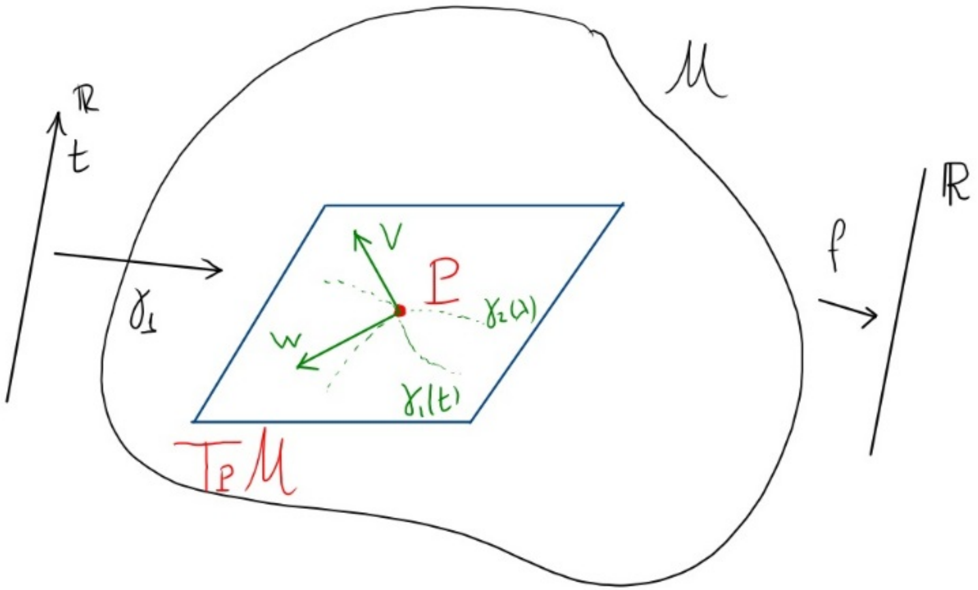
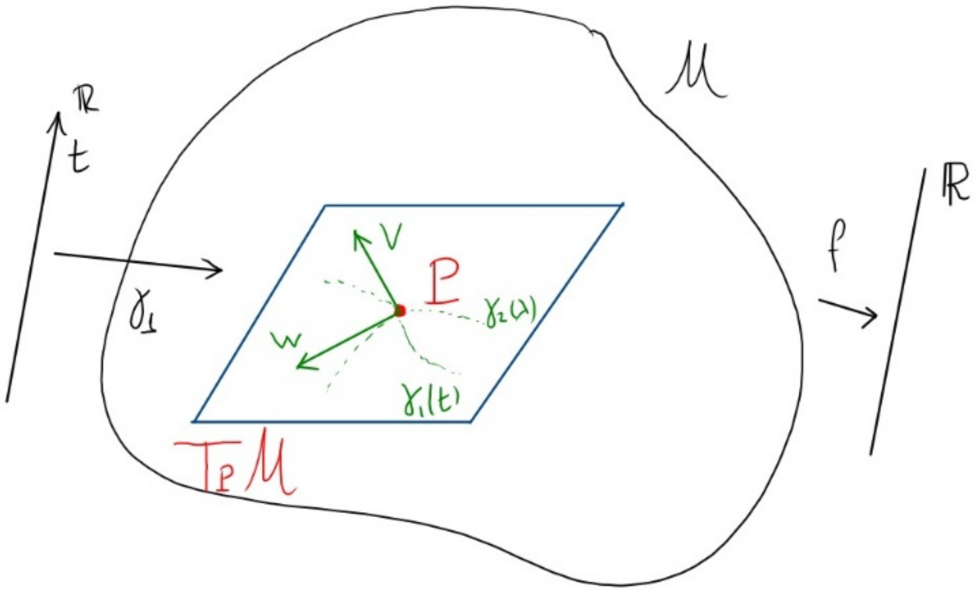
- Vectors : tangent to curves



- Vectors: tangent to curves

- Measure rate of change of functions on M along a curve:

$$V = \frac{d}{dt}\bigg|_p \quad : \quad f \mapsto V(f) = \frac{df}{dt}\bigg|_p$$



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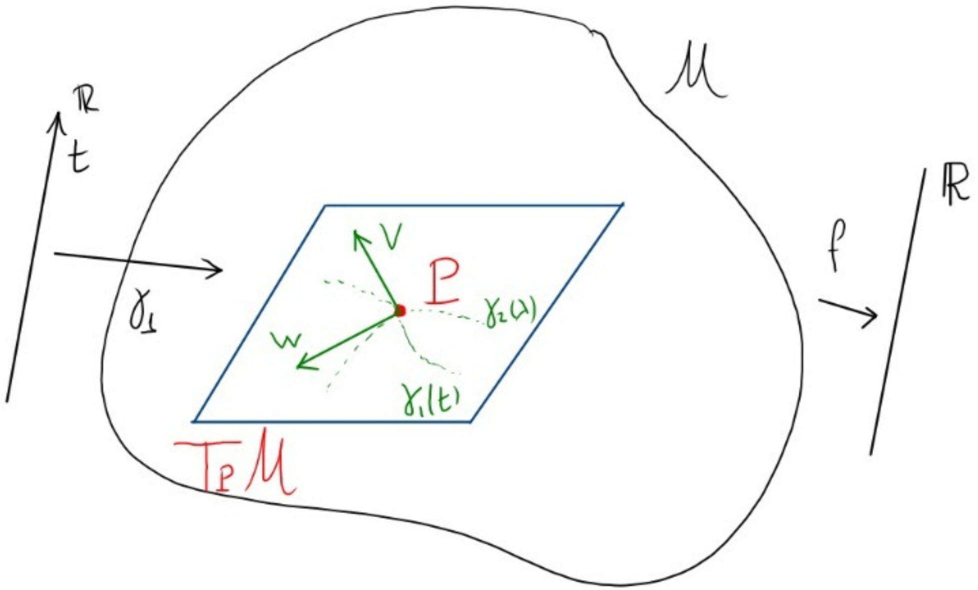

- Measure rate of change of functions on $\mathcal{M}$ along a curve:

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- They are derivations:

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$$V(f \cdot g) = V(f) \cdot g + f \cdot V(g)$$

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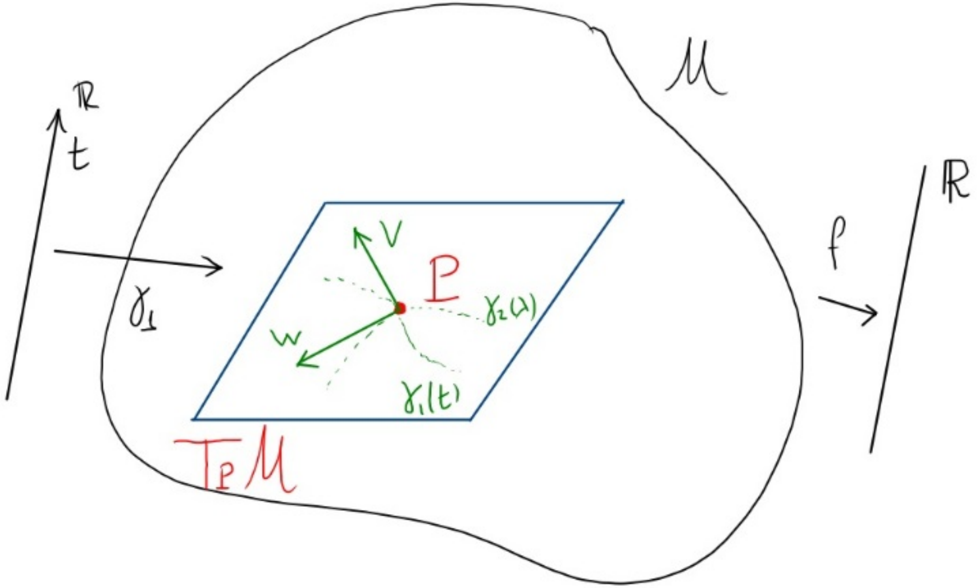
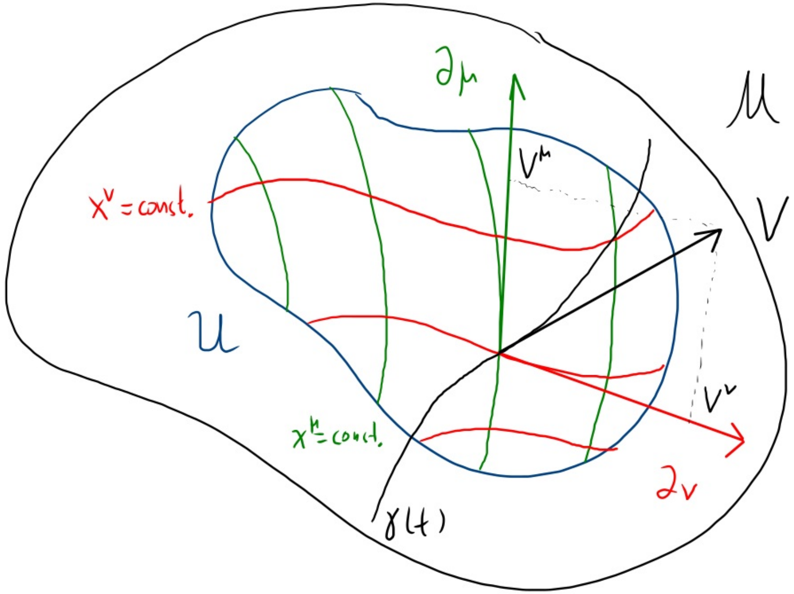


- They are derivations
- They form an n-dim vector space $T_P\mathcal{M}$

Different at each P!

