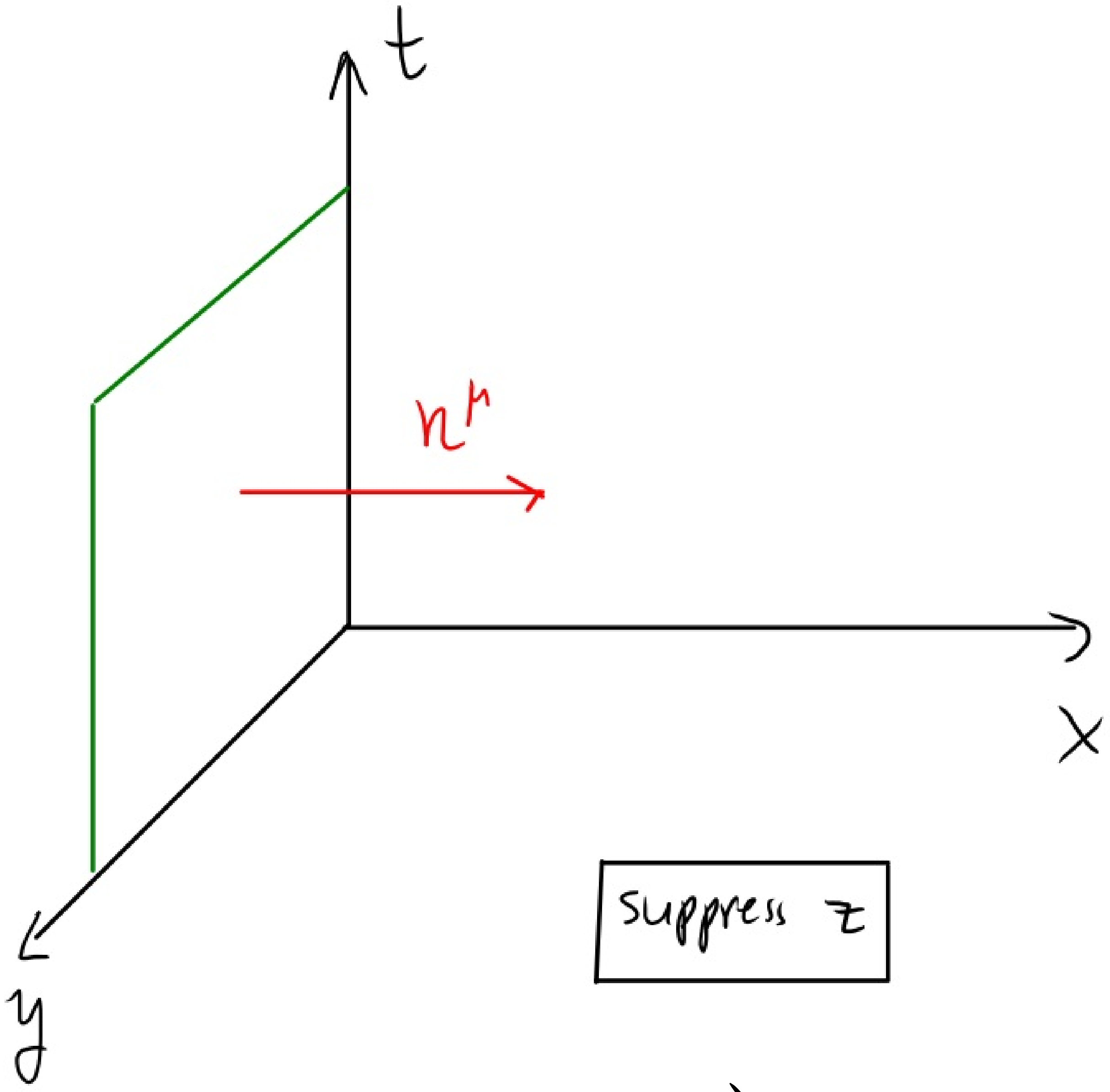

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$\Rightarrow T^{ij}$ stress tensor

Stress-Energy Tensor in Special Relativity

$$T^{\mu\nu} = \left(\begin{array}{c} \text{stress-energy} \\ \text{tensor} \end{array} \right) = \left(\begin{array}{c|c} \begin{array}{c} \text{energy} \\ \text{density} \end{array} & \text{energy flux} \\ \hline \begin{array}{c} \text{momentum} \\ \text{density} \end{array} & \begin{array}{c} \text{stress} \\ \text{tensor} \end{array} \end{array} \right)$$

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$$T^{\mu\nu} = T^{\nu\mu} \Rightarrow (\text{energy flux}) = (\text{momentum density})$$

Stress - Energy Tensor in Classical Field Theories

Let $\quad S = \int d^4x \, \mathcal{L}(\varphi_n, \partial_\mu \varphi_n)$

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conserved!
currents

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Stress-Energy Tensor in Classical Field Theories

$$4\pi T^{00} = 4\pi \rho = \frac{1}{2}\left(\mathcal{E}^2 + B^2\right)$$

$$4\pi T^{0i} = 4\pi S^i = \left(\vec{\mathcal{E}} \times \vec{B}\right)^i$$

$$4\pi T^{ij} = \left(-\mathcal{E}^i \mathcal{E}^j + \frac{1}{2}\delta^{ij}\mathcal{E}^2\right) + \left(-B^i B^j + \frac{1}{2}\delta^{ij} B^2\right)$$

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Conservation of Energy - Momentum (special relativity only...)

Since $\Delta p^\mu = - T^{\mu\nu} n_\nu \Delta V$ is conserved, then

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$$\frac{d}{dt}\int_V \pi^i \, dV = \frac{dp^i}{dt} \qquad\qquad + \int T^{ij}(-dA_j)$$

(force exerted on volume by outside stresses)

Perfect Fluids

perfect = negligible
- heat conduction
- viscosity
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Perfect Fluids

perfect = negligible $\left.\begin{cases} \bullet \text{ heat conduction} \\ \bullet \text{ viscosity} \\ \bullet \text{ transport + dissipative processes} \end{cases}\right\} \Rightarrow T^{\mu\nu}$ diagonal

At inertial frame:

$$T^{\mu\nu} = \mathrm{diag}(\rho, p, p, p) = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

$\rho, p = \left(\begin{array}{c} \text{energy density} \\ + \\ \text{pressure @} \\ \text{rest frame} \end{array}\right)$

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Equation of state: $p = w\rho$

$w = 0$ dust
$w = 1/3$ radiation
$w = -1$ vacuum energy

Perfect Fluids

$$T^{\mu\nu} = (\rho + p)\, u^\mu u^\nu + p\, \eta^{\mu\nu}$$

a tensor equation, reduces to $T^{\mu\nu}$ when at rest frame $u^\mu = (1,0,0,0)$
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Perfect Fluids

Non relativistic limit: $\partial_t \rho + \vec{\nabla}\cdot(\rho\vec{v}) = 0$ continuity eq.

$$\rho\left[\partial_t\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right] = -\vec{\nabla}p$$ Euler eq.

(see Carroll § 1.9)

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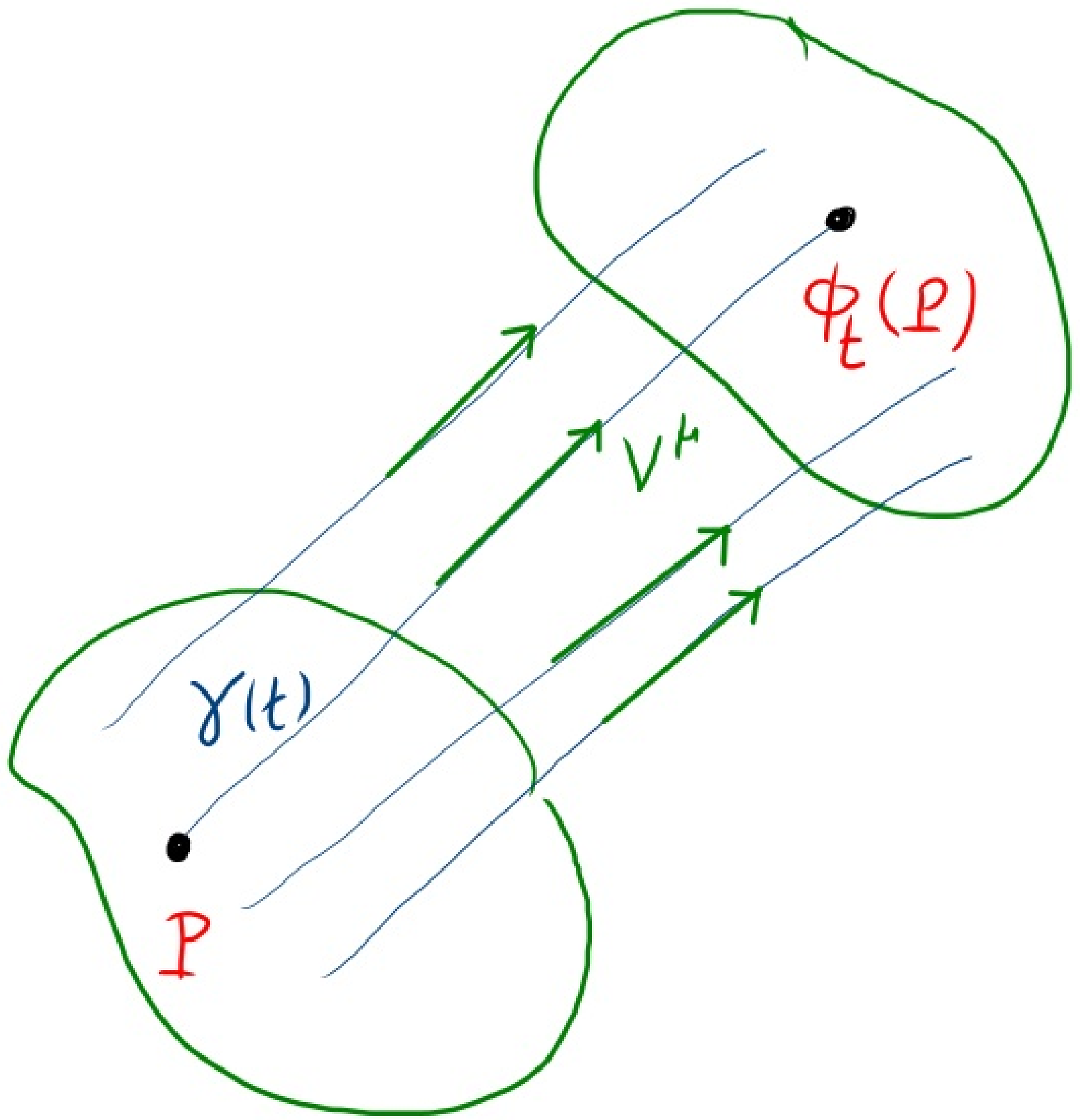
Note: this is a local conservation law

spacetime dynamics $\rightsquigarrow$ no conservation (e.g. CMB in expanding space)

