

Stress - Energy Tensor

Conservation Laws

Hartle ch 22

Ferrari ch 5, 7.4.1

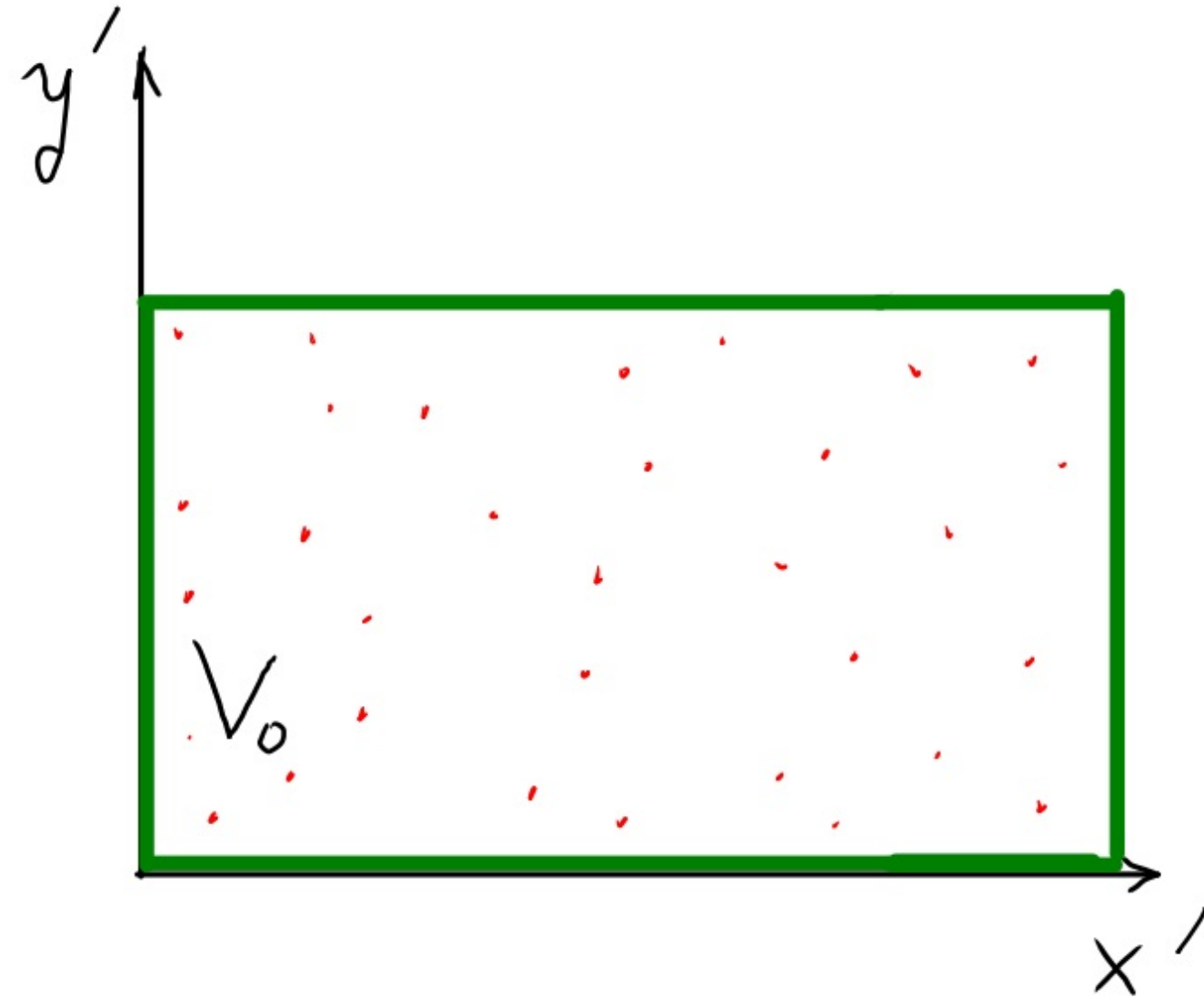
Misner et al p. 130 - 160

# Stress - Energy Tensor in Special Relativity

Number Density:

$N$  particles

$V_0$  volume of box in their  
common rest frame



# Stress - Energy Tensor in Special Relativity

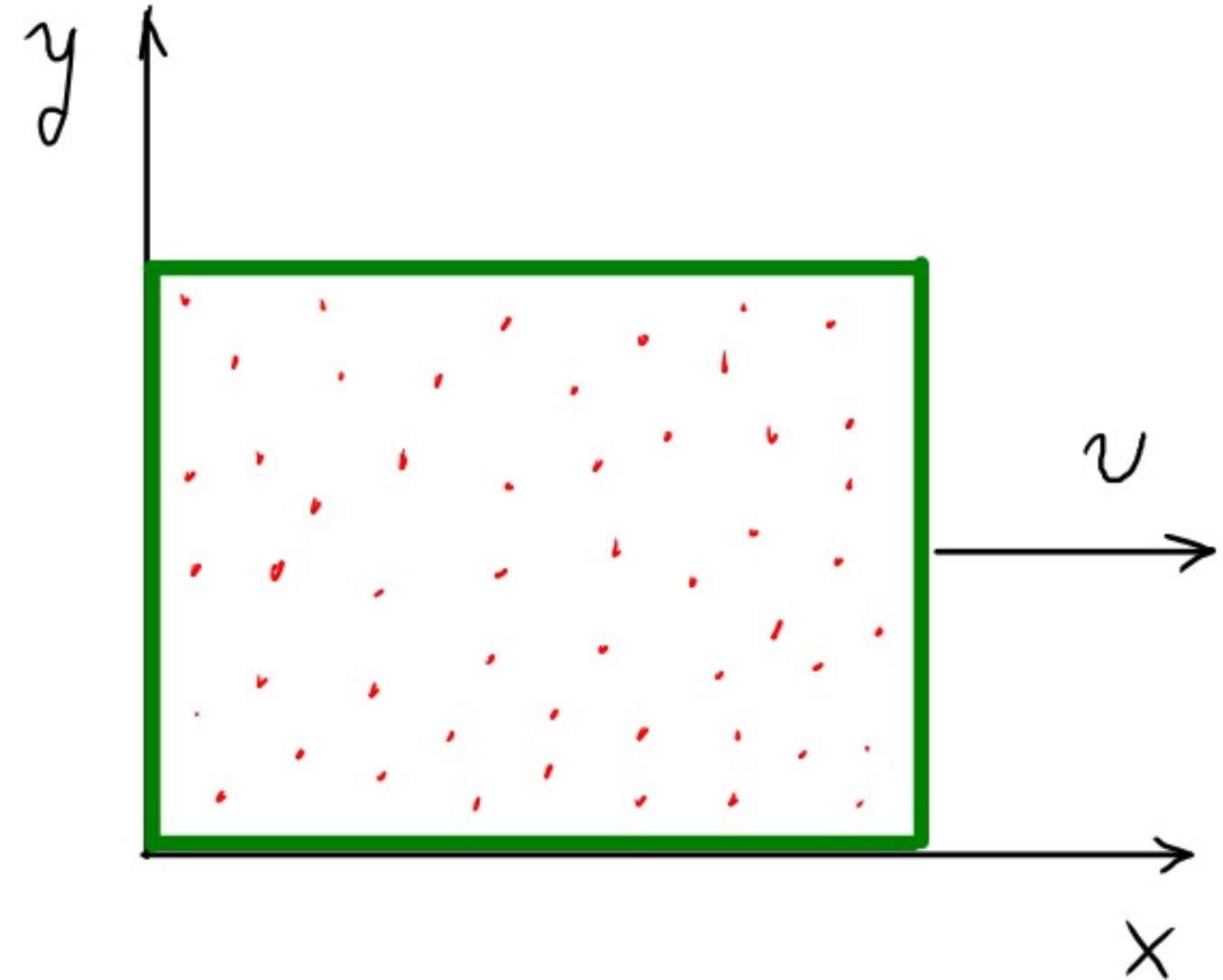
Number Density:

$N$  particles

$V_0$  volume of box in their common rest frame

$U$  common velocity of particles in  $(x, y)$  inertial frame

$V = \frac{V_0}{\gamma} = (1 - v^2)^{1/2} V_0$  volume of box in  $(x, y)$  frame



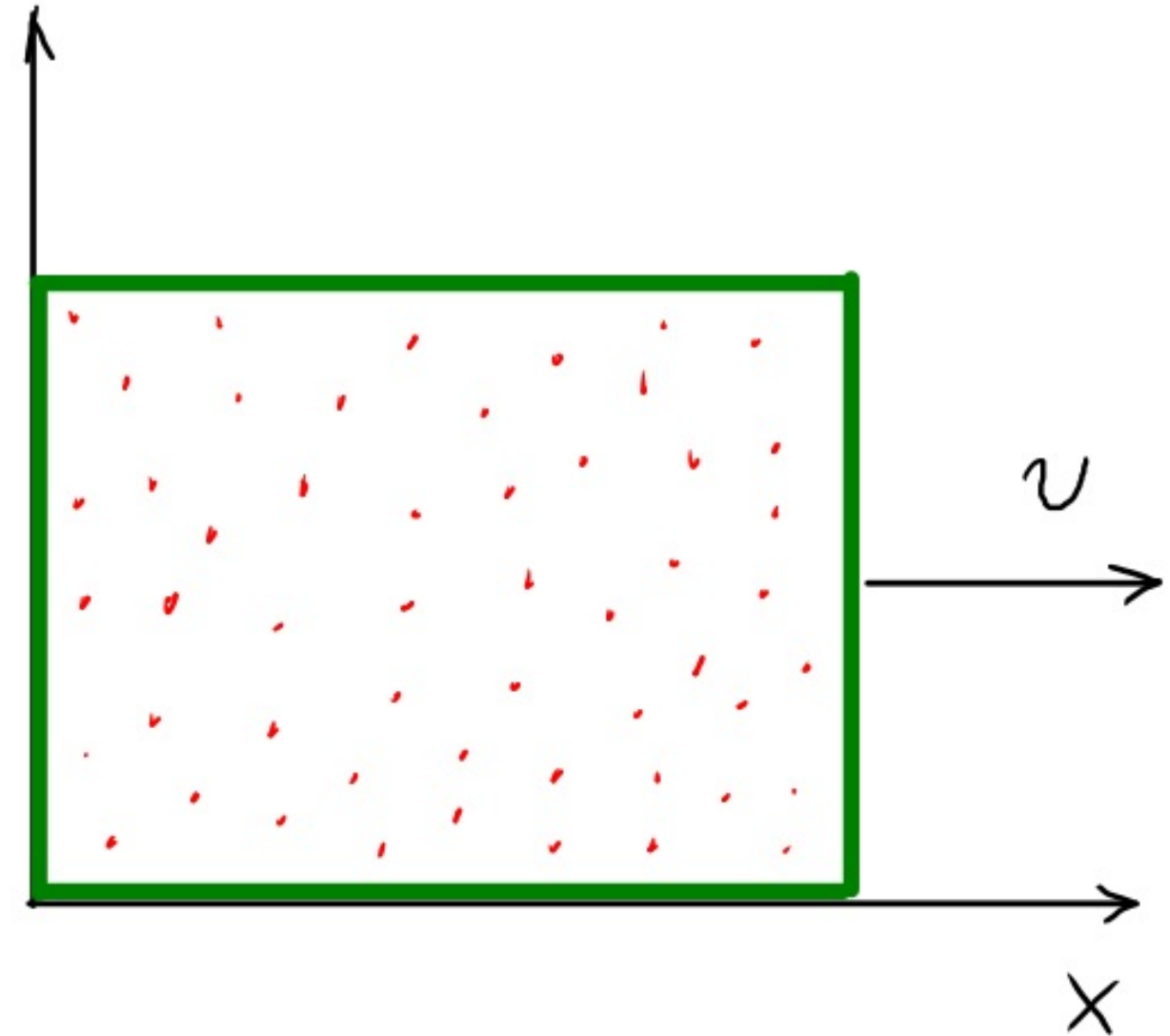
# Stress - Energy Tensor in Special Relativity

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$$n_0 = \frac{N}{V_0} \quad \text{number density in rest frame } y$$

$$n = \frac{N}{V} \quad \text{" " " " (x, y) " "}$$

same



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$v$  common velocity of particles in  $(x, y)$  inertial frame

$$V = \frac{V_0}{\gamma} = (1 - v^2)^{1/2} V_0 \quad \text{volume of box in } (x, y) \text{ frame}$$



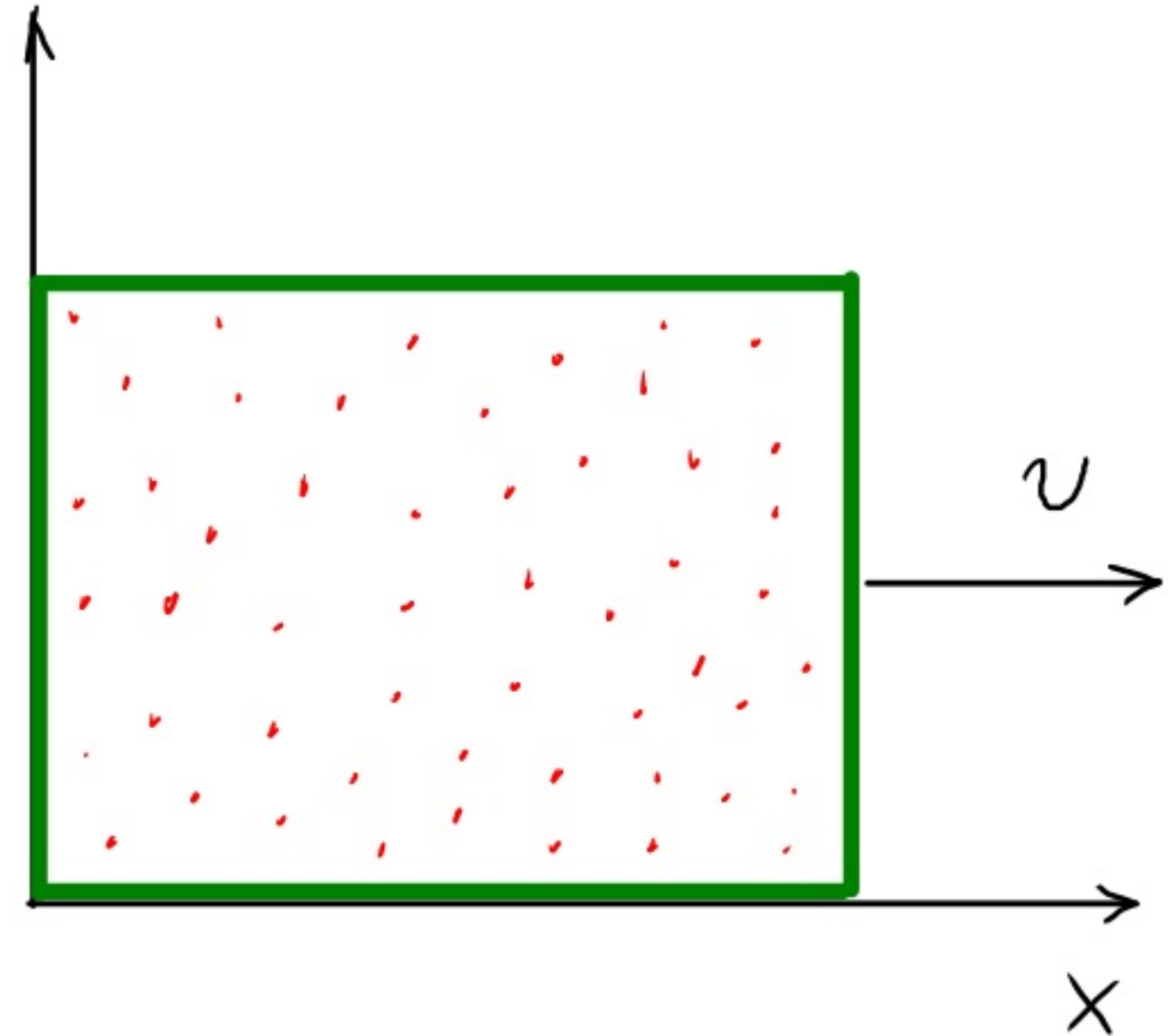
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$$n = \frac{N}{V} \quad \text{" " " " (x, y) " "}$$

$$n = \frac{N}{V_0/\gamma} = \gamma \frac{N}{V_0} = \gamma n_0$$

same



# Stress - Energy Tensor in Special Relativity

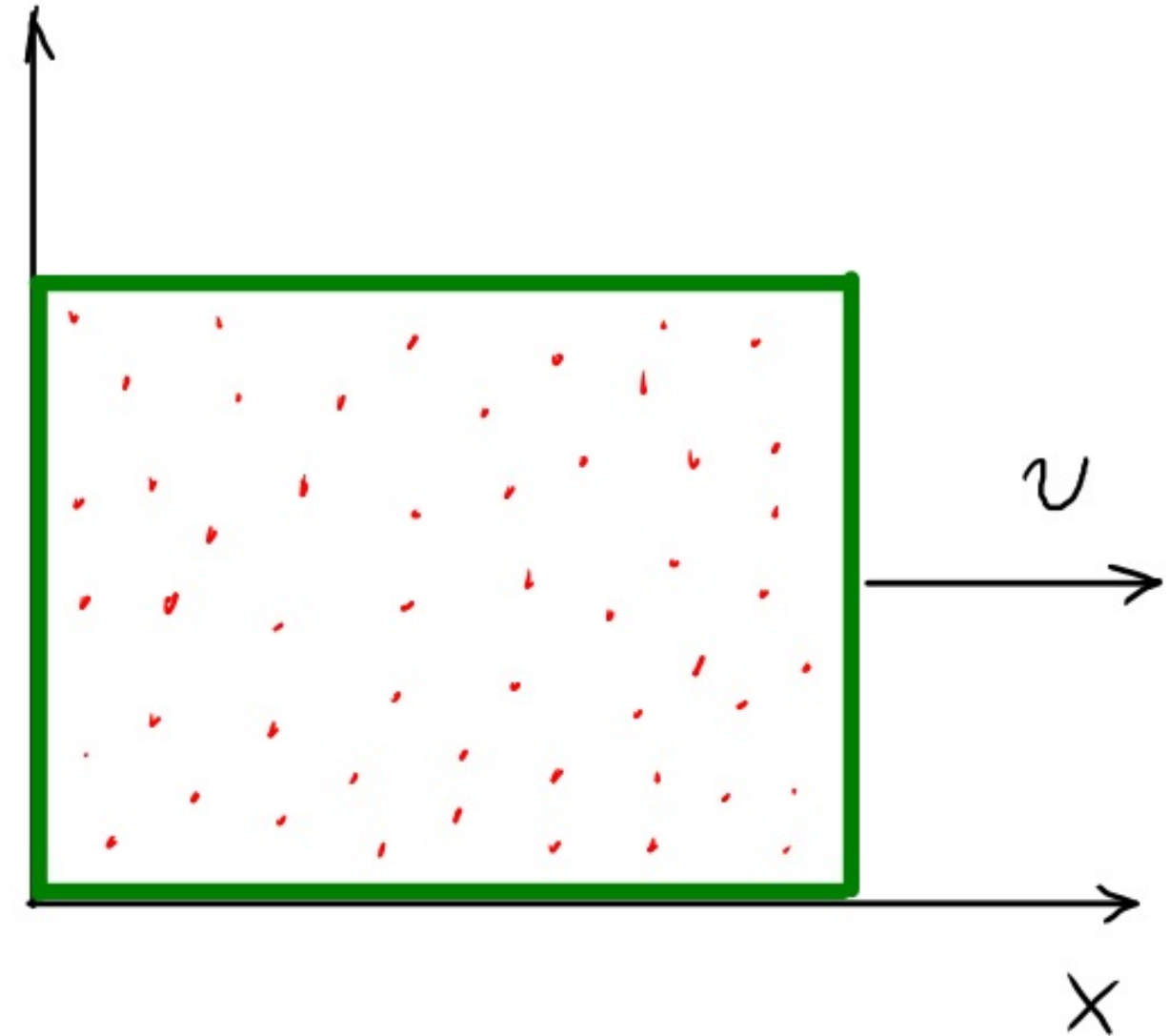
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$$n = \frac{N}{V_0/\gamma} = \gamma \frac{N}{V_0} = \gamma n_0$$

$$\text{But } u^0 = \gamma, \text{ so } n = n_0 u^0$$

$$u^\mu = (\gamma, \gamma \vec{v})$$



# Stress - Energy Tensor in Special Relativity

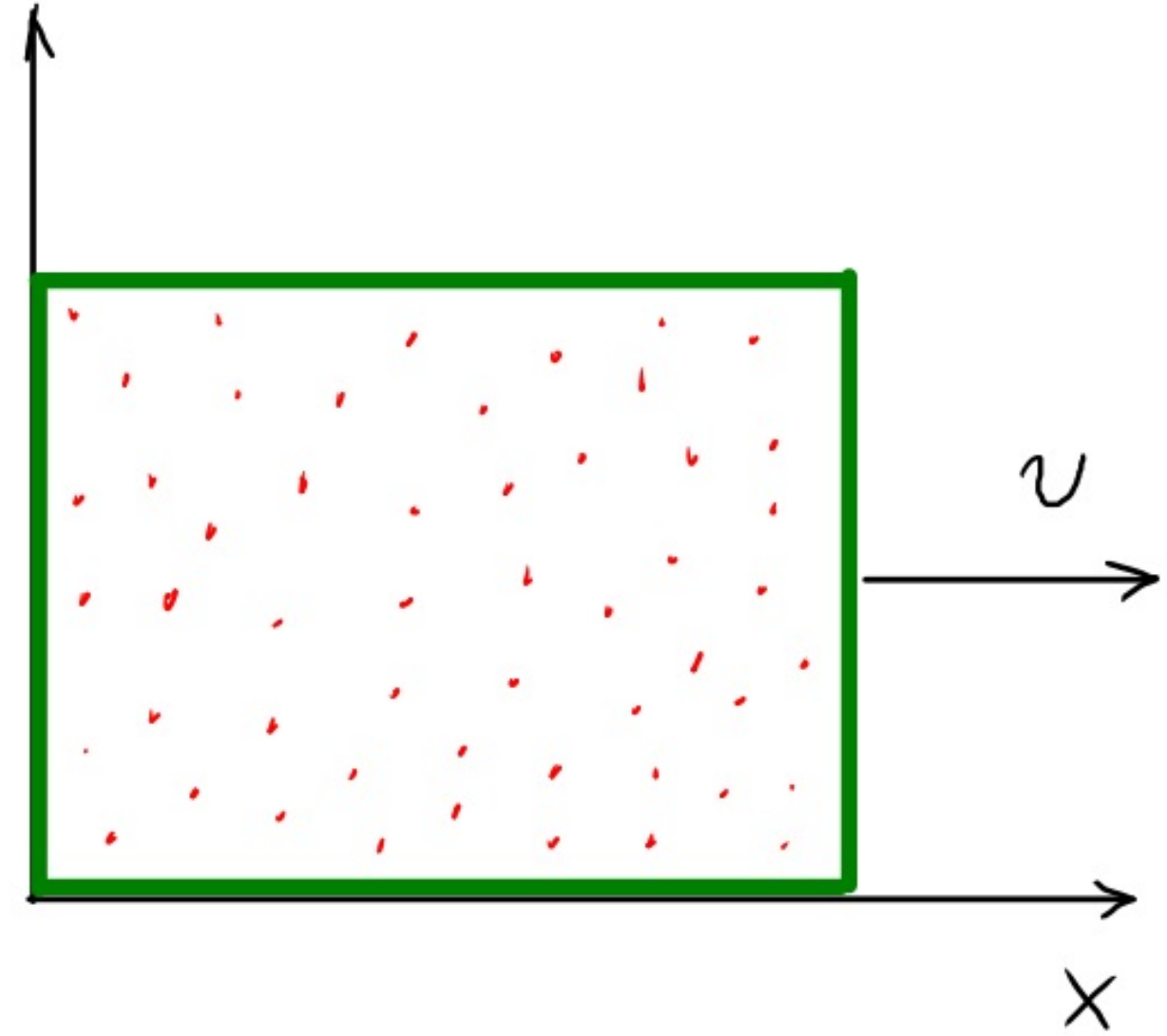
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$$n = \frac{N}{V_0/\gamma} = \gamma \frac{N}{V_0} = \gamma n_0$$

But  $\gamma^0 = \gamma$ , so  $n = n_0 \gamma^0$

We define  $j^\mu = n_0 u^\mu$ , the number current density

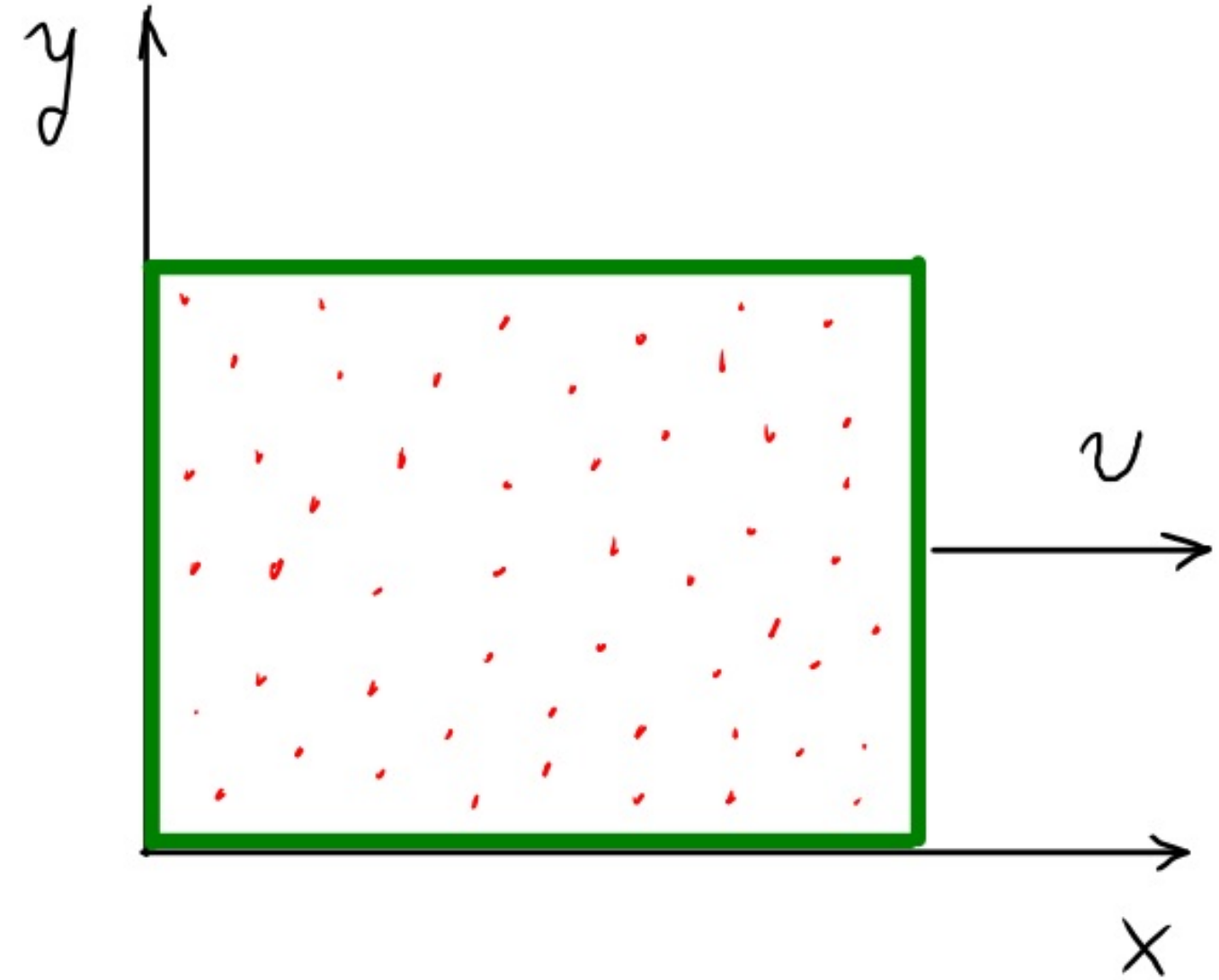


# Stress - Energy Tensor in Special Relativity

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$$j^\mu = n_0 (\gamma, \gamma \vec{v})$$

$$j^0 = n_0 \gamma \quad j^i = n_0 \gamma v^i$$



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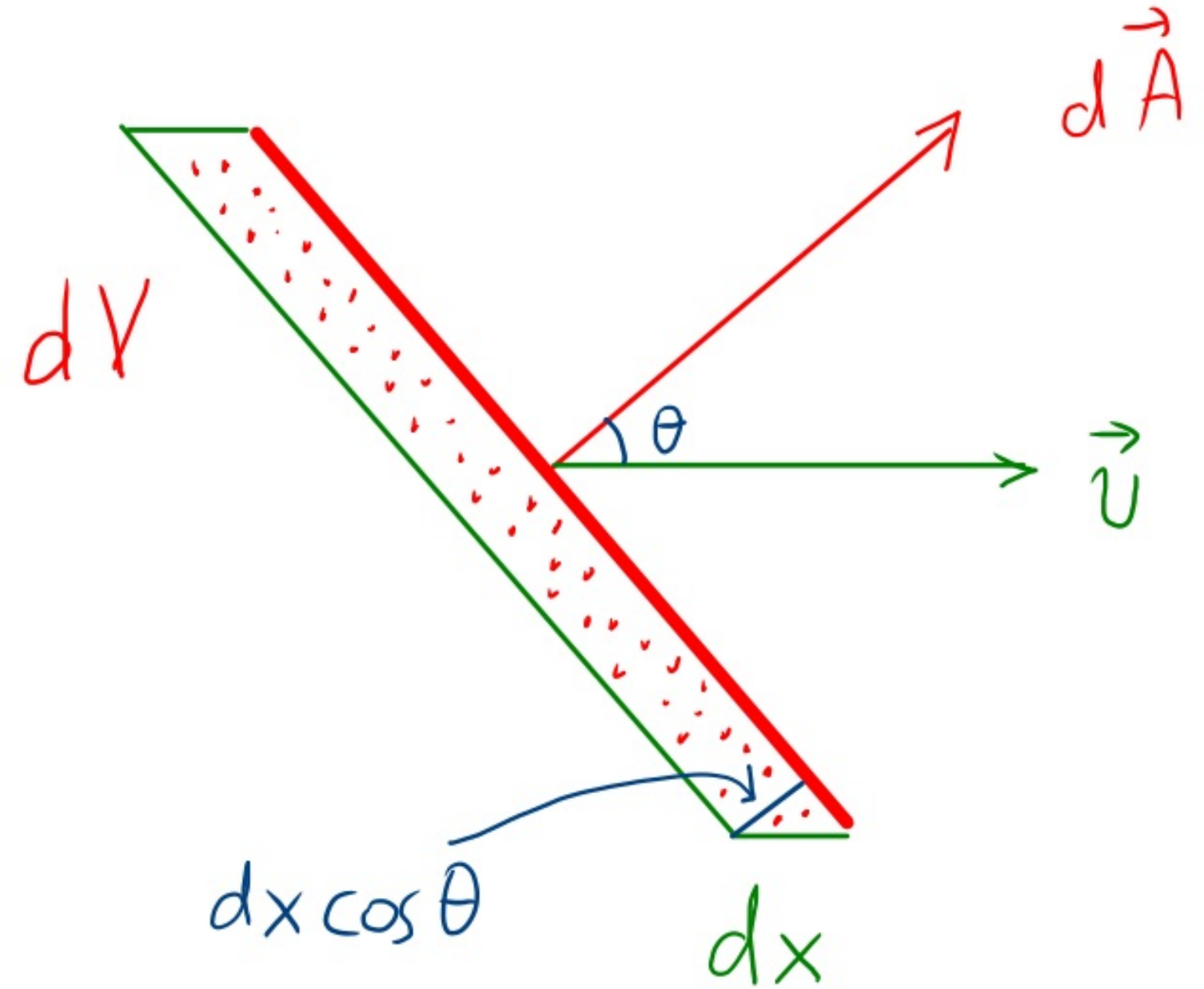
# Stress - Energy Tensor in Special Relativity