- Vectors: tangent to curves



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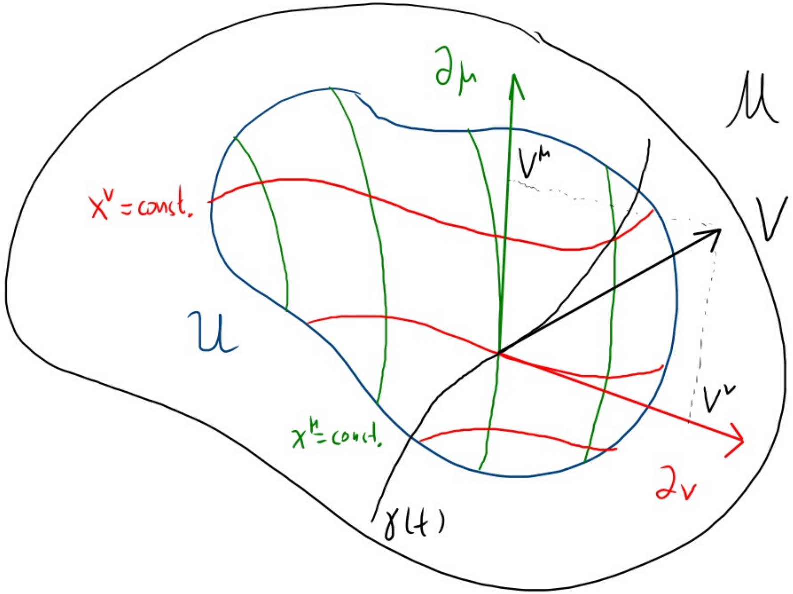
Choice of coordinate system x^μ selects a coordinate basis $\{\partial_\mu\}$ s.t. $V = V^\mu \partial_\mu$

$$V = \frac{dx^\mu}{dt}\partial_\mu \quad , \quad V^\mu = \underbrace{\frac{dx^\mu}{dt}}_{\text{rate of change of } x^\mu \text{ along } \gamma}$$



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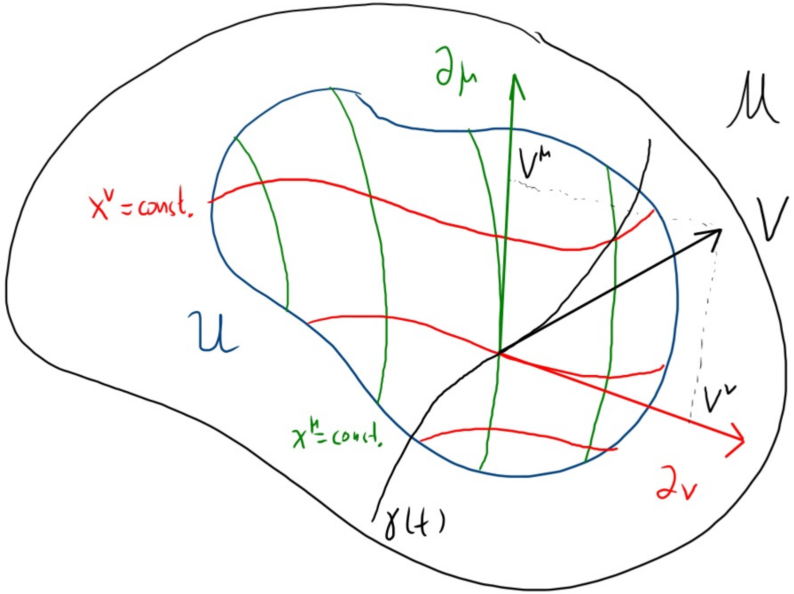

- coordinate xfm: $x^\mu \to x^{\mu'}$

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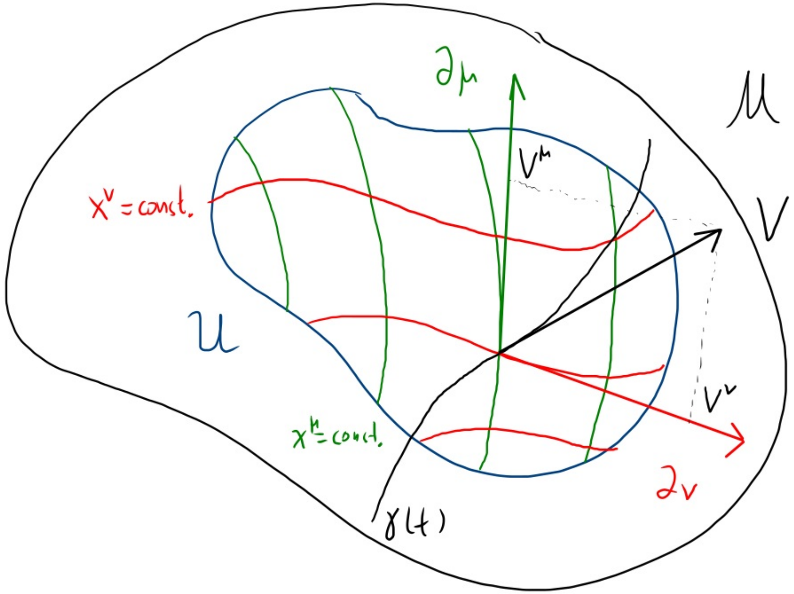

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summation convention: up/down

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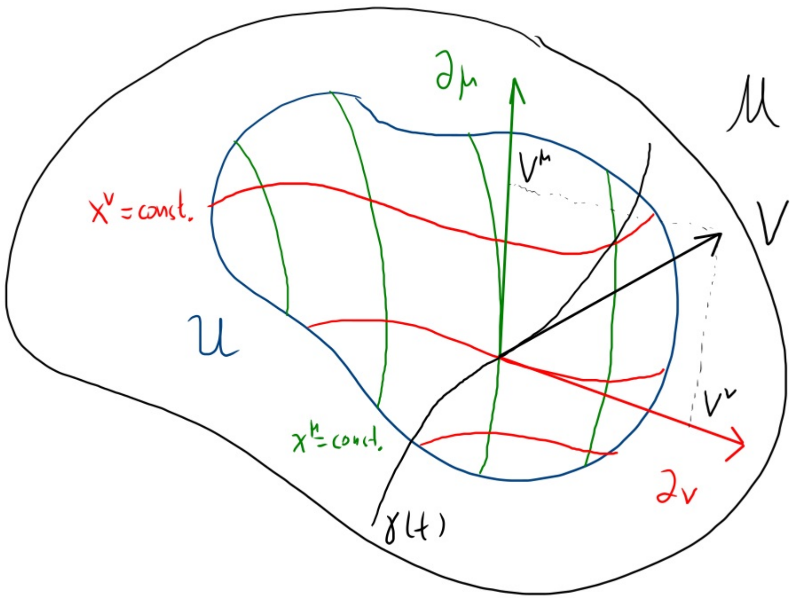

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invertible/non-singular

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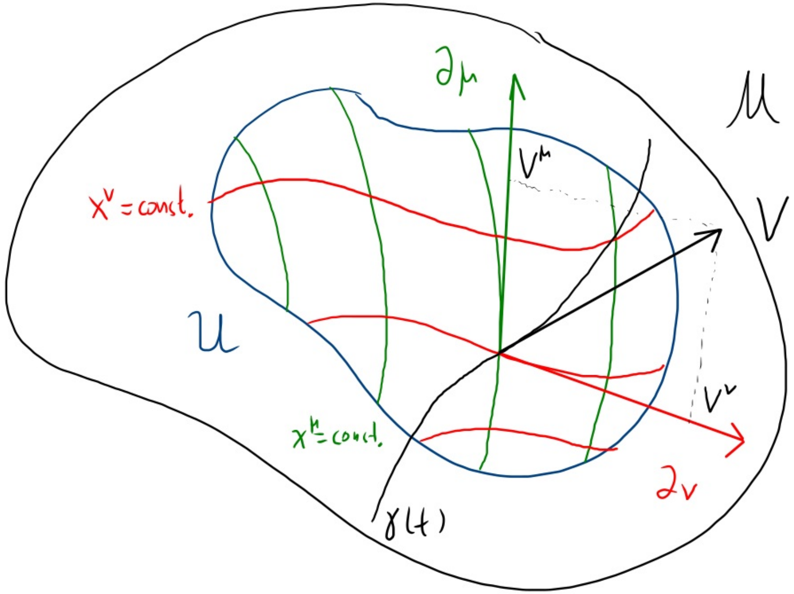

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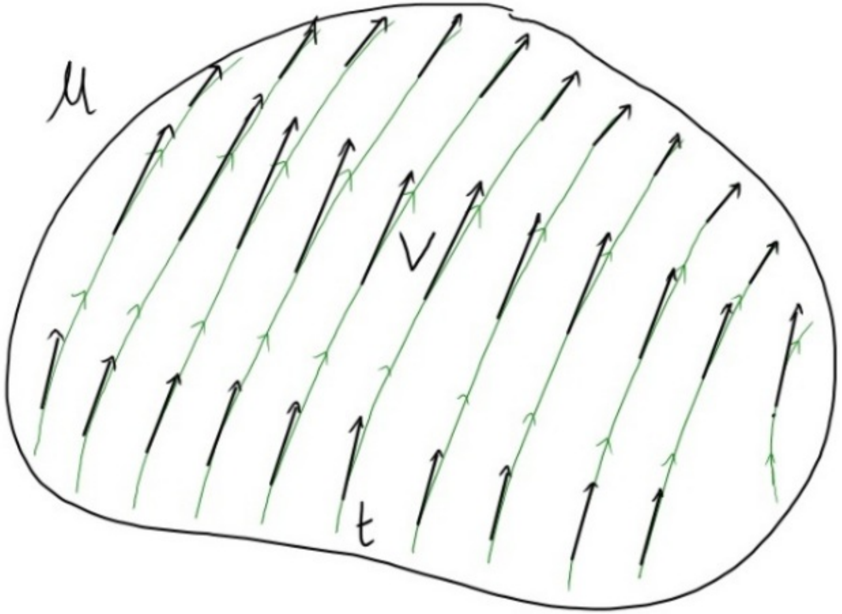
row column

- Vector fields:

smoothly defined vectors $\forall p \in \mathcal{U}$



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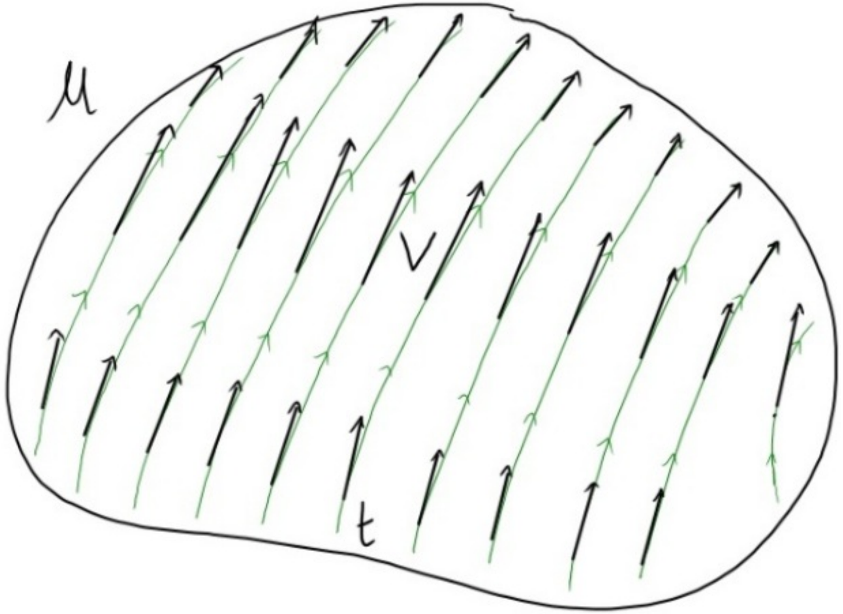
$\Leftrightarrow\ V(f) = \dfrac{df}{dt}$ a smooth function $\forall\, f \in \mathcal{F}(\mathcal{M})$



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$$\Leftrightarrow \quad V(f) = \frac{df}{dt} \quad \text{a smooth function } \forall f \in \mathcal{F}(\mathcal{M})$$