Conservation Laws + Isometries

- A vector field V^μ generates a 1-parameter family of diffeomorphisms by moving points on its integral curves

$$P \to \phi_t(P)$$



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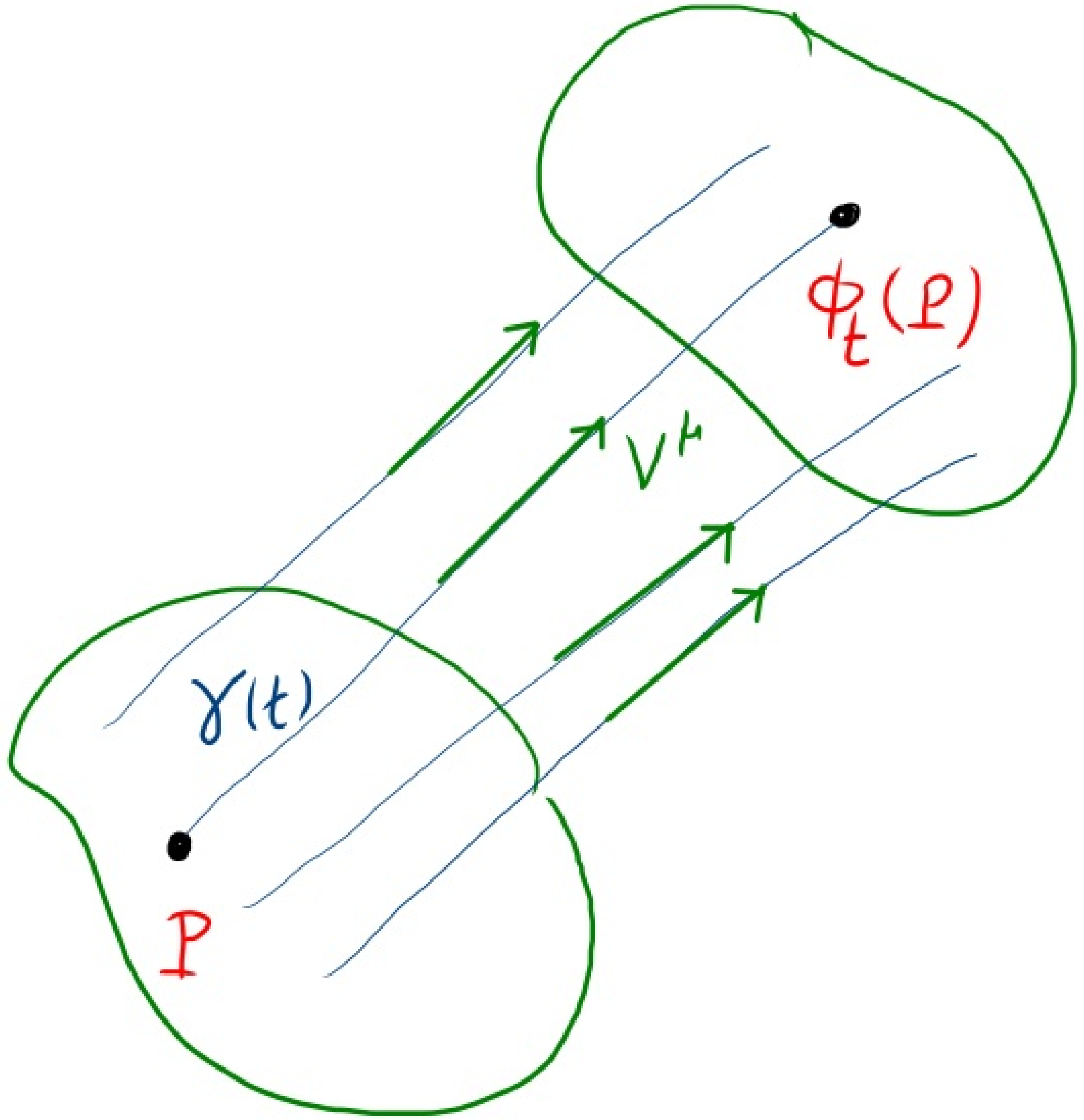

- Tensors defined at $\phi_t(P)$ can be pulled back at P: If $P \to x^\mu$, $\phi_t(P) \to x^\mu(t)$, then

$$\phi_t^* g_{\mu\nu}(x^\sigma) = \frac{\partial x^{\mu'}(t)}{\partial x^\mu} \frac{\partial x^{\nu'}(t)}{\partial x^\nu} g_{\mu'\nu'}(x^{\sigma'})$$

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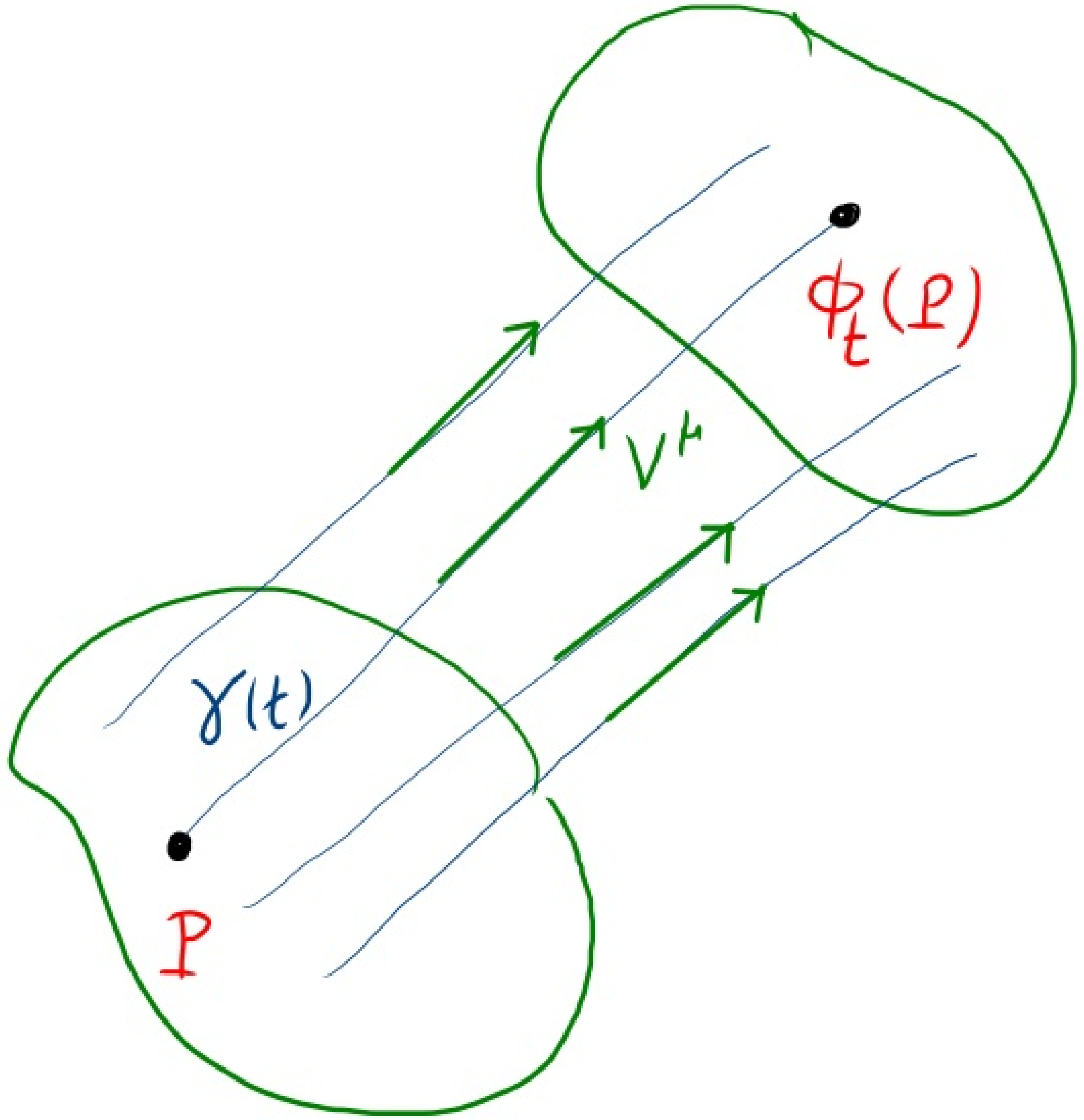

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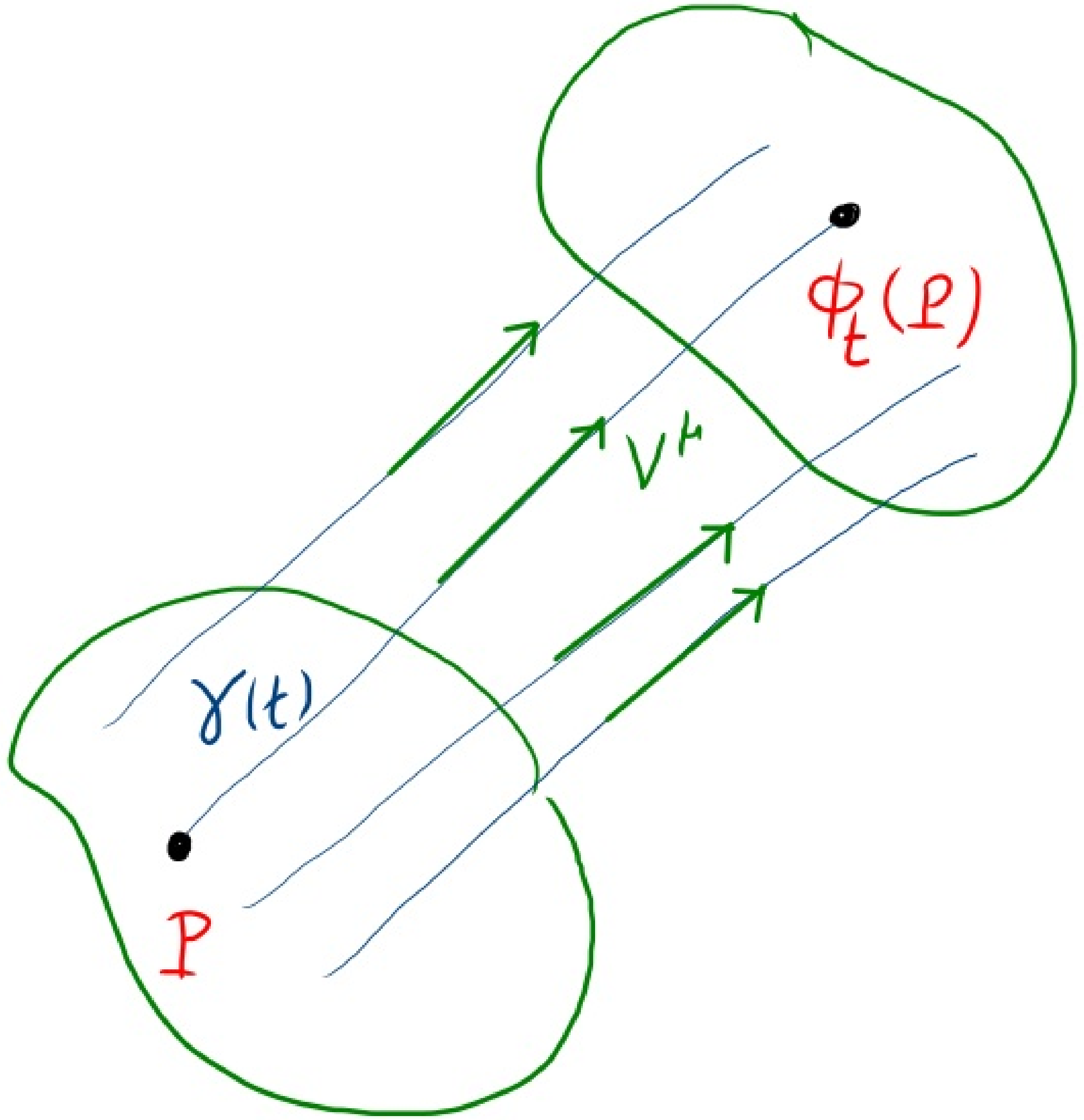

Then

$$(\mathcal{L}_V g)_{\mu\nu} = V^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu} \partial_\mu V^\rho + g_{\mu\rho} \partial_\nu V^\rho$$

$$= V^\rho \nabla_\rho g_{\mu\nu} + g_{\rho\nu} \nabla_\mu V^\rho + g_{\mu\nu} \nabla_\nu V^\rho$$