$$j^\mu = n_0(\gamma, \gamma \vec{v})$$

$$j^0 = n_0 \gamma \quad j^i = n_0 \gamma v^i$$

In time  $dt$ , all particles in  $dV$   
cross surface  $d\vec{A}$ :

$$dN = n \cdot dV$$



$$dx = v dt$$

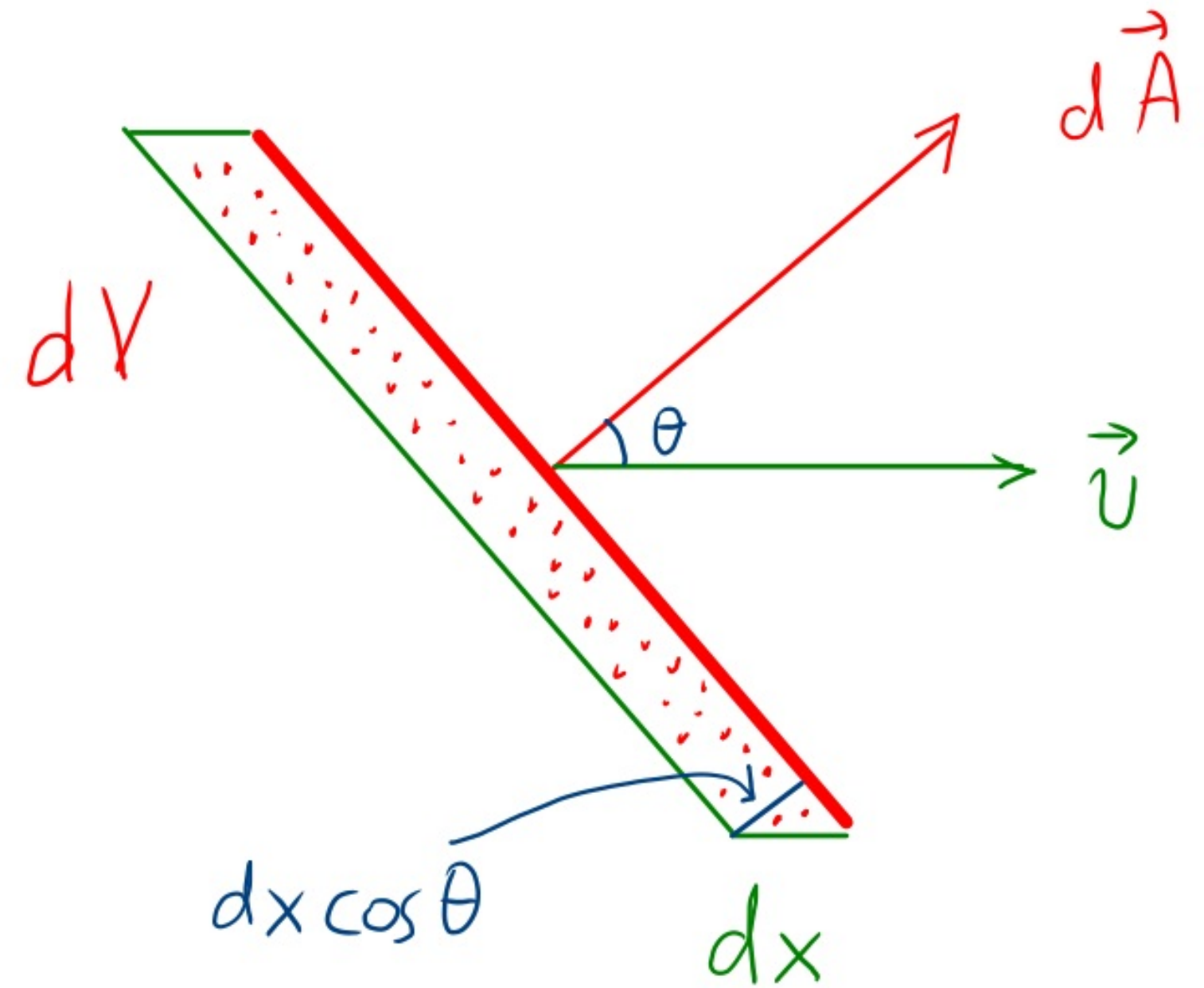
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In time  $dt$ , all particles in  $dV$   
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$$dN = n \cdot dV = n(dx \cos \theta) dA$$





# Stress - Energy Tensor in Special Relativity

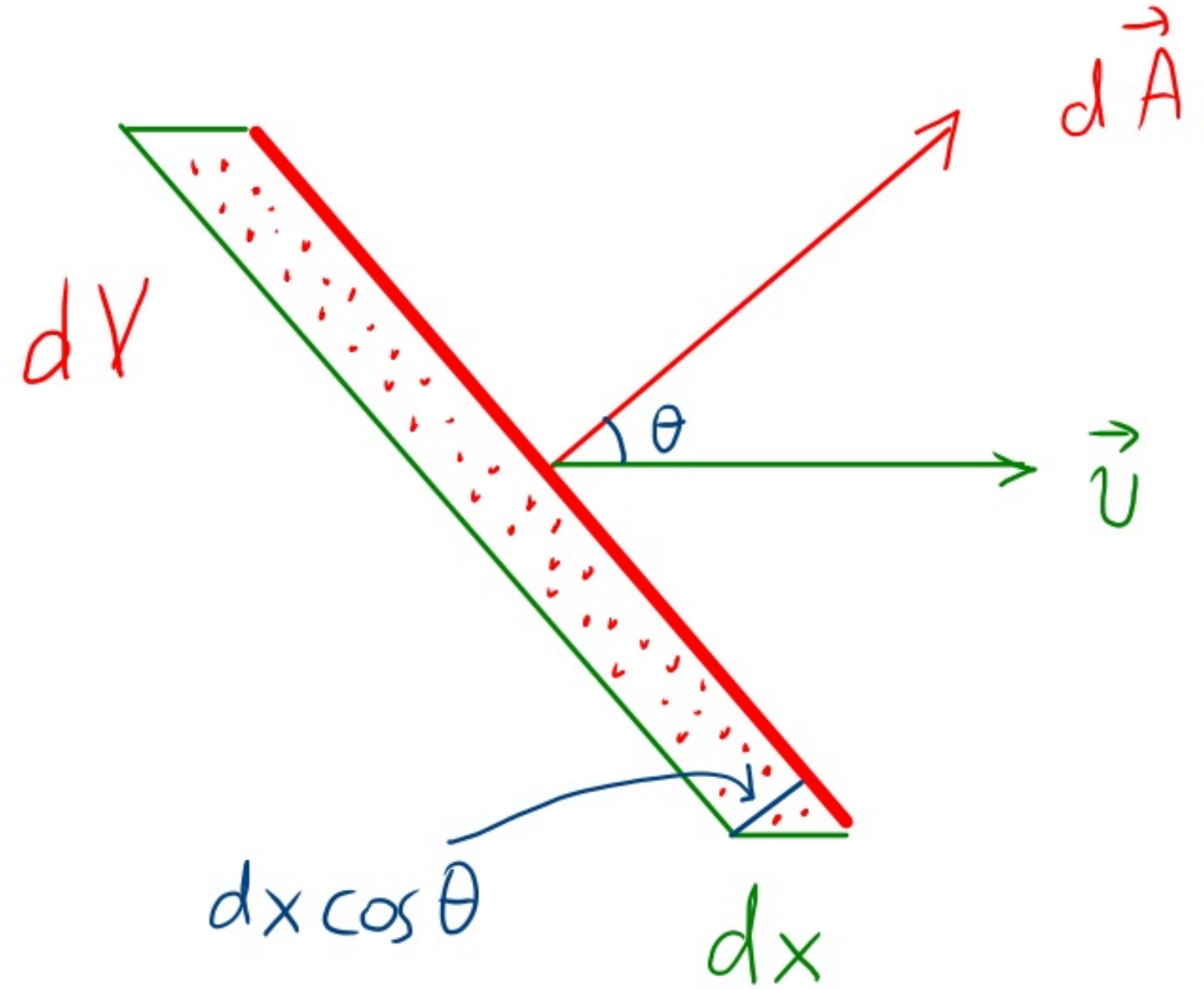
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In time  $dt$ , all particles in  $dV$   
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$$dN = n \cdot dV = n(dx \cos \theta) dA \Rightarrow$$

$$\frac{dN}{dt} = n \frac{dx}{dt} dA \cos \theta$$



$$v = \frac{dx}{dt}$$

# Stress - Energy Tensor in Special Relativity

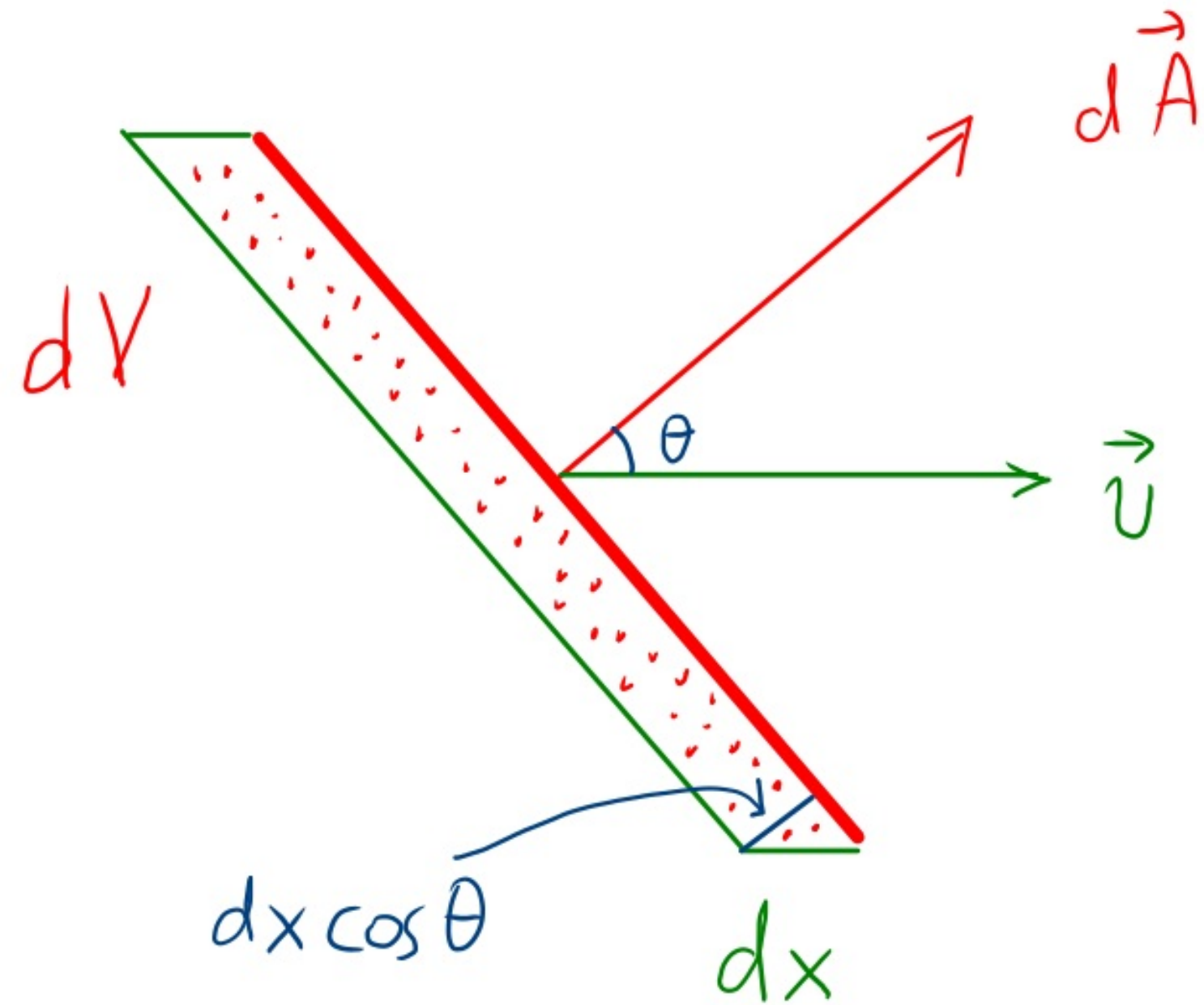
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# Stress - Energy Tensor in Special Relativity

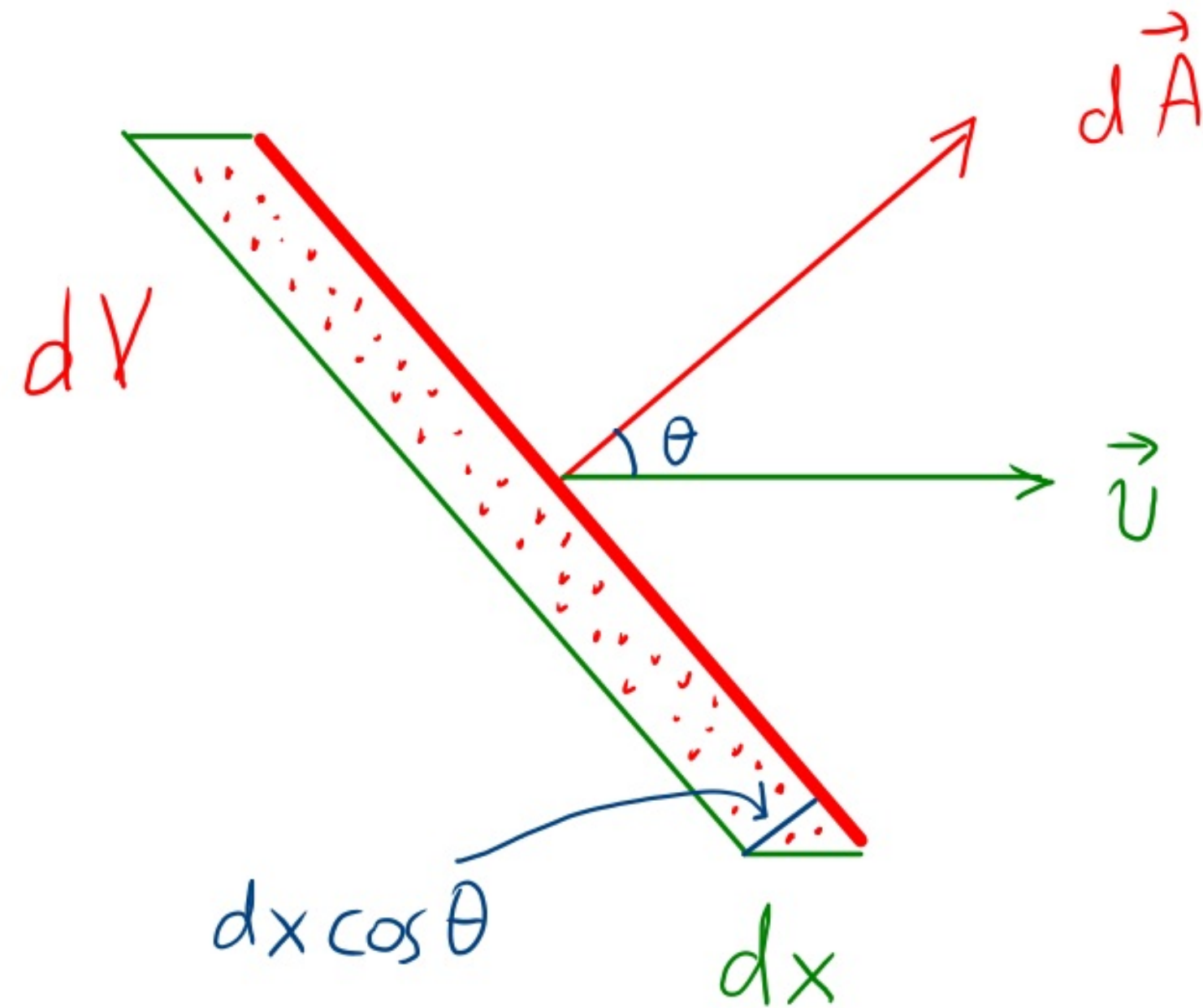
$$j^\mu = n_0(\gamma, \gamma \vec{v})$$

$$j^0 = n_0 \gamma = n, \quad j^i = n_0 \gamma v^i = n v^i$$

In time  $dt$ , all particles in  $dV$   
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$$dN = n \cdot dV = n(dx \cos \theta) dA \Rightarrow$$

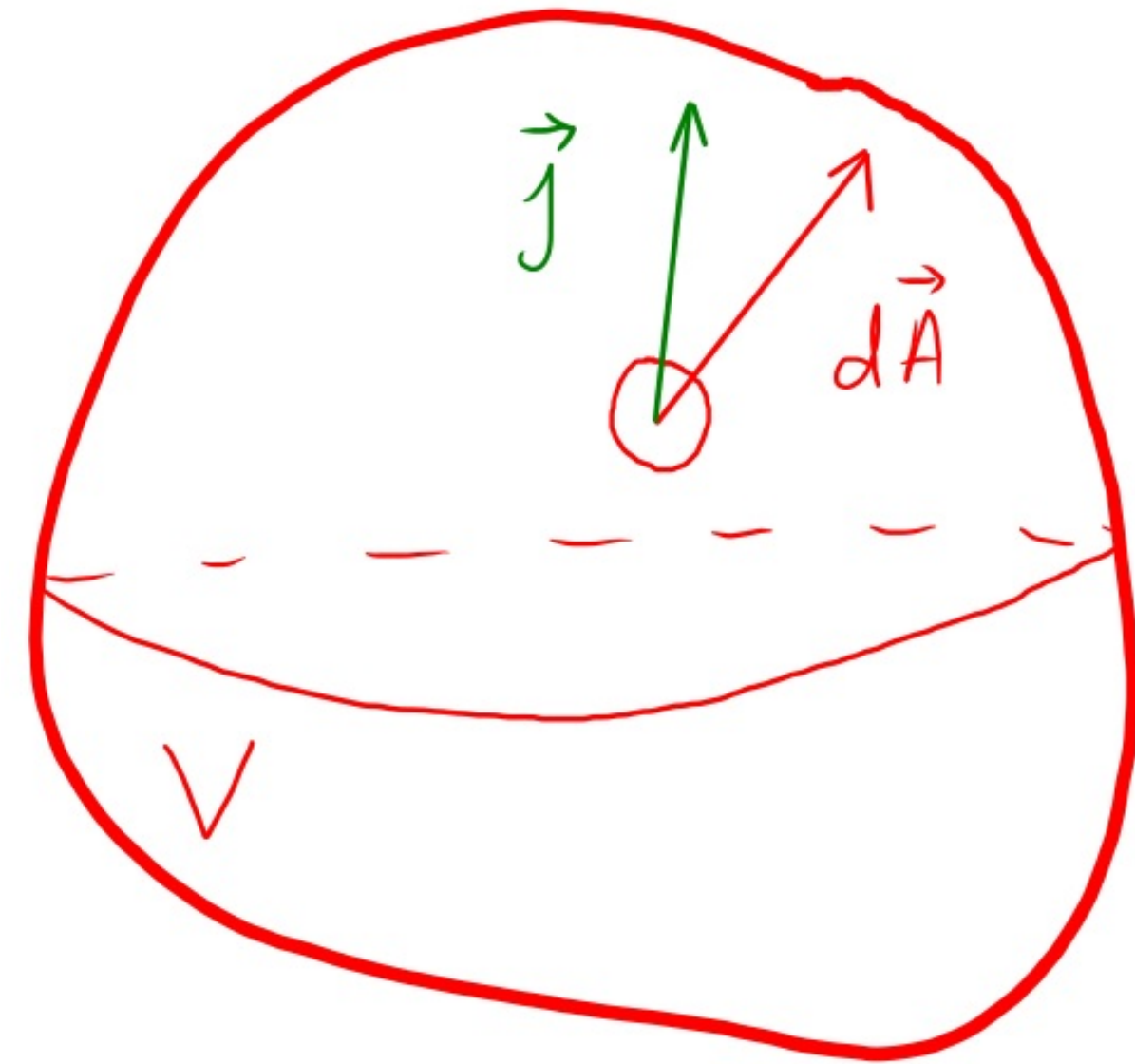
$$\begin{aligned} \frac{dN}{dt} &= n \frac{dx}{dt} dA \cos \theta = n v \cdot dA \cos \theta = n \vec{v} \cdot d\vec{A} \\ &= \vec{j} \cdot d\vec{A} \end{aligned}$$



# Stress - Energy Tensor in Special Relativity

Therefore if we have a closed surface  $S = \partial V$ ,  
the rate of change of particles enclosed is:

$$\frac{dN}{dt} = - \int_{\partial V} \vec{j} \cdot d\vec{A} \quad \left( (-) \text{ because } \vec{j} \cdot d\vec{A} > 0 \Rightarrow \right. \\ \left. N \text{ decreases} \right)$$



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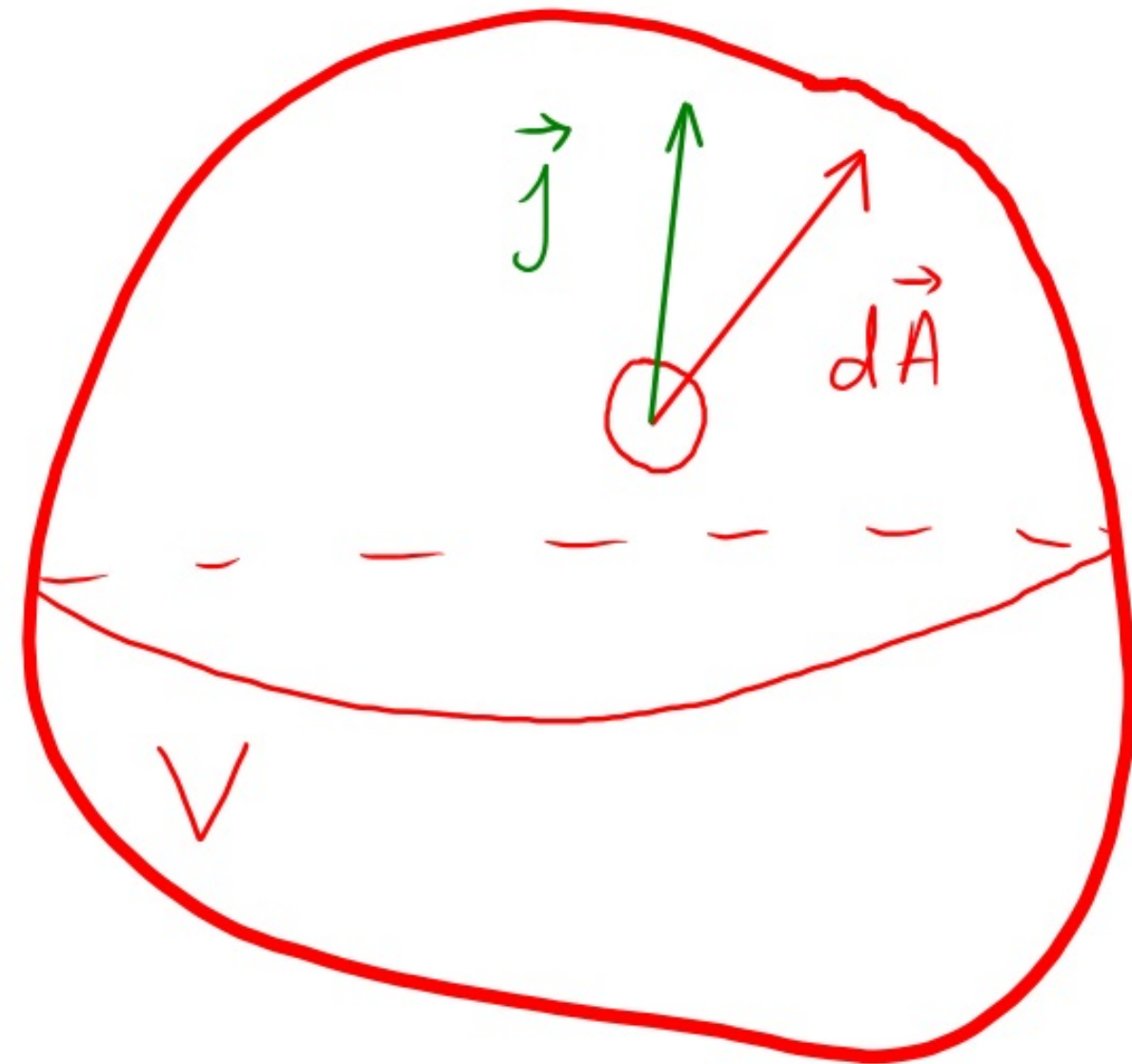
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$$\frac{dN}{dt} = \frac{d}{dt} \int_V n \cdot dV = \frac{d}{dt} \int_V j^0 dV = \int_V \frac{\partial j^0}{\partial t} dV$$

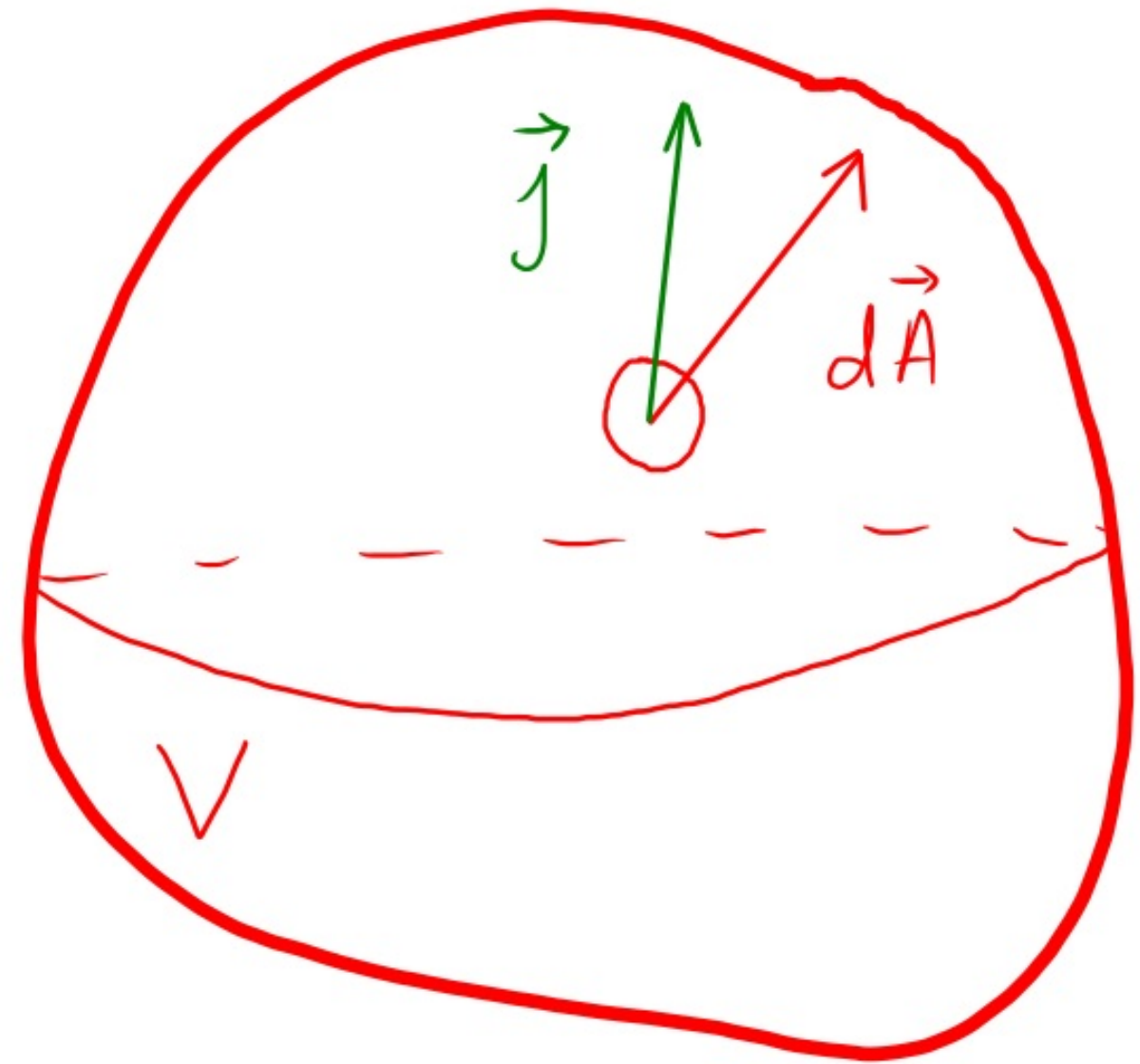


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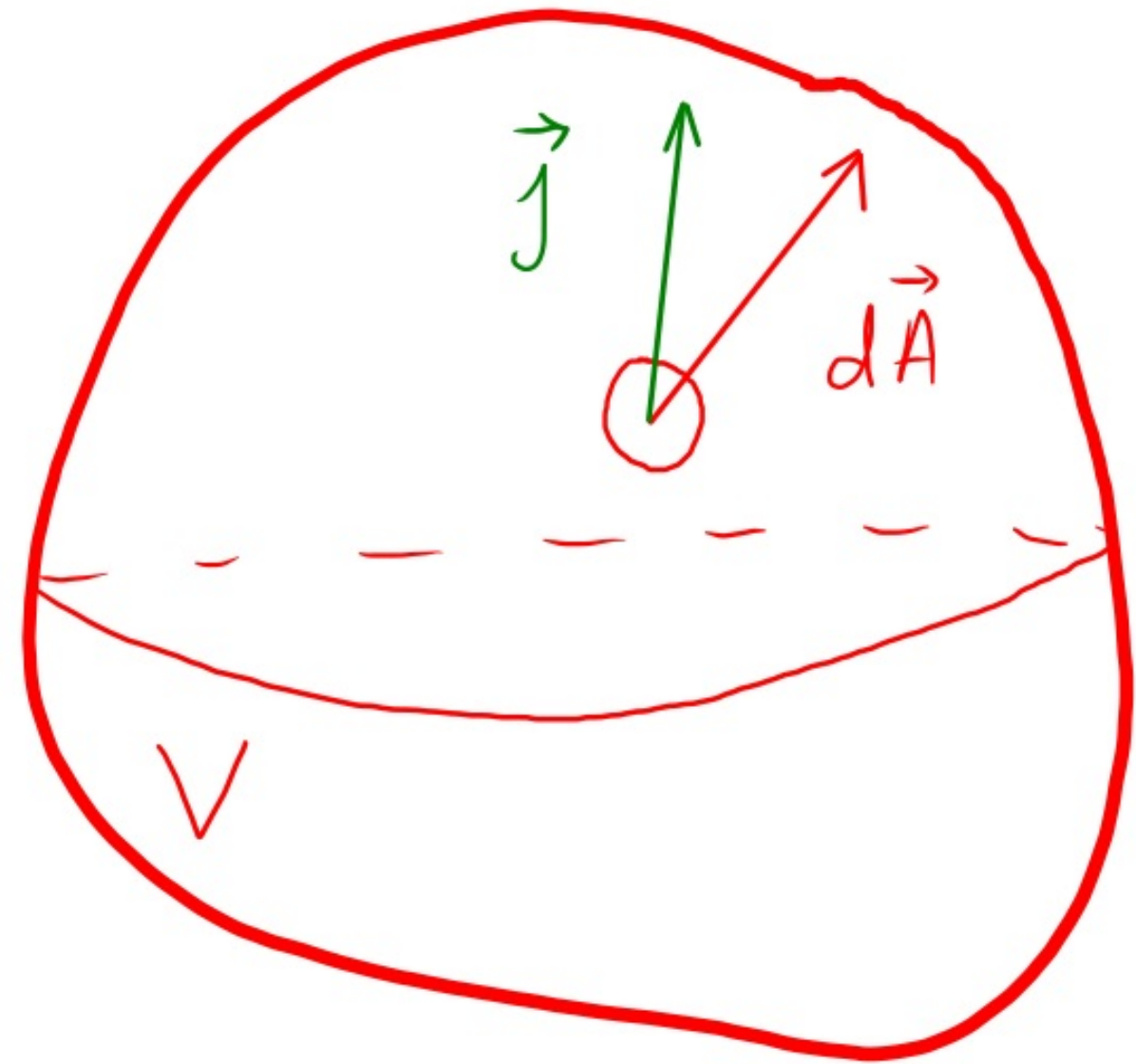
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$$\Rightarrow \frac{\partial j^0}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$$