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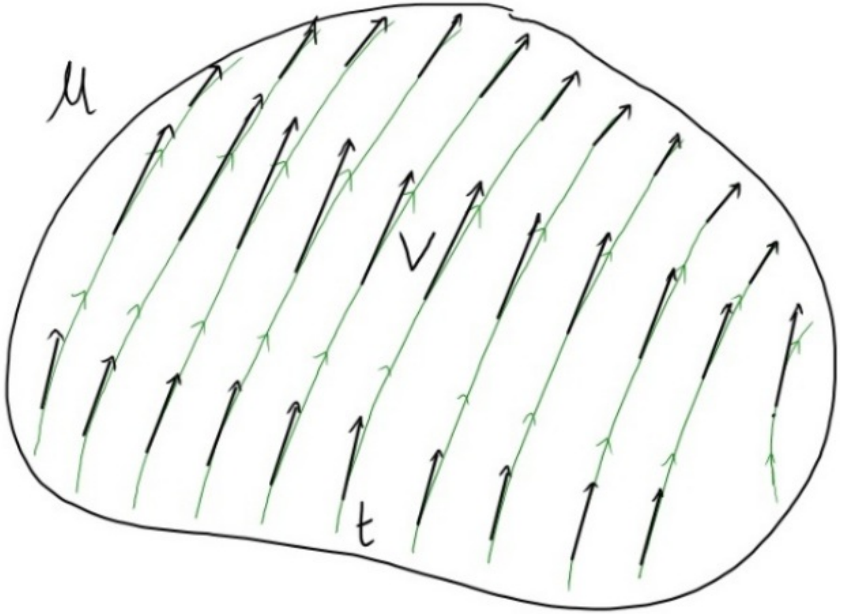

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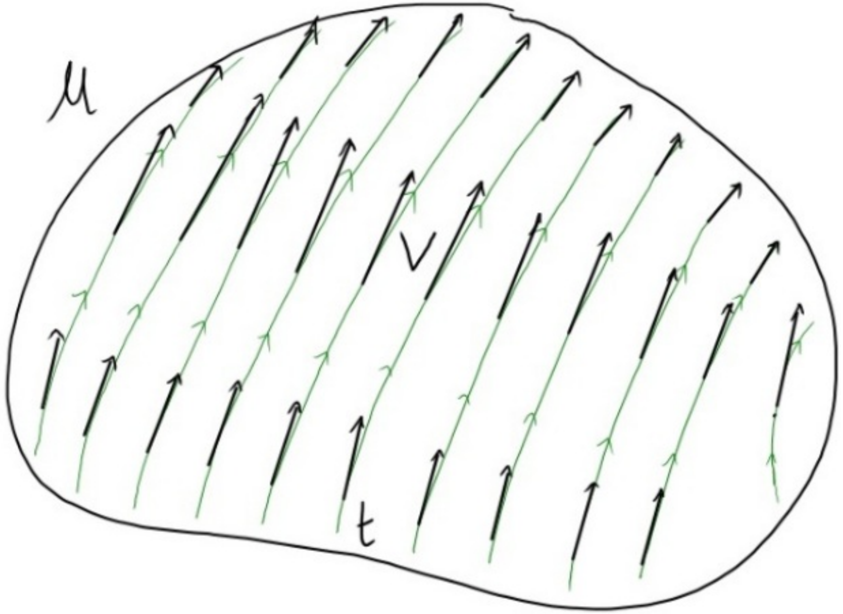

$\Leftrightarrow\; V(f) = \dfrac{df}{dt}$ a smooth function $\forall f \in \mathcal{F}(\mathcal{M})$

$\Leftrightarrow\; V^{\mu} = \dfrac{dx^{\mu}}{dt}$ smooth functions

- Integral curves of nonzero v.f. in $\mathcal{U} \in \mathcal{M}$ form a congruence:
  - a unique i.c. passing through each $P \in \mathcal{U}$
  - never cross

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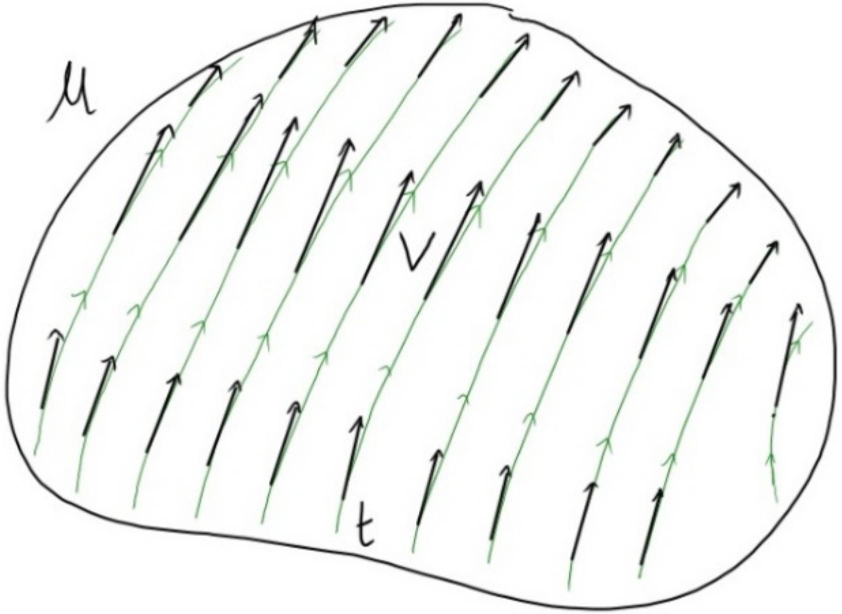

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Derivatives of f along integral curves

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One-forms:

- defined at a point $P \in \mathcal{M}$ like vectors
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$\Rightarrow$ form a dual vector space $T^*_p\mathcal{M}$

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$$V \to \omega(V)$$

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Exercise: Prove $\alpha\omega + \beta\sigma$ is a 1-form

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  Let $\{e_\alpha\}$ be a basis in T_pM

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Define its dual $\{e^\alpha\}$ from

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$$\underbrace{e^\alpha}_{\text{1-form}} (\underbrace{e_\beta}_{\text{vector}}) = \underbrace{\delta^\alpha{}_\beta}_{\text{number}}$$

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- These are not orthonormality conditions
- Not an inner product
- e^α and e_α are different objects

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components in $\{e_\alpha\}$

$$e^\alpha(V) = e^\alpha(V^\beta e_\beta)$$

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use linearity of the e^α maps

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$$\forall e_\alpha$$

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linearity

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the two maps are the same

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$\omega(V) = \omega_\alpha V^\alpha \in \mathbb{R}$ is the contraction of ω and V

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exercise: prove that the map is linear

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In fact we can build vectors on top of 1-forms s.t.
1-forms are the fundamental geometric objects on $\mathcal{M}$

An Important 1-form: The gradient df

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For $f \in \mathcal{F}(M)$, define the 1-form

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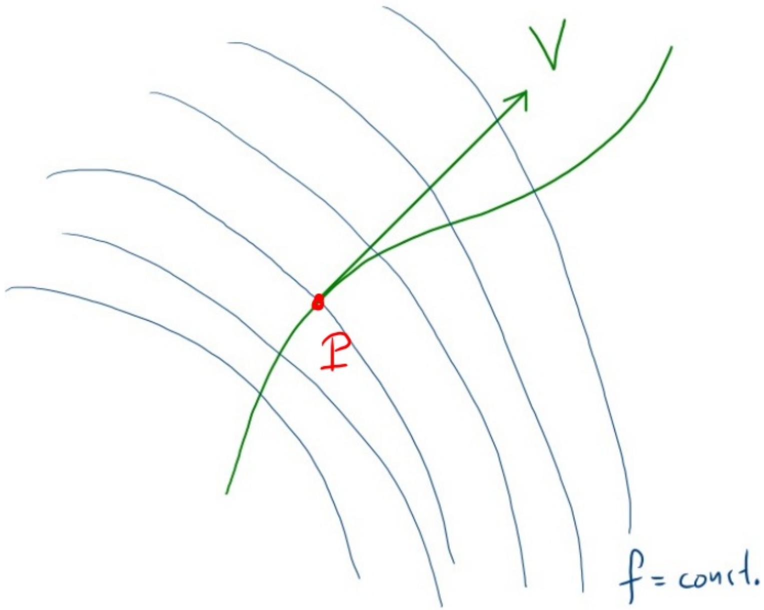
$$df : T_p M \to \mathbb{R}$$

$$V \mapsto df(V) = V(f) = \underbrace{\frac{df}{dt}}_{\text{along the curve defining } V}$$

choice of V

$\Updownarrow$

choice of equivalence classes of curves through P



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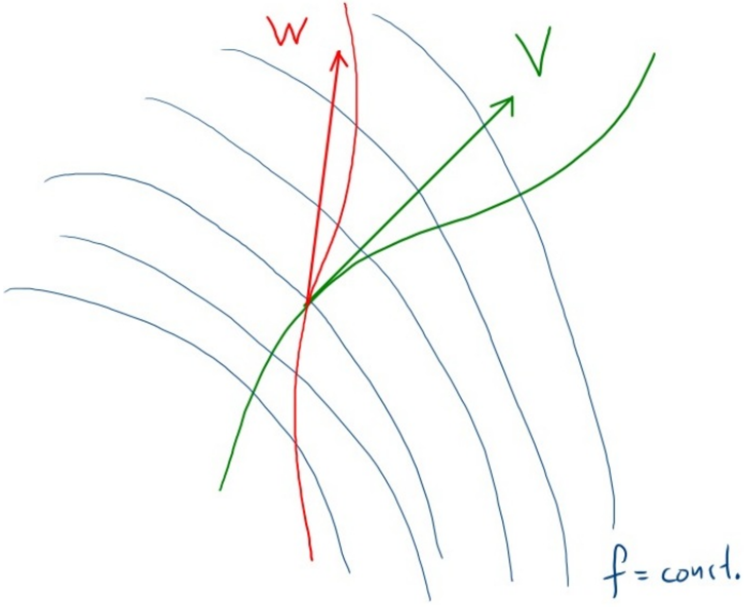
$$
\begin{aligned}
df : \quad T_p M &\to \mathbb{R} \\
V &\mapsto df(V) = V(f) \\
W &\mapsto df(W) = W(f) \\
&\qquad\qquad\qquad\quad\;\; = \frac{df}{d\lambda}
\end{aligned}
$$

the same function,
different vectors



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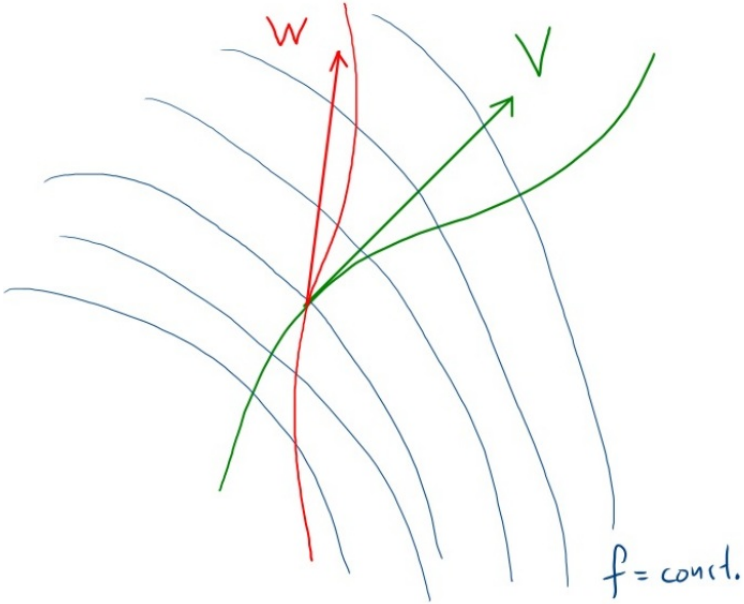