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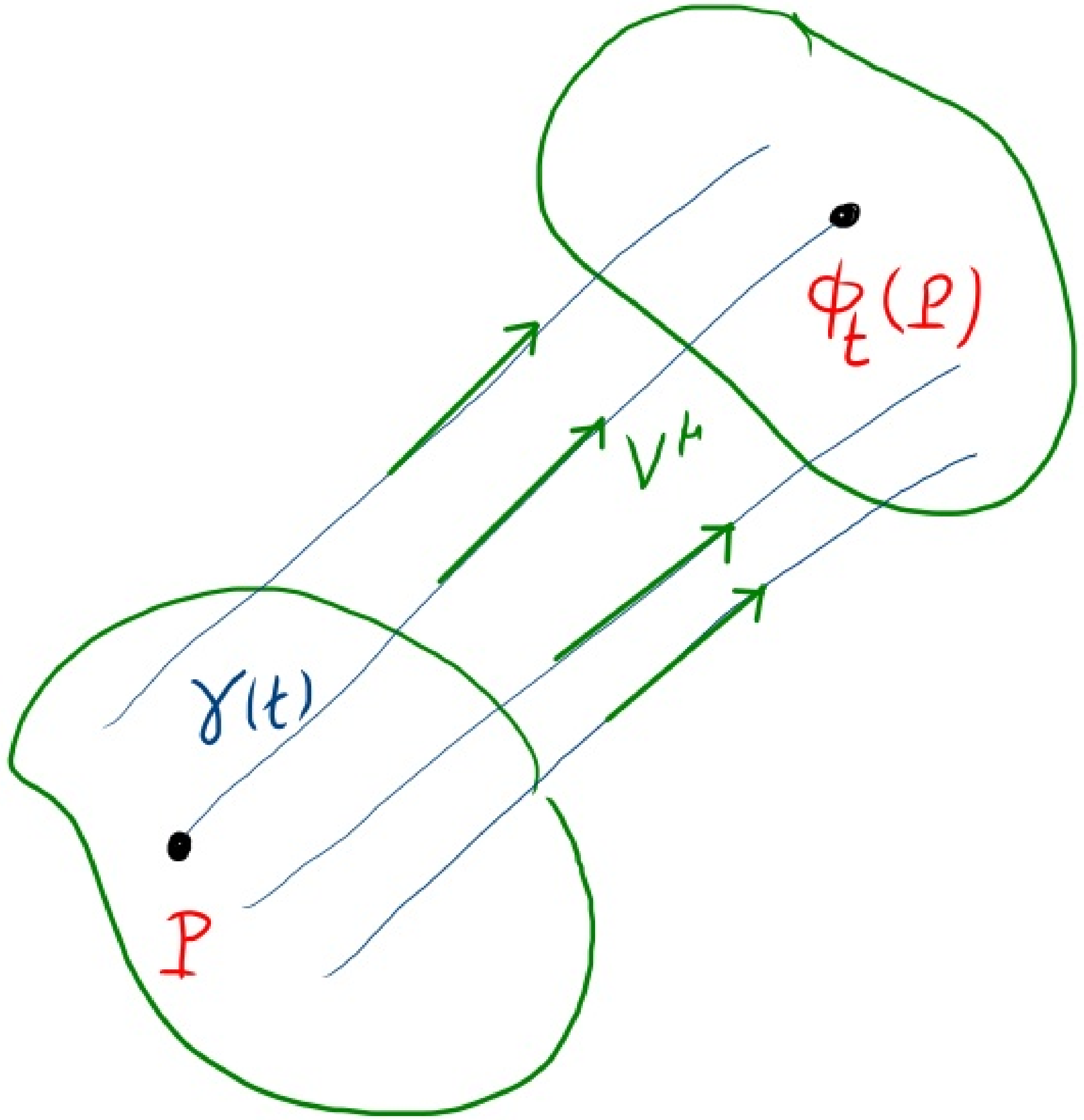

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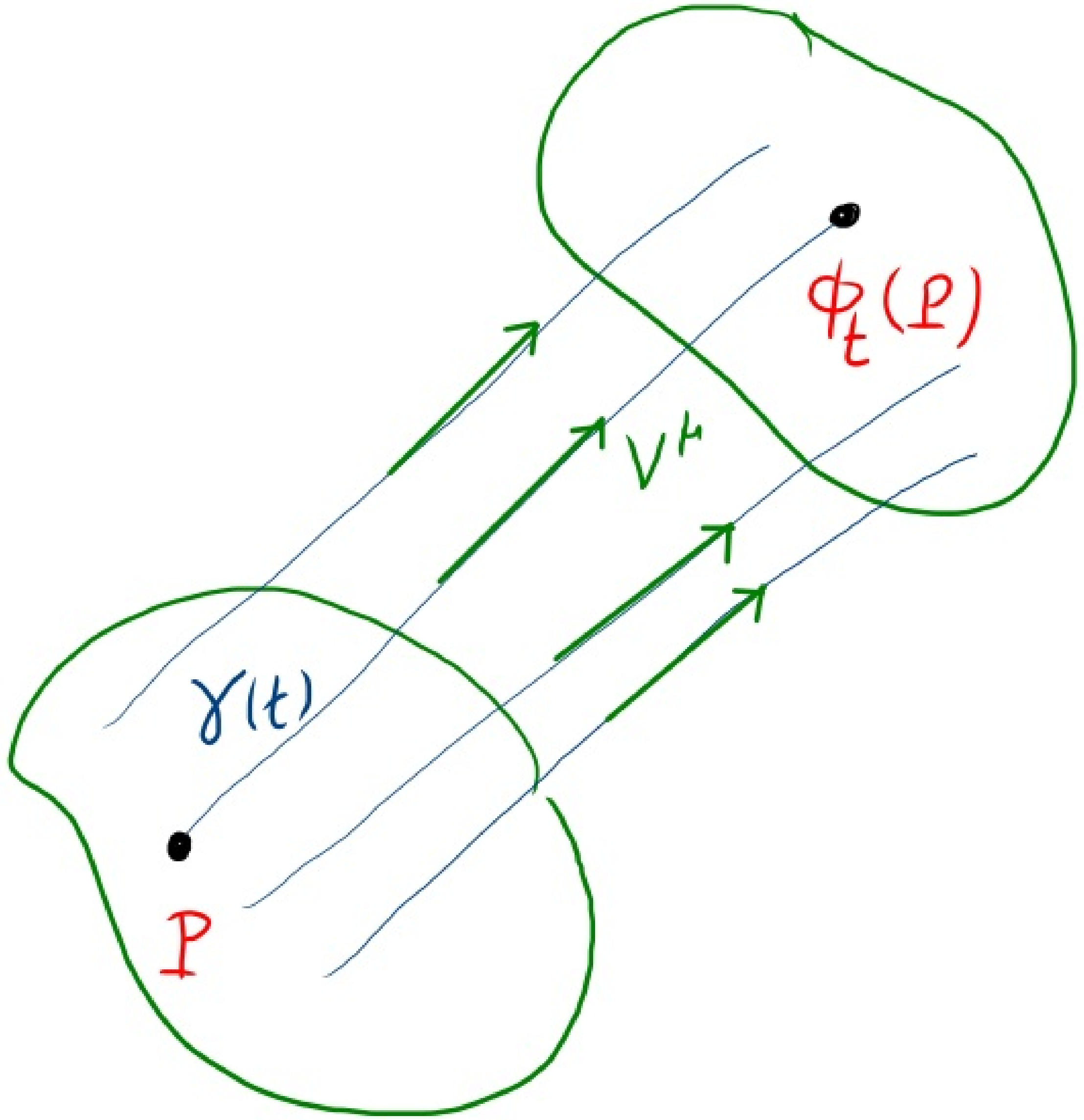
$\Rightarrow \mathcal{L}_V g_{\mu\nu} = 0$



Then

$$(\mathcal{L}_V g)_{\mu\nu} = V^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu} \partial_\mu V^\rho + g_{\mu\rho} \partial_\nu V^\rho$$

$$= V^\rho \nabla_\rho g_{\mu\nu} + g_{\rho\nu} \nabla_\mu V^\rho + g_{\mu\nu} \nabla_\nu V^\rho$$

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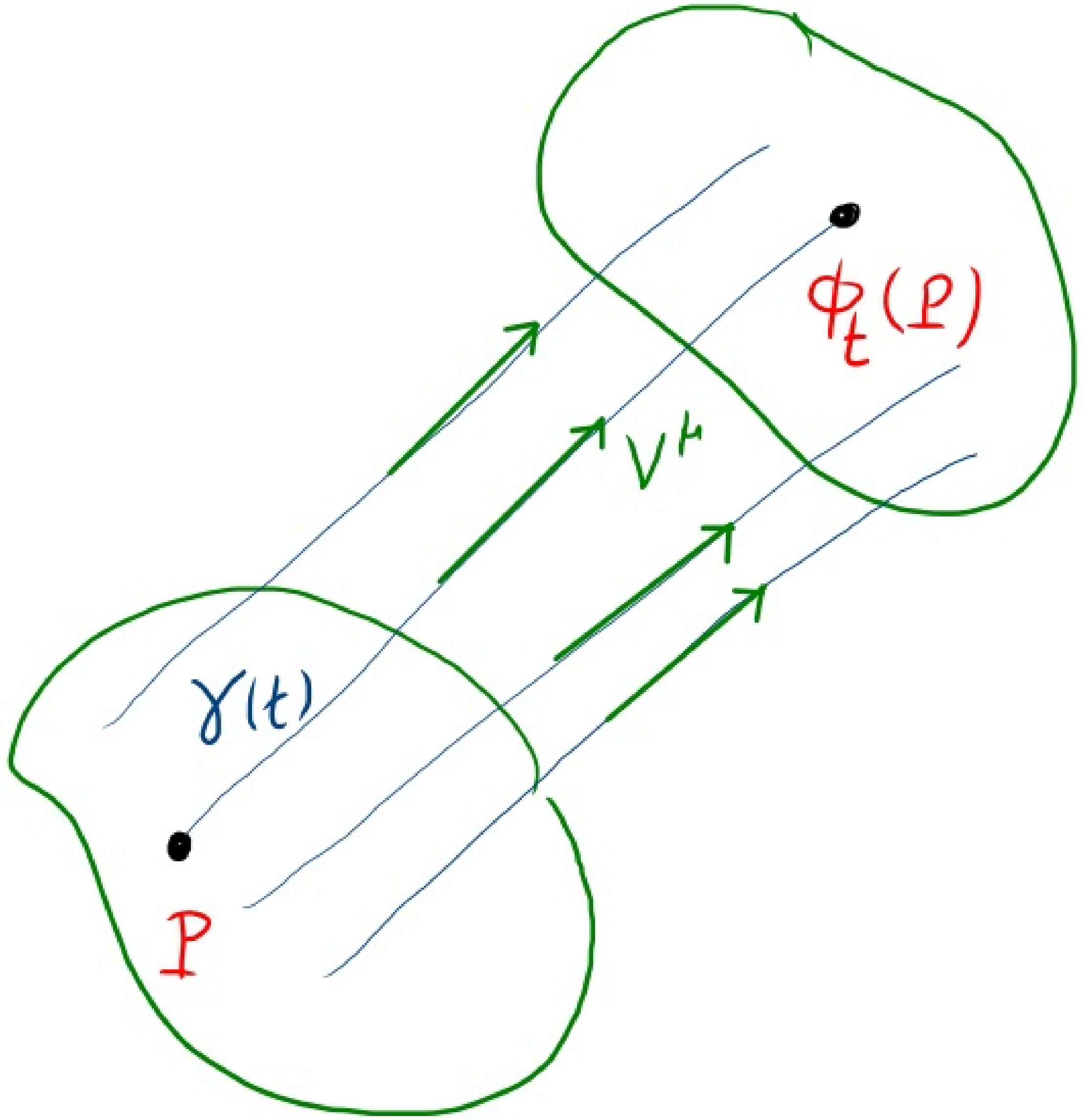

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metric compatibility

$g_{\rho\nu}\nabla_\mu V^\rho = \nabla_\mu(g_{\rho\nu}V^\rho) = \nabla_\mu V_\nu$

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Conservation Laws + Isometries

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Because $\mathcal{L}_\xi \chi = [\xi, \chi]$, and $\mathcal{L}_{[\xi,\chi]} = [\mathcal{L}_\xi, \mathcal{L}_\chi]$, so

$$\mathcal{L}_{[\xi,\chi]} g = [\mathcal{L}_\xi, \mathcal{L}_\chi] g = \mathcal{L}_\xi (\cancelto{0}{\mathcal{L}_\chi g}) - \mathcal{L}_\chi (\cancelto{0}{\mathcal{L}_\xi g}) = 0$$