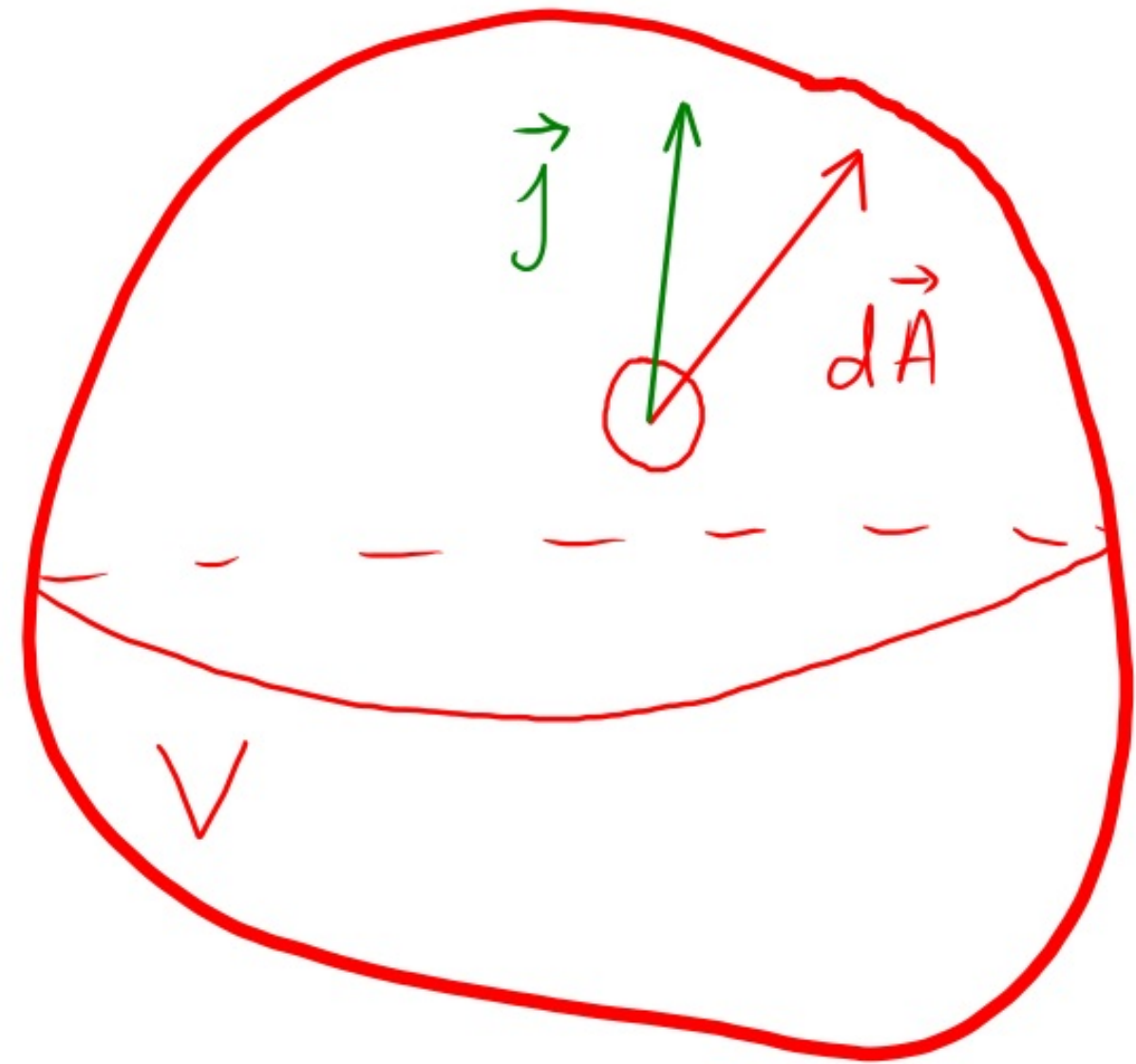
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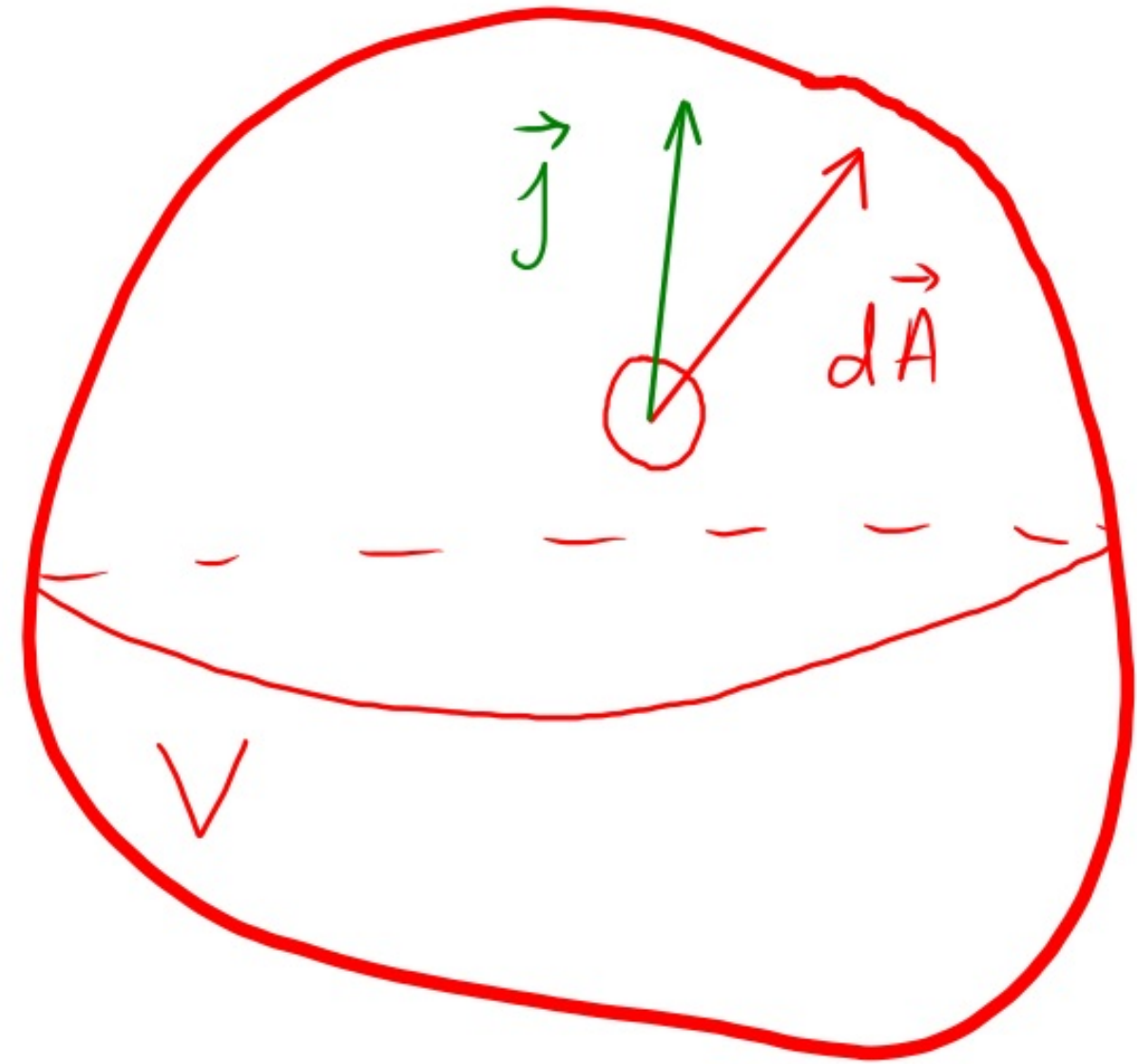
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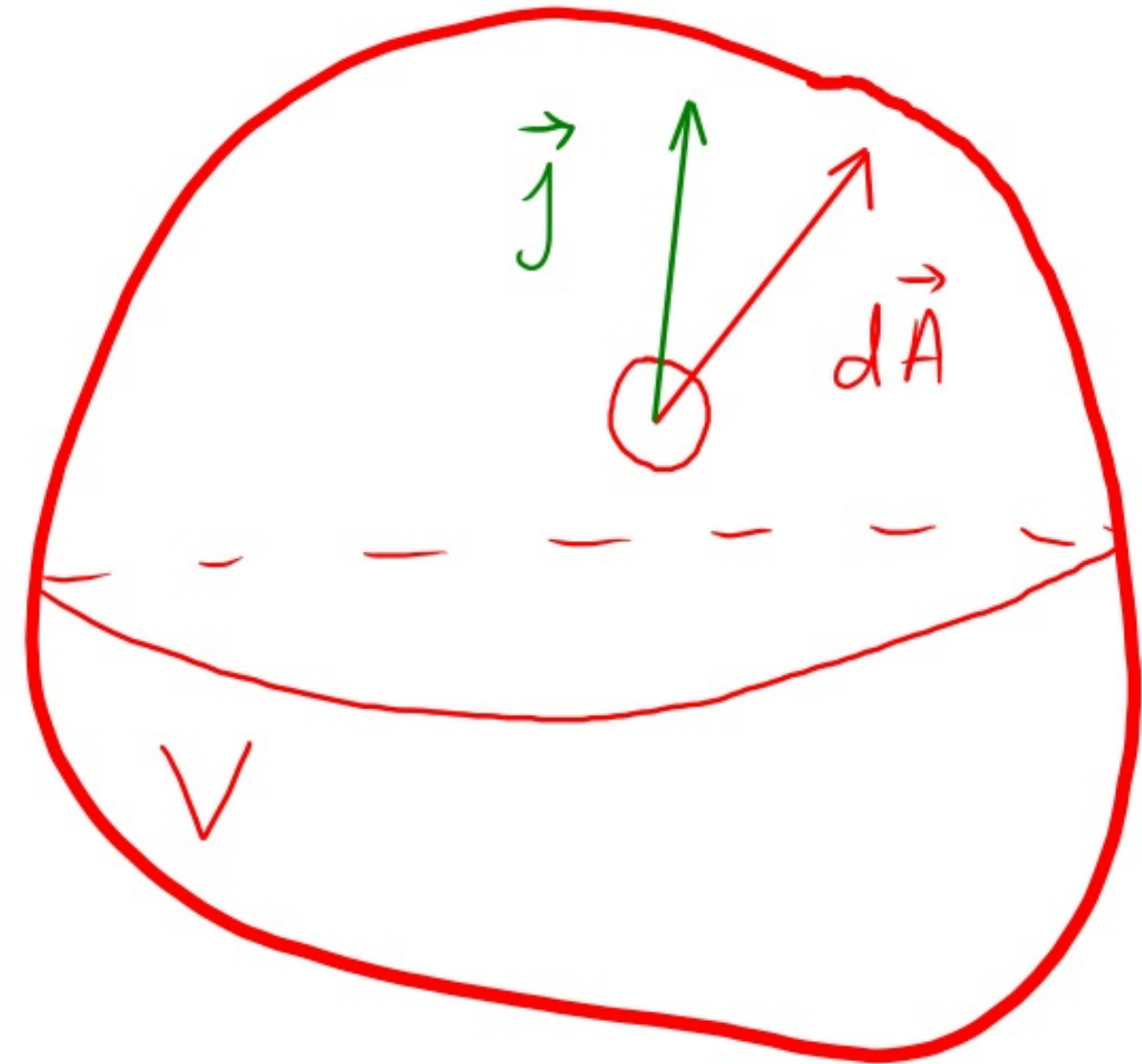
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continuity equation

→ we say that

$j^\mu$  "conserved"



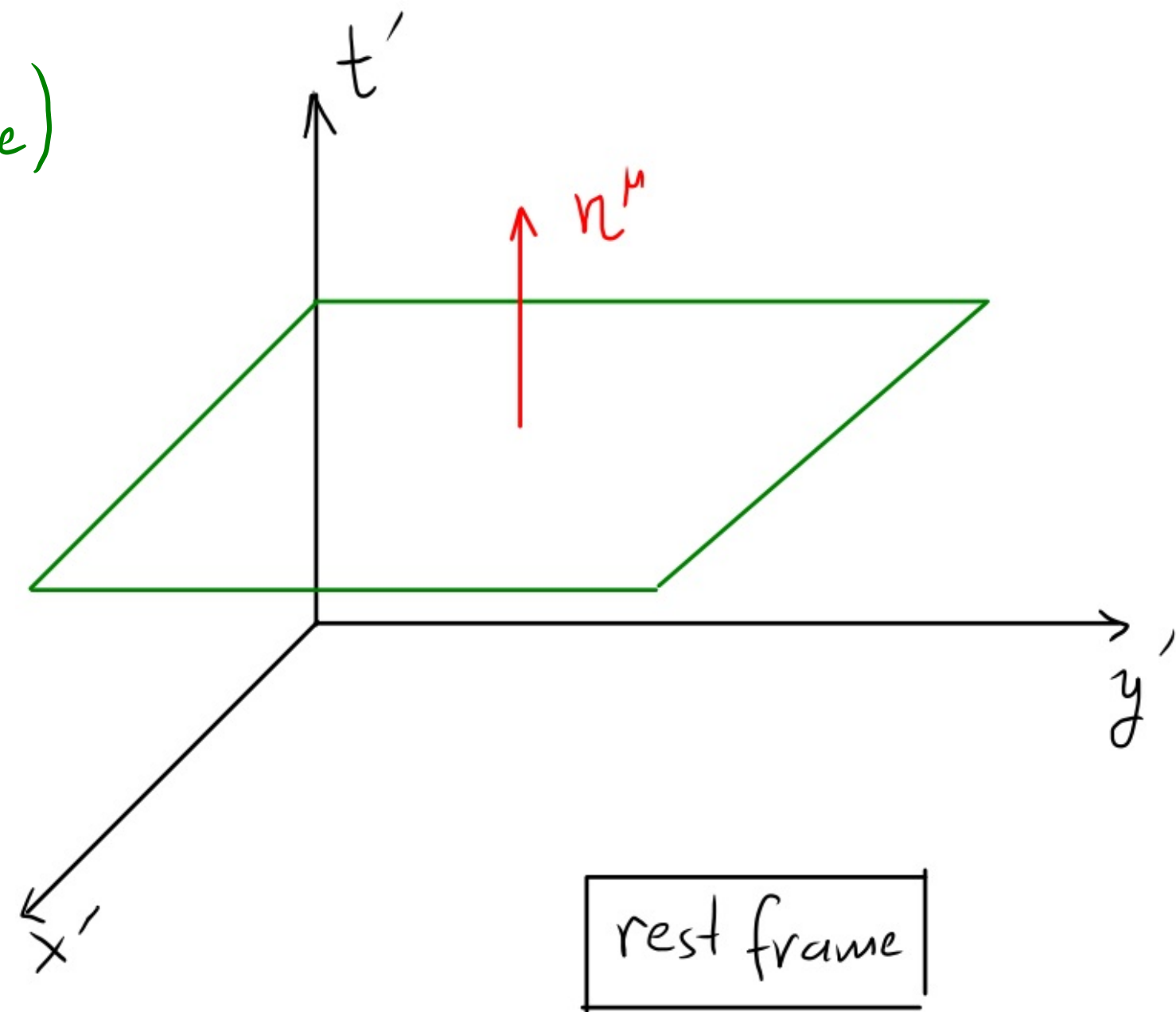


# Stress - Energy Tensor in Special Relativity

Consider the spacelike 3-surface (3 volume)  
 $t' = \text{const.}$  in rest frame of particles:

$$n^\mu = (1, 0, 0, 0) \quad \text{normal to surface}$$

$$u^\mu = (1, 0, 0, 0) \quad \text{4-velocity of particles}$$



# Stress - Energy Tensor in Special Relativity

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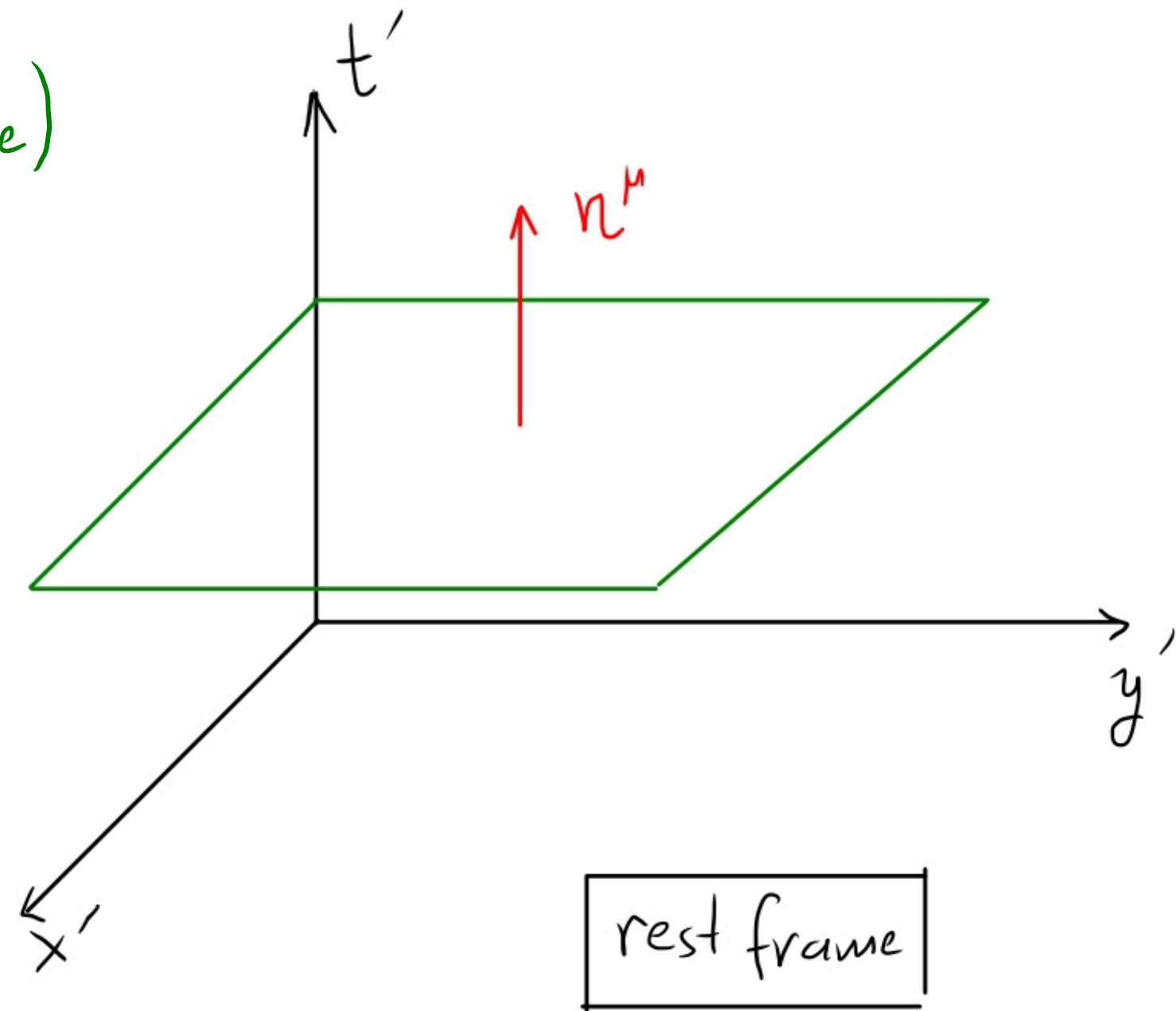
$t' = \text{const.}$  in rest frame of particles:

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timelike:  $\eta_\mu \eta^\mu = \eta_{\mu\nu} \eta^\mu \eta^\nu = -1$

$$u_\mu u^\mu = \eta_{\mu\nu} u^\mu u^\nu = -1$$



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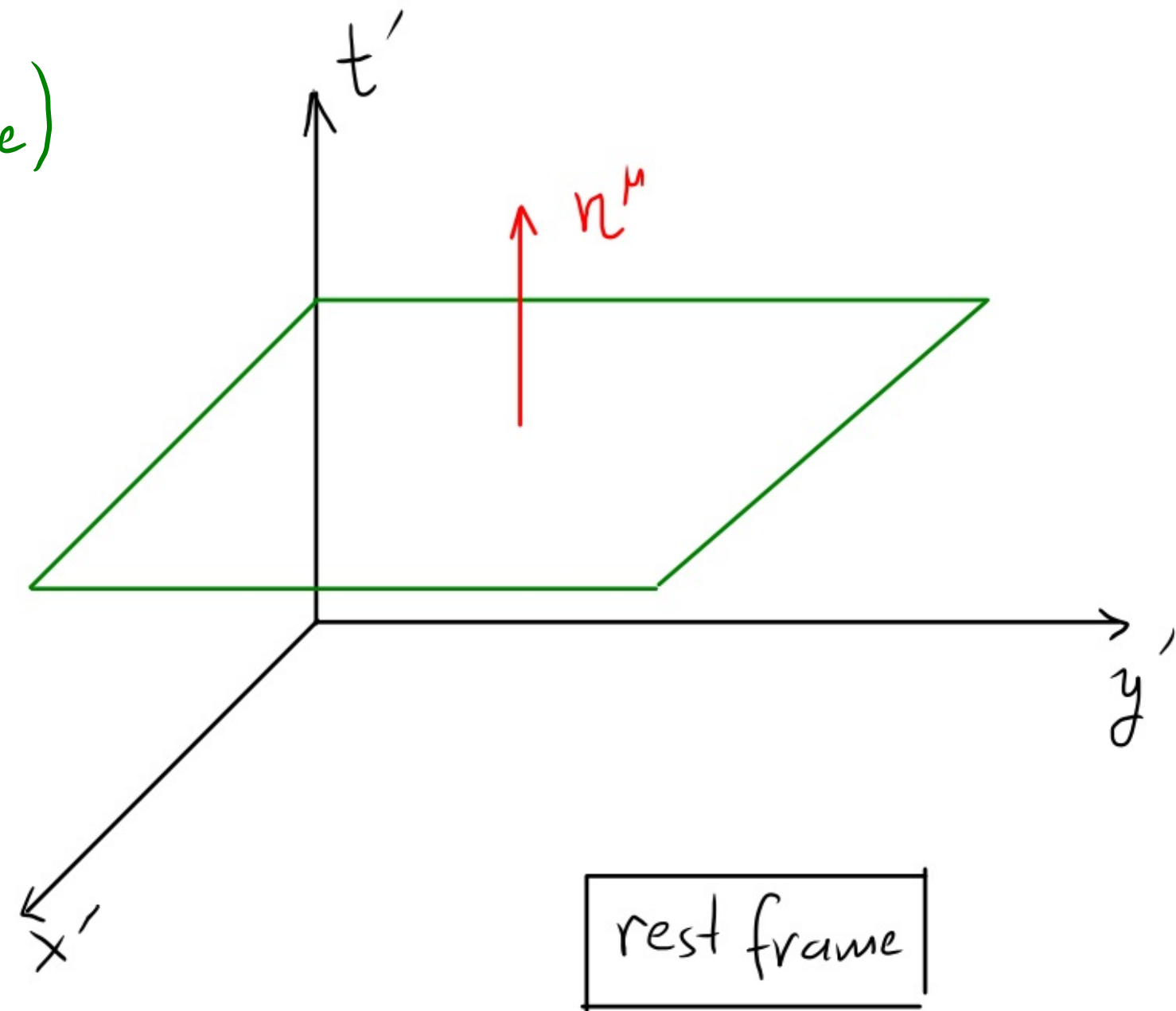
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then  $j_\mu \eta^\mu = \eta_0 \eta_{\mu\nu} u^\mu \eta^\nu$



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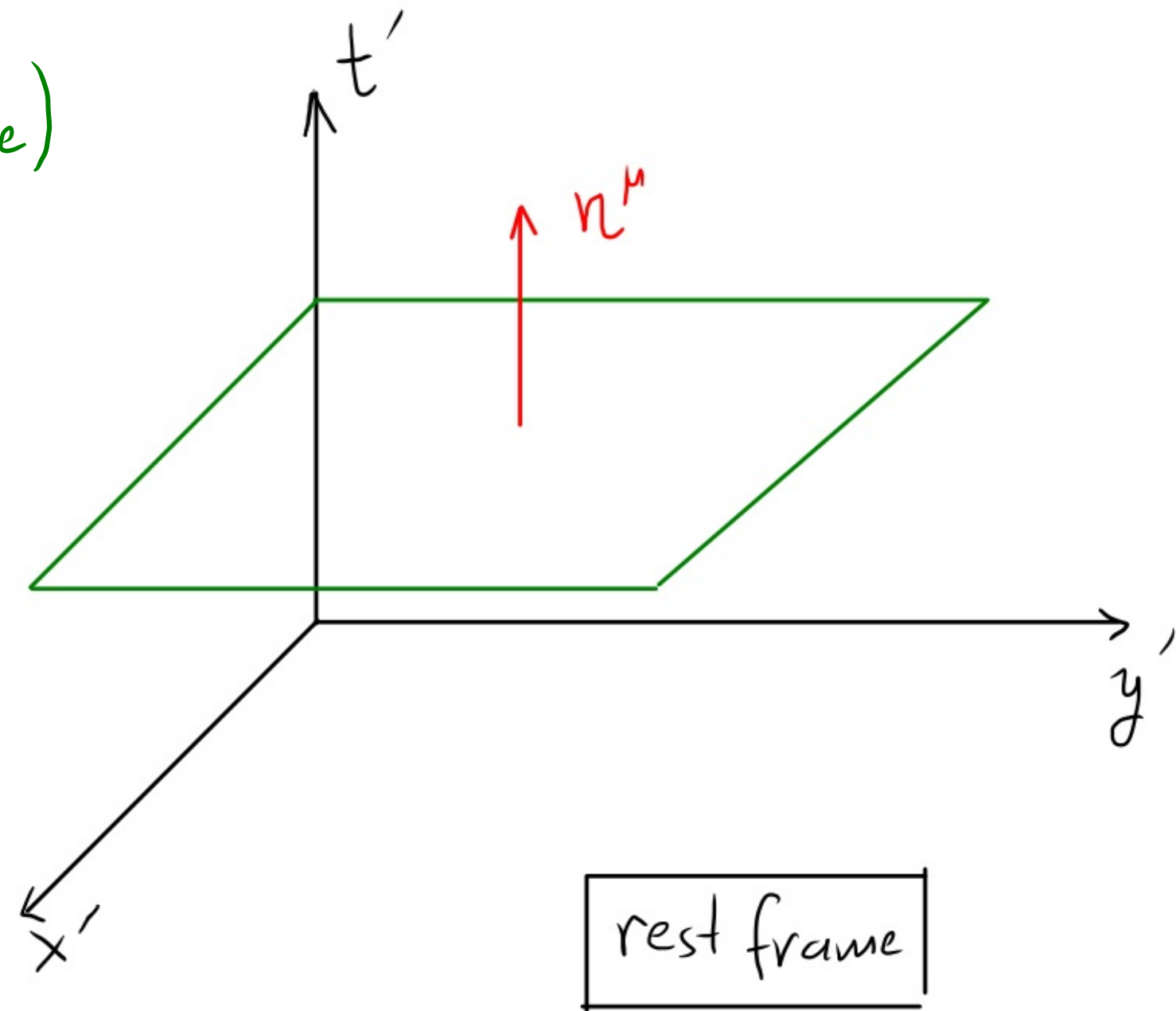
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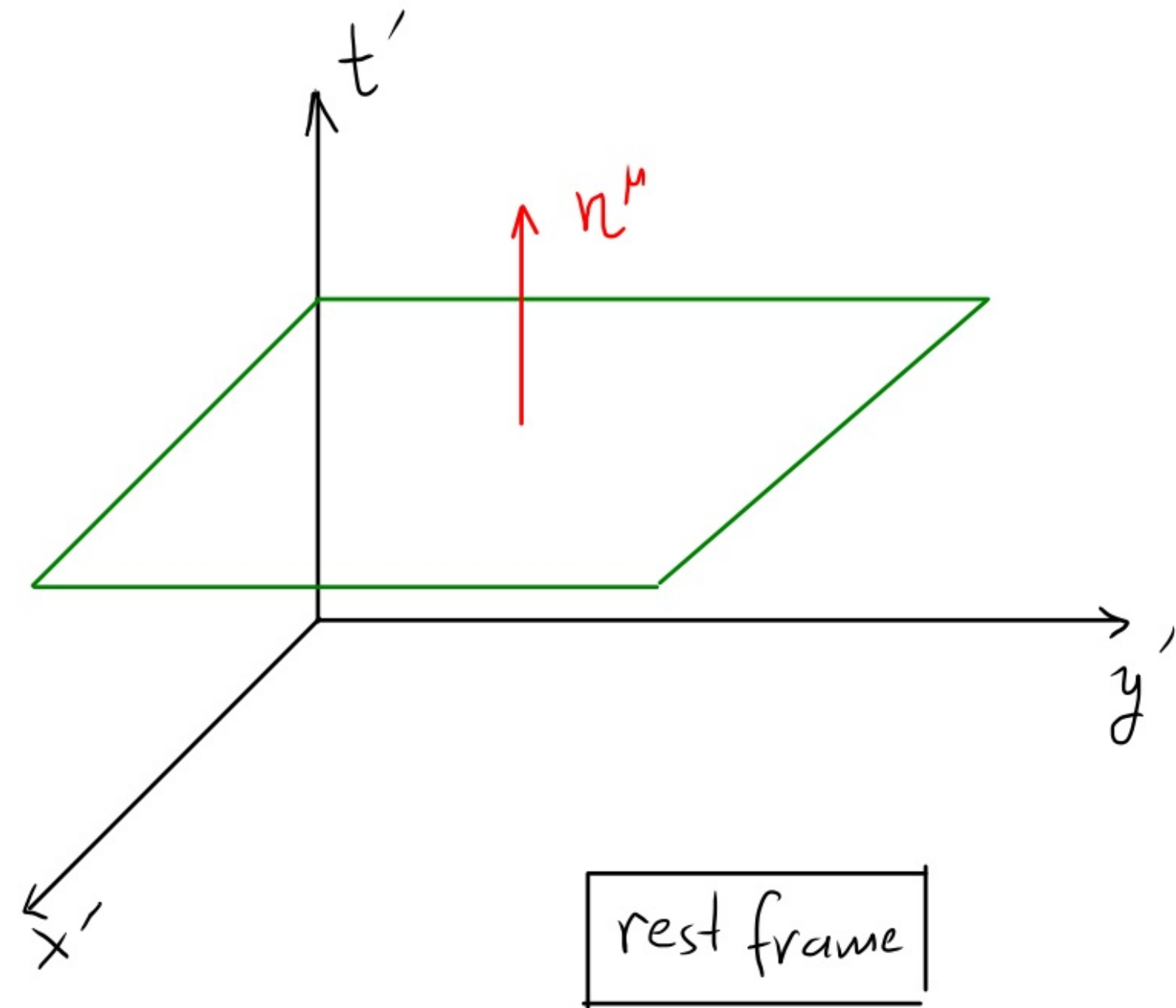
$$u_\mu u^\mu = \eta_{\mu\nu} u^\mu u^\nu = -1$$

then  $j_\mu \eta^\mu = \eta_0 \eta_{\mu\nu} u^\mu \eta^\nu = \eta_0 \eta_{00} u^0 \eta^0 = -\eta_0$