Linear:

$$df(\alpha V + \beta W) = (\alpha V + \beta W)(f)$$

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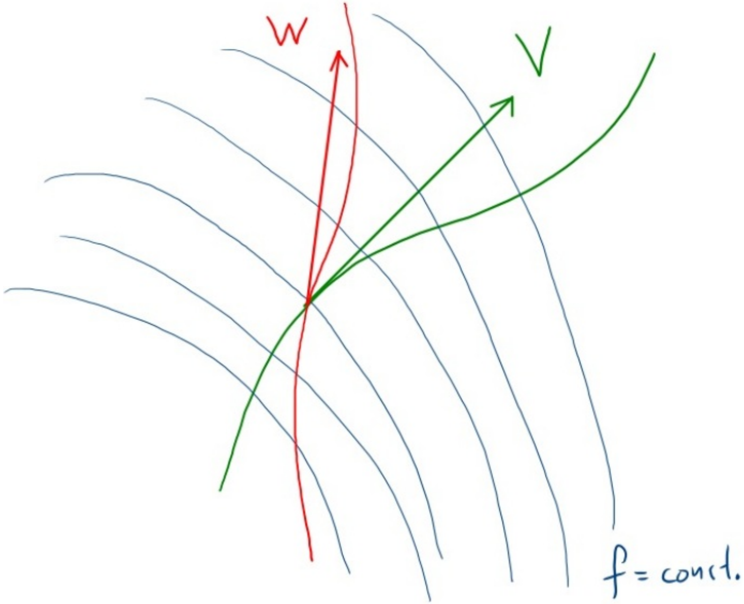

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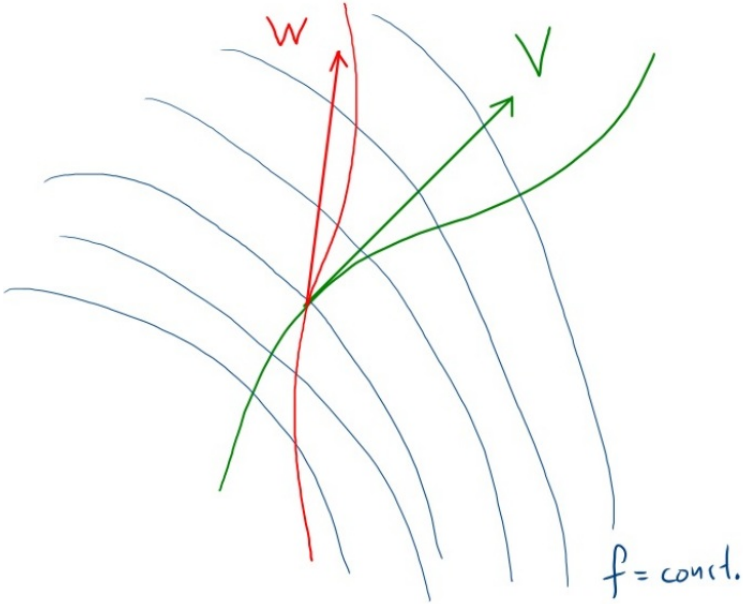

Linear:

$$df(\alpha V + \beta W) = (\alpha V + \beta W)(f)$$

$$= \alpha V(f) + \beta W(f)$$

$$= \alpha\, df(V) + \beta\, df(W)$$

If $\{\partial_\mu\}$ is a coordinate basis:

$$df(\partial_\mu) = \partial_\mu f$$

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So

$$\omega = \omega_\mu dx^\mu \Rightarrow \omega_\mu = \omega(\partial_\mu)$$

$$= dx^\mu(\omega)$$

Coordinate xfm $\{x^\mu\} \to \{x^{\mu'}\}$

then:

$$dx^\mu \to dx^{\mu'}$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \to \partial_{\mu'} = \frac{\partial}{\partial x^{\mu'}}$$

Coordinate xfm $\{x^\mu\} \to \{x^{\mu'}\}$

$$\omega = \omega_{\mu'}\, dx^{\mu'} = \omega_\mu\, dx^\mu$$

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Coordinate xfm $\{x^\mu\} \to \{x^{\mu'}\}$

$$\omega = \omega_{\mu'}\, dx^{\mu'} = \omega_\mu\, dx^\mu$$

components of $dx^{\mu'}$ in $\{dx^\nu\}$ basis

$$dx^{\mu'}(\partial_\nu) = \partial_\nu(x^{\mu'}) = \frac{\partial x^{\mu'}}{\partial x^\nu}$$

$$\Rightarrow dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\nu}\, dx^\nu$$

Coordinate xfm $\{x^\mu\} \to \{x^{\mu'}\}$

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Coordinate xfm $\{x^\mu\} \to \{x^{\mu'}\}$

$$\omega = \omega_{\mu'}\, dx^{\mu'} = \left(\omega_{\mu'} \frac{\partial x^{\mu'}}{\partial x^\nu}\right) dx^\nu$$

$$\Rightarrow \omega_\mu = \frac{\partial x^{\mu'}}{\partial x^\mu}\, \omega_{\mu'}$$

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(invert linear system)

$$\Rightarrow \omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \omega_\mu$$

Coordinate xfm $\{x^\mu\} \to \{x^{\mu'}\}$

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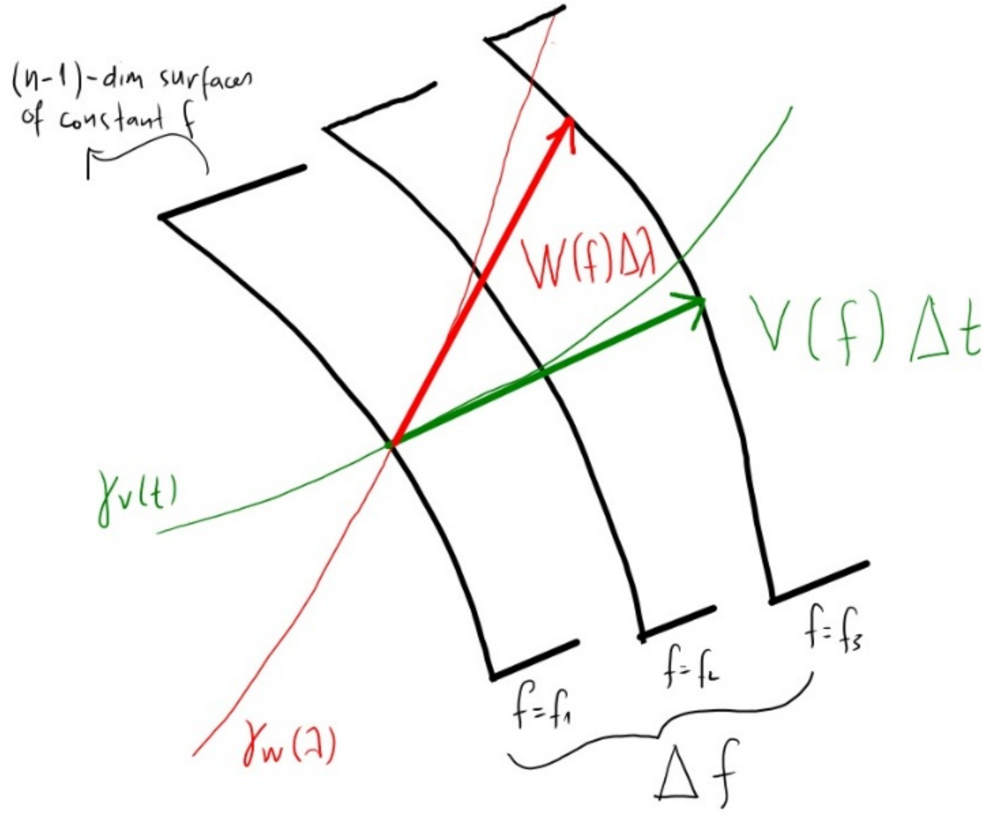
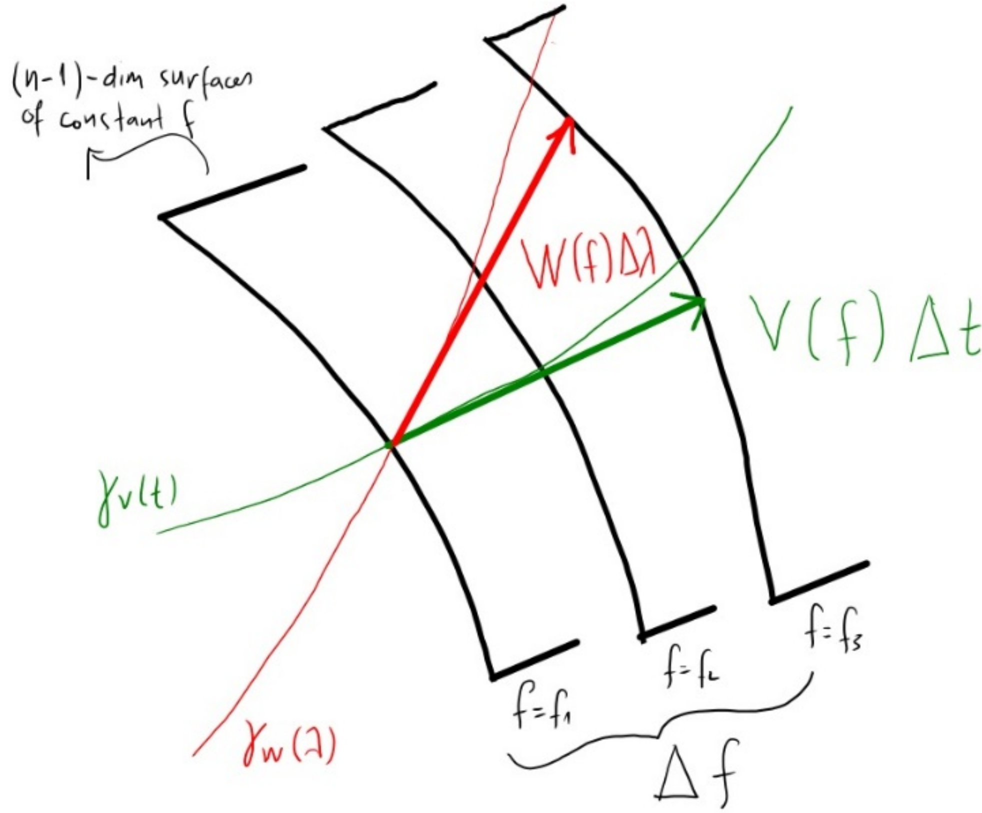
The geometry of df