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Then $\xi = \partial_\sigma$ generates an isometry:

In that coordinate system $\xi^\mu = \delta^\mu{}_\sigma$, and

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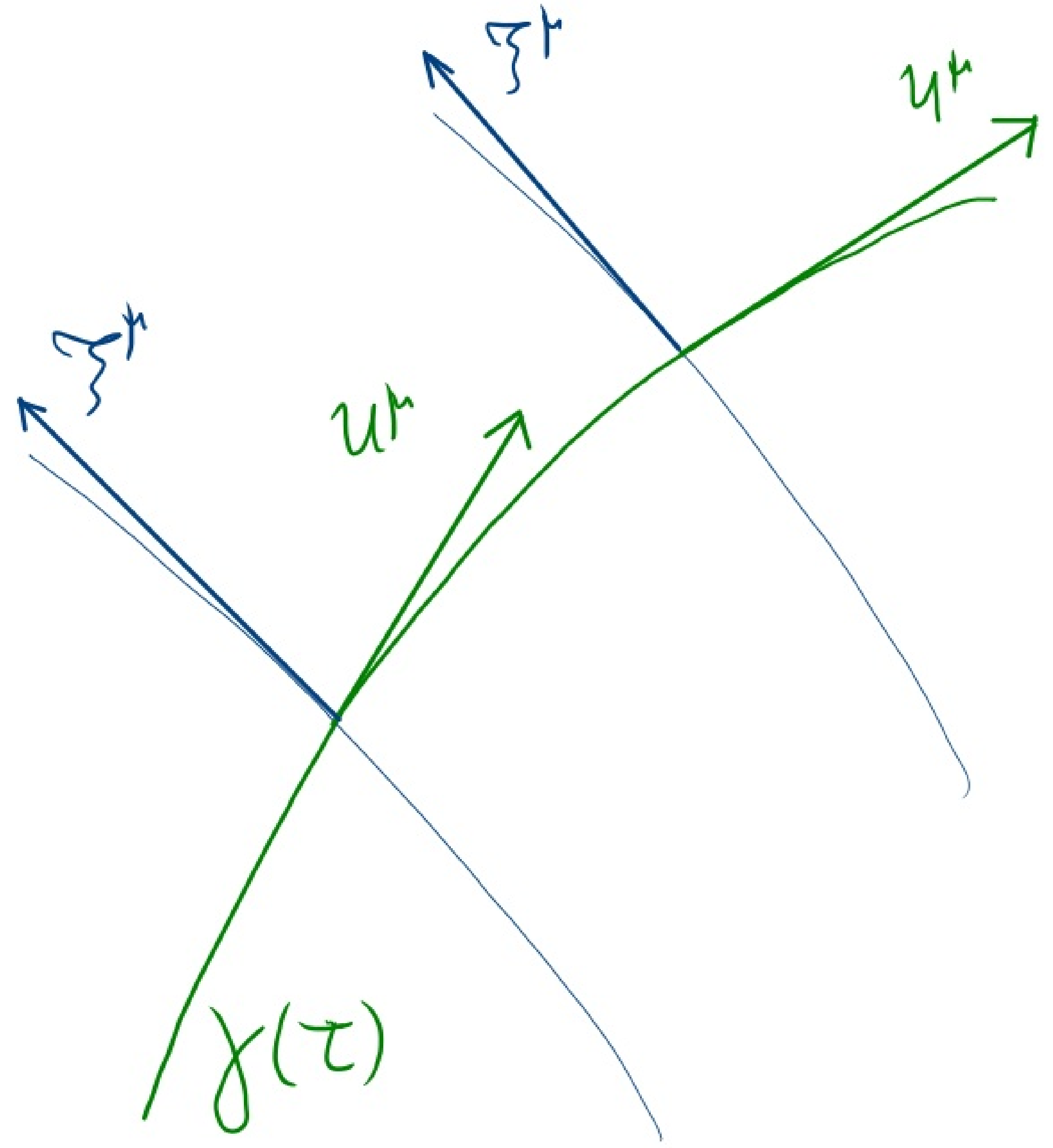
$$\begin{aligned}
\mathcal{L}_\xi g_{\mu\nu} &= \xi^\lambda \partial_\lambda g_{\mu\nu} + g_{\mu\lambda} \partial_\nu \xi^\lambda + g_{\lambda\nu} \partial_\mu \xi^\lambda \\
&= \delta^\lambda{}_\sigma \partial_\lambda g_{\mu\nu} + g_{\mu\lambda} \partial_\nu \underbrace{\cancel{(\delta^\lambda{}_\sigma)}}_{0} + g_{\lambda\mu} \partial_\mu \underbrace{\cancel{(\delta^\lambda{}_\sigma)}}_{0} \\
&= \partial_\sigma g_{\mu\nu} = 0 \Rightarrow \xi^\mu \text{ a KVF s.t. } \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0
\end{aligned}$$

Conserved Quantities Along Geodesics

Let:
- $\gamma(\tau)$ a geodesic
- τ affine parameter
- u^μ tangent vector
- ξ^μ a KVF



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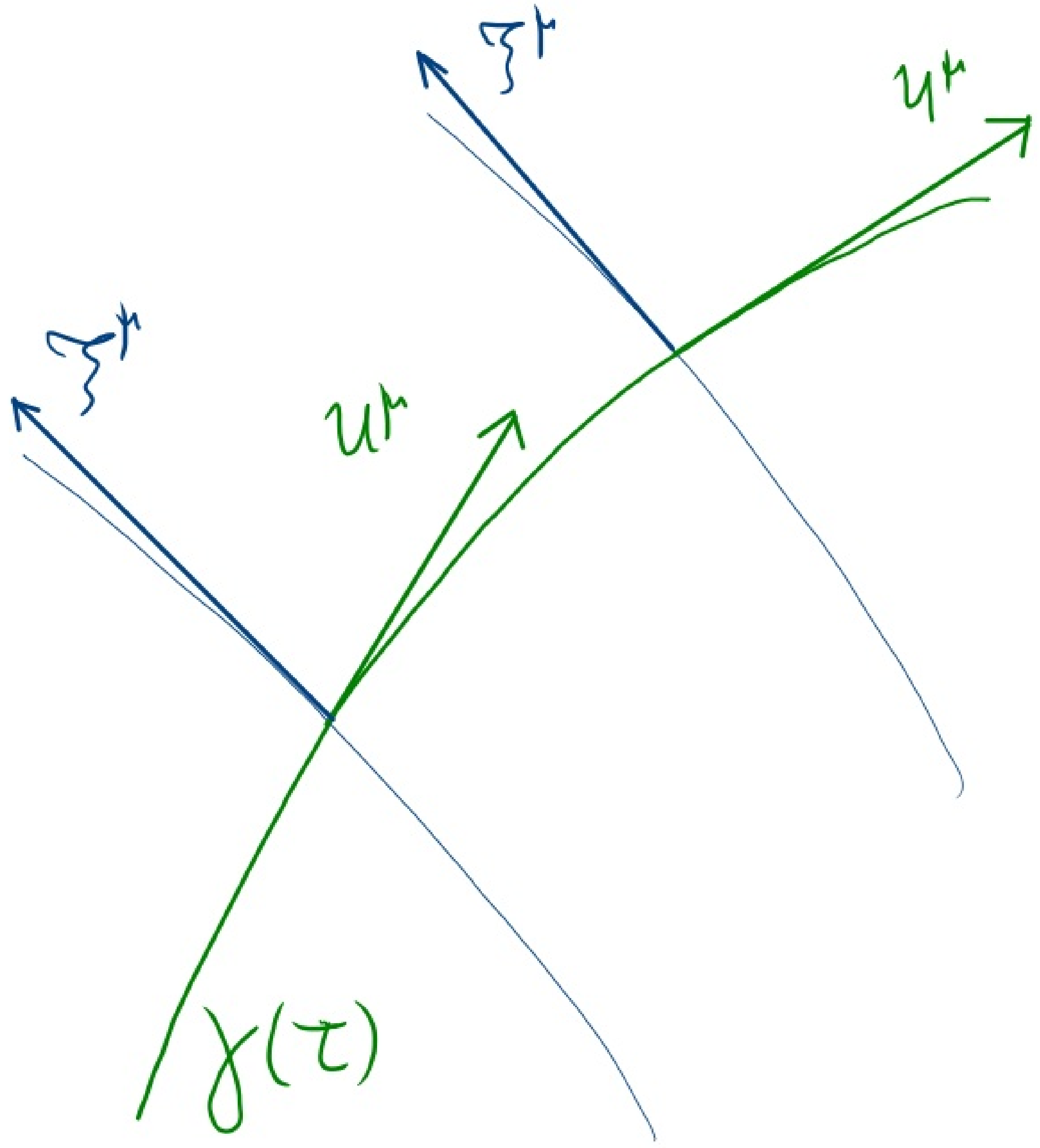


$\Rightarrow \quad \frac{d}{d\tau}\left(\xi^\mu u_\mu\right) = 0$

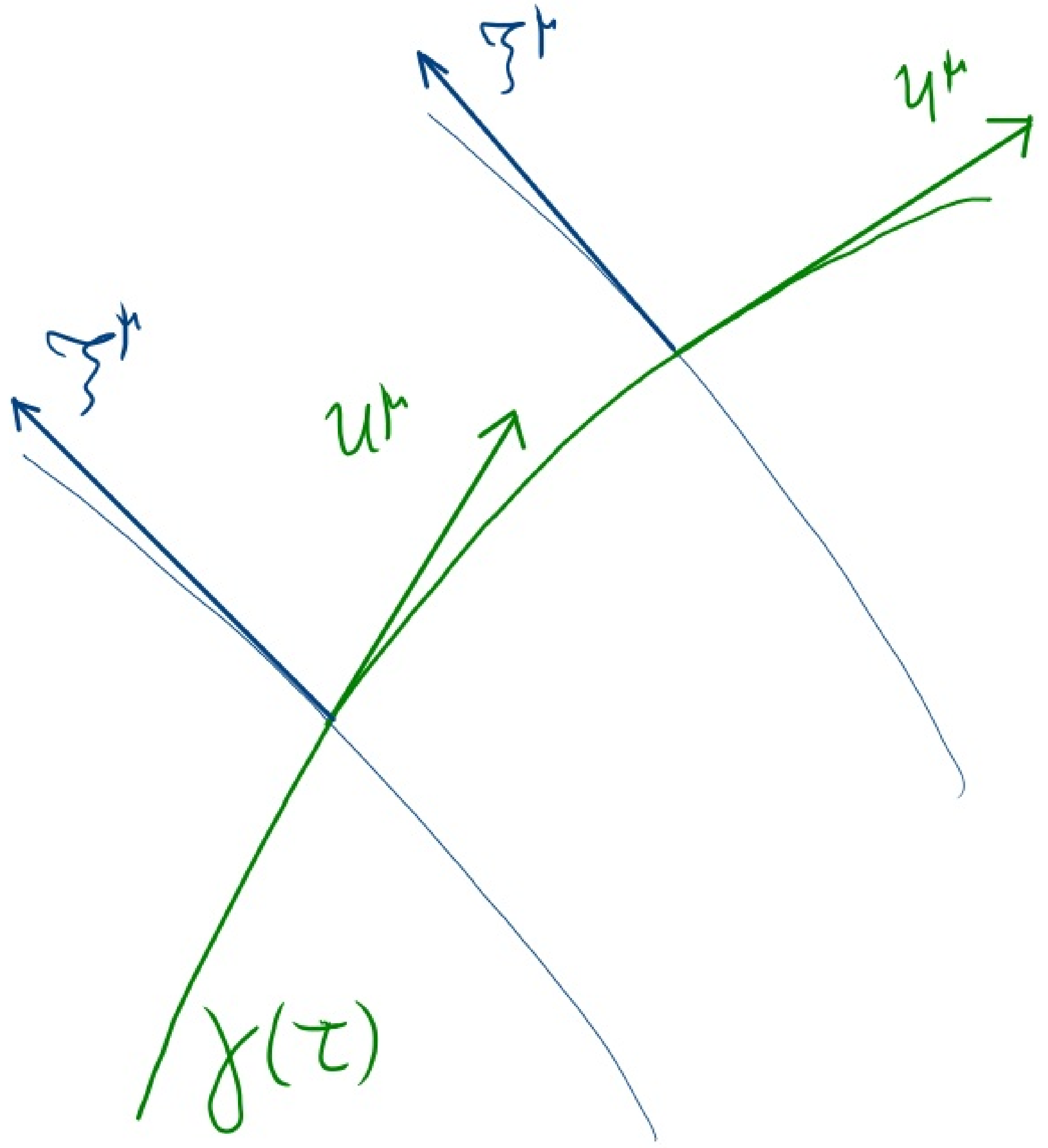
$\Rightarrow \quad \xi^\mu u_\mu$ conserved along geodesic