# Stress - Energy Tensor in Special Relativity

Then  $dN_0 = n_0 dV_0 = -j_\mu \eta^\mu dV_0$

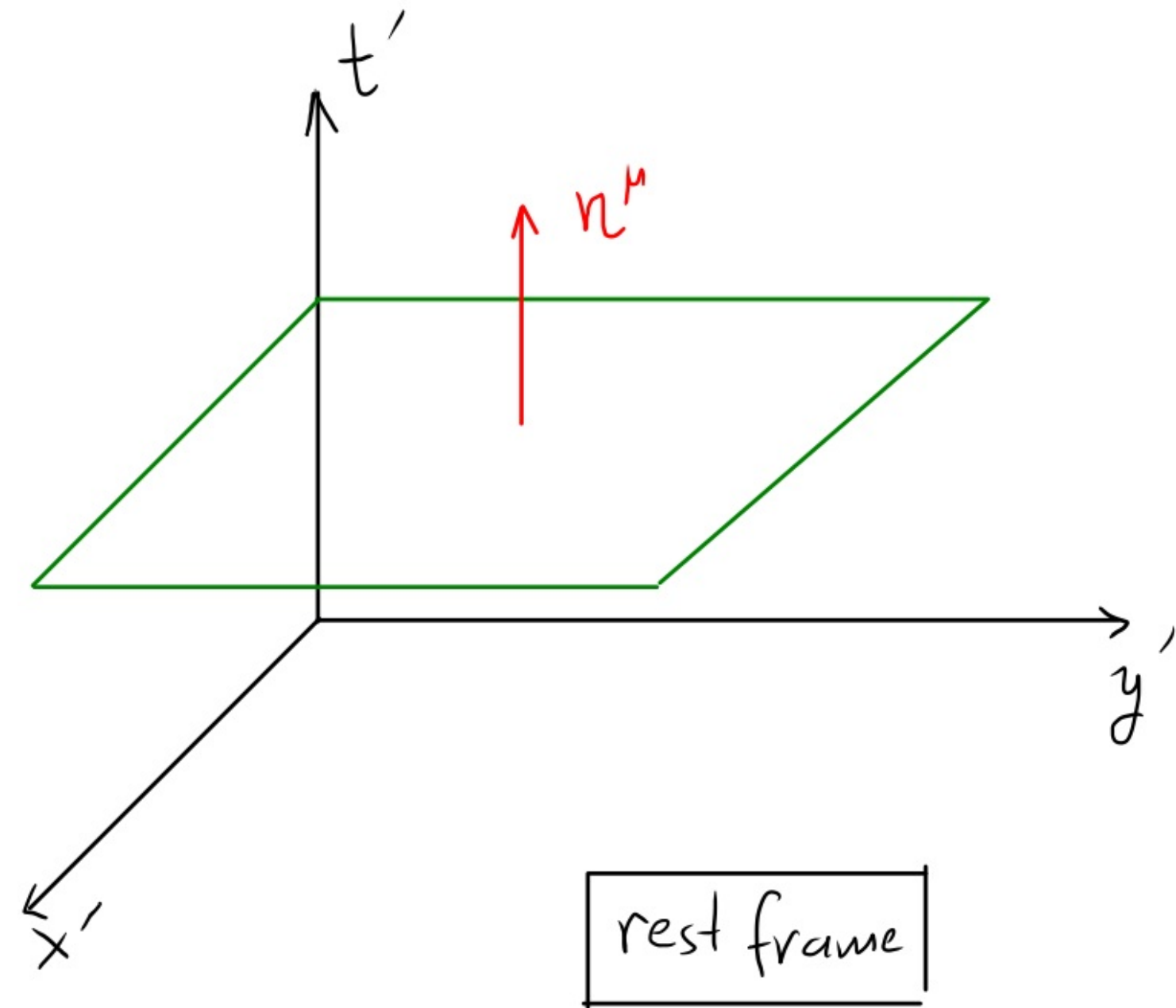


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# Stress - Energy Tensor in Special Relativity

Then  $dN_0 = n_0 dV_0 = -j_\mu \underbrace{\eta^\mu}_{\substack{\downarrow \\ \text{"area" element of} \\ \text{3-surface}}} dV_0$



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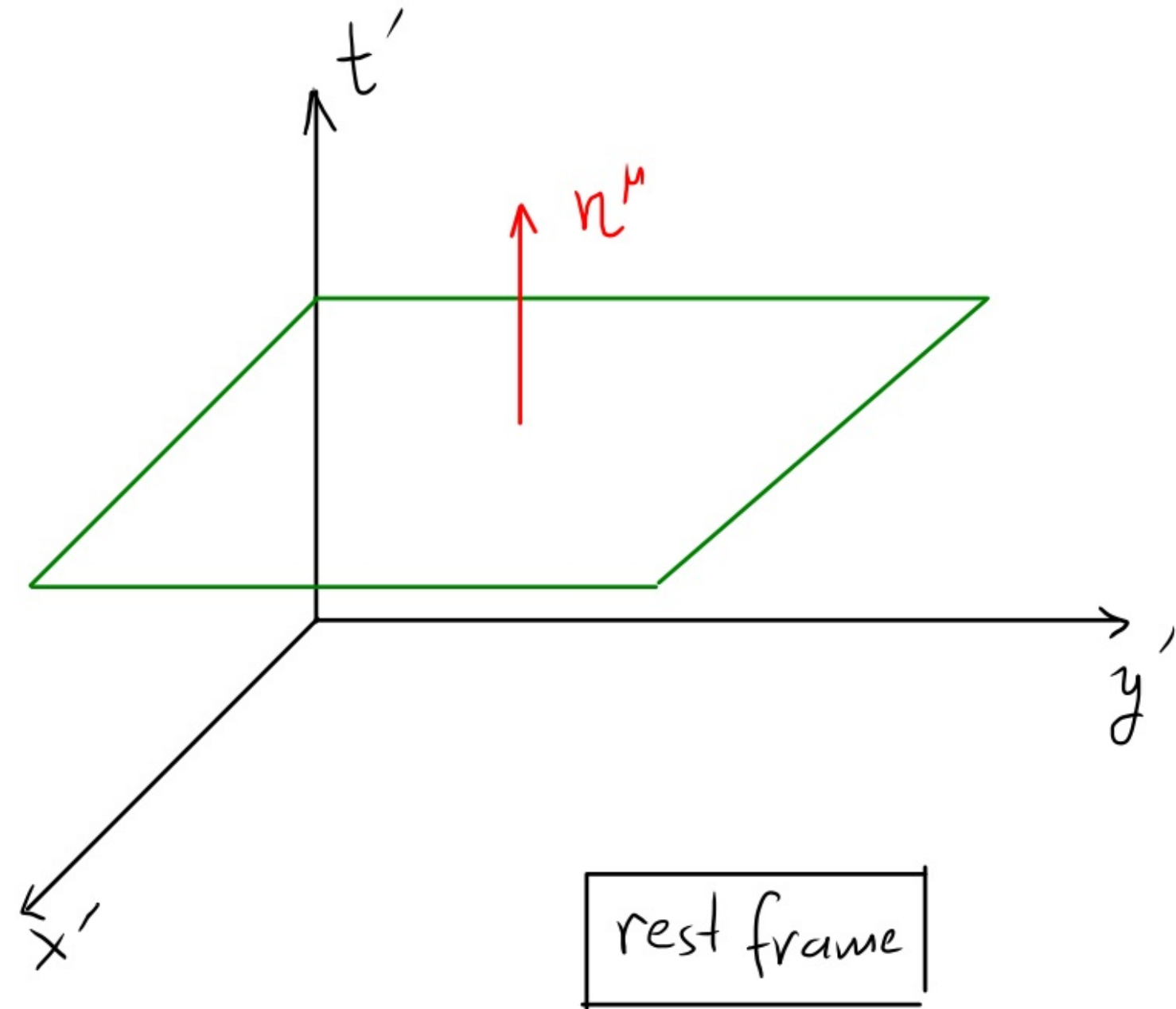
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# Stress - Energy Tensor in Special Relativity

Then  $dN_0 = n_0 dV_0 = -j_\mu \eta^\mu dV_0$

"area" element of  
3-surface

flux of number  
current through  
space like surface



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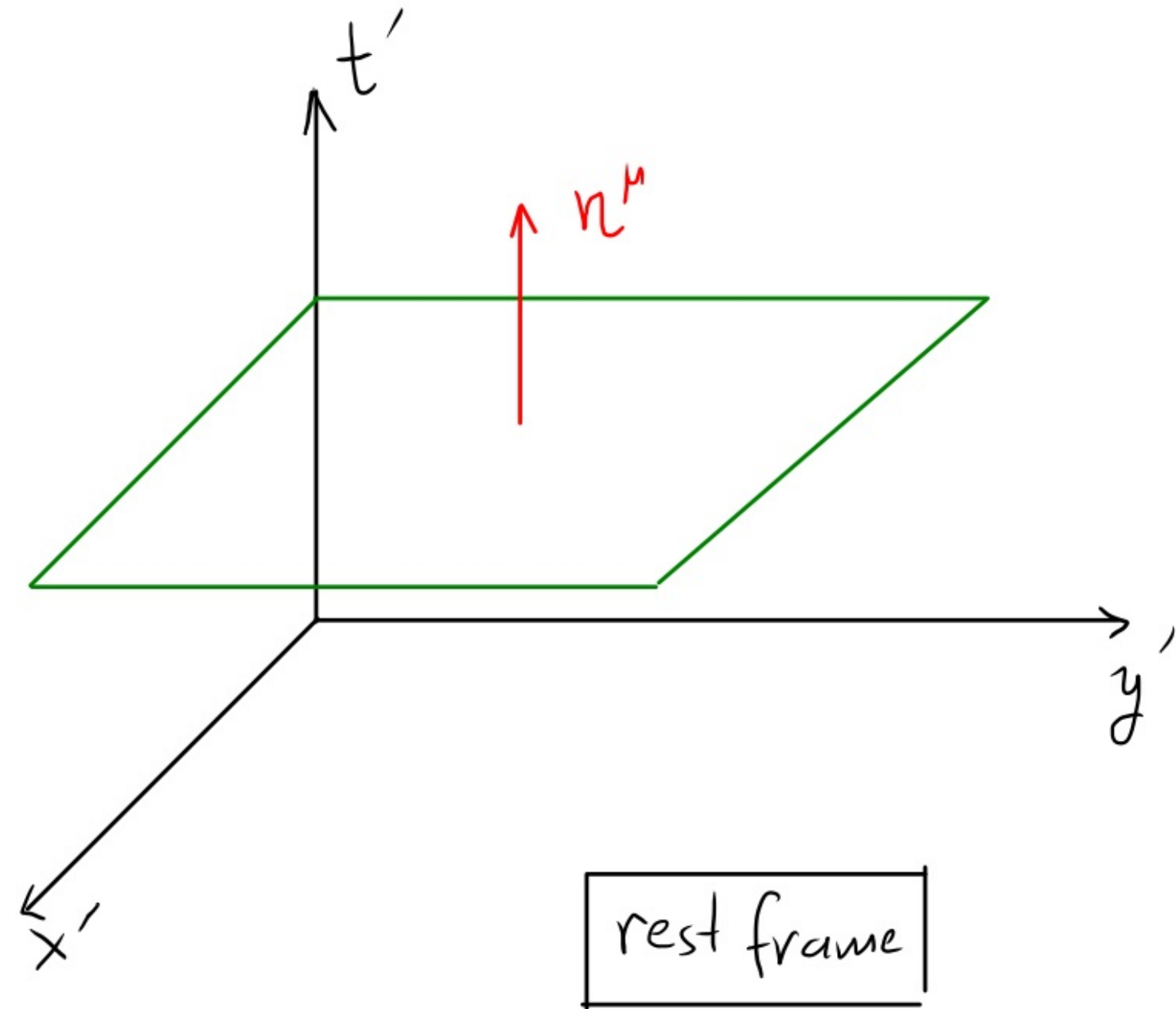
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"area" element of  
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flux of number  
current through  
space like surface

- density  $n_0$  is a flux of number current through spacelike surfaces



then  $j_\mu \eta^\mu = n_0 \eta_{\mu\nu} u^\mu \eta^\nu = n_0 \eta_{00} u^0 \eta^0 = -n_0$



# Stress - Energy Tensor in Special Relativity

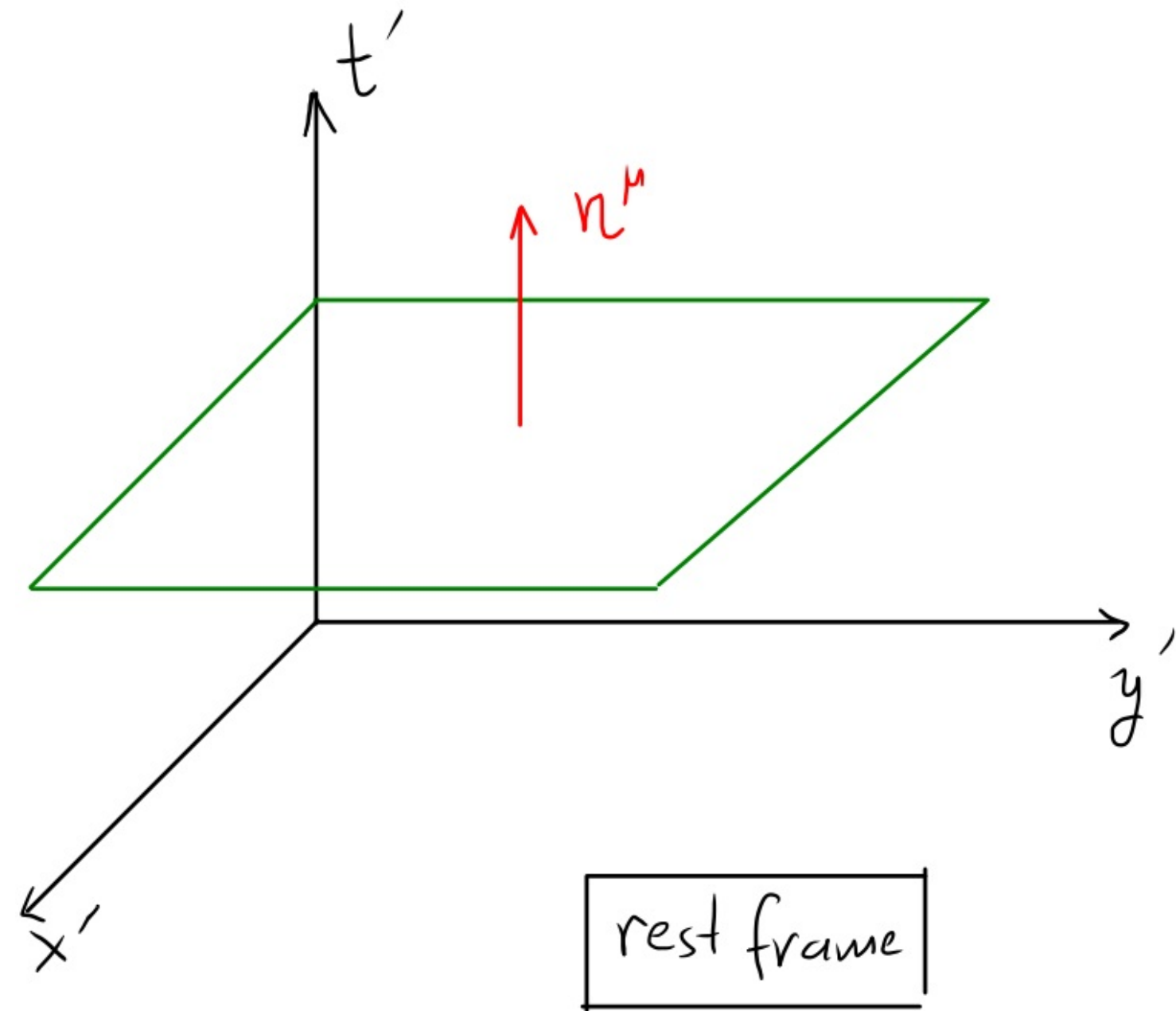
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"area" element of  
3-surface

flux of number  
current through  
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• density  $n_0$  is a flux of number  
current through spacelike surfaces

• likewise, currents  $\vec{j}$  are fluxes in spacelike directions  
through timelike surfaces (spacelike normals)



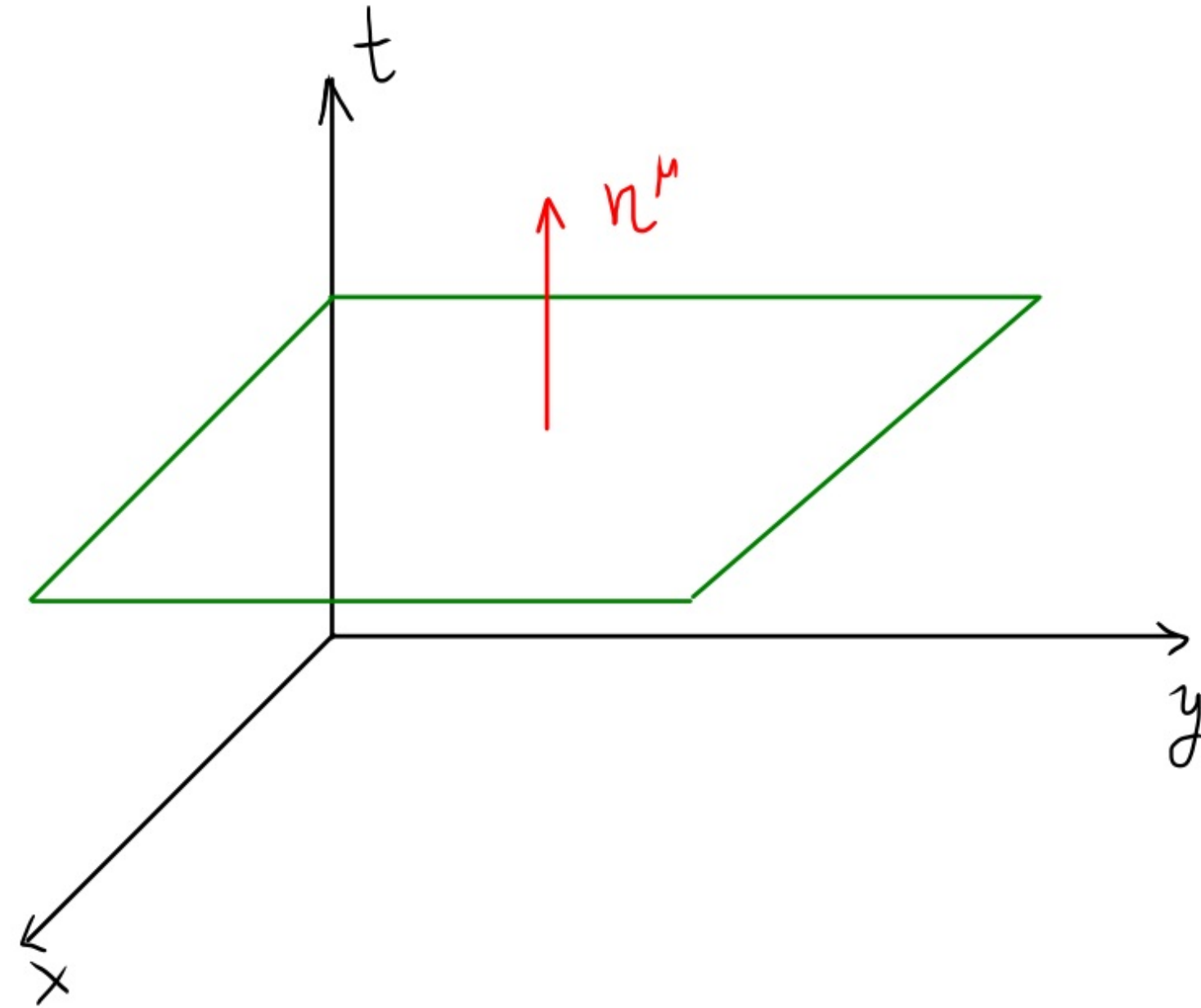
# Stress - Energy Tensor in Special Relativity

Similarly in  $(t, x, y, z)$  frame:

$$j^{\mu} = (j^0, \vec{j})$$

$$n^{\mu} = (1, 0, 0, 0)$$

↳ constant  $t$  surface



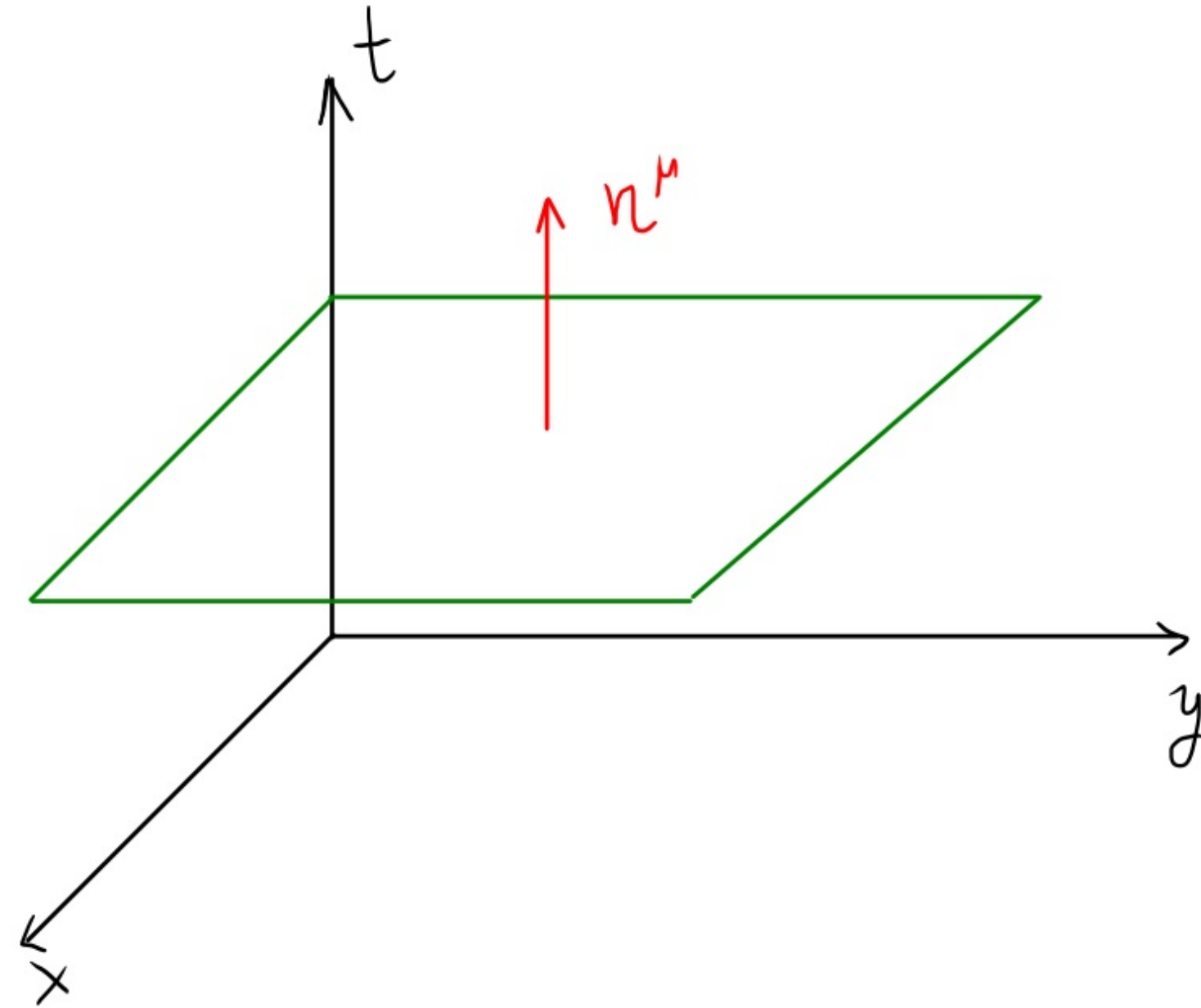
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$$j^{\mu} = (j^0, \vec{j}) \quad n^{\mu} = (1, 0, 0, 0)$$

↳ constant  $t$  surface

$$dN = n dV = j^0 dV = -j^{\mu} \eta_{\mu} dV$$



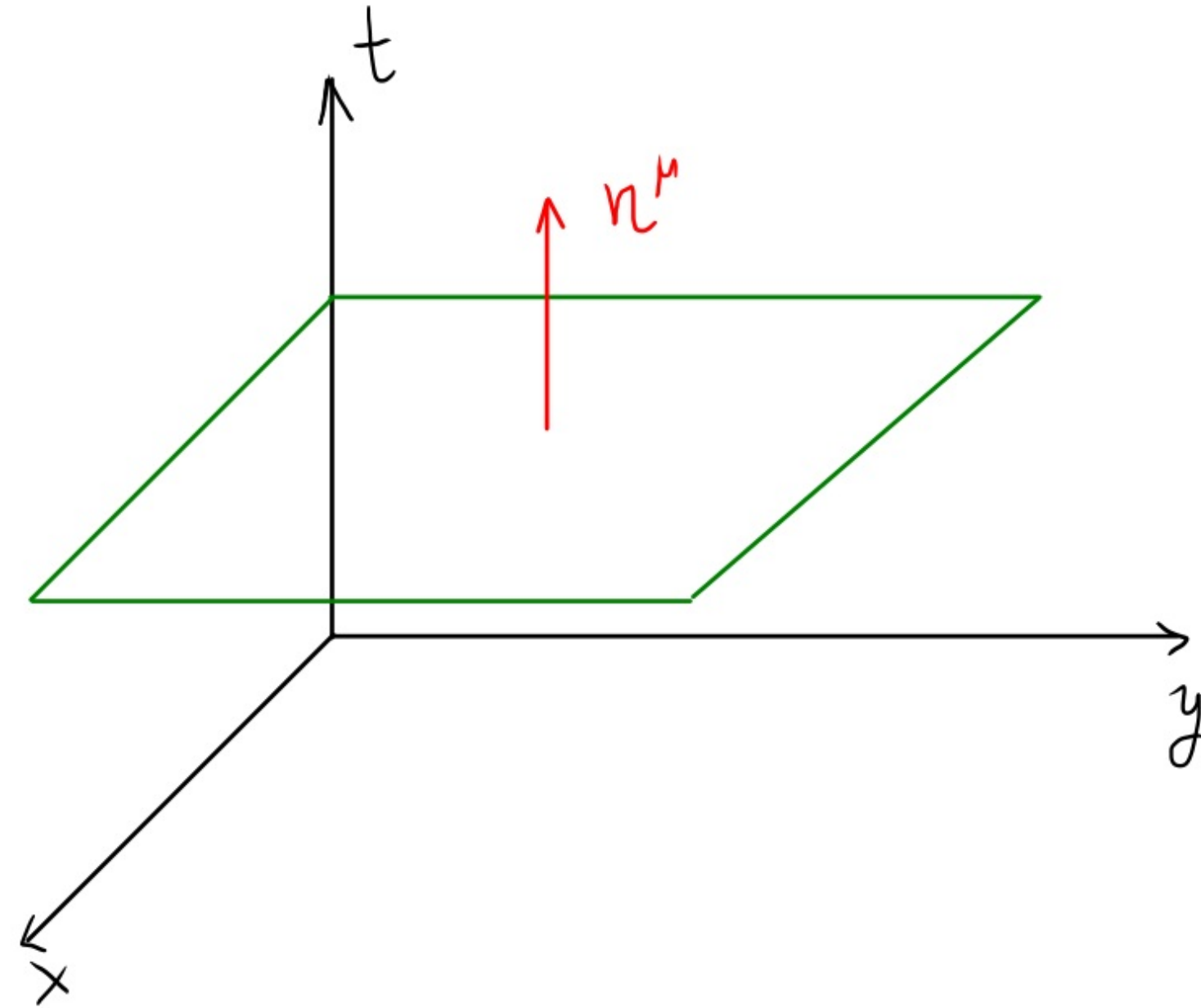
# Stress - Energy Tensor in Special Relativity