The geometry of df

$$\Delta f = V(f)\,\Delta t$$

$$\Delta f = W(f)\,\Delta\lambda$$

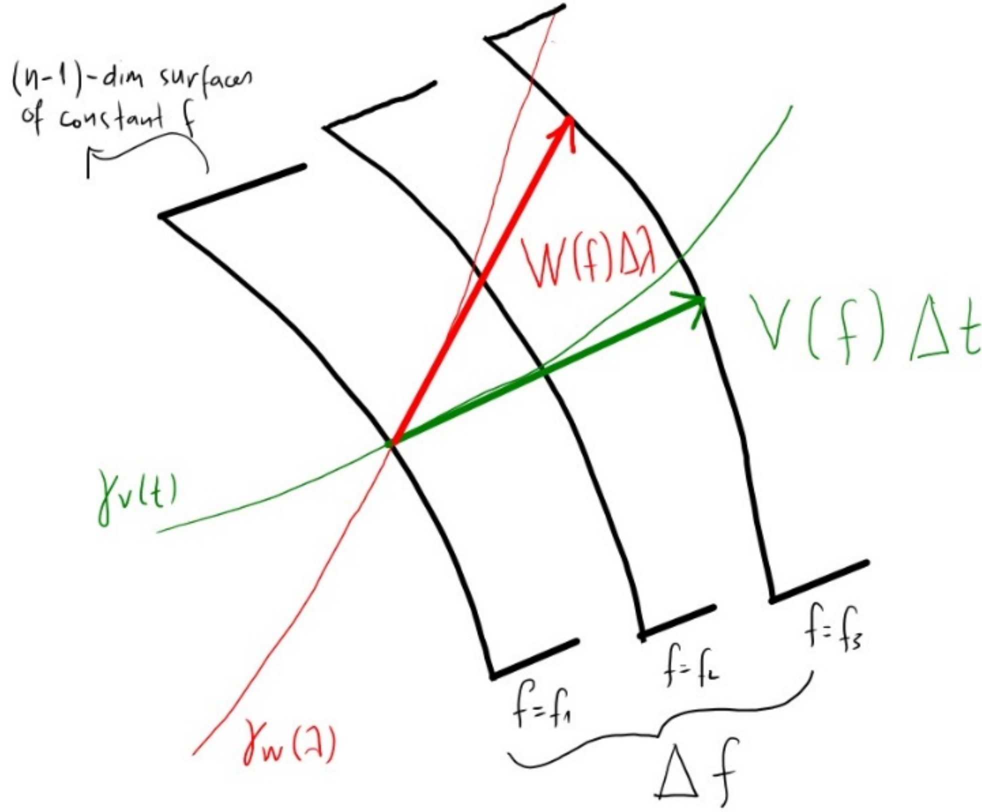


The geometry of df

$$\Delta f = V(f)\,\Delta t$$

$$\Delta f = W(f)\,\Delta\lambda$$

$$V(f) = \frac{df}{dt} \qquad W(f) = \frac{df}{d\lambda}$$



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For all vectors

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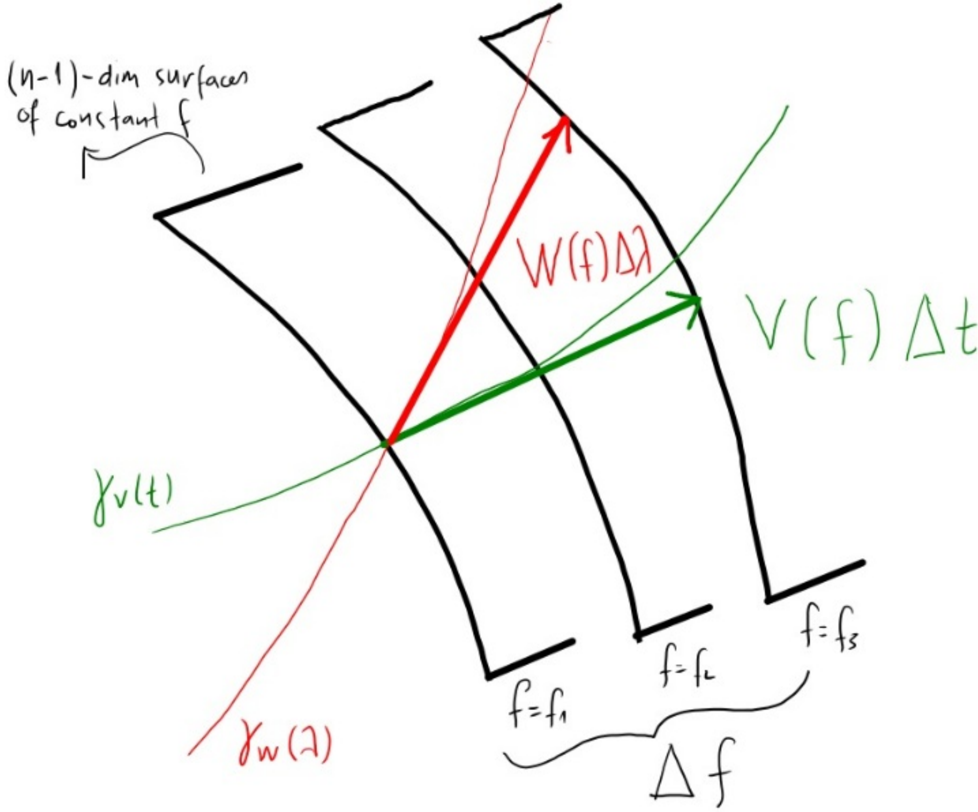

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For all vectors

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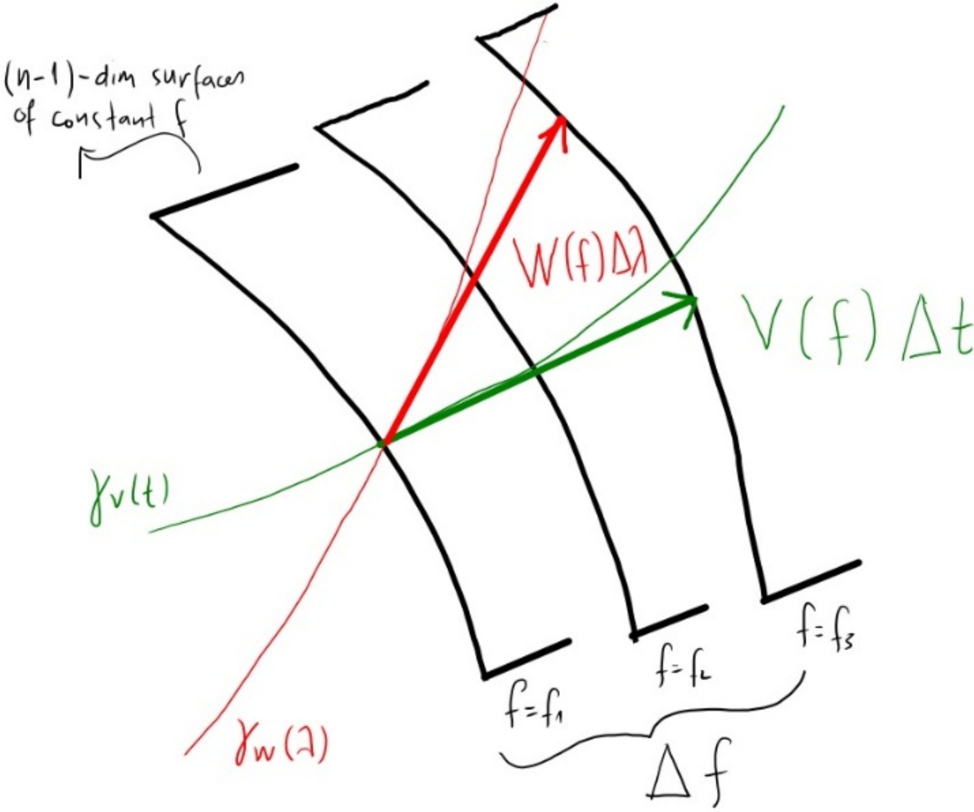
$$= \frac{\partial f}{\partial x^r} V^r \,\Delta t$$



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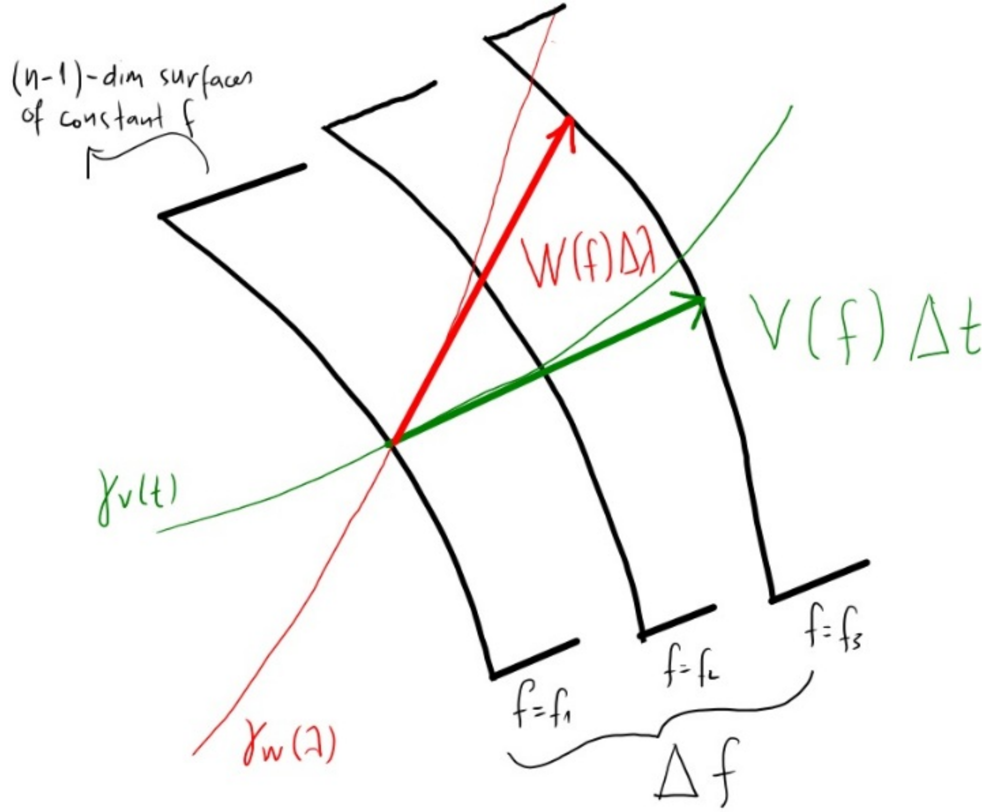

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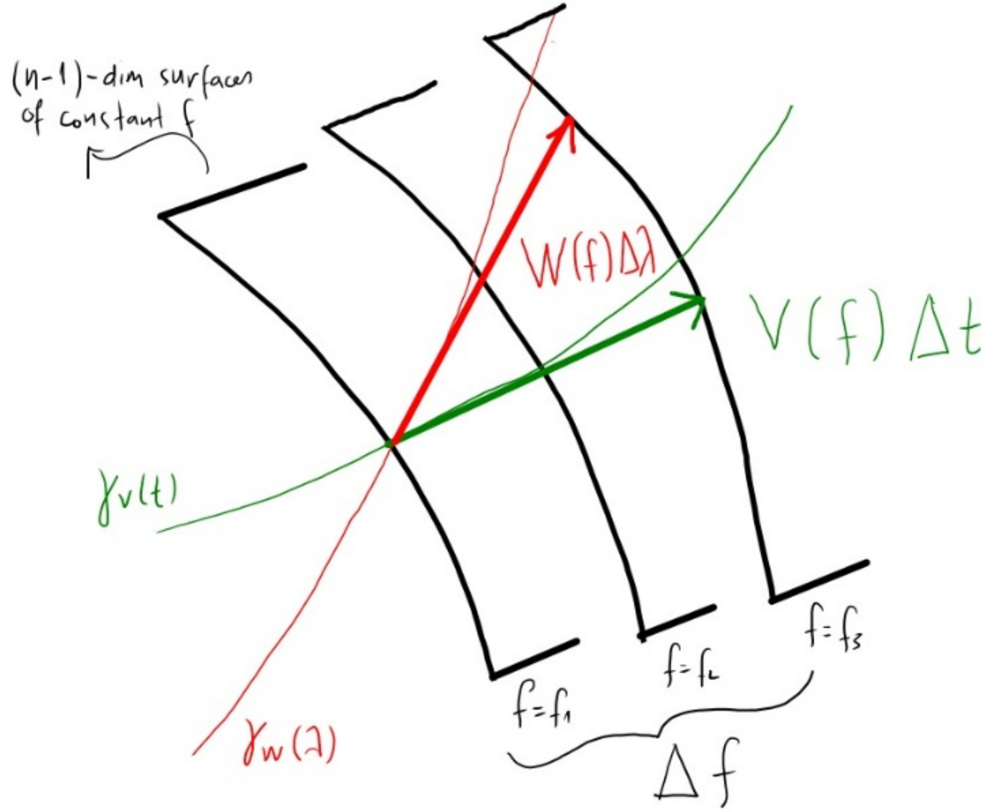