Conserved Quantities Along Geodesics

Indeed:

$$\frac{d}{d\tau}(\xi^\mu u_\mu) = D_u(\xi^\mu u_\mu) = u^\nu \nabla_\nu(\xi^\mu u_\mu)$$

a function: $\frac{d}{d\tau} = D_u$



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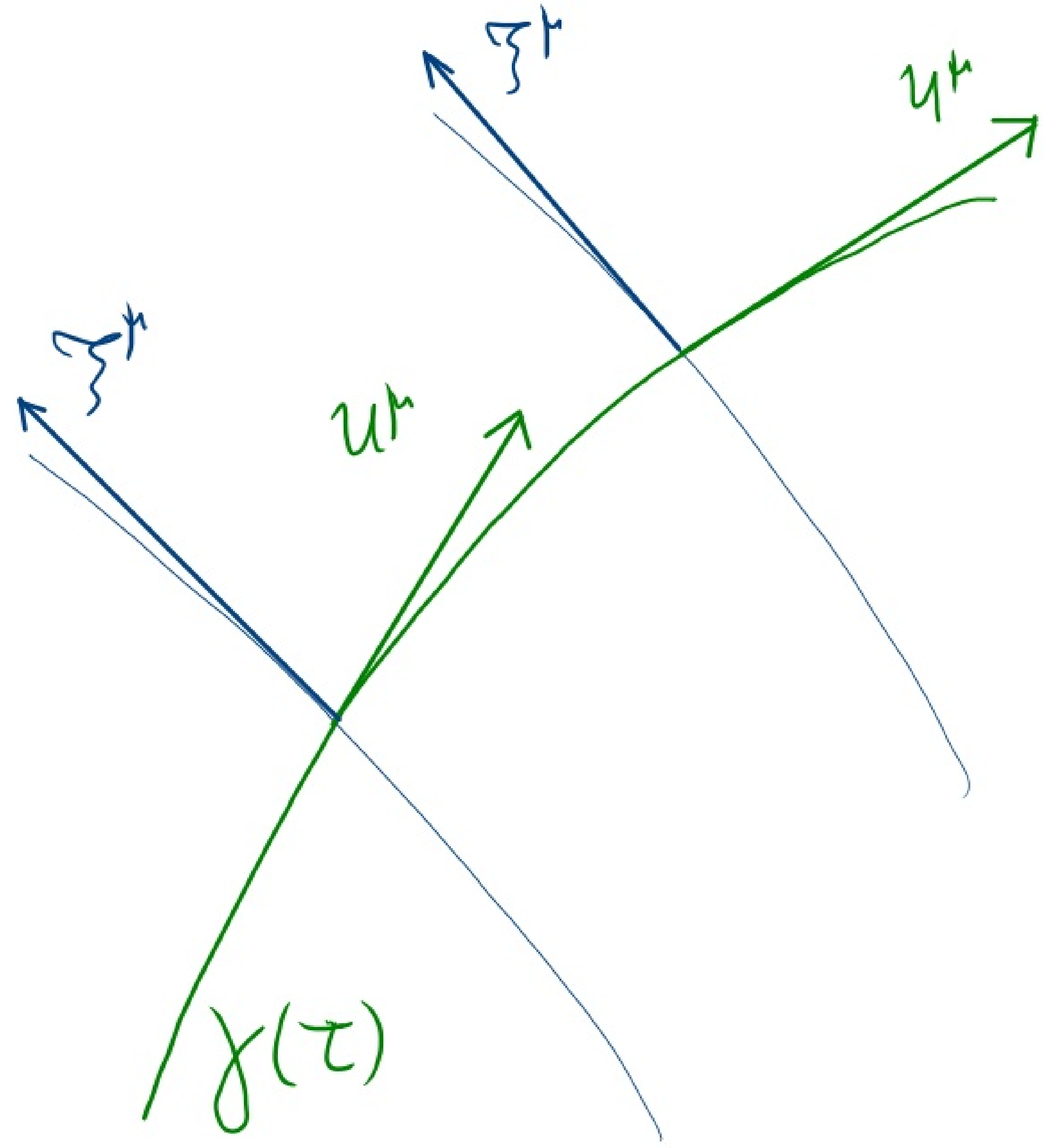

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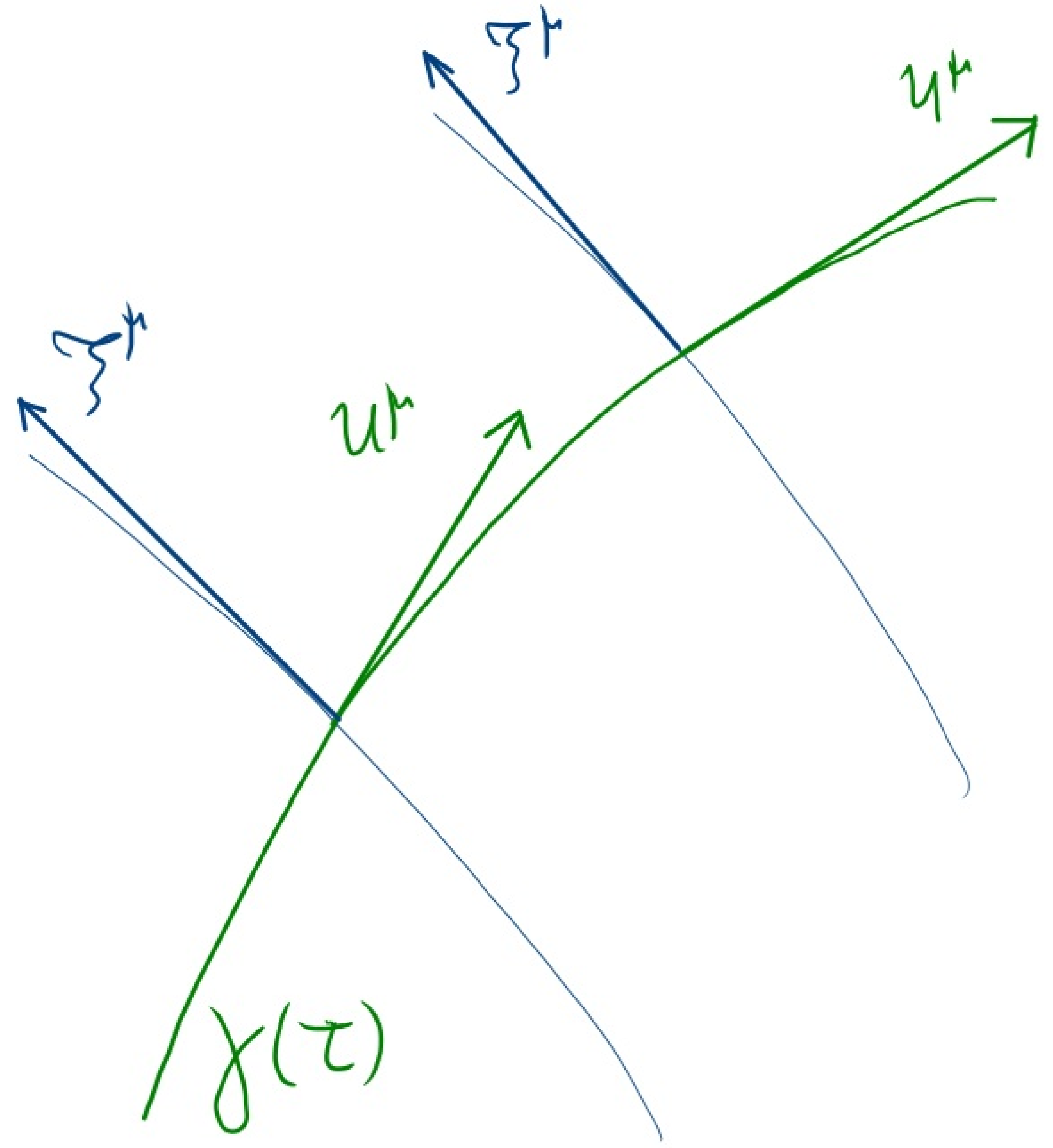

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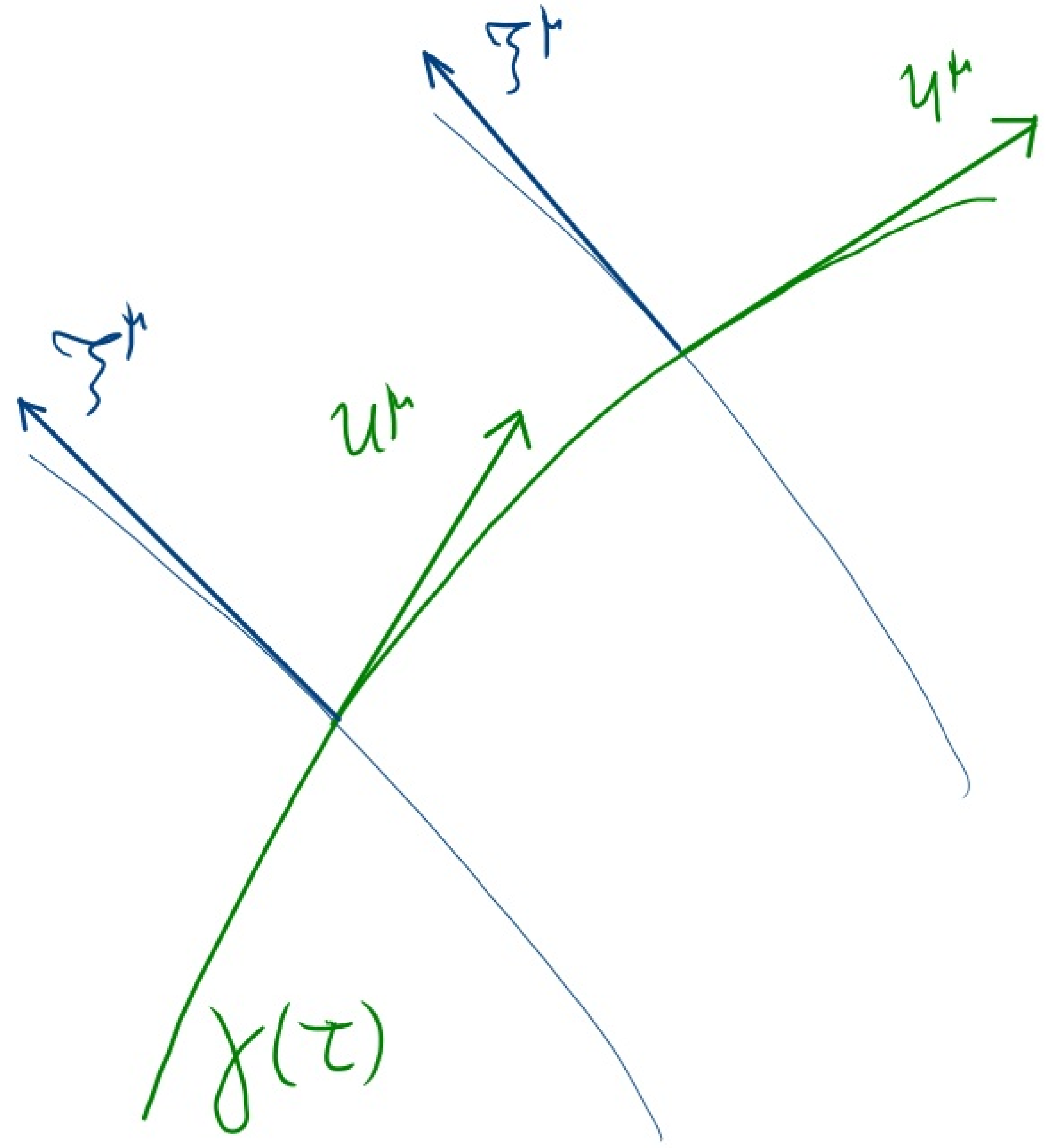