Consider 4-momentum in volume  $\Delta V$

$$\Delta p^{\mu} = - \underbrace{T^{\mu\nu}}_{\text{"current"}} n_{\nu} \Delta V$$

compare with:

$$\Delta N = - j^{\mu} n_{\mu} \Delta V$$



# Stress - Energy Tensor in Special Relativity

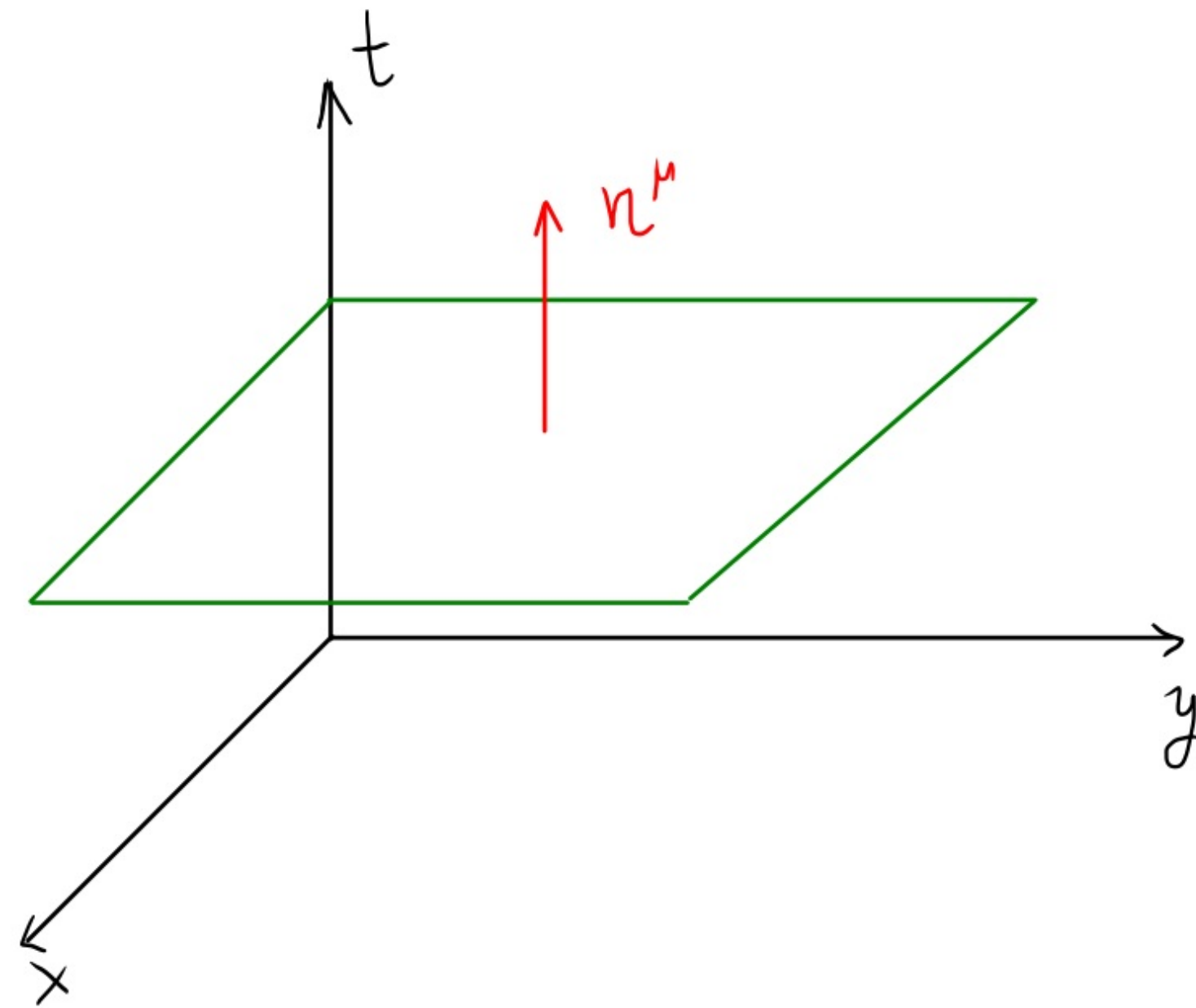
Consider 4-momentum in 3-volume  $\Delta V$

$$\Delta p^\mu = - T^{\mu\nu}$$

need extra  
index

$$n_\nu \Delta V$$

↓ spacelike 3-surface  
"area"  
timelike

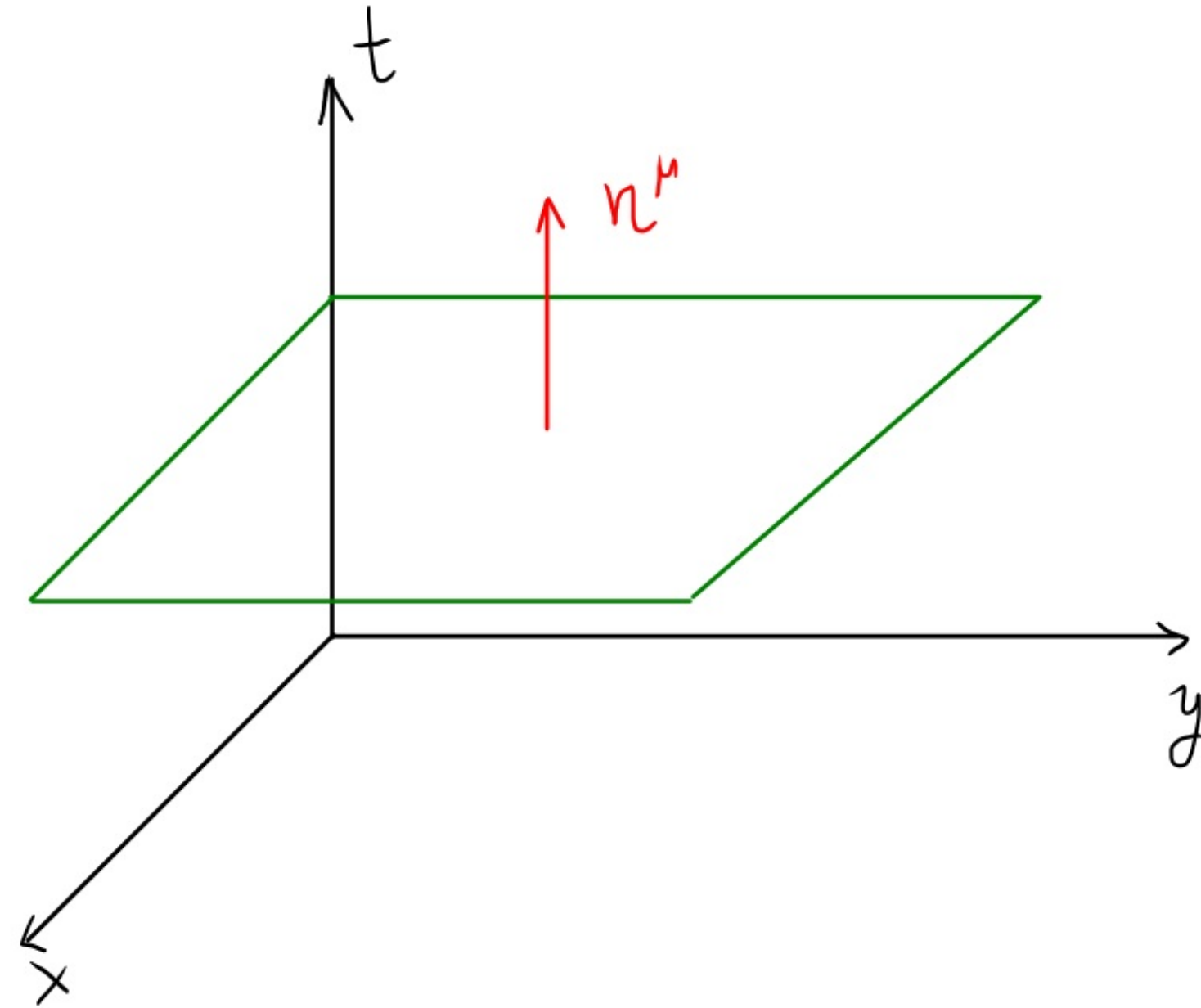


# Stress - Energy Tensor in Special Relativity

Consider 4-momentum in 3-volume  $\Delta V$

$$\Delta p^{\mu} = - \underbrace{T^{\mu\nu}}_{\text{4-momentum density}} n_{\nu} \Delta V$$

4-momentum density

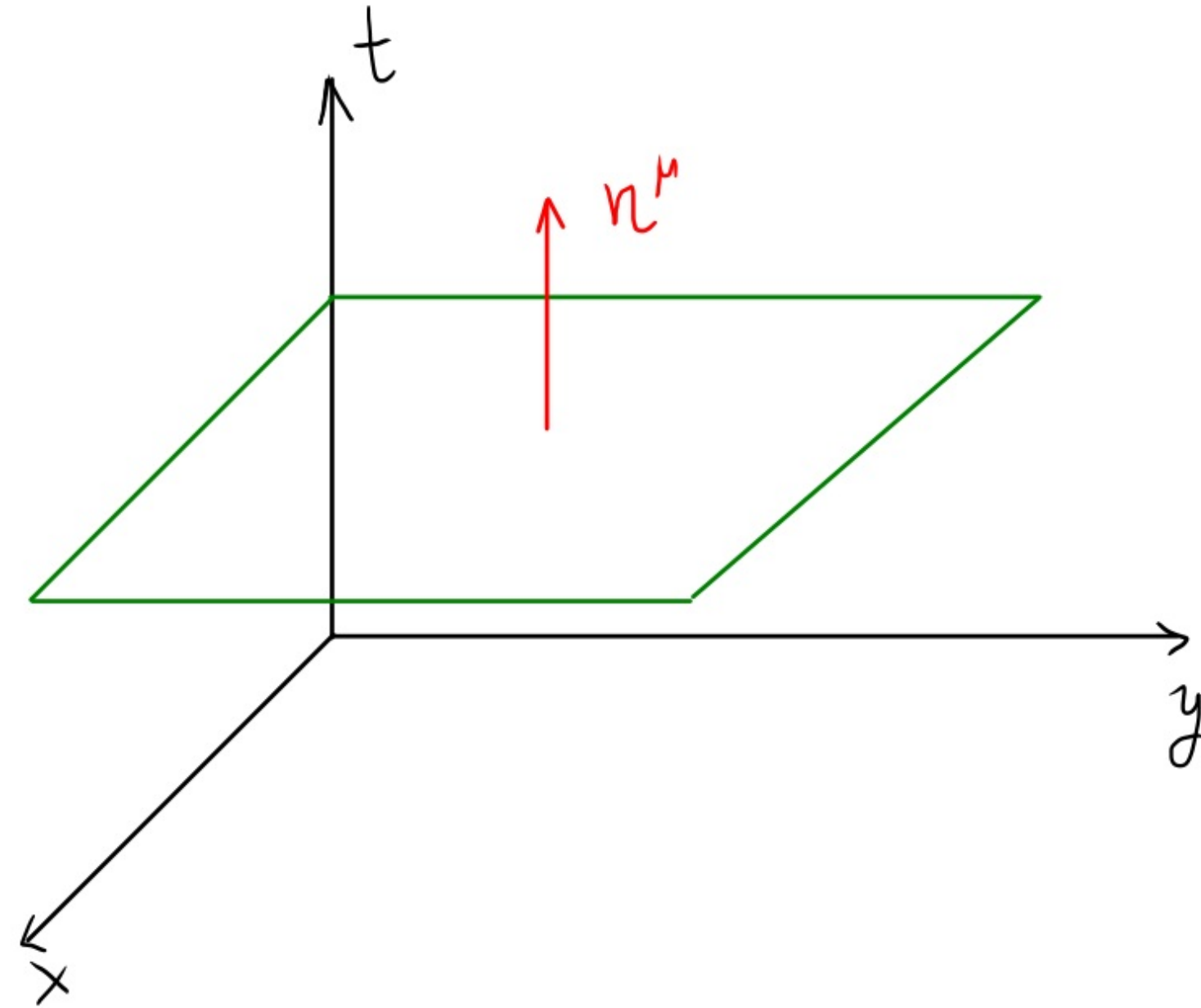


# Stress - Energy Tensor in Special Relativity

Consider 4-momentum in volume  $\Delta V$

$$\Delta p^\mu = - T^{\mu\nu} n_\nu \Delta V$$

If  $\Delta V$  at rest,  $n^\mu = (1, 0, 0, 0)$



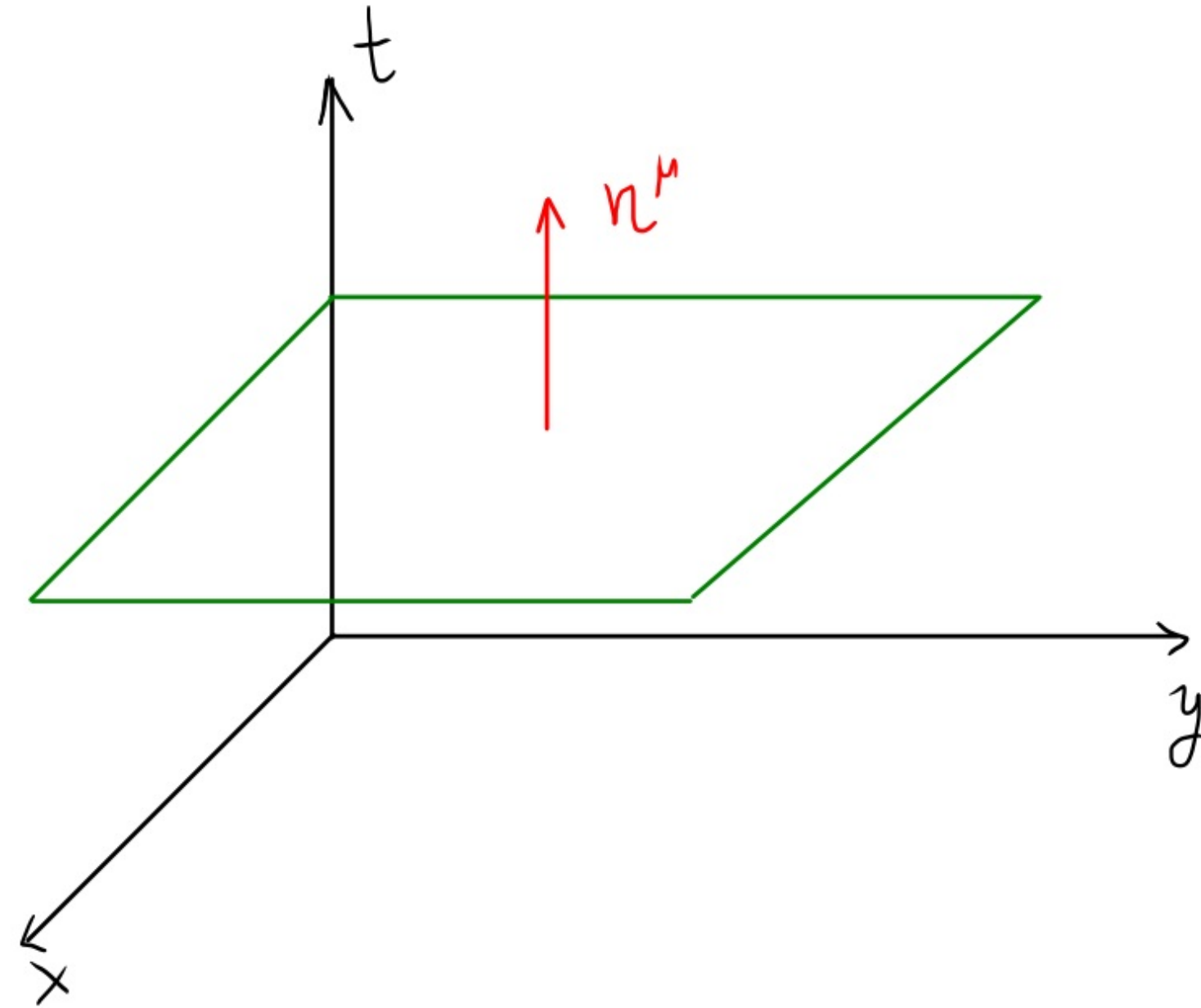
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If  $\Delta V$  at rest,  $n^\mu = (1, 0, 0, 0)$

$$\Rightarrow \Delta p^\mu = + T^{\mu 0} \Delta V \quad n_\mu = (-1, 0, 0, 0)$$





# Stress - Energy Tensor in Special Relativity

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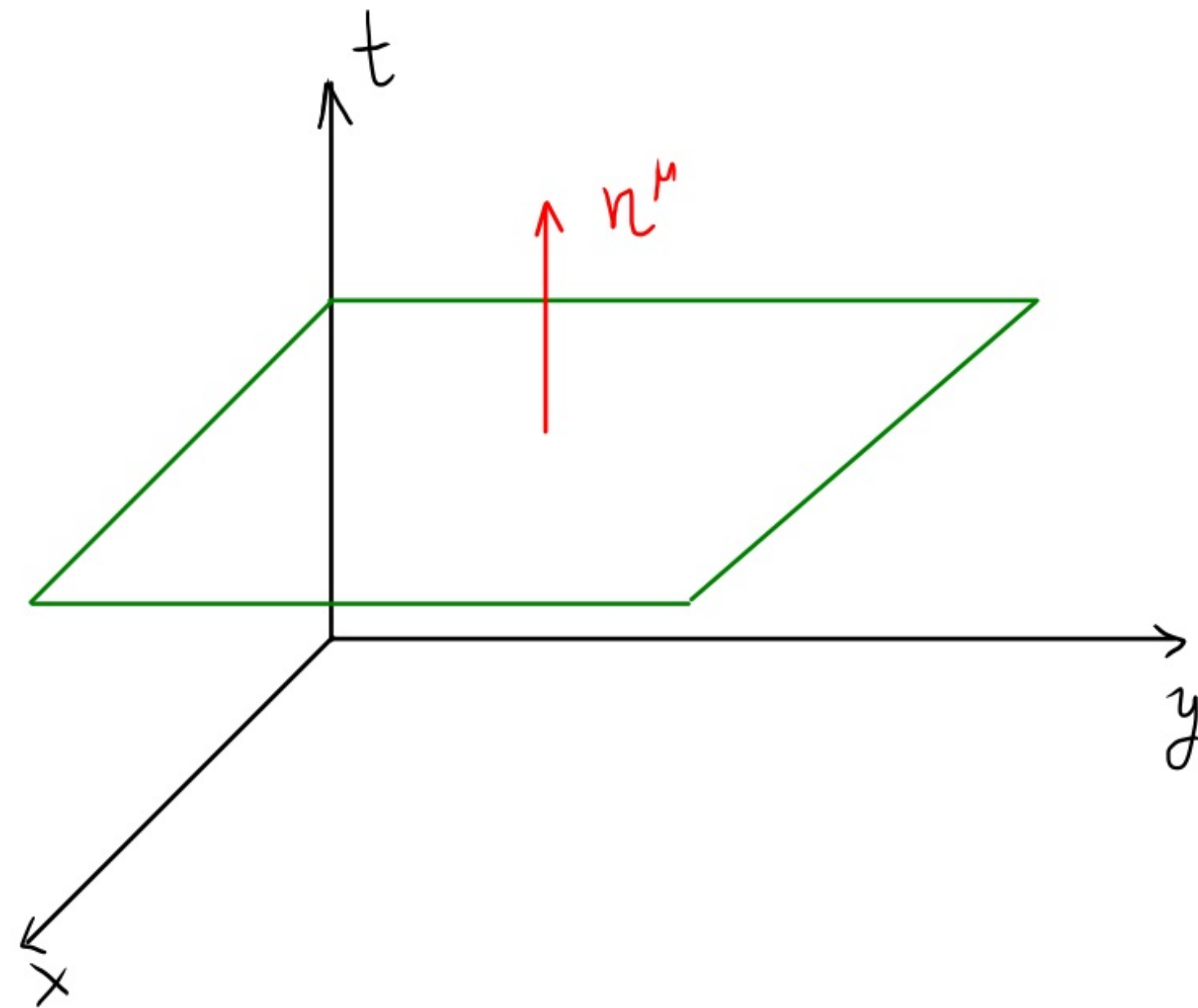
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If  $\Delta V$  at rest,  $n^\mu = (1, 0, 0, 0)$

$$\Rightarrow \Delta p^\mu = + T^{\mu 0} \Delta V \quad n_\mu = (-1, 0, 0, 0)$$

$$\rho = \frac{\Delta p^0}{\Delta V} = T^{00} \quad \text{energy density}$$

$$\pi^i = \frac{\Delta p^i}{\Delta V} = T^{i0} \quad \text{momentum density}$$

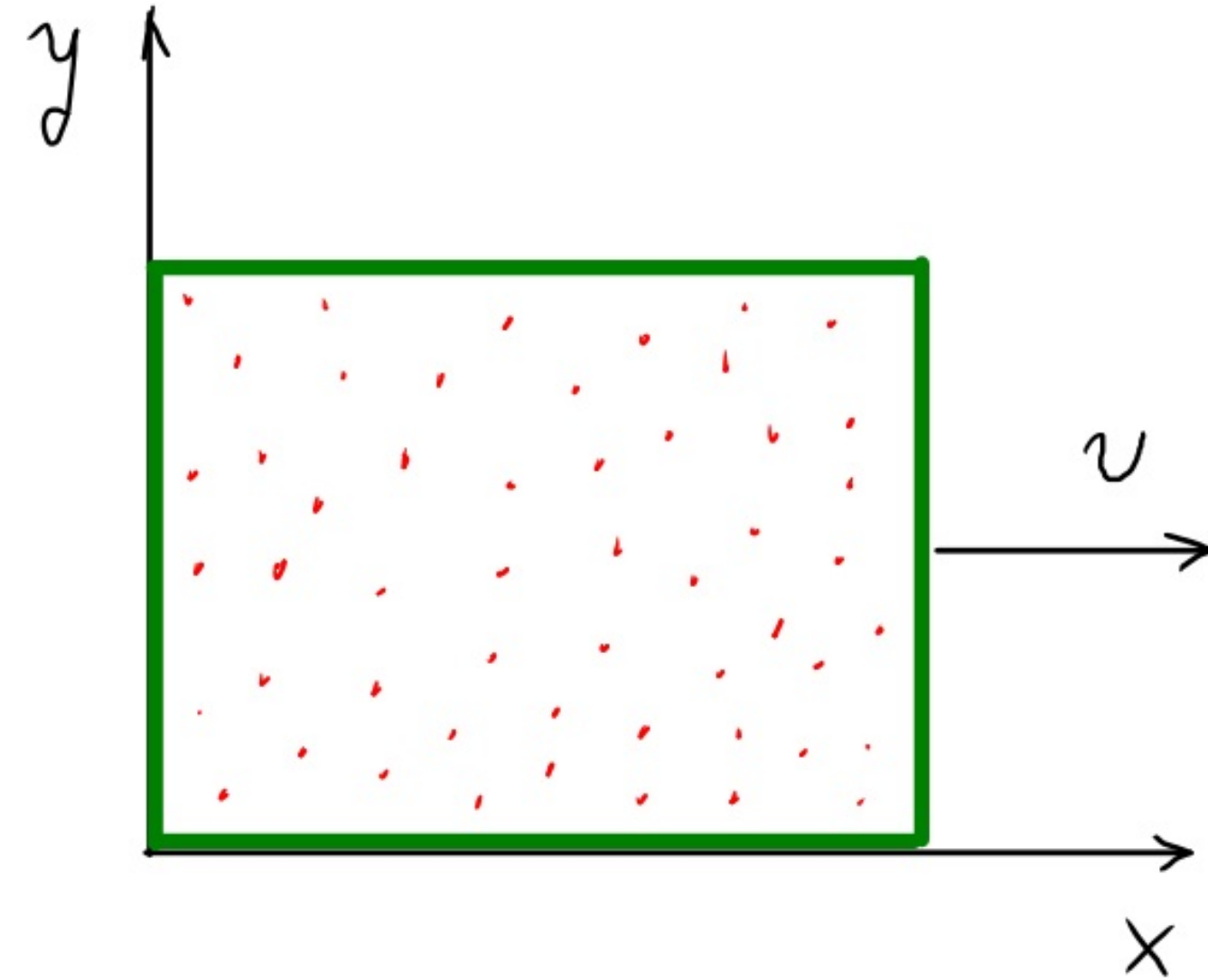


# Stress - Energy Tensor in Special Relativity

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Example:

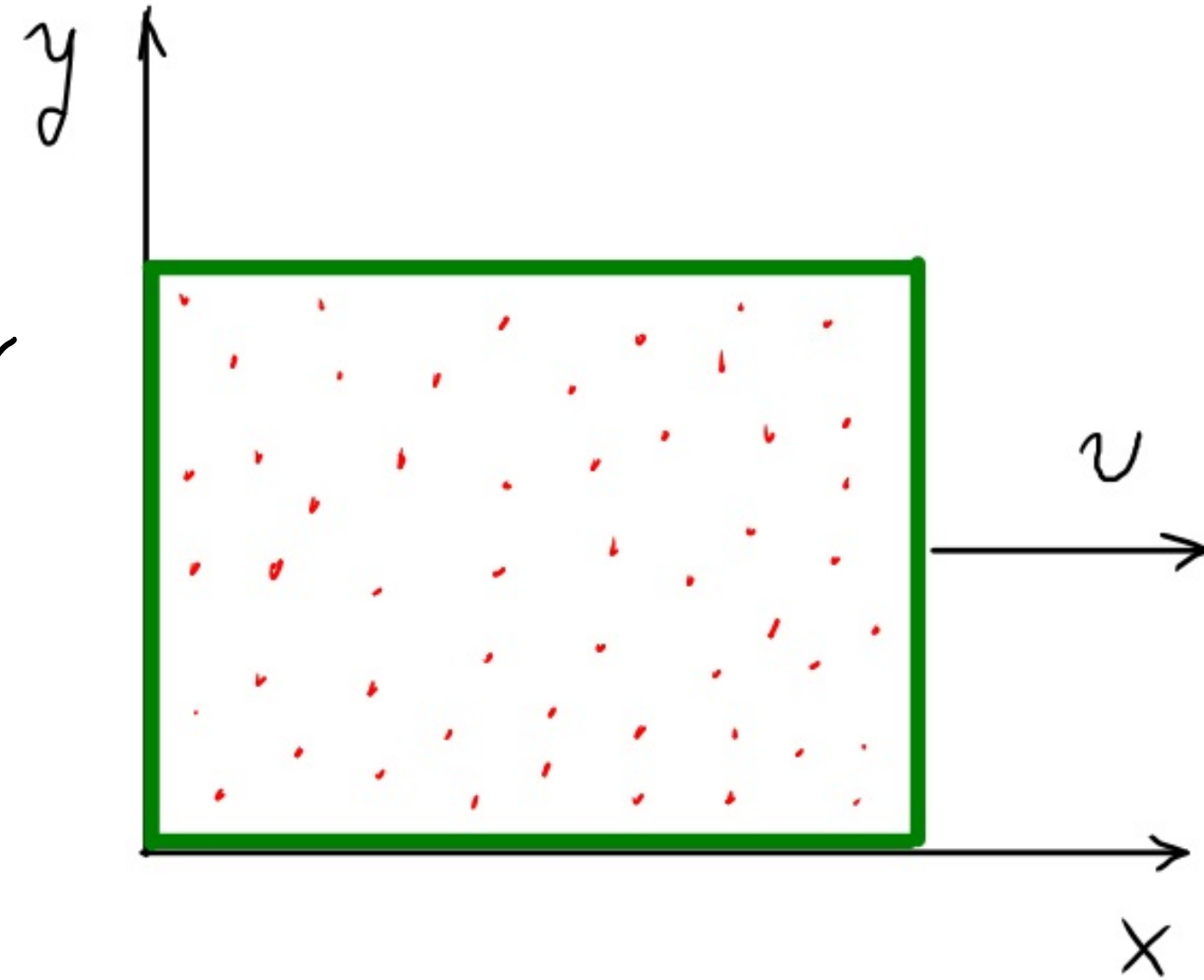
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# Stress - Energy Tensor in Special Relativity

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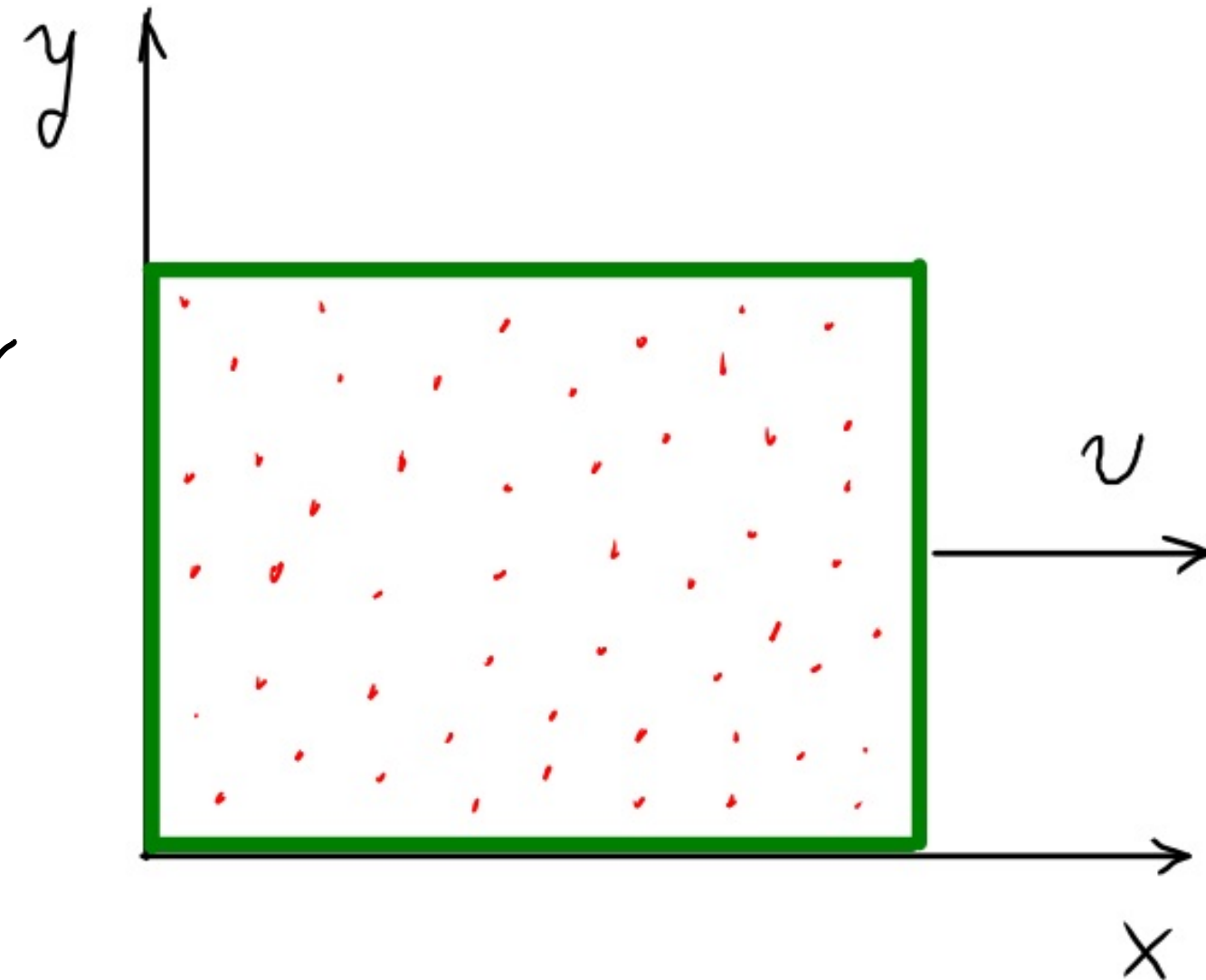