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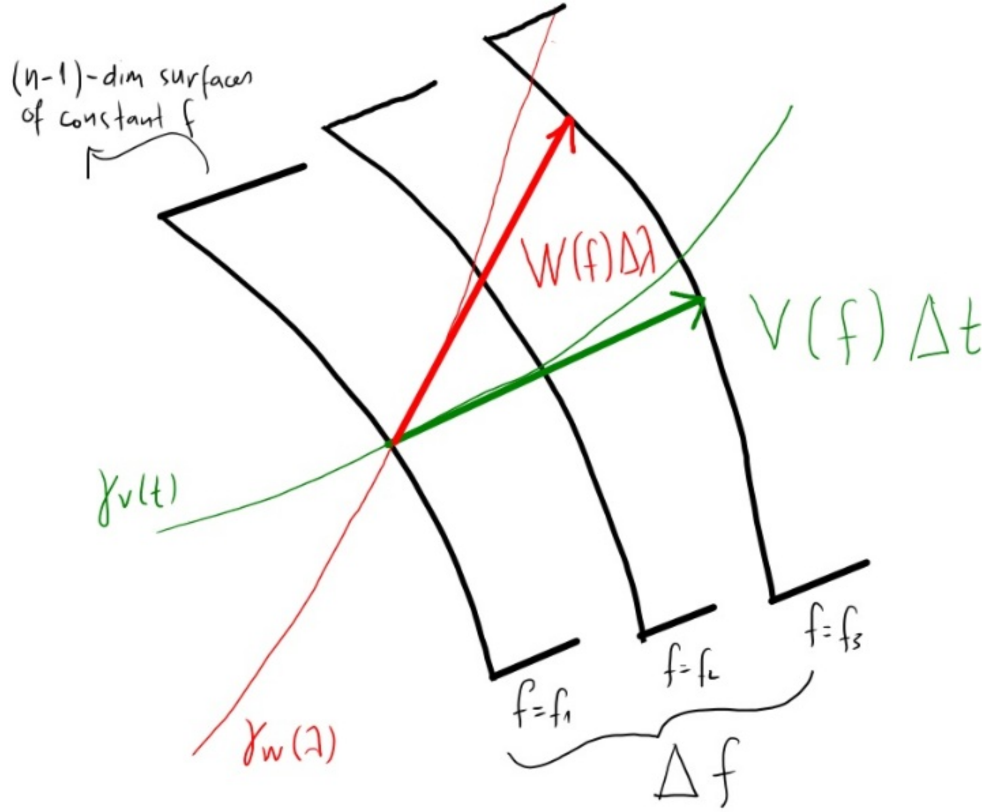


For *all* vectors

$$\Delta f = V(f)\,\Delta t = df(V)\,\Delta t$$

$$= \frac{\partial f}{\partial x^r} V^r \Delta t = \frac{\partial f}{\partial x^r}\frac{dx^r}{dt}\Delta t = \frac{\partial f}{\partial x^r}\Delta x^r$$

No reference to the curve, just choose $\{\Delta x^r\}$

The geometry of df

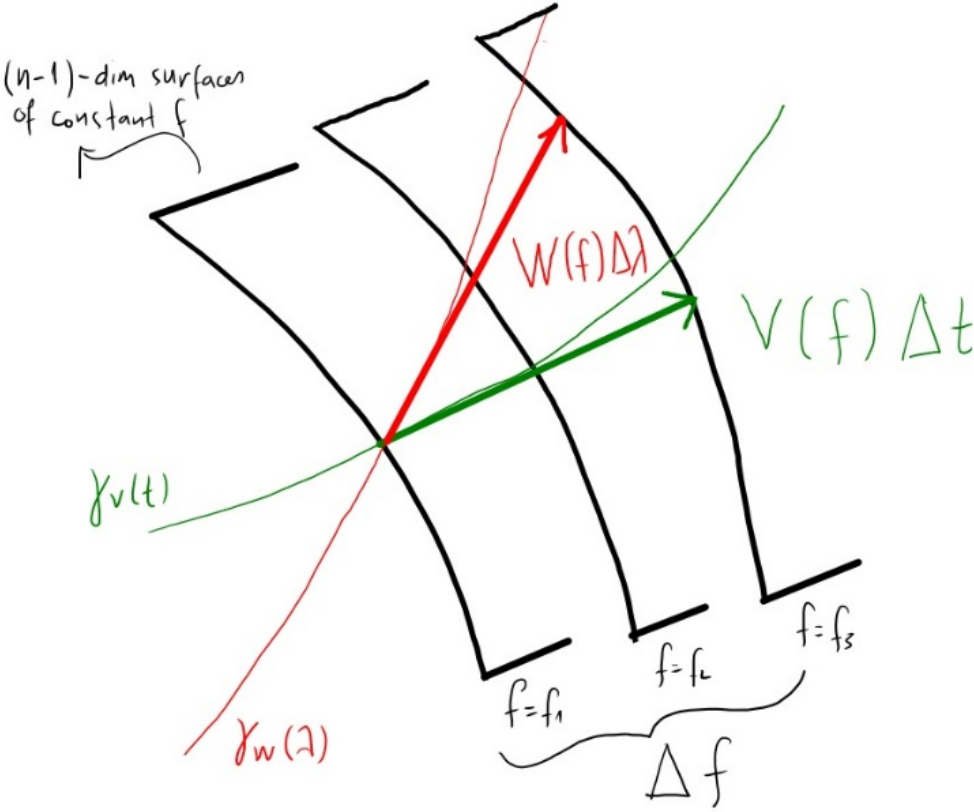
$$\Delta f = V(f)\,\Delta t$$

$$\Delta f = W(f)\,\Delta\lambda$$

For all vectors

$$\Delta f = V(f)\,\Delta t = \underbrace{df(V)}\,\Delta t$$

$\left(\begin{array}{l} \text{\# of pierced surfaces of constant } f \\ \text{by } V \text{ per unit parameter} \end{array} \right) \times \Delta t$



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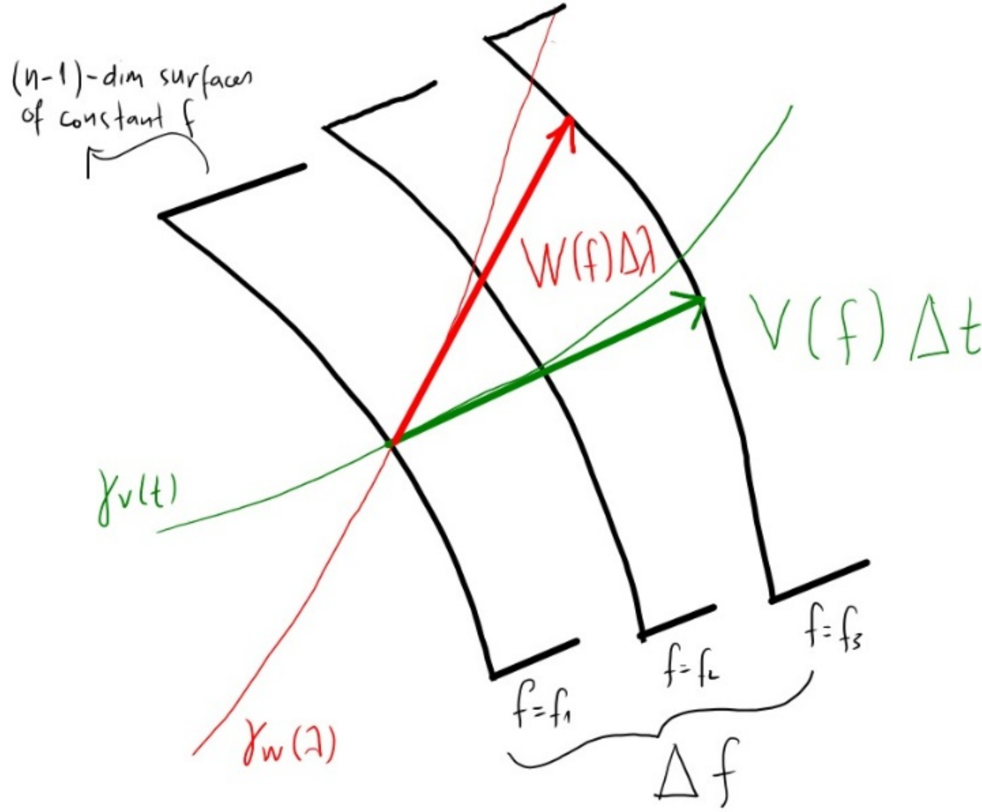


For all vectors

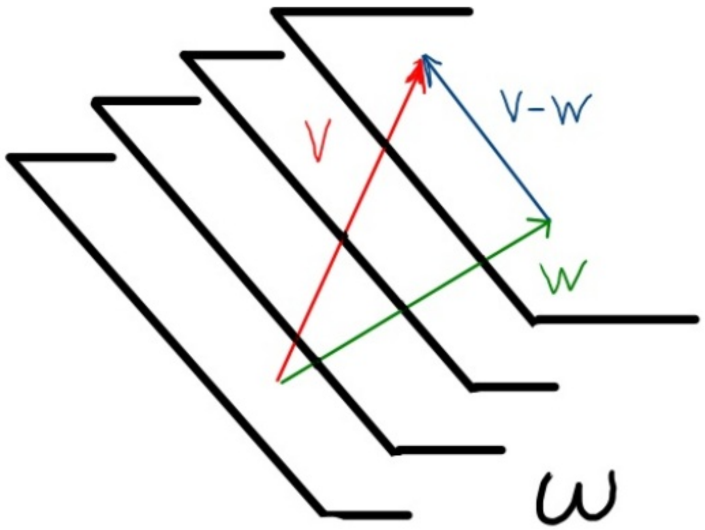
$$\Delta f = V(f)\,\Delta t = \underbrace{df(V)}\,\Delta t$$

independent of V, if V dt end on same surface

$$\left(\begin{array}{l}\text{\# of pierced surfaces of constant } f \\ \text{by } V \text{ per unit parameter}\end{array}\right) \times \Delta t$$

- Vectors:

Drawn as arrows in $\mathbb{R}^n$



- One-forms:

Drawn as $(n-1)$ dim parallel hyperplanes in $\mathbb{R}^n$

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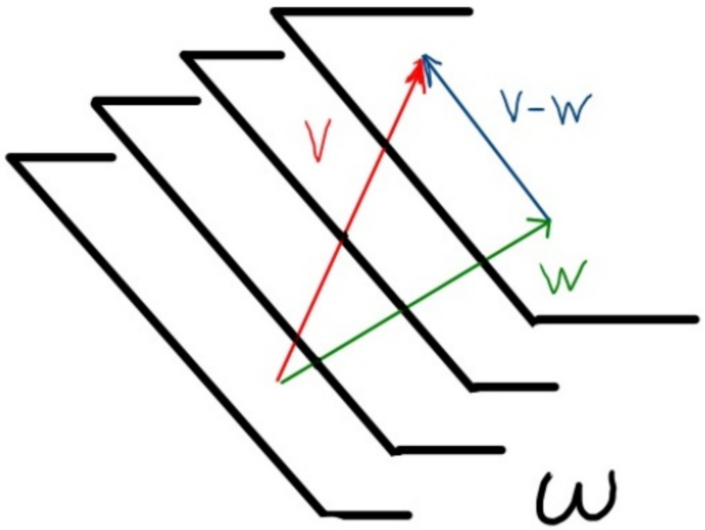

$$\omega(V) = \left(\begin{array}{c} \#\text{ pierced hyperplanes} \\ \text{by } V \end{array} \right)$$

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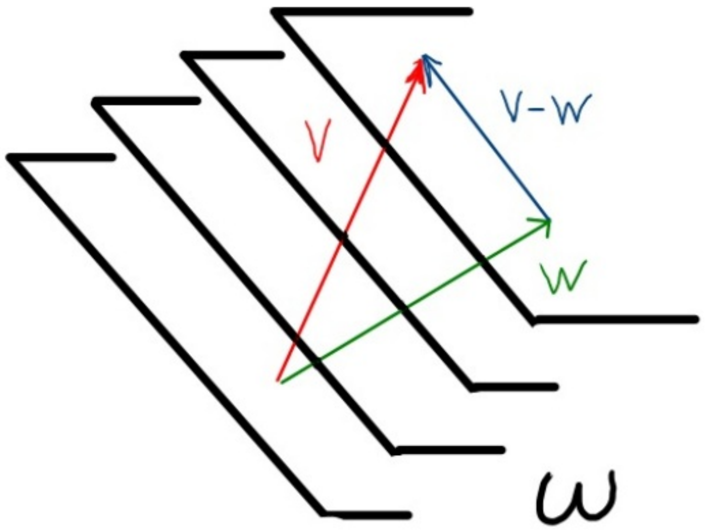
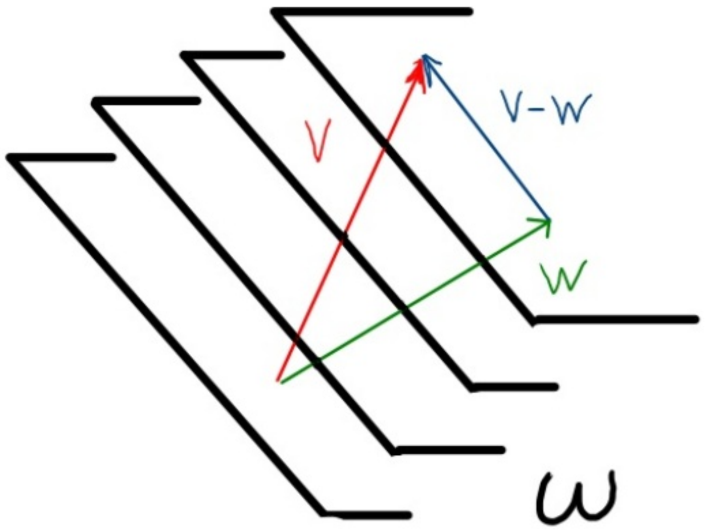


$$\omega(v) = \left(\begin{array}{c} \#\text{ pierced hyperplanes} \\ \text{by } v \end{array} \right)$$

$\rightarrow$ longer arrows $\Rightarrow$ larger $\omega(v)$

- Vectors:

Drawn as arrows in $\mathbb{R}^n$



- One-forms:

Drawn as $(n-1)$ dim parallel hyperplanes in $\mathbb{R}^n$

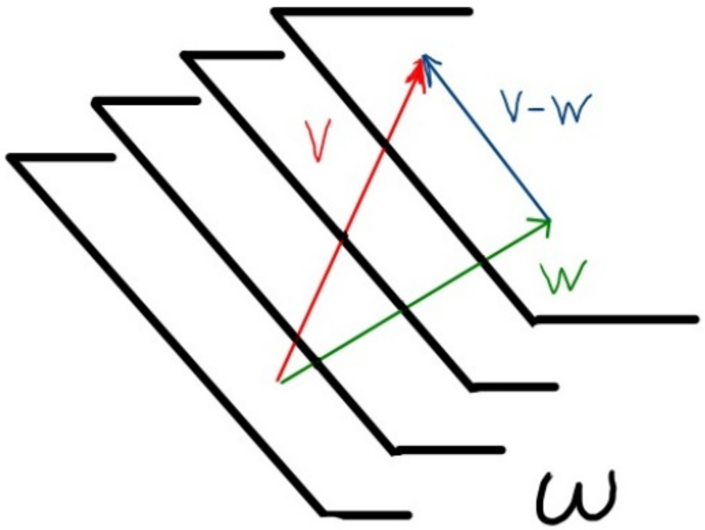
$$\omega(V) = \left(\begin{array}{c} \#\text{ pierced hyperplanes} \\ \text{by } V \end{array} \right)$$

$\rightarrow$ longer arrows $\Rightarrow$ larger $\omega(V)$

$\rightarrow$ denser planes $\Rightarrow$ larger $\omega(V)$

- Vectors:

Drawn as arrows in $\mathbb{R}^n$



- One-forms:

Drawn as $(n-1)$ dim parallel hyperplanes in $\mathbb{R}^n$

$$\omega(V) = \left(\begin{array}{c}\#\text{ pierced hyperplanes} \\ \text{by } V\end{array}\right)$$

If $\omega(V) = \omega(W) \Leftrightarrow \omega(V-W) = 0$

- Vectors:

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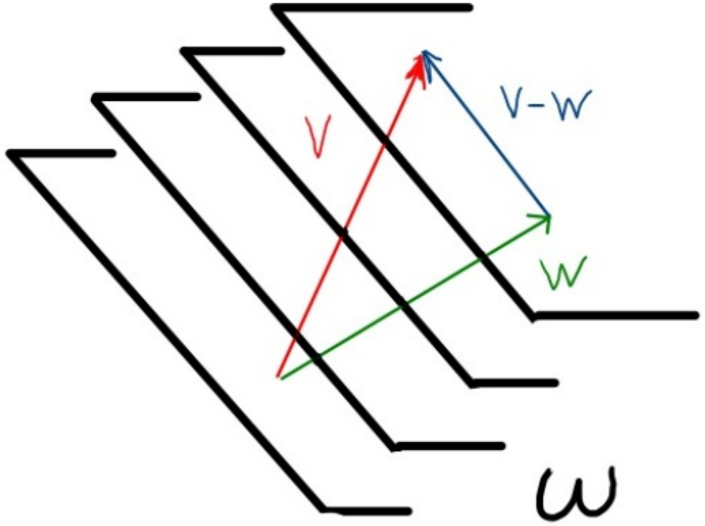

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Drawn as $(n-1)$ dim parallel hyperplanes in $\mathbb{R}^n$

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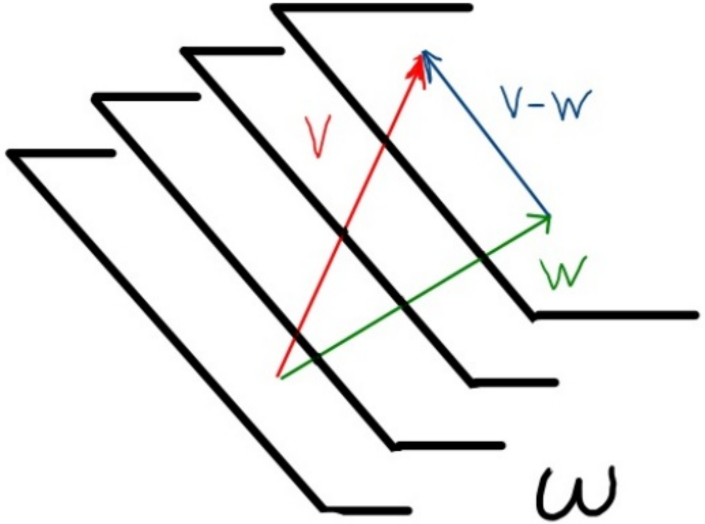
$V - W \parallel$ hyperplanes

If $\omega(V) = \omega(W) \Leftrightarrow \omega(V - W) = 0$

↳ does not pierce

- Vectors:

Drawn as arrows in $\mathbb{R}^n$



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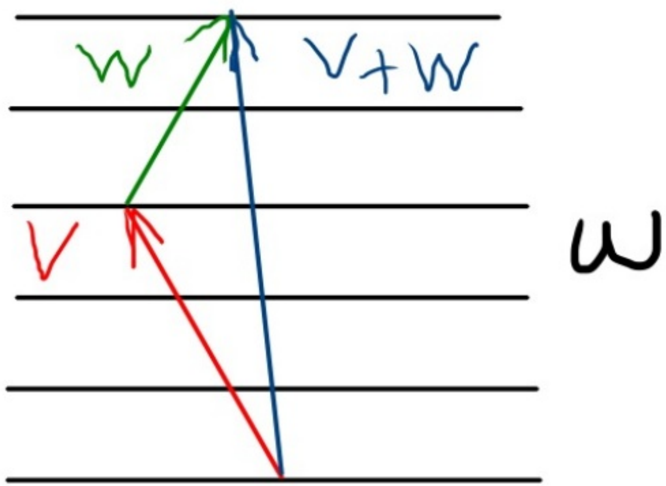

Linearity: $\omega(\alpha V) = \alpha\, \omega(V)$

$$\omega(V+W) = \omega(V) + \omega(W)$$

One-form fields

- An assignment of a 1-form $\forall \underline{p} \in M$ in a smooth way

  $\omega(V)$ a smooth function in $\mathcal{F}(M)$

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  $$df(V) = V(f) = V^\mu \partial_\mu f$$

- $\omega = \omega_\mu dx^\mu$ smooth $\Leftrightarrow$ ω_μ smooth functions

Tensors

$$\underbrace{T_p^*M \times \dots \times T_p^*M}_{k\text{-times}} \times \underbrace{T_pM \times \dots \times T_pM}_{l\text{-times}}$$

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$\rightarrow$ a vector space

$\rightarrow \dim(T_p^*M \times \dots \times T_p^*M \times T_pM \times \dots \times T_pM) = n^k \cdot n^l = n^{k+l}$

Tensors

$$T : \underbrace{T_p^*\mathcal{M} \times \dots \times T_p^*\mathcal{M}}_{k\text{-times}} \times \underbrace{T_p\mathcal{M} \times \dots \times T_p\mathcal{M}}_{l\text{-times}} \to \mathbb{R}$$