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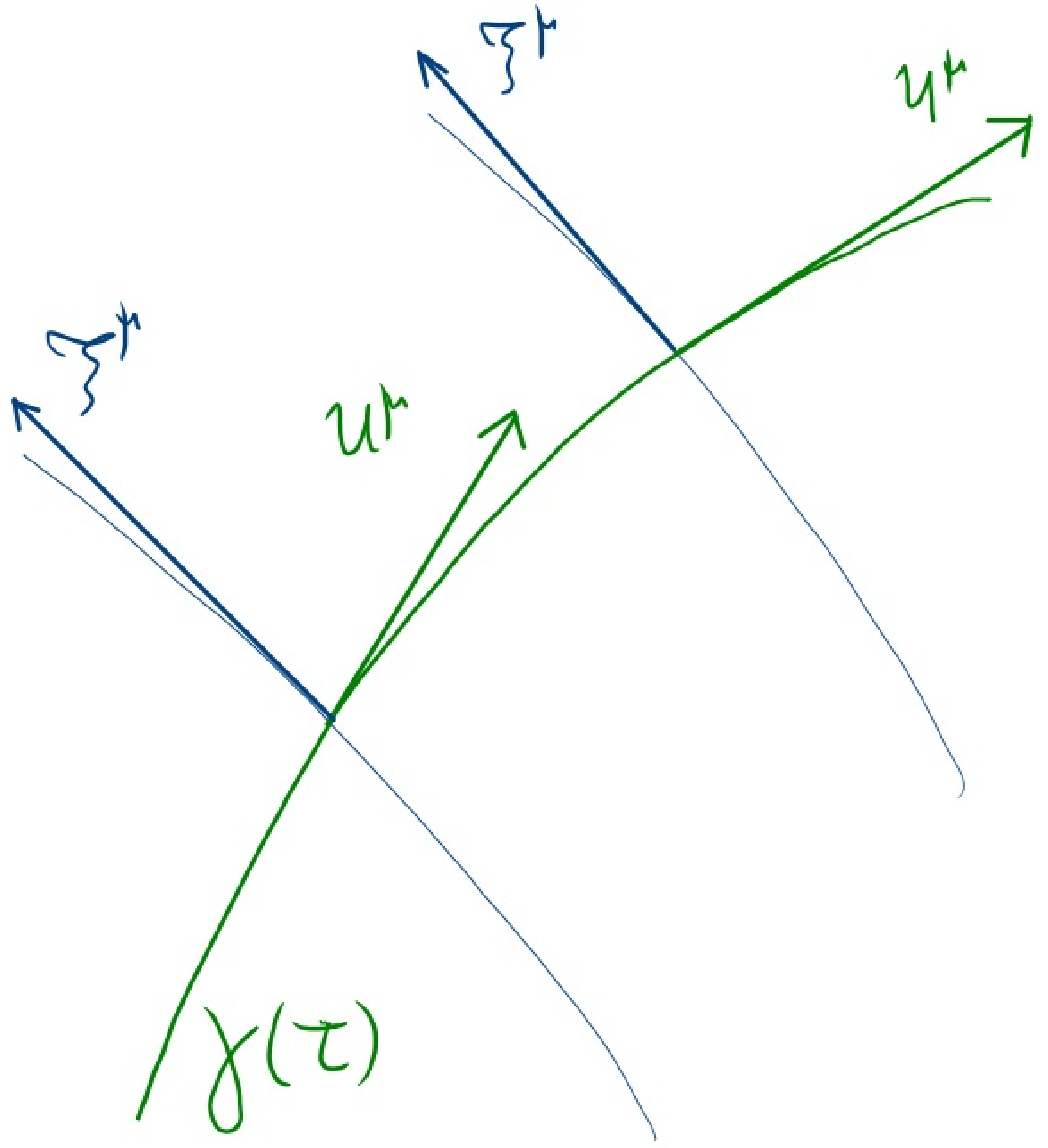

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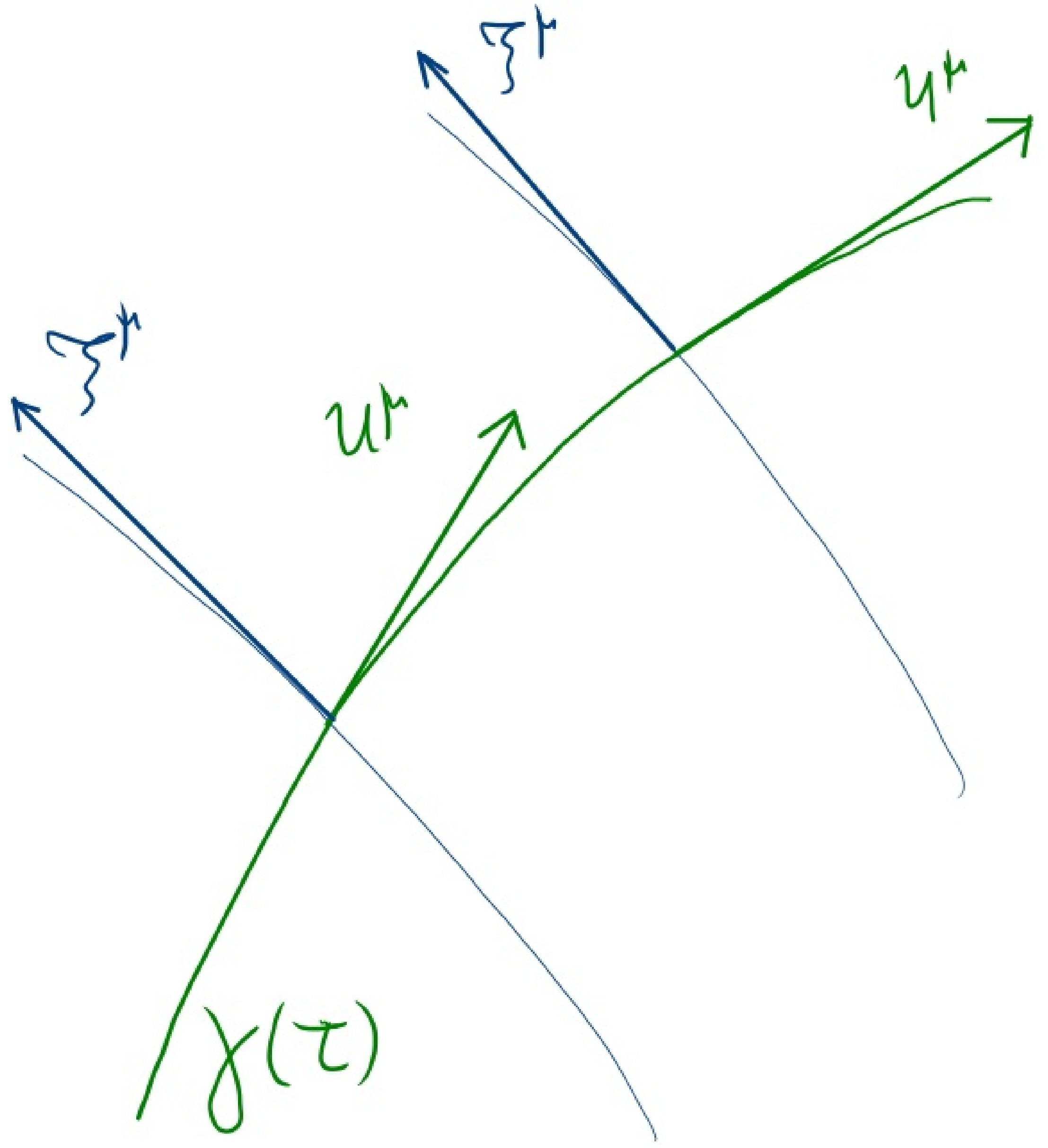
$$= 0$$



Stress - Energy Tensor

$$T^{\mu\nu} = T^{\nu\mu} \quad \text{and} \quad \nabla_\mu T^{\mu\nu} = 0$$

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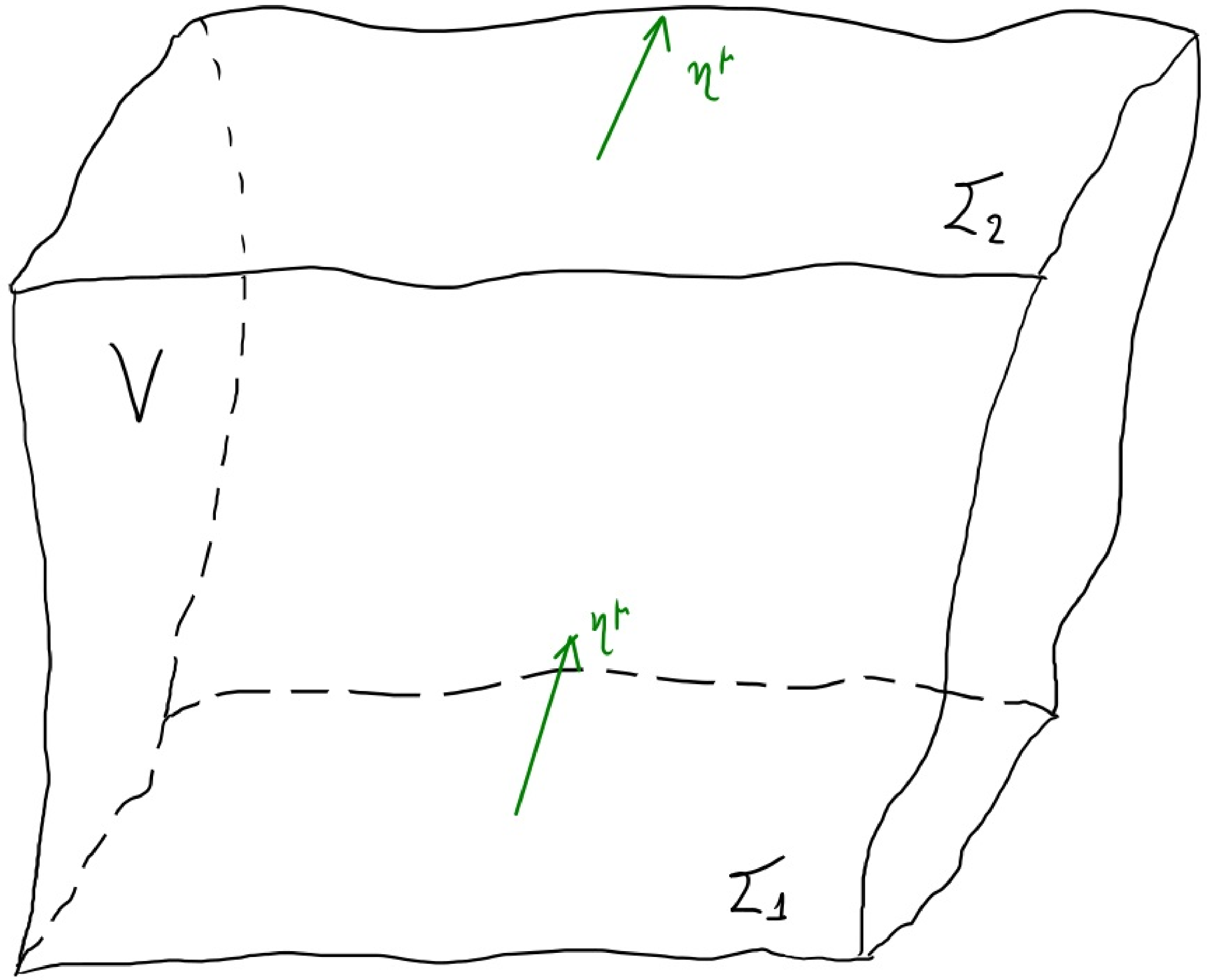
If ξ^μ timelike $\rightarrow$ energy conservation