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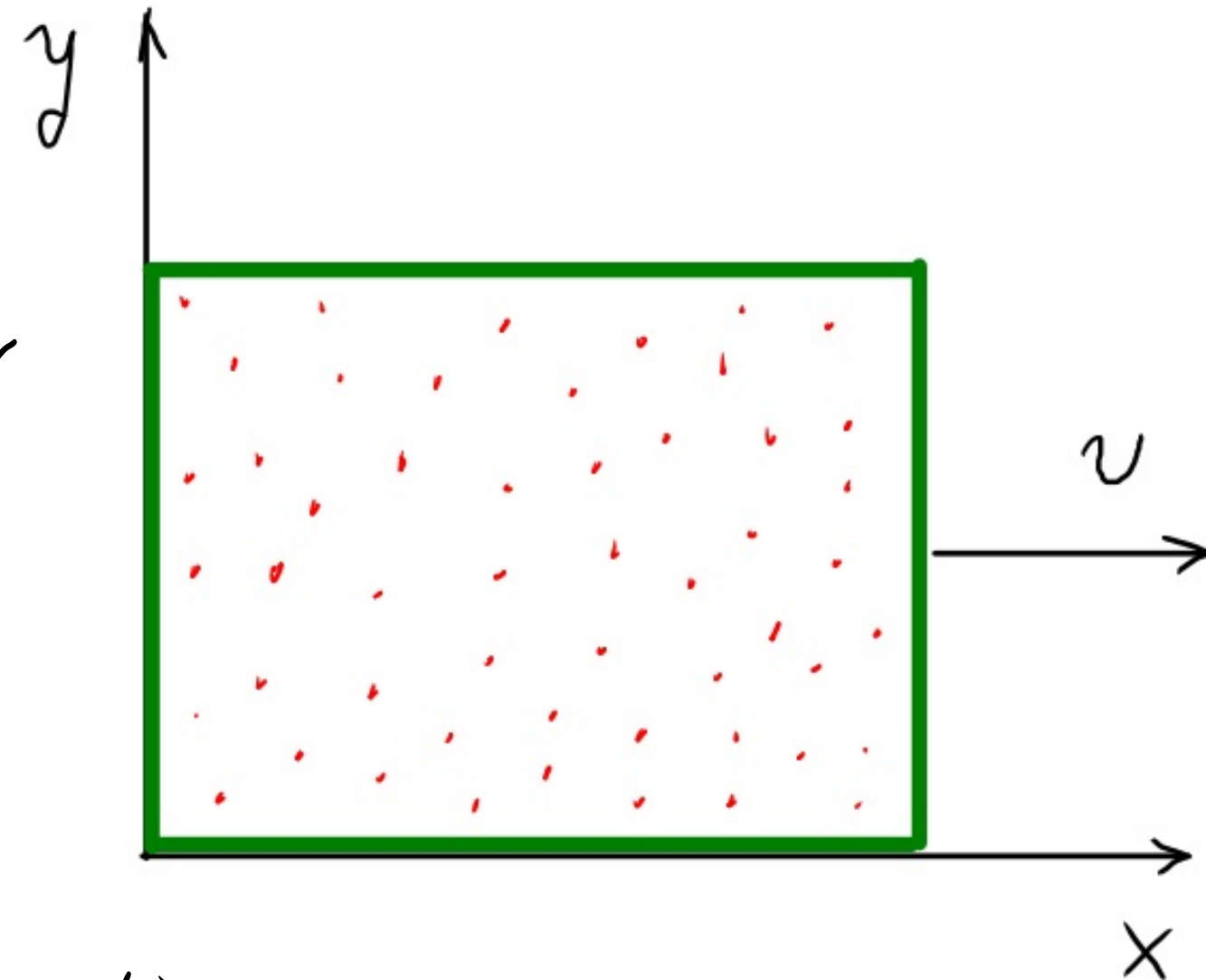
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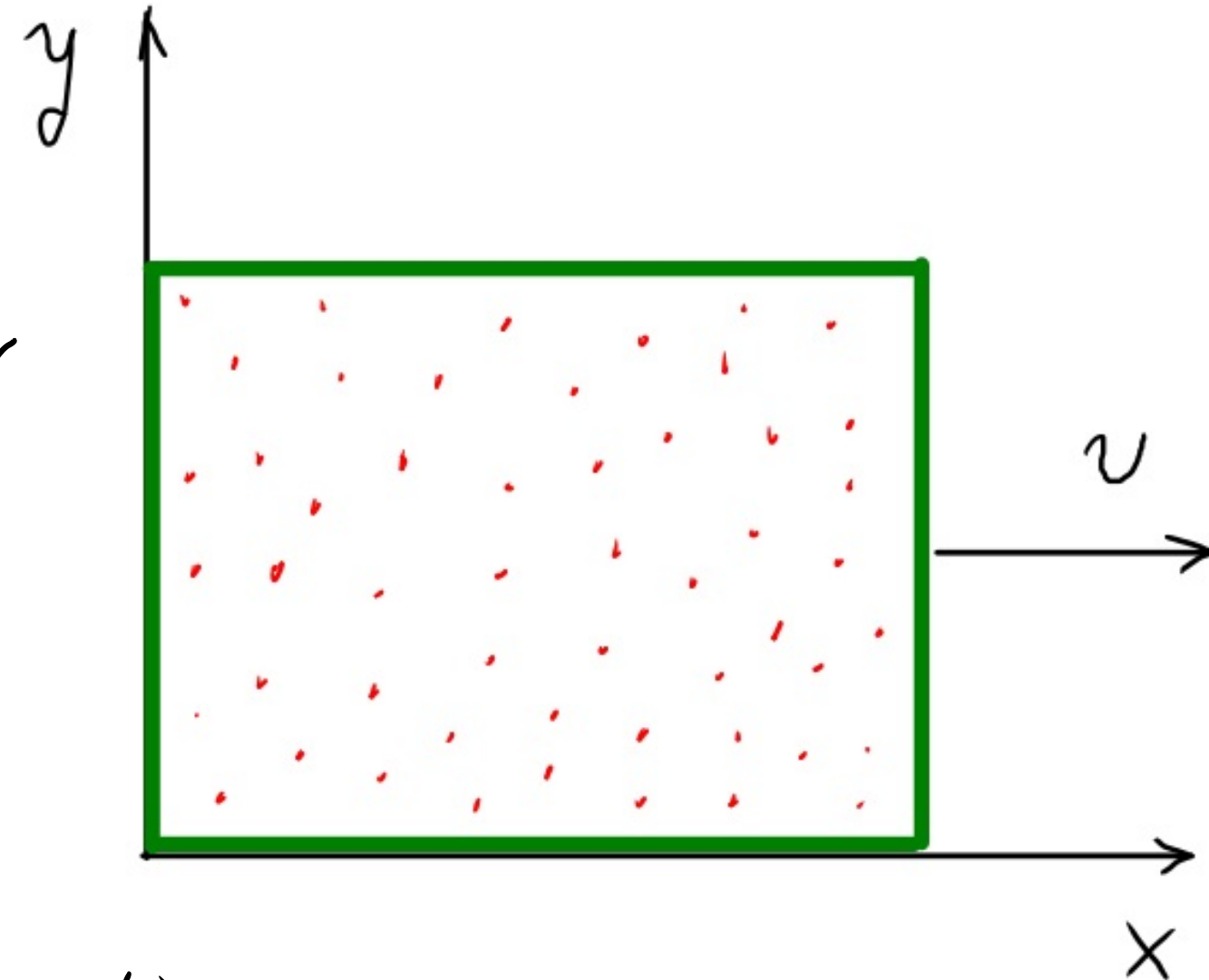
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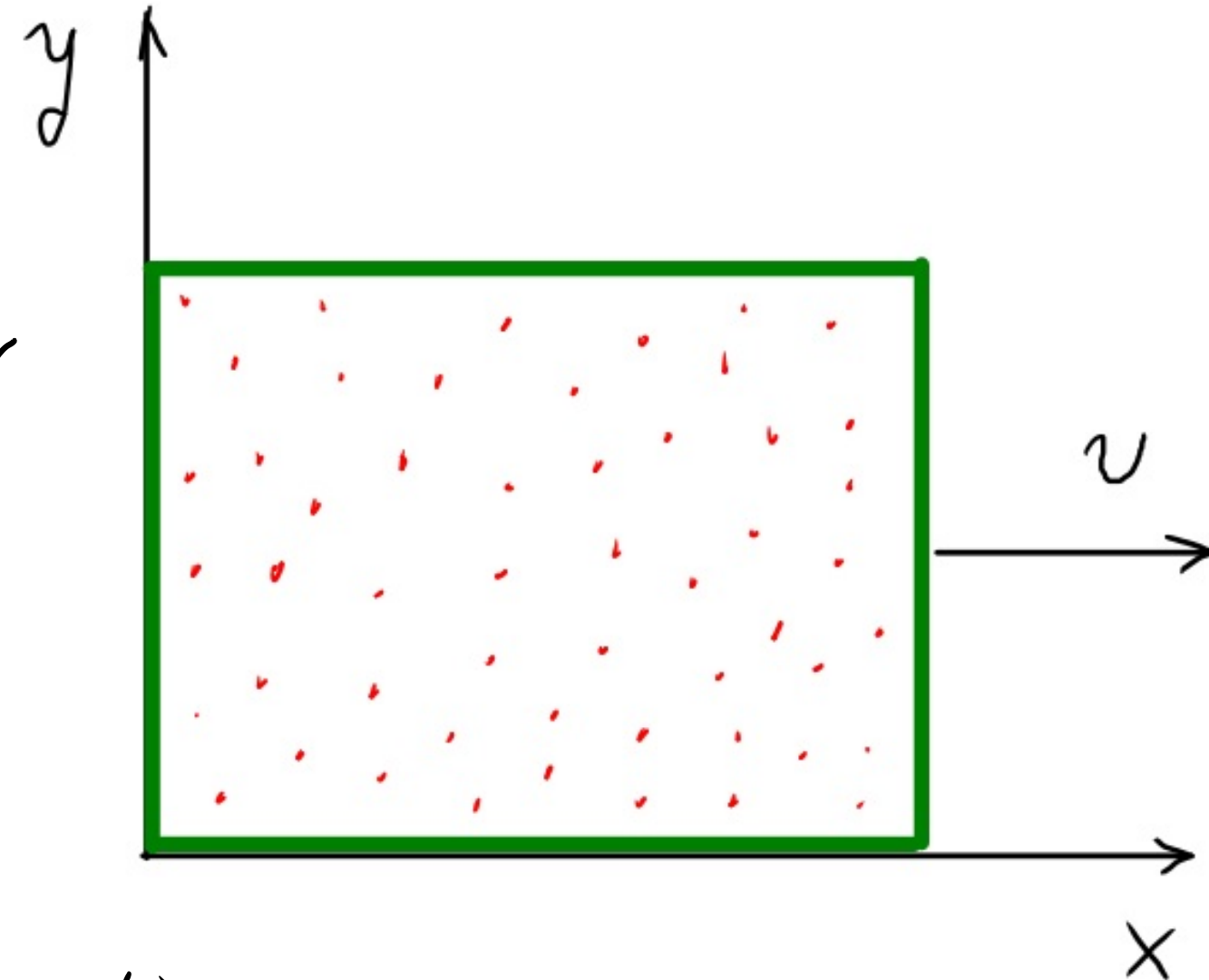
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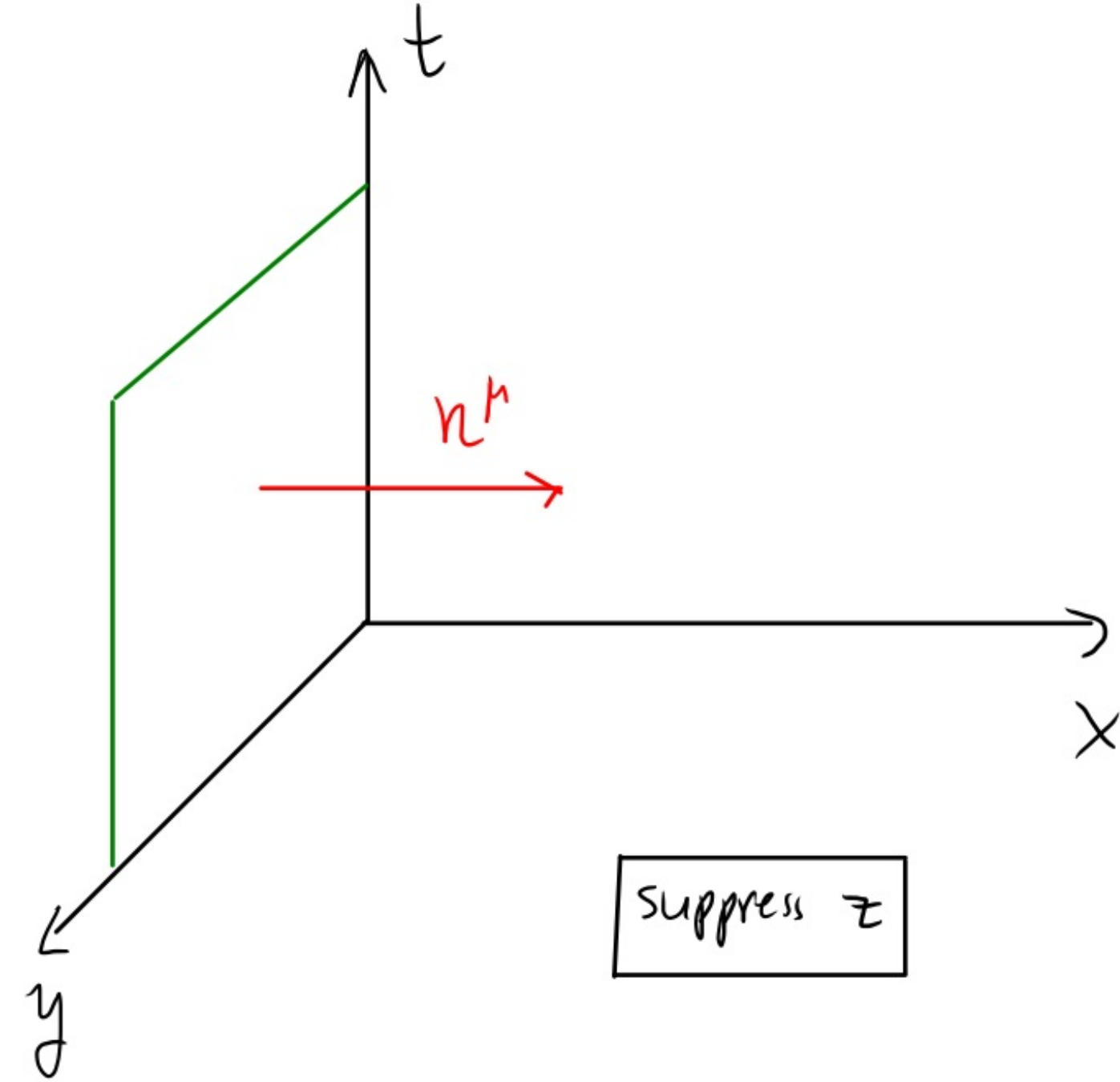
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Now consider  $T^{mj}$ , e.g.  $T^{m1}$

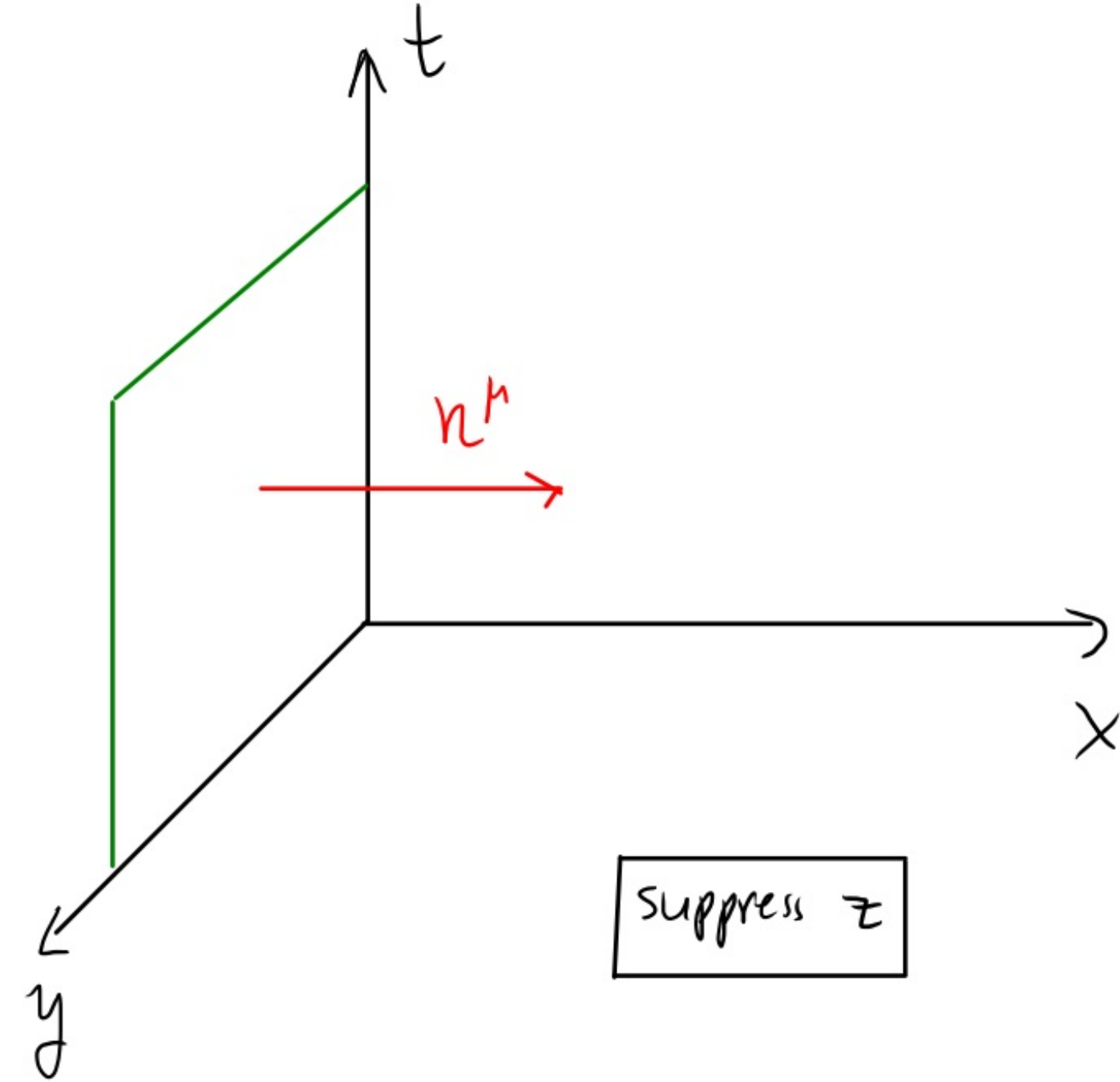


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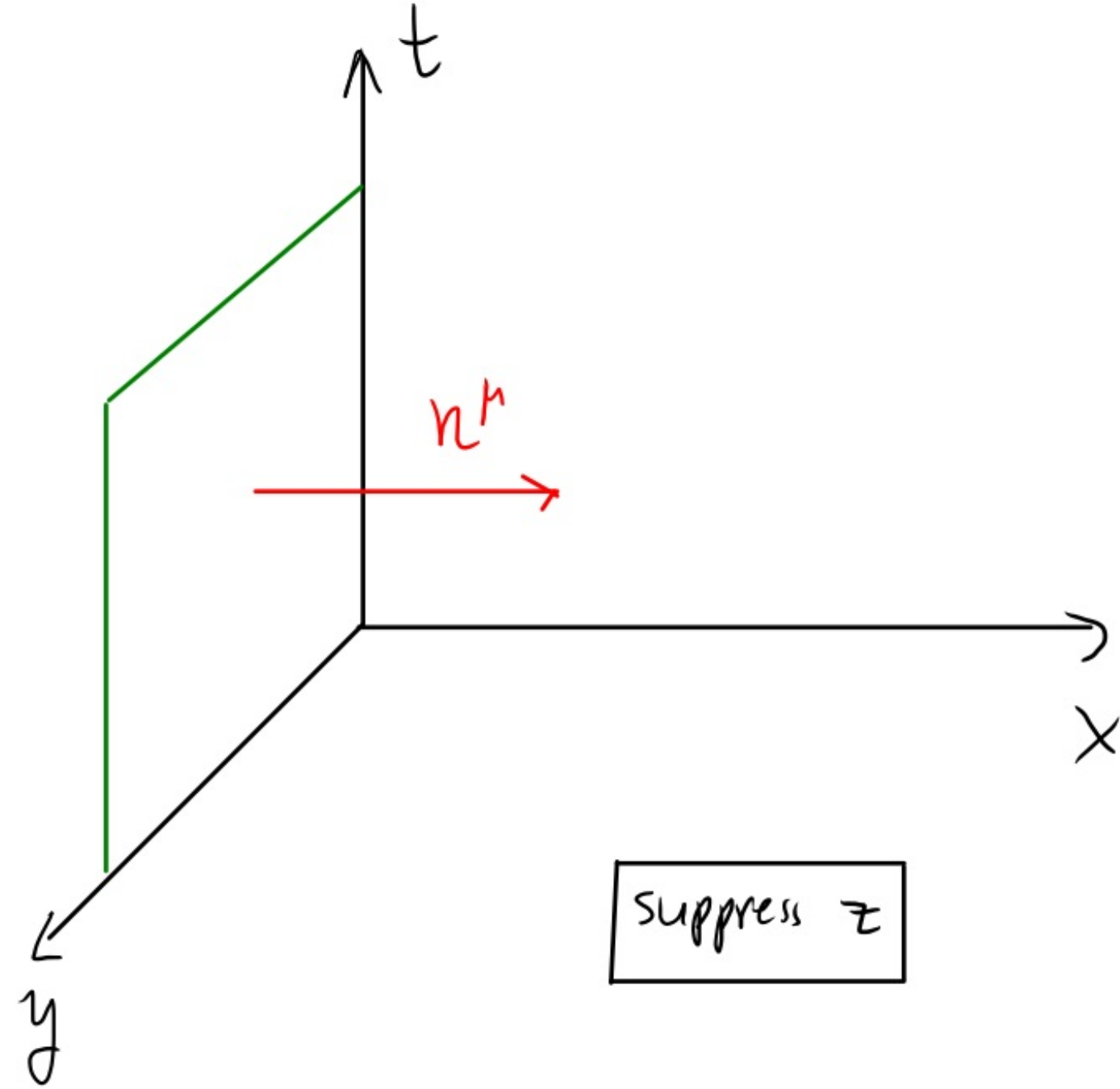
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$\Delta V$  of 3-surface



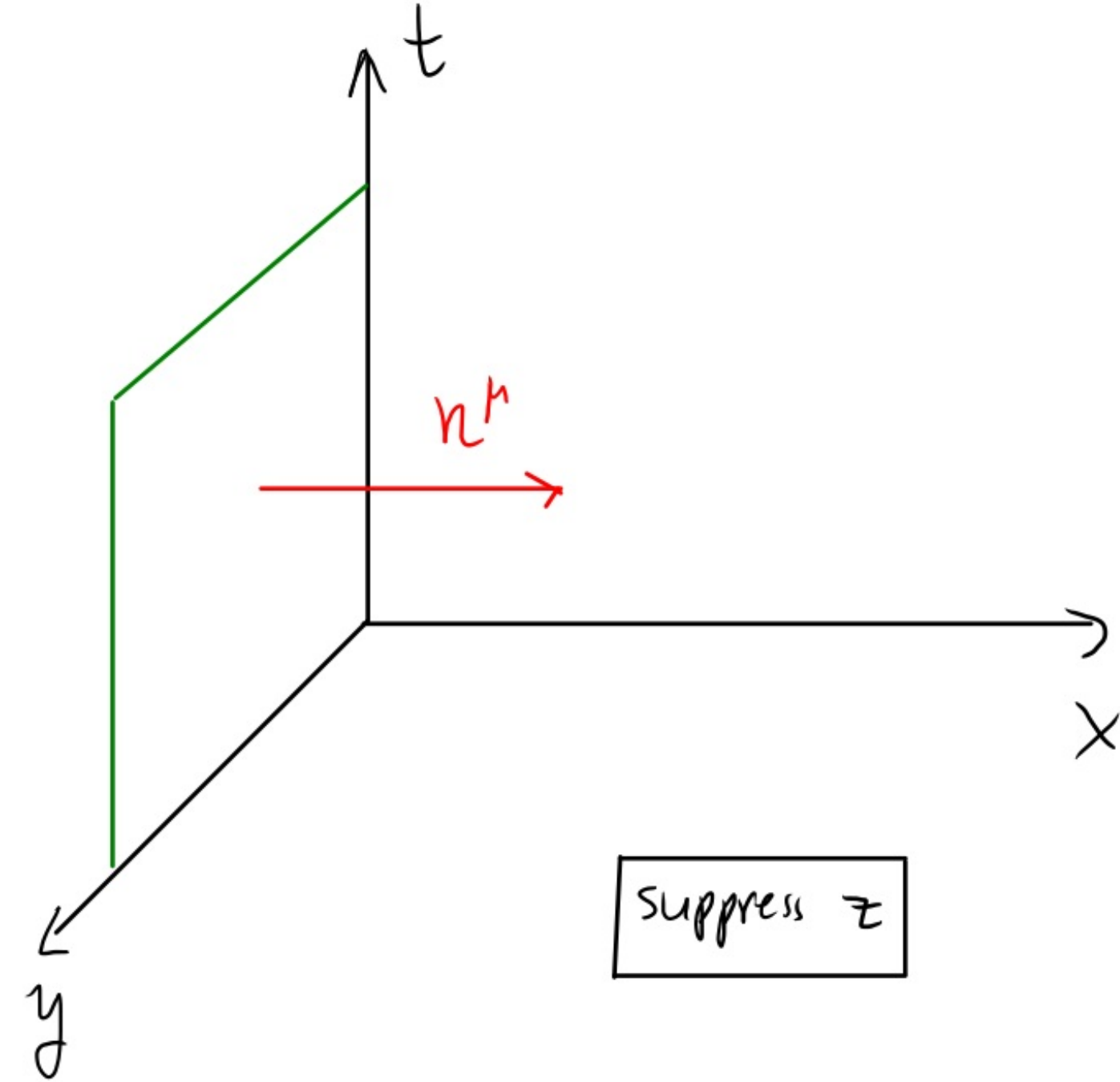
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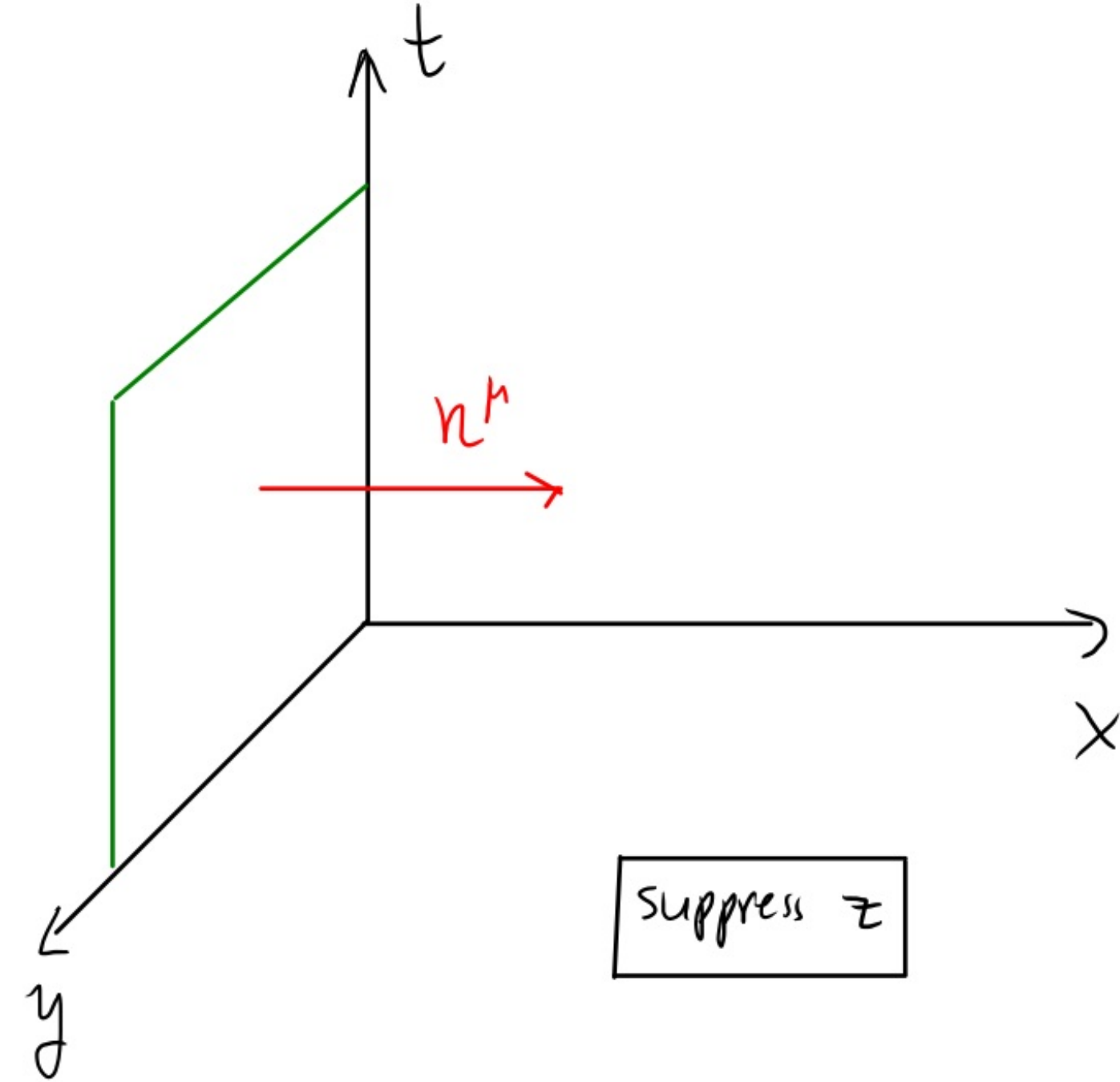
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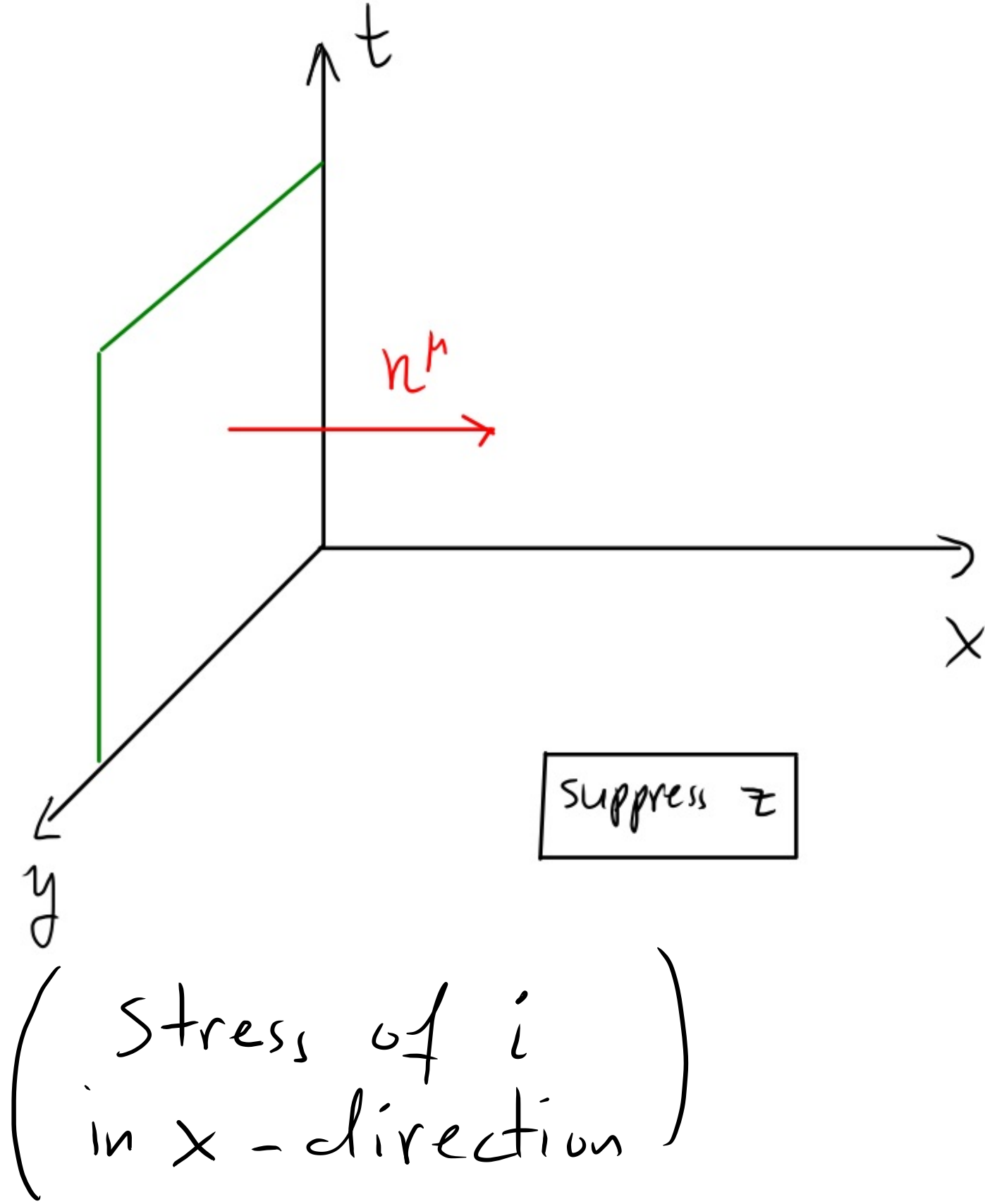
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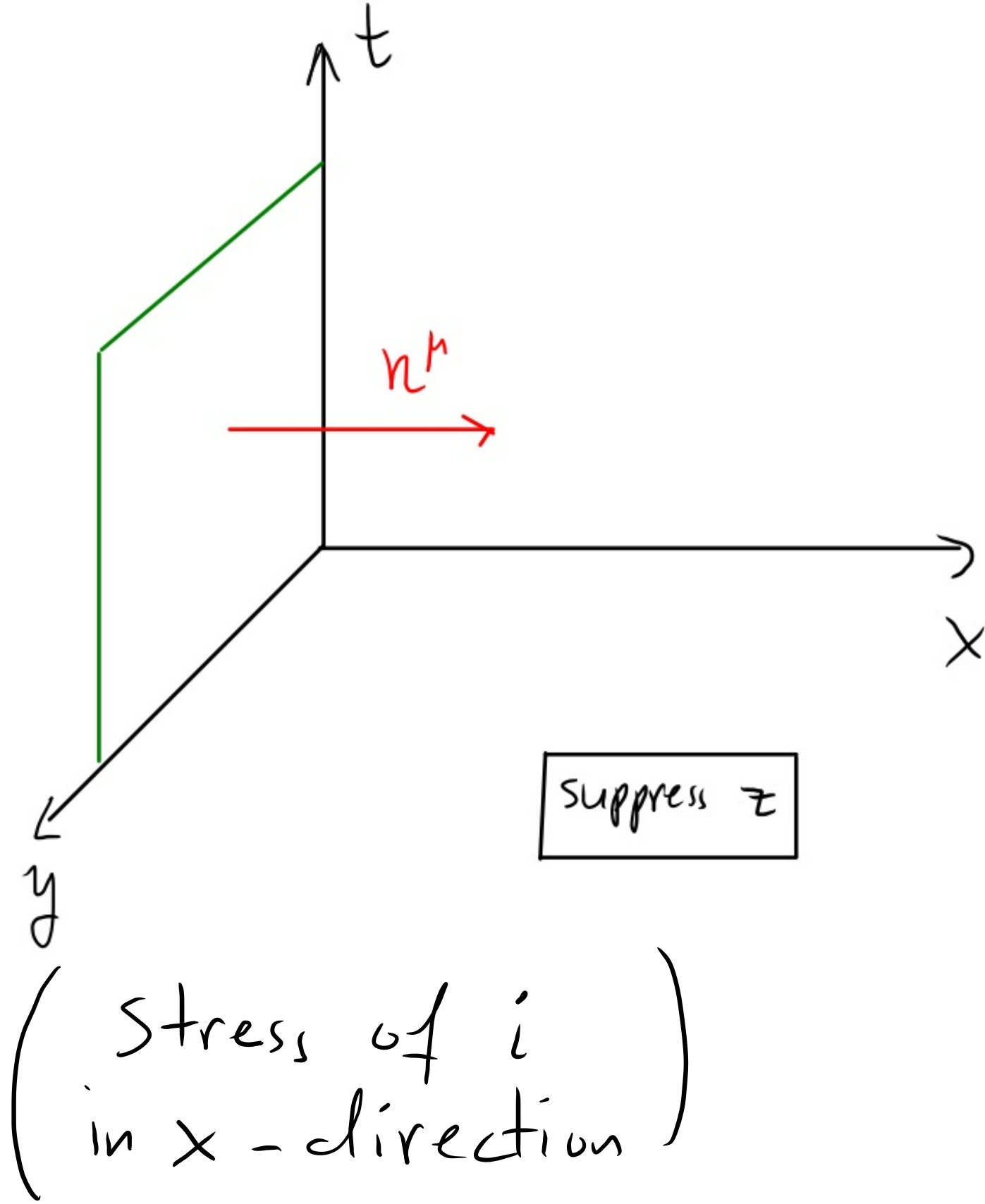
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$\Rightarrow T^{ij}$  stress tensor