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of rank (k, l)

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of $\begin{pmatrix} \text{rank} \\ \text{type} \\ \text{order} \\ \text{valence} \\ \text{degree} \end{pmatrix}$ (k, l)

Tensors

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(k, l) tensors form the vector space $T_p^{(k,l)} M$

vectors are of type $(1, 0)$

1-forms " " " $(0, 1)$

Example: a $(1,1)$ tensor

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Linearity:

$$T(\alpha\omega + \beta\sigma; V) = \alpha T(\omega; V) + \beta T(\sigma; V)$$
$$T(\omega; \alpha V + \beta W) = \alpha T(\omega; V) + \beta T(\omega; W)$$

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Define $T^\mu{}_\nu = T(dx^\mu; \partial_\nu)$, the components of T in $\{dx^\mu\}, \{\partial_\nu\}$

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$$T(\omega; V) = T(\omega_\mu dx^\mu; V^\nu \partial_\nu) = \omega_\mu V^\nu T(dx^\mu; \partial_\nu)$$

$$T(\omega; V) = T^{\mu}{}_{\nu}\, \omega_{\mu} V^{\nu} \quad , \quad T^{\mu}{}_{\nu} \equiv T(dx^{\mu}; \partial_{\nu})$$

$$T(\omega; V) = T^{\mu}{}_{\nu}\, \omega_\mu V^\nu \quad , \quad T^{\mu}{}_{\nu} \equiv T(dx^\mu; \partial_\nu)$$

For a (k,l) tensor:

$$T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \equiv T(dx^{\mu_1}, \ldots, dx^{\mu_k}; \partial_{\nu_1}, \ldots, \partial_{\nu_l})$$

$$T(\omega; V) = T^{\mu}{}_{\nu}\, \omega_{\mu} V^{\nu} \quad , \quad T^{\mu}{}_{\nu} \equiv T(dx^{\mu}; \partial_{\nu})$$

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$$T(\omega, \ldots; V, \ldots) = T^{\mu_1 \ldots}{}_{\nu_1 \ldots}\, \omega_{\mu_1} \ldots V^{\nu_1} \ldots$$

$$T(\omega; V) = T^{\mu}{}_{\nu}\, \omega_{\mu} V^{\nu} \quad , \quad T^{\mu}{}_{\nu} \equiv T(dx^{\mu}; \partial_{\nu})$$

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Tensor Product

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T is of type (k_2, l_2)

$S \otimes T$ is of type $(k_1 + k_2, l_1 + l_2)$

Tensor Product

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$S \otimes T$ is of type $(k_1 + k_2, l_1 + l_2)$

$$S \otimes T \left(\omega^{(1)}, \dots, \omega^{(k_1)}, \sigma^{(1)}, \dots, \sigma^{(k_2)}; V_{(1)}, \dots, V_{(l_1)}, W_{(1)}, \dots, W_{(l_2)} \right)$$

$$= S\left(\omega^{(1)}, \dots, \omega^{(k_1)}; V_{(1)}, \dots, V_{(l_1)} \right) \cdot T\left(\sigma^{(1)}, \dots, \sigma^{(k_2)}; W_{(1)}, \dots, W_{(l_2)} \right)$$

Tensor Product

- $S \otimes T \neq T \otimes S$

e.g.

$$S \otimes T(\omega^{(1)}, \omega^{(2)}, \omega^{(3)}; V_{(1)}, V_{(2)}) = S(\omega^{(1)}, \omega^{(2)}; V_{(1)})\, T(\omega^{(3)}; V_{(3)})$$

$$T \otimes S(\omega^{(1)}, \omega^{(2)}, \omega^{(3)}; V_{(1)}, V_{(2)}) = T(\omega^{(1)}; V_{(1)}) \cdot S(\omega^{(2)}, \omega^{(3)}; V_{(2)})$$

$$S \otimes T(\omega^{(1)}, \dots, \omega^{(k_1)}, \sigma^{(1)}, \dots, \sigma^{(k_2)}; V_{(1)}, \dots, V_{(\ell_1)}, W_{(1)}, \dots, W_{(\ell_2)})$$

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- $S \otimes T \neq T \otimes S$
- $S \otimes T$ is a linear function of its arguments (prove!)

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- $S \otimes T$ is a linear function of its arguments
- $(S \otimes T)^{\mu_1 \dots \mu_{k_1} \nu_1 \dots \nu_{k_2}}{}_{\lambda_1 \dots \lambda_{l_1} \rho_1 \dots \rho_{l_2}} =$ (also prove!)

$$= S^{\mu_1 \dots \mu_{k_1}}{}_{\lambda_1 \dots \lambda_{l_1}} \cdot T^{\nu_1 \dots \nu_{k_2}}{}_{\rho_1 \dots \rho_{l_2}}$$

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Example:

$\{dx^\mu \otimes dx^\nu\}$ a coordinate basis of $T_p^{(0,2)}M$

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ω, σ 1-forms

V vector

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Coordinate change $\Rightarrow$ $\{dx^\mu\}, \{\partial_\mu\} \to \{dx^{\mu'}\}, \{\partial_{\mu'}\}$

<u>Example:</u> $T \in T_p^{(2,1)} M$

$$T(\omega, \sigma; V) \in \mathbb{R}$$

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we need better notation... (abstract index notation)

Contractions

Example: $T = T^{\mu\nu}{}_{\rho}\ \partial_\mu \otimes \partial_\nu \otimes dx^\rho$

a (2,1) tensor

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empty slot for
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Define the (1,0) tensor:

$T(\ldots, dx^\lambda; \partial_\lambda)$

Independent of choice of basis!
Same as $T(\ldots, dx^{\lambda'}; \partial_{\lambda'})$

(Prove!)

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$$T(\ldots, dx^\lambda; \partial_\lambda) = T^{\mu\nu}{}_{\rho}\, \partial_\nu(dx^\lambda) \cdot dx^\rho(\partial_\lambda)\, \partial_\mu$$

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ν, ρ indices have been contracted

Contractions

Contracting two indices xfms a

$(k, l) \to (k-1, l-1)$ tensor

with components

$$T^{\mu_1 \dots \lambda \dots \mu_k}{}_{\nu_1 \dots \lambda \dots \nu_l}$$

Contractions

Contracting two indices in $\otimes$:

$T = T^{\mu\nu}{}_{\rho}\, \partial_\mu \otimes \partial_\nu \otimes dx^\rho$ (2, 1) tensor

$R = R^{\lambda}{}_{\sigma}\, \partial_\lambda \otimes dx^\sigma$ (1, 1) tensor

Contractions

Contracting two indices in $\otimes$:

$$T = T^{\mu\nu}{}_{\rho}\, \partial_\mu \otimes \partial_\nu \otimes dx^\rho \qquad (2,1) \text{ tensor}$$

$$R = R^{\lambda}{}_{\sigma}\, \partial_\lambda \otimes dx^\sigma \qquad (1,1) \text{ tensor}$$

$$T \otimes R = T^{\mu\nu}{}_{\rho}\, R^{\lambda}{}_{\sigma}\, \partial_\mu \otimes \partial_\nu \otimes \partial_\lambda \otimes dx^\rho \otimes dx^\sigma$$

$(3,2)$ tensor

Contractions

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contract ρ, λ

$$S = T^{\mu\nu}{}_{\lambda} R^{\lambda}{}_{\sigma}\, \partial_\mu \otimes \partial_\nu \otimes dx^\sigma \qquad (2,1) \text{ tensor}$$

Symmetries of Tensors

$g_{\mu\nu} = g_{\nu\mu}$ totally symmetric

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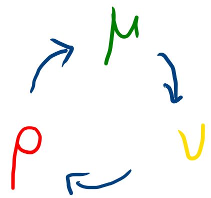
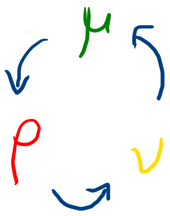
$$\begin{aligned} S_{\mu\nu\rho} = \; & S_{\rho\mu\nu} \\ & S_{\nu\rho\mu} \\ & S_{\mu\rho\nu} \\ & S_{\nu\mu\rho} \\ & S_{\rho\nu\mu} \end{aligned}$$





totally symmetric

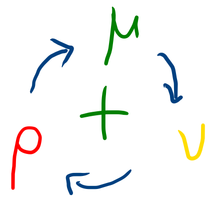
$3! = 6$ permutations of indices

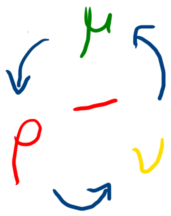
Symmetries of Tensors

$g_{\mu\nu} = -g_{\nu\mu}$ totally antisymmetric

$A_{\mu\nu\rho} = -A_{\nu\mu\rho}$ antisymmetric in its first two indices

$$S_{\mu\nu\rho} = +S_{\rho\mu\nu} + S_{\nu\rho\mu} - S_{\mu\rho\nu} - S_{\nu\mu\rho} - S_{\rho\nu\mu}$$





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$$S_{(\mu_1\mu_2\ldots\mu_k)} = \frac{1}{k!}\sum_{\sigma} S_{\sigma(\mu_1)\sigma(\mu_2)\ldots\sigma(\mu_k)}$$

$$\sigma = \begin{pmatrix} \mu_1 & \mu_2 & \ldots & \mu_k \\ \sigma(\mu_1) & \sigma(\mu_2) & \ldots & \sigma(\mu_k) \end{pmatrix}$$ 1-1 map of k-integers

Antisymmetrization

$$\mathrm{sign}(\sigma) = (-1)^{(\#\text{permutations})}$$

$$g_{[\mu\nu]} = \frac{1}{2}\left(g_{\mu\nu} - g_{\nu\mu}\right)$$

$$A_{[\mu\nu]\rho} = \frac{1}{2!}\left(A_{\mu\nu\rho} - A_{\nu\mu\rho}\right)$$

$$A_{[\mu\nu\rho]} = \frac{1}{3!}\left(A_{\mu\nu\rho} + A_{\rho\mu\nu} + A_{\nu\rho\mu} - A_{\mu\rho\nu} - A_{\nu\mu\rho} - A_{\rho\nu\mu}\right)$$

$$\begin{array}{ccc} & \mu & \\ & + & \\ \rho & & \nu \end{array} \qquad \begin{array}{ccc} & \mu & \\ & - & \\ \rho & & \nu \end{array}$$

$$S_{[\mu_1\mu_2\cdots\mu_k]} = \frac{1}{k!}\sum_{\sigma}\mathrm{sign}(\sigma)\, S_{\sigma(\mu_1)\sigma(\mu_2)\ldots\sigma(\mu_k)}$$

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$R^{\mu}{}_{[\nu\rho\lambda]}$ antisymetrize ν, ρ, λ

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$R^{\mu}{}_{[\nu\rho\lambda]}$ antisymetrize $\nu, \rho\, \lambda$

$C_{(\mu\nu|\rho|\lambda)}$ symmetrize μ, ν, λ , exclude ρ

$$T^{\overbrace{(\mu\nu\rho)}^{\text{symm}}\lambda\overbrace{[\sigma\tau]}^{\text{antisymm}}}$$

$$[\alpha|\overbrace{\beta\gamma}^{\text{exclude}}|\delta](\underbrace{\varepsilon\zeta}_{\text{symm}})$$

antisymm