" spacelike momentum "



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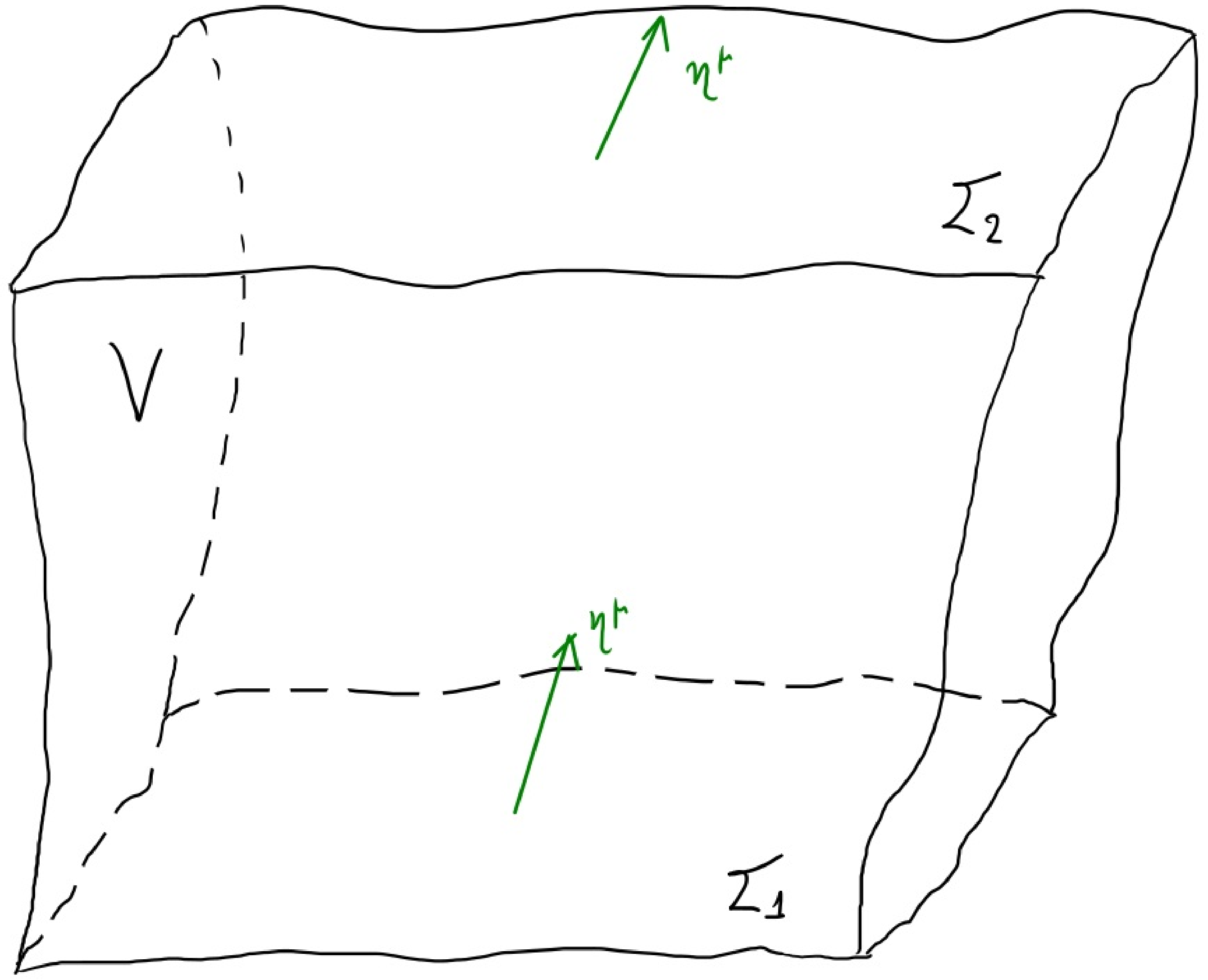
<u>E.g. ξ^μ timelike:</u>

Consider $\begin{cases} \text{4-d volume } V \\ \Sigma_1, \Sigma_2 \text{ spacelike} \\ \eta^\mu \quad \text{timelike} \end{cases}$



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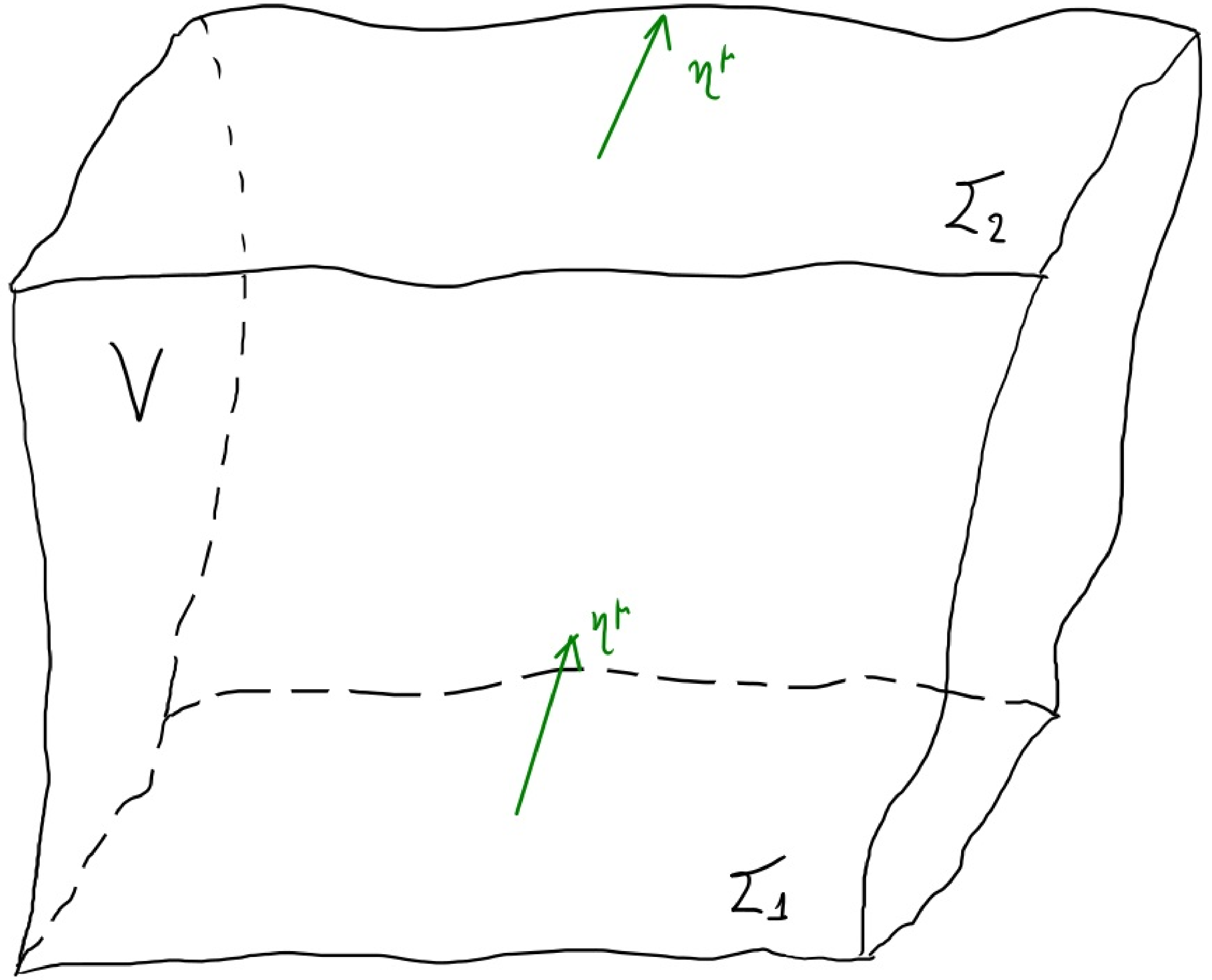

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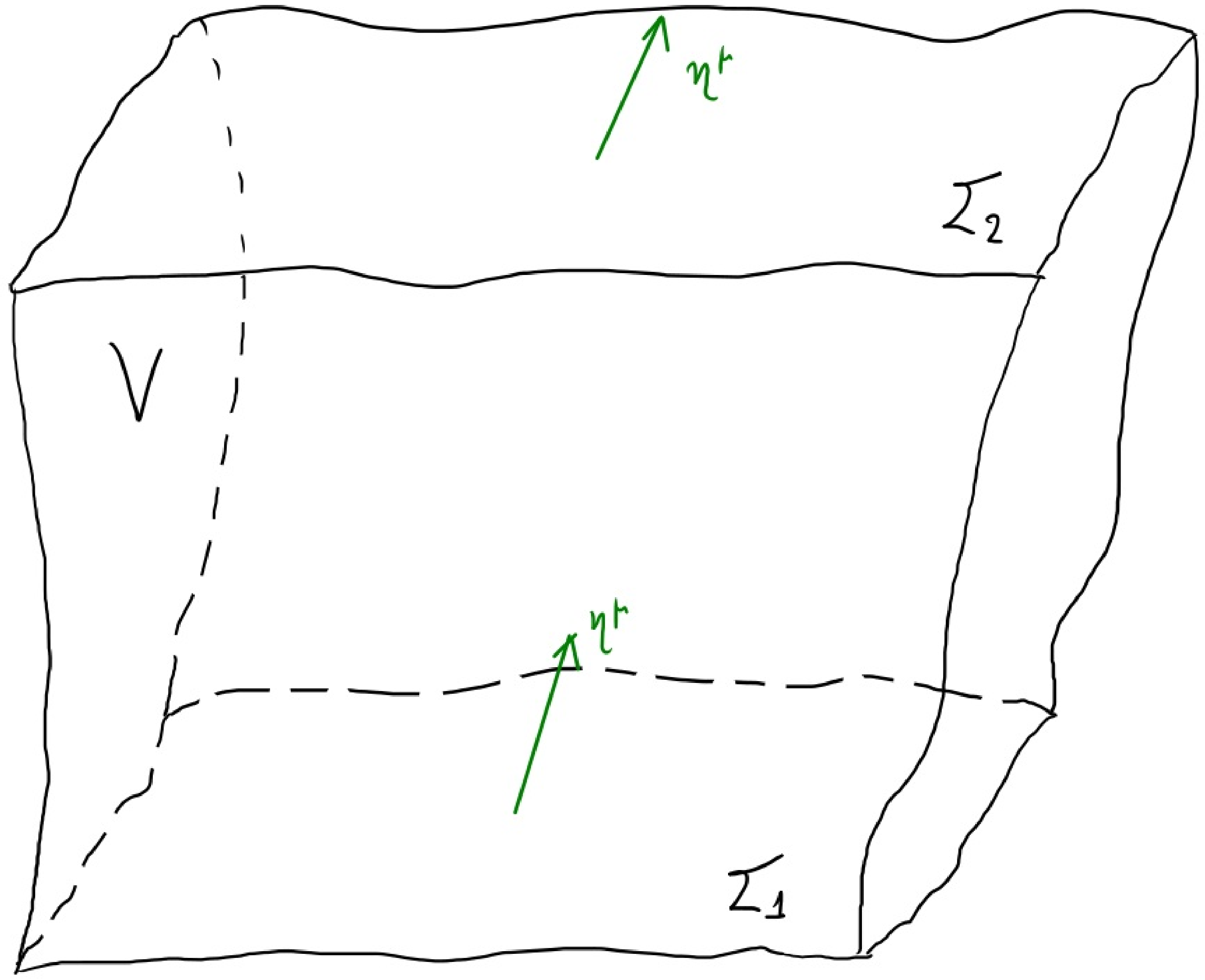