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$$T^{\mu\nu} = T^{\nu\mu} \Rightarrow (\text{energy flux}) = (\text{momentum density})$$

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conserved!  
currents



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$$4\pi T^{00} = 4\pi \rho = \frac{1}{2} (\mathcal{E}^2 + \mathcal{B}^2)$$

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$$4\pi T^{ij} = \left( -\mathcal{E}^i \mathcal{E}^j + \frac{1}{2} \delta^{ij} \mathcal{E}^2 \right) + \left( -\mathcal{B}^i \mathcal{B}^j + \frac{1}{2} \delta^{ij} \mathcal{B}^2 \right)$$

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$$\parallel$$
$$\frac{d}{dt} \int_V \pi^i dV = \frac{dP^i}{dt}$$

$$\parallel$$
$$+ \int T^{ij} (-dA_j)$$

$\parallel$   
(force exerted on  
volume by outside  
stresses)

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perfect = negligible

- heat conduction
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At inertial frame:

$$T^{\mu\nu} = \text{diag}(\rho, P, P, P) = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

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Equation of state:  $P = w \rho$

- $w = 0$  dust
- $w = 1/3$  radiation
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# Perfect Fluids

Non relativistic limit:  $\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0$  continuity eq.

$$\rho [\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{v}] = - \vec{\nabla} p \quad \text{Euler eq.}$$

(see Carroll § 1.9)

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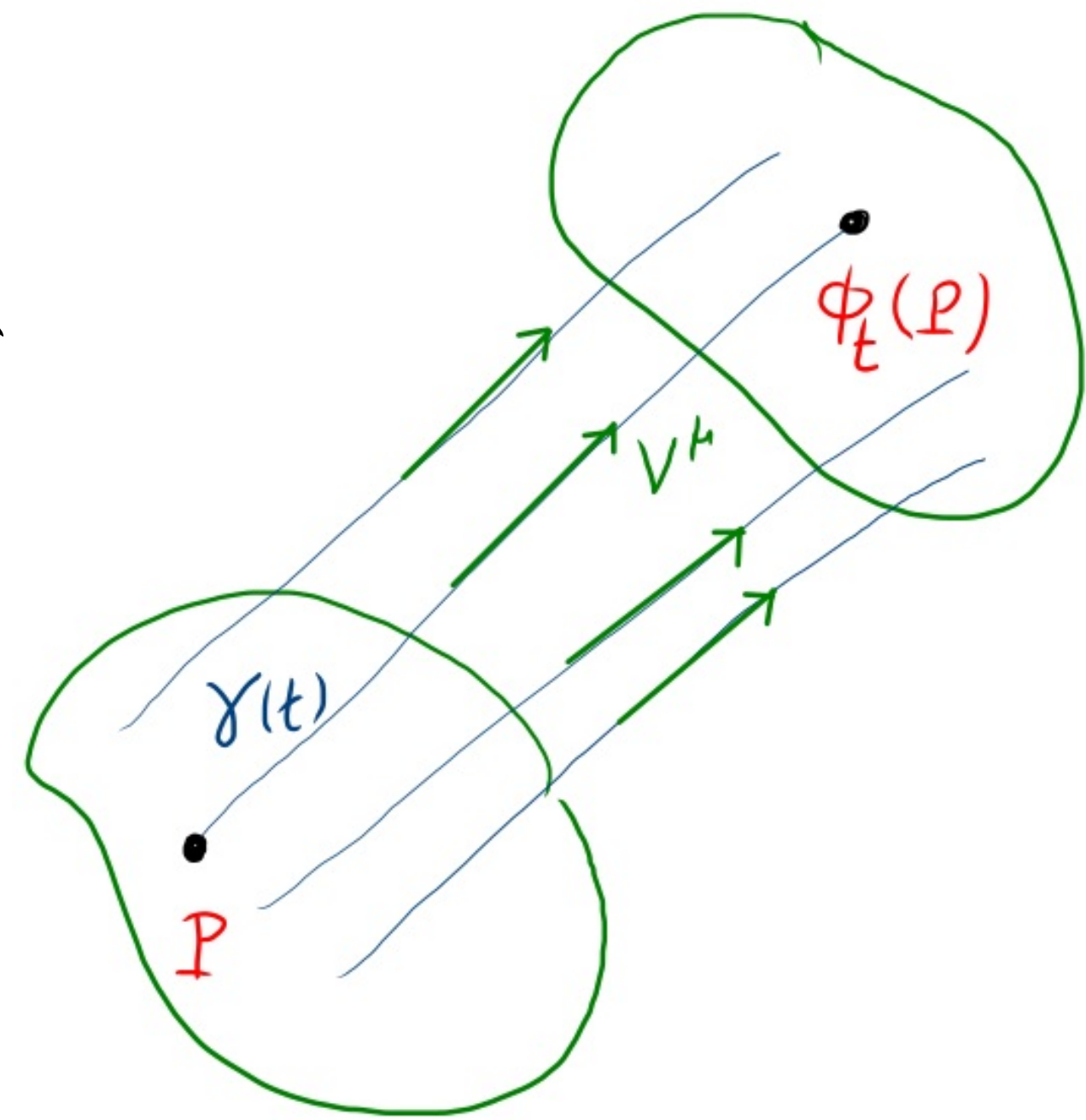
Note: this is a local conservation law

spacetime dynamics  $\leadsto$  no conservation (e.g. CMB in expanding space)

# Conservation Laws + Isometries

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$$P \rightarrow \phi_t(P)$$



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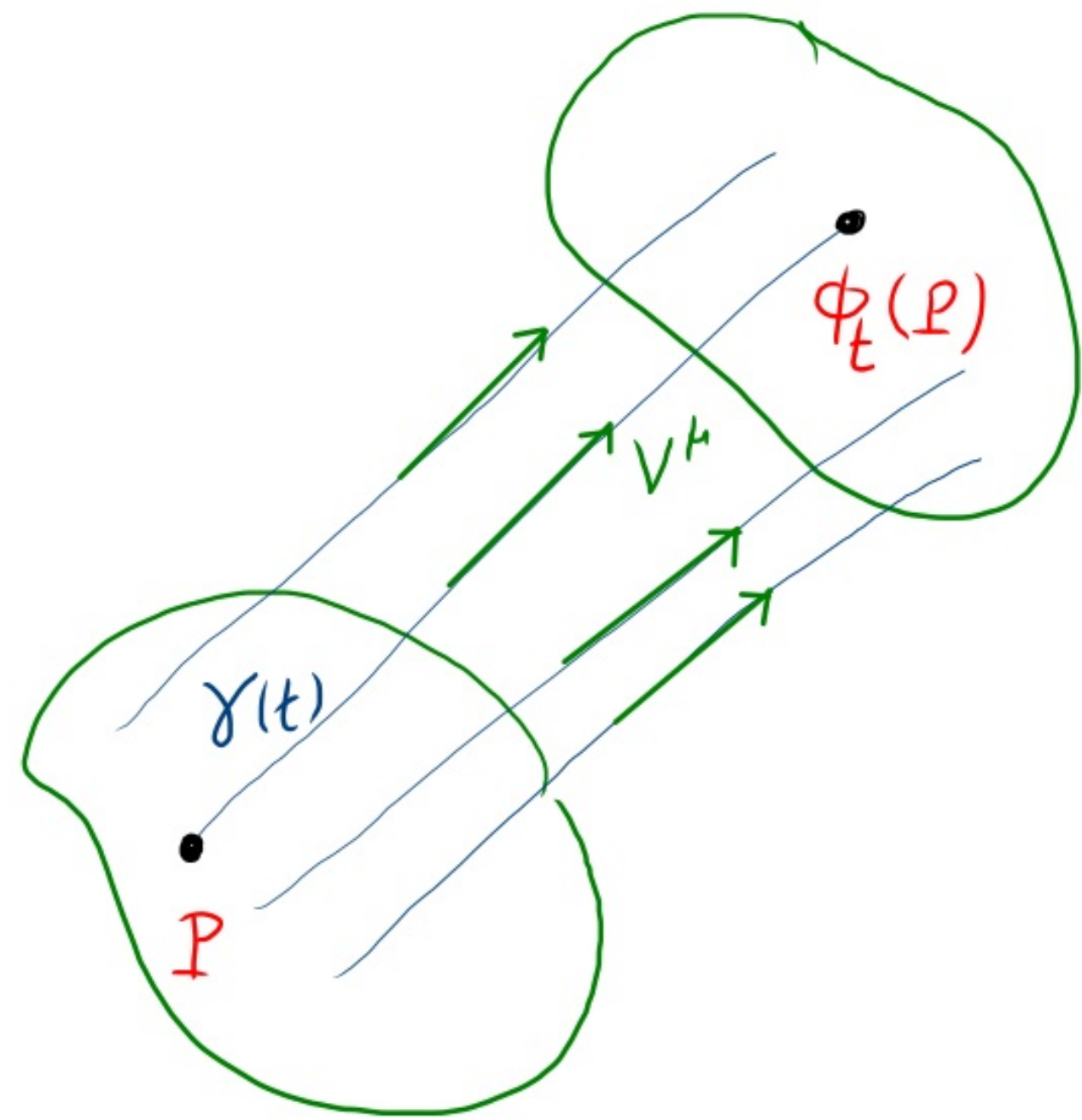
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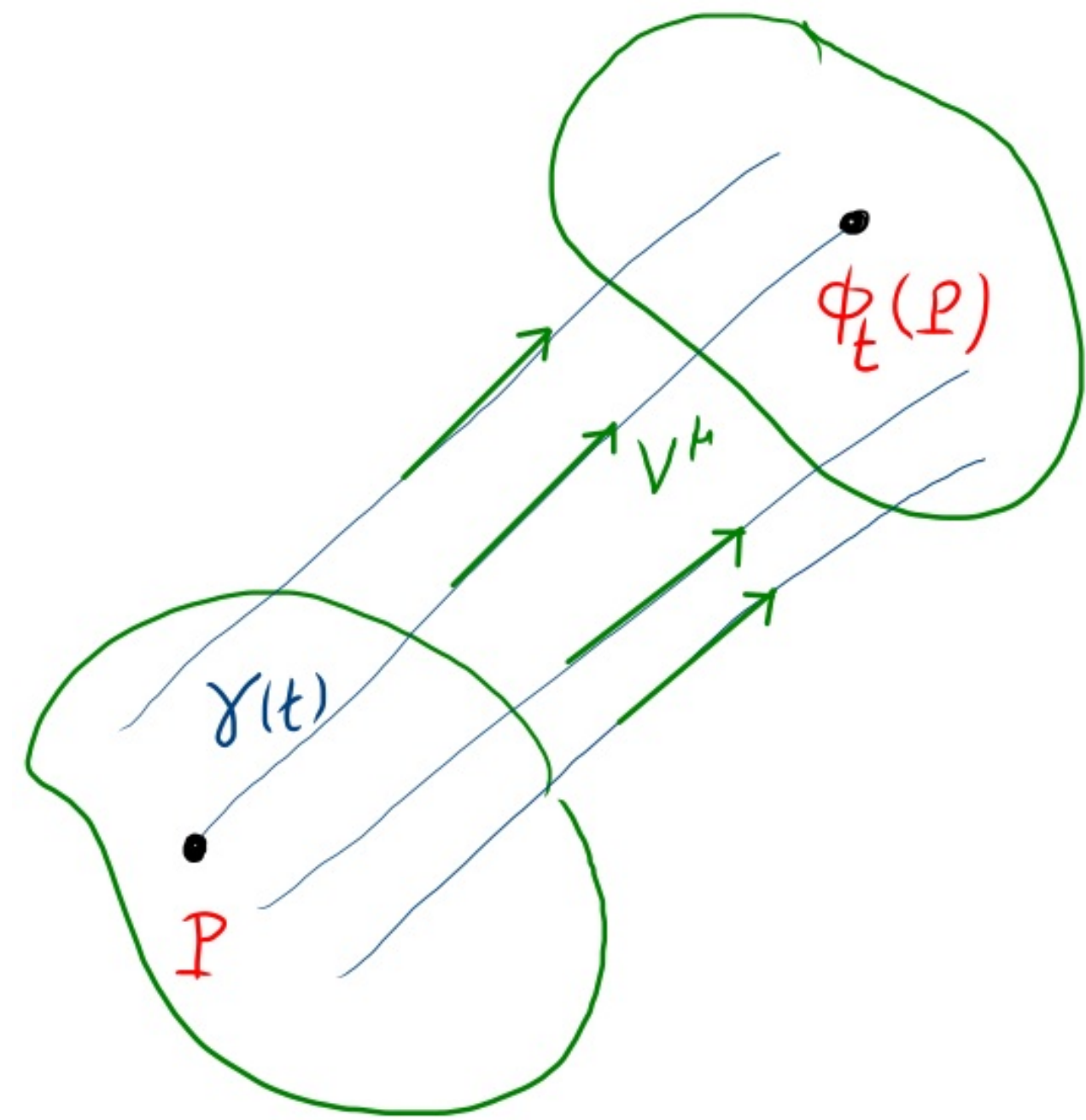
$$\phi_t^* g_{\mu\nu}(x^{\sigma}) = \frac{\partial x^{\mu}(t)}{\partial x^{\rho}} \frac{\partial x^{\nu}(t)}{\partial x^{\sigma}} g_{\rho\sigma}(x^{\sigma})$$



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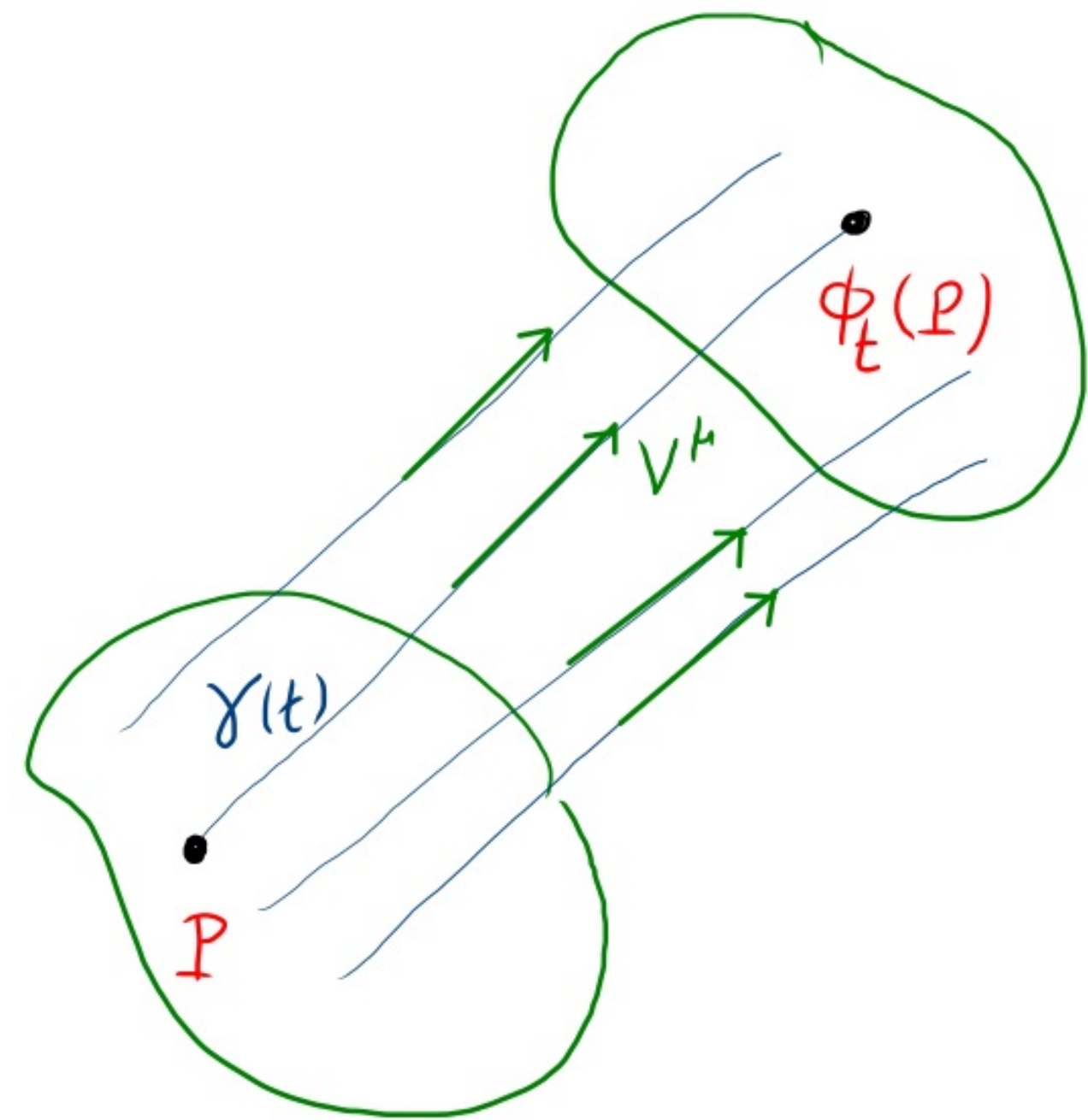
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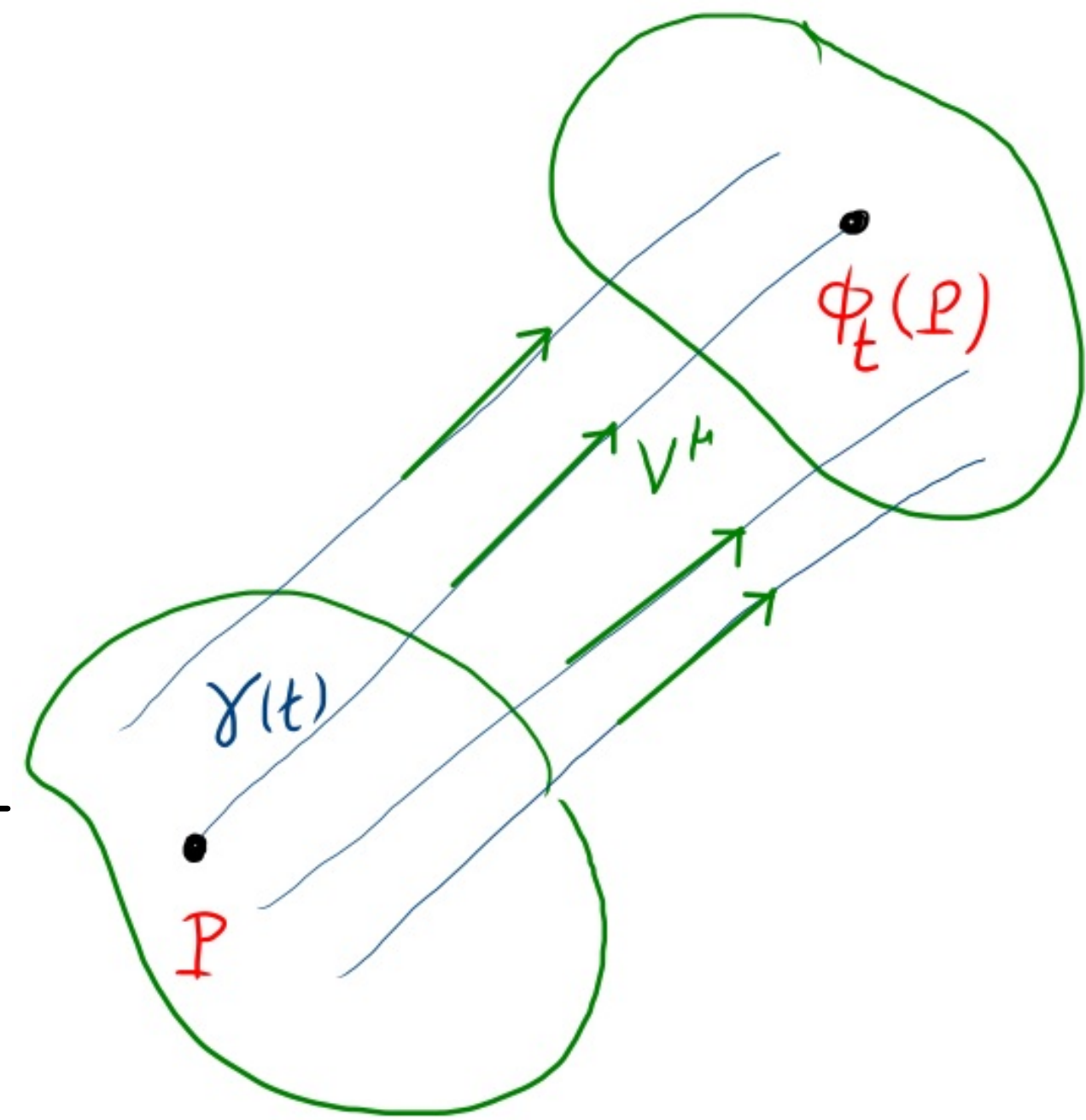
$$\begin{aligned} (\mathcal{L}_V g)_{\mu\nu} &= V^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu} \partial_\mu V^\rho + g_{\mu\rho} \partial_\nu V^\rho \\ &= V^\rho \nabla_\rho g_{\mu\nu} + g_{\rho\nu} \nabla_\mu V^\rho + g_{\mu\rho} \nabla_\nu V^\rho \end{aligned}$$





# Conservation Laws + Isometries

$\phi_t$  is an isometry if  $\phi_t^* g = g$



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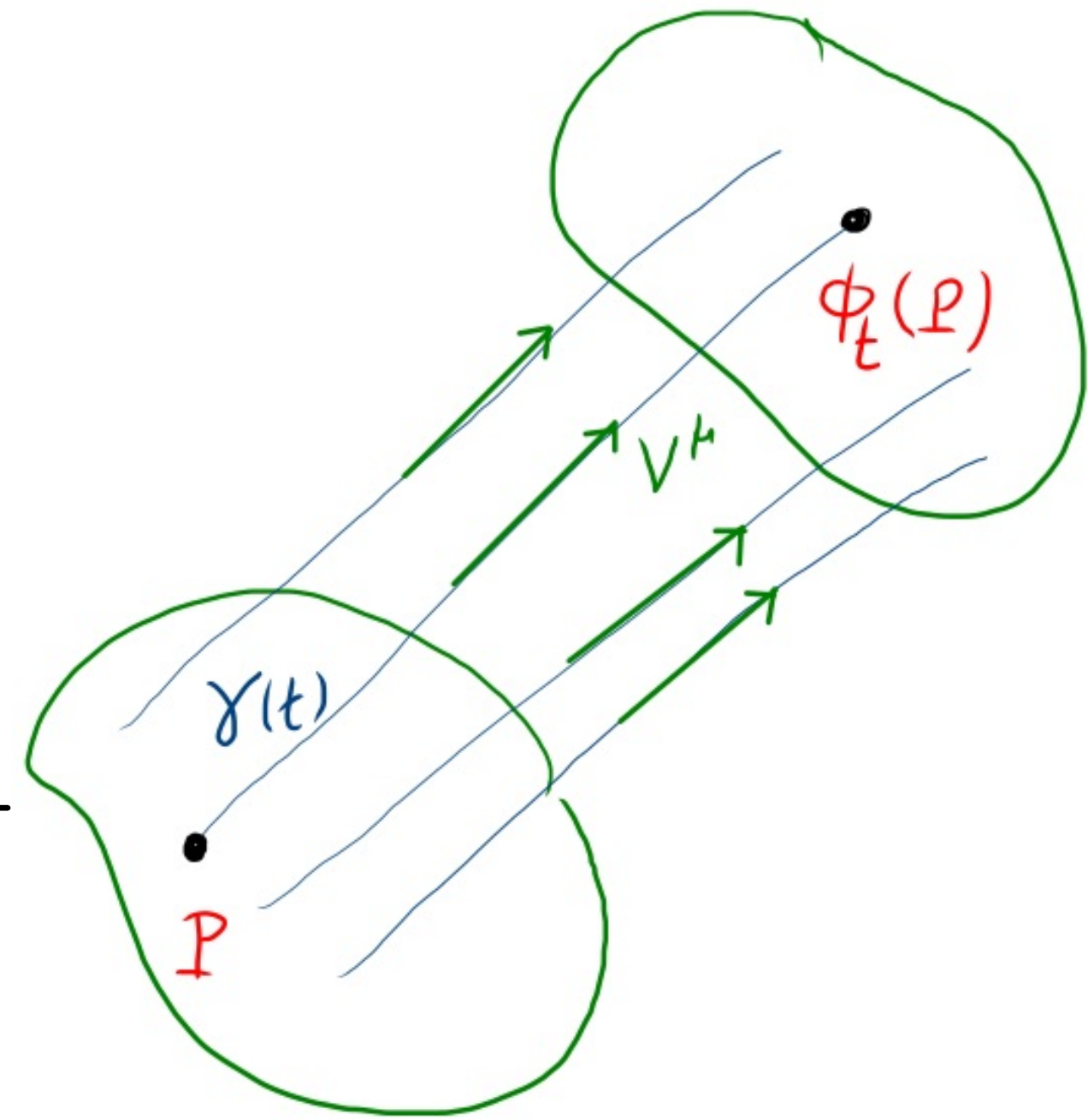
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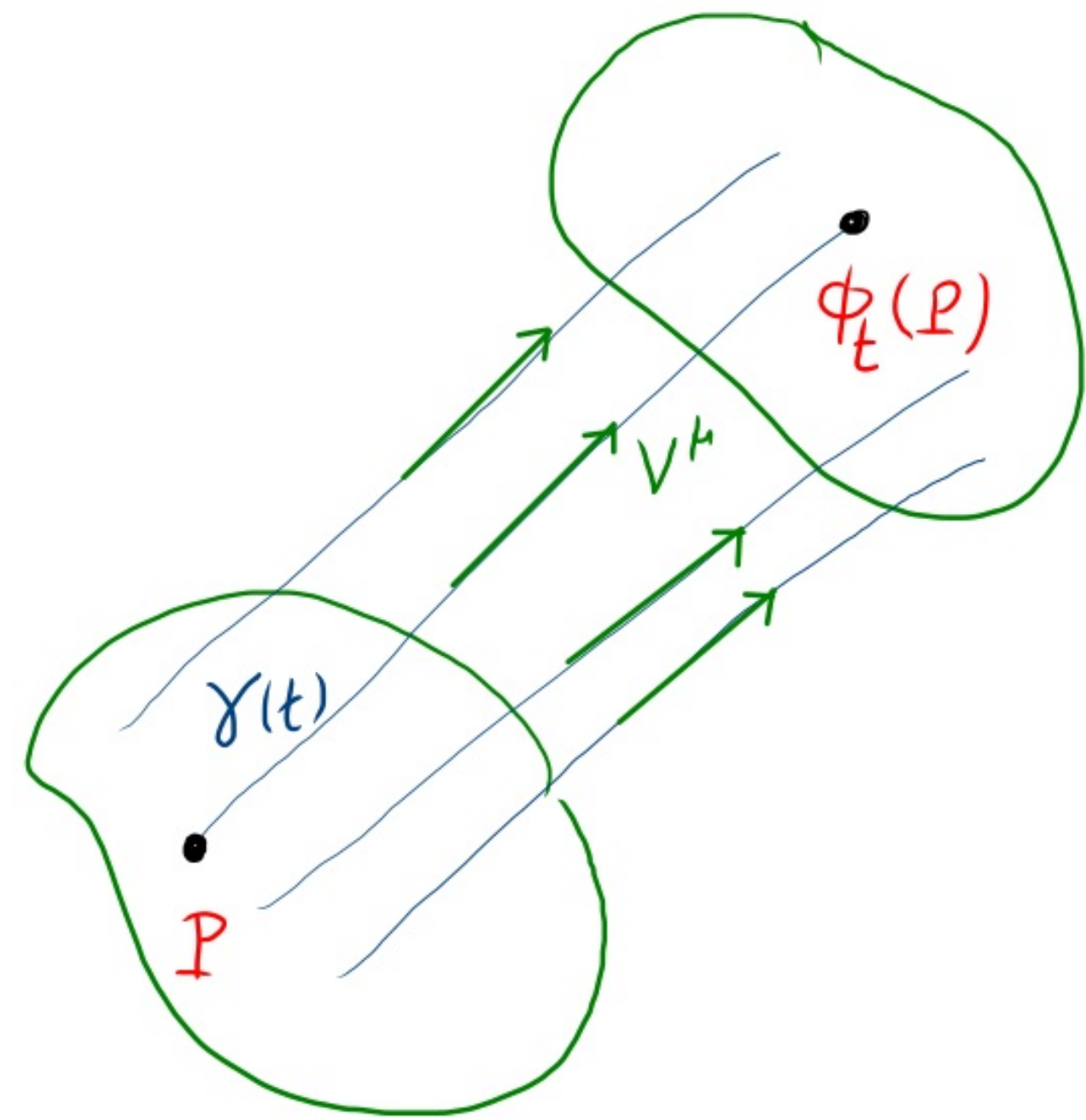
$$\Rightarrow \nabla_\mu V_\nu + \nabla_\nu V_\mu = 0$$

$$(\mathcal{L}_V g)_{\mu\nu} = V^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu} \partial_\mu V^\rho + g_{\mu\rho} \partial_\nu V^\rho$$

$$= \cancel{V^\rho \nabla_\rho g_{\mu\nu}} + g_{\rho\nu} \nabla_\mu V^\rho + g_{\mu\nu} \nabla_\nu V^\rho$$

metric  
compatibility

$$\nabla_\mu (g_{\rho\nu} V^\rho) = \nabla_\mu V_\nu$$



$$\nabla_\nu (g_{\mu\rho} V^\rho) = \nabla_\nu V_\mu$$



# Conservation Laws + Isometries

If  $\xi^\mu$  generates an isometry

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Because  $\mathcal{L}_\zeta \chi = [\zeta, \chi]$ , and  $\mathcal{L}_{[\zeta, \chi]} = [\mathcal{L}_\zeta, \mathcal{L}_\chi]$ , so

$$\mathcal{L}_{[\zeta, \chi]} g = [\mathcal{L}_\zeta, \mathcal{L}_\chi] g = \mathcal{L}_\zeta(\cancel{\mathcal{L}_\chi g}) - \mathcal{L}_\chi(\cancel{\mathcal{L}_\zeta g}) = 0$$

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In that coordinate system  $\xi^\mu = \delta^\mu_\sigma$ , and

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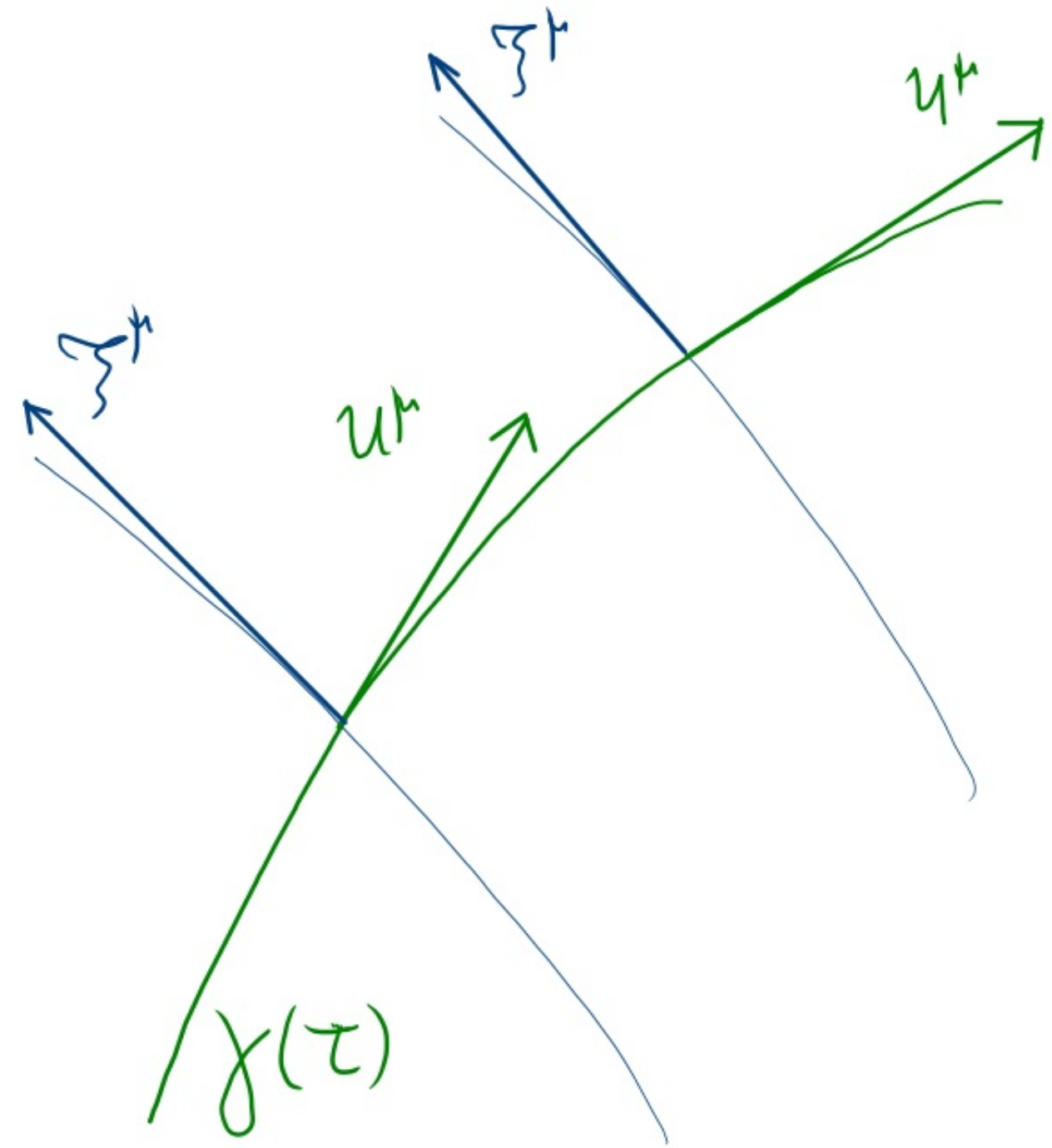
$$\mathcal{L}_\mathcal{L} g_{\mu\nu} = \mathcal{L}^\lambda \partial_\nu g_{\mu\lambda} + g_{\mu\lambda} \partial_\nu \mathcal{L}^\lambda + g_{\lambda\nu} \partial_\mu \mathcal{L}^\lambda$$

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$$= \partial_\sigma g_{\mu\nu} = 0 \Rightarrow \mathcal{L}^\mu \text{ a KVF s.t. } \nabla_\mu \mathcal{L}_\nu + \nabla_\nu \mathcal{L}_\mu = 0$$

# Conserved Quantities Along Geodesics

- Let:
- $\gamma(\tau)$  a geodesic
  - $\tau$  affine parameter
  - $u^\mu$  tangent vector
  - $\xi^\mu$  a KVF

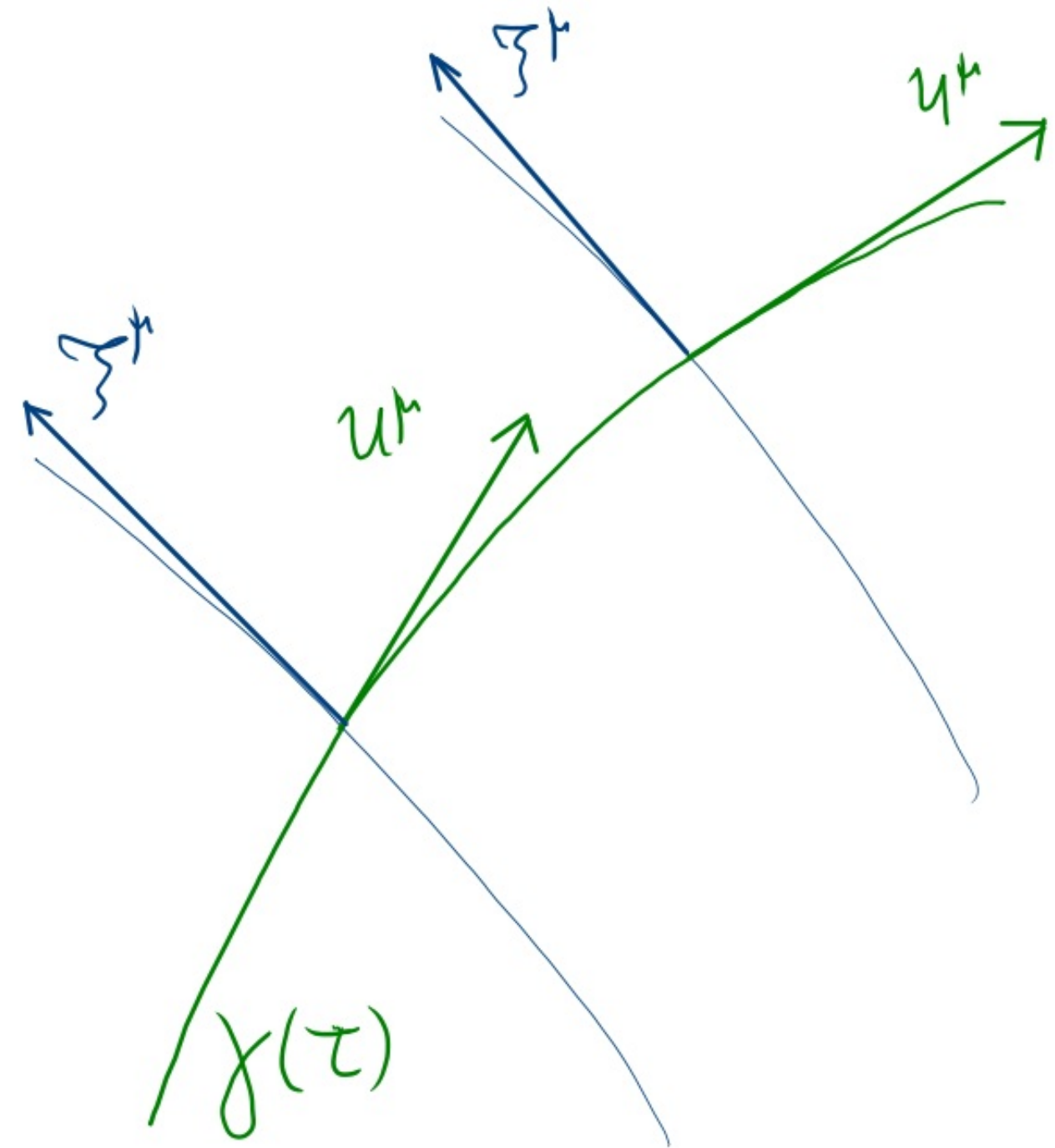


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$$\Rightarrow \frac{d}{d\tau} (\zeta^\mu u_\mu) = 0$$

$\Rightarrow \zeta^\mu u_\mu$  conserved along geodesic



# Conserved Quantities Along Geodesics

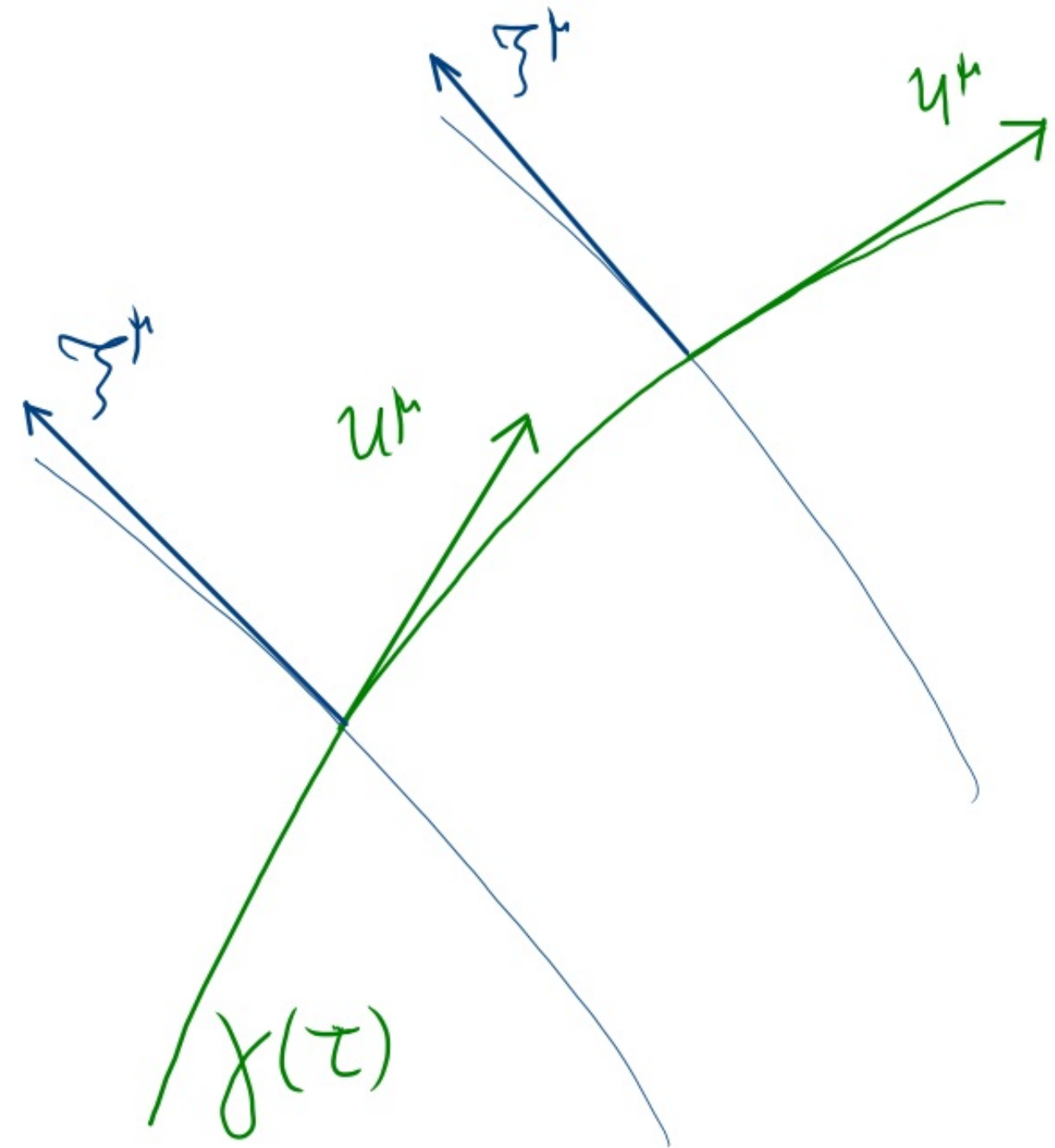
Indeed:

$$\frac{d}{dz}(\xi^\mu u_\mu) = D_u(\xi^\mu u_\mu) = u^\nu \nabla_\nu(\xi^\mu u_\mu)$$

a function:  $\frac{d}{dz} = D_u$

---

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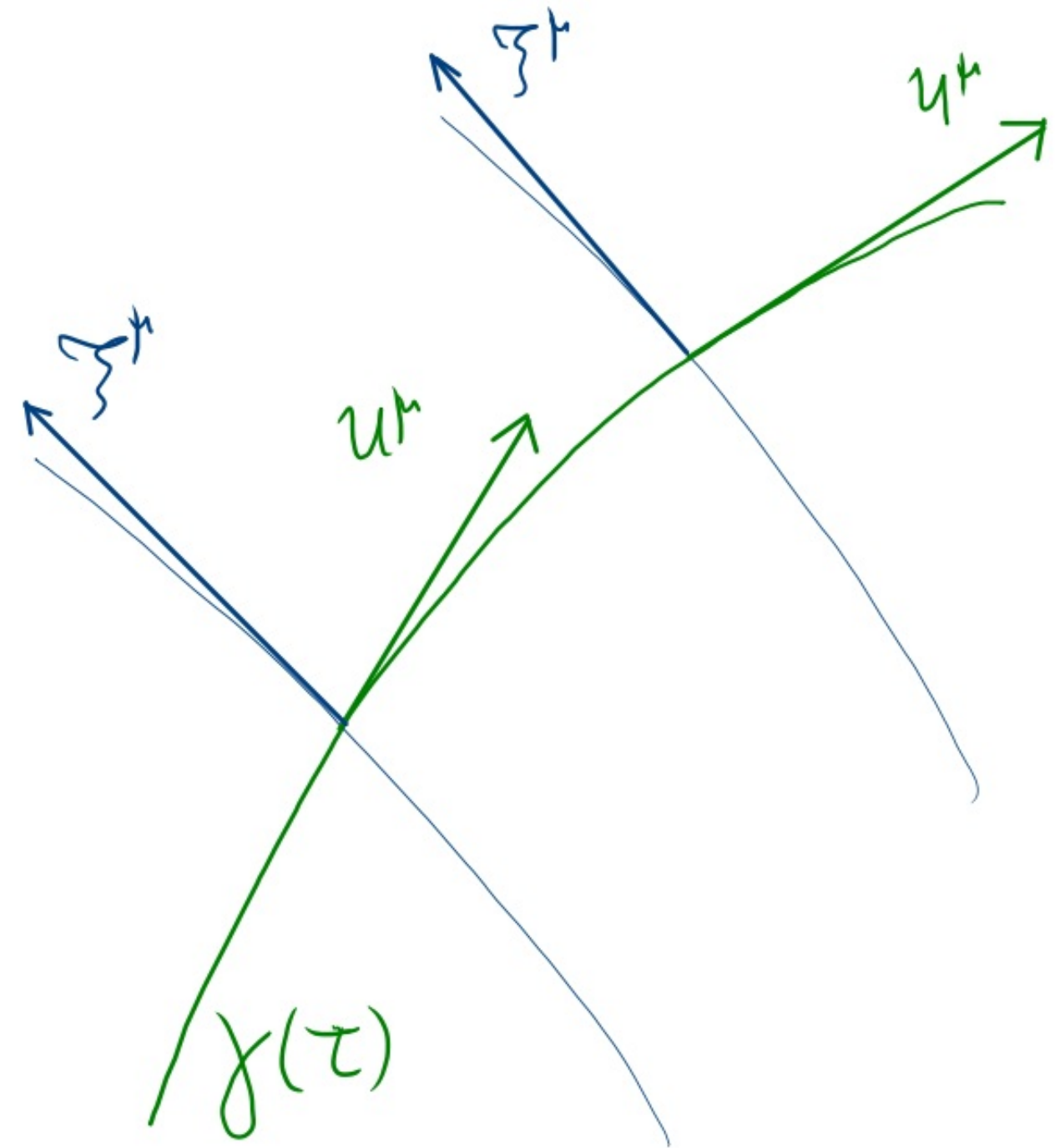
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---

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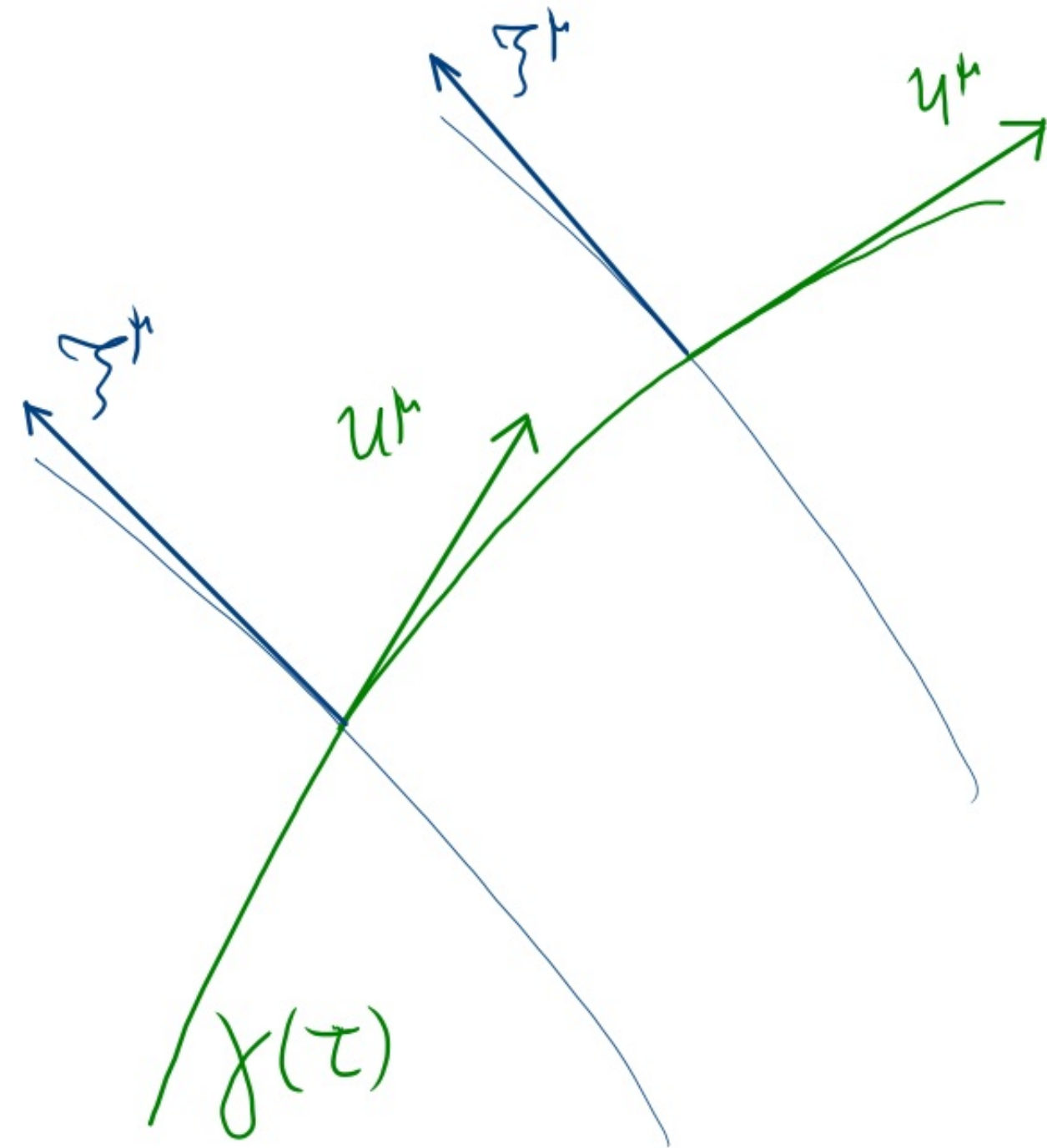
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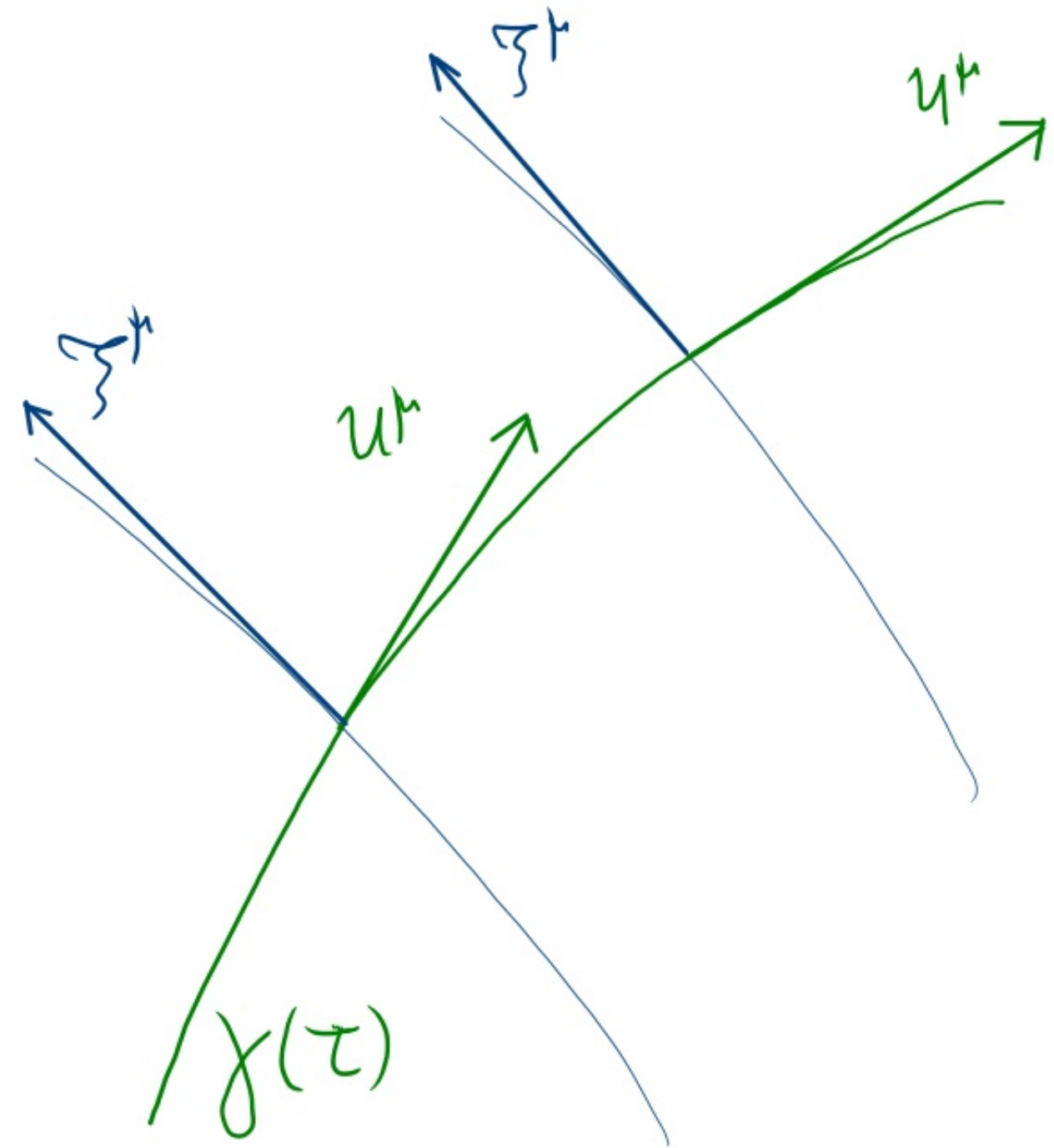
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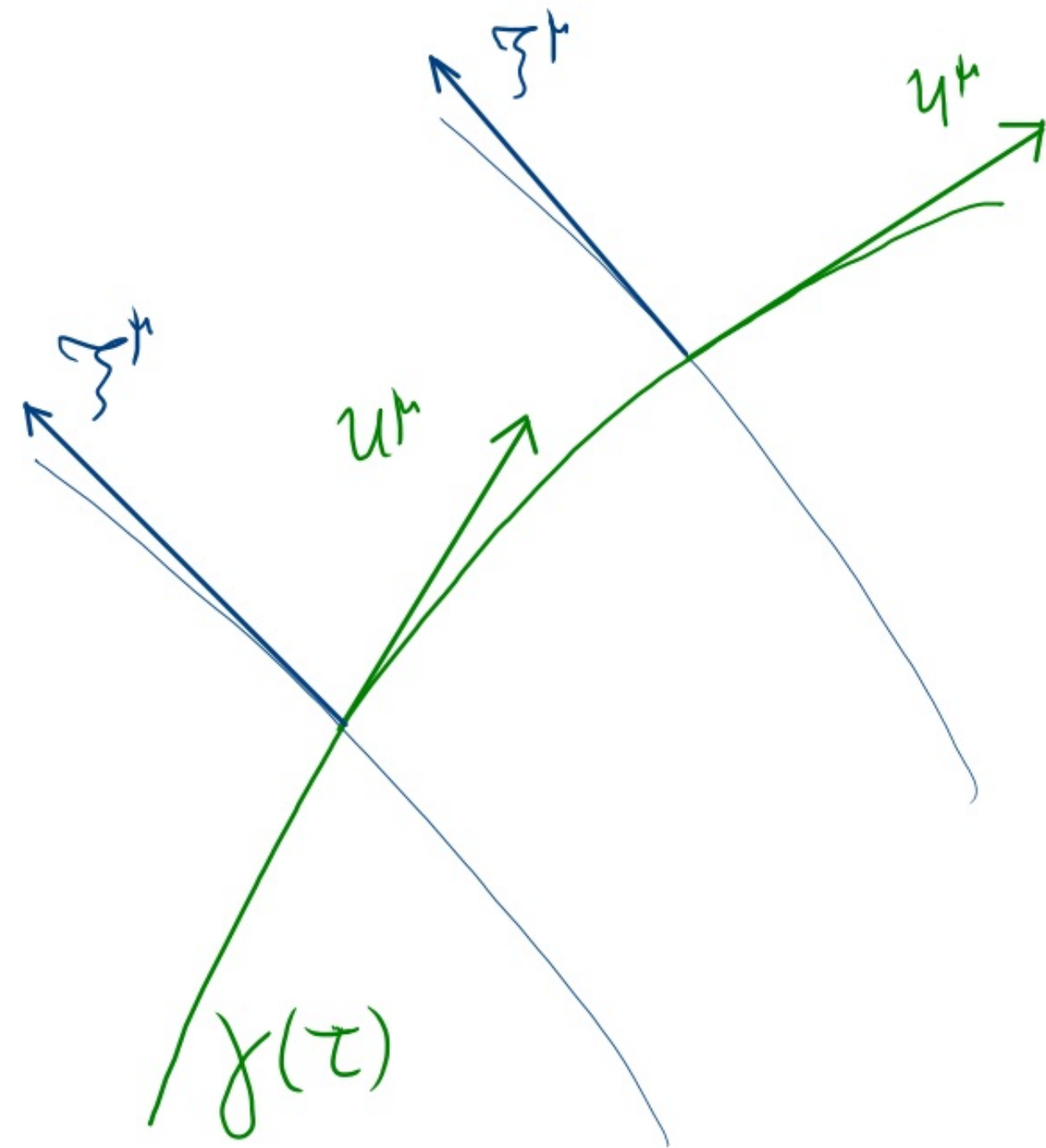
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