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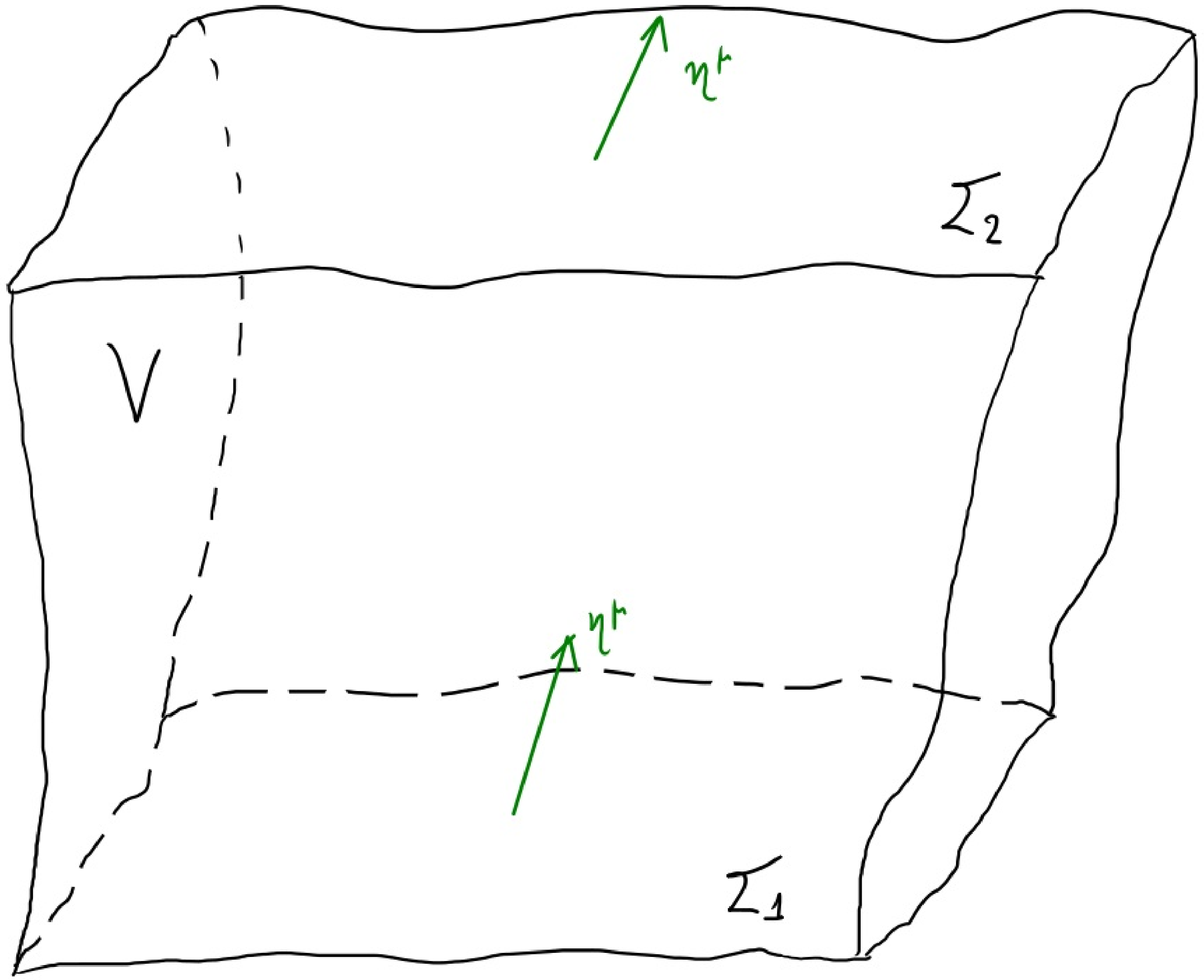

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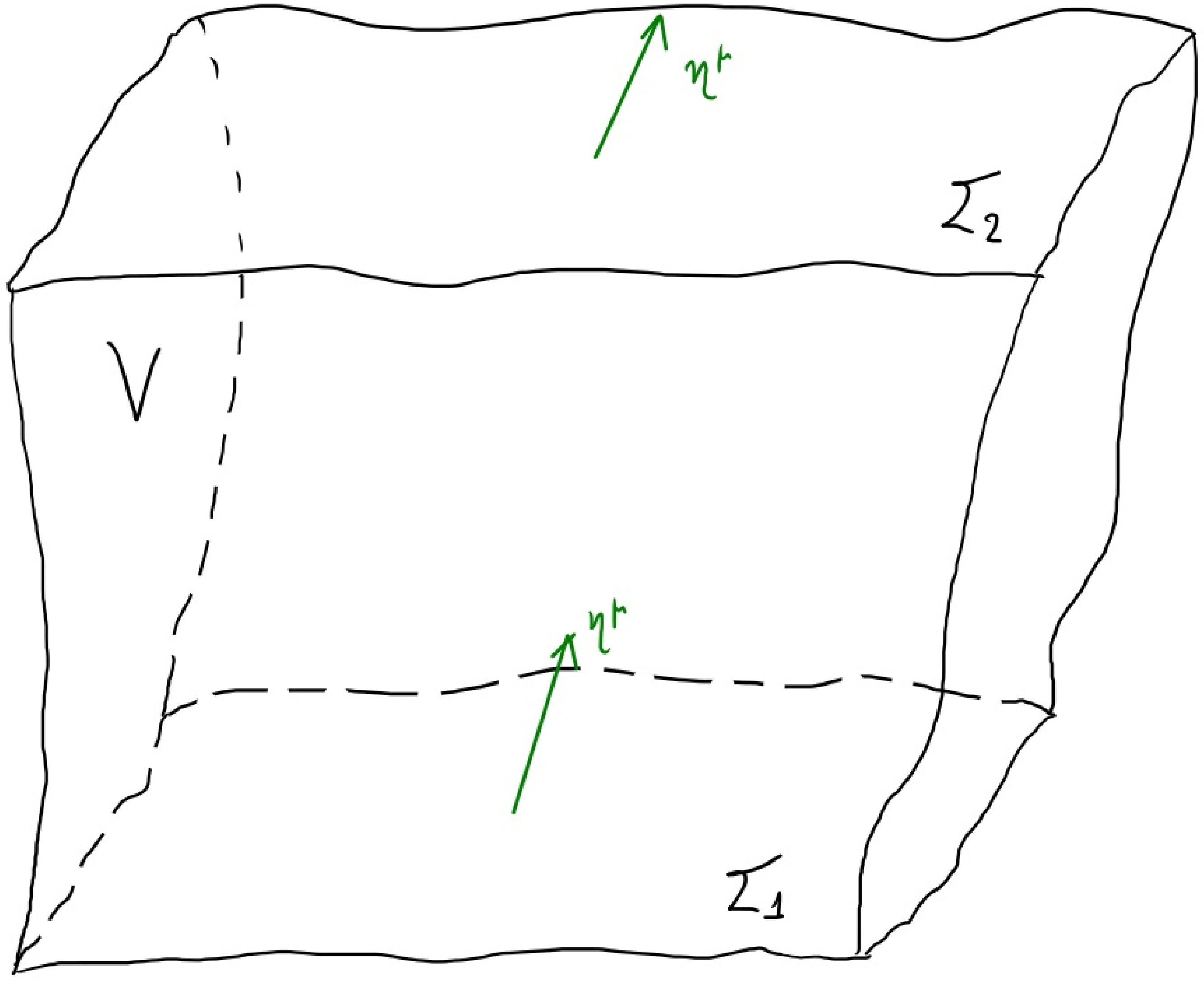

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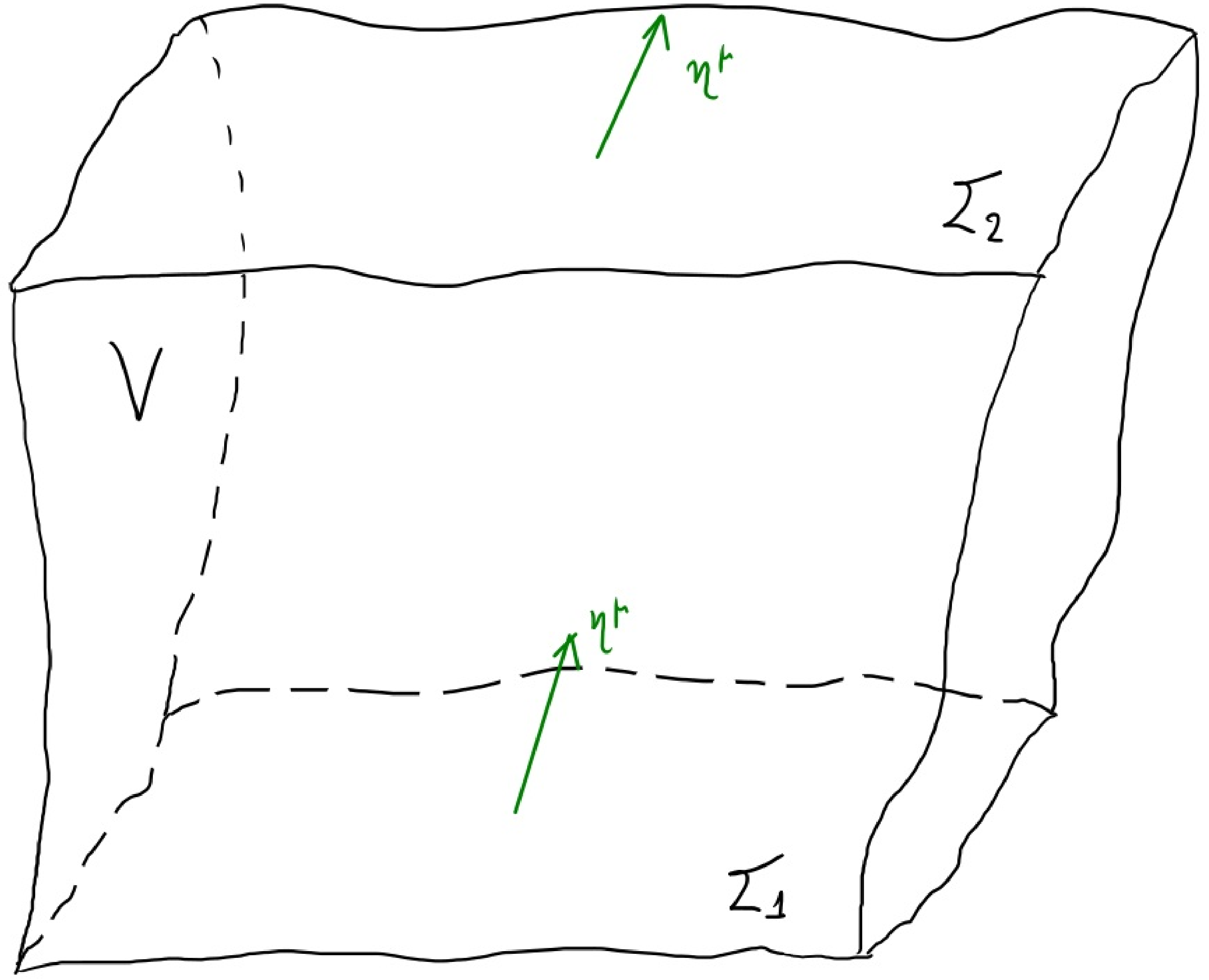

Choose Σ_1, Σ_2 s.t. $\xi^\mu = \eta^\mu = (\partial_0)^\mu$

$g_{00} = -1$

$$\int_{\Sigma} J_\mu \eta^\mu \sqrt{\gamma}\, d^3x = \int_{\Sigma} T^{00} \sqrt{\gamma}\, d^3x = E = \text{const}$$

Example: Schwarzschild metric

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

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KVFs

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K^μ time translations $\rightarrow$ energy conservation

R^μ φ - translations $\rightarrow$ angular momentum conservation

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Conserved quantities along geodesics:

$$E = -K_\mu u^\mu = \left(1-\frac{2M}{r}\right)\frac{dt}{d\tau}$$

$$L = R_\mu u^\mu = r^2 \sin^2\theta \frac{d\varphi}{d\lambda}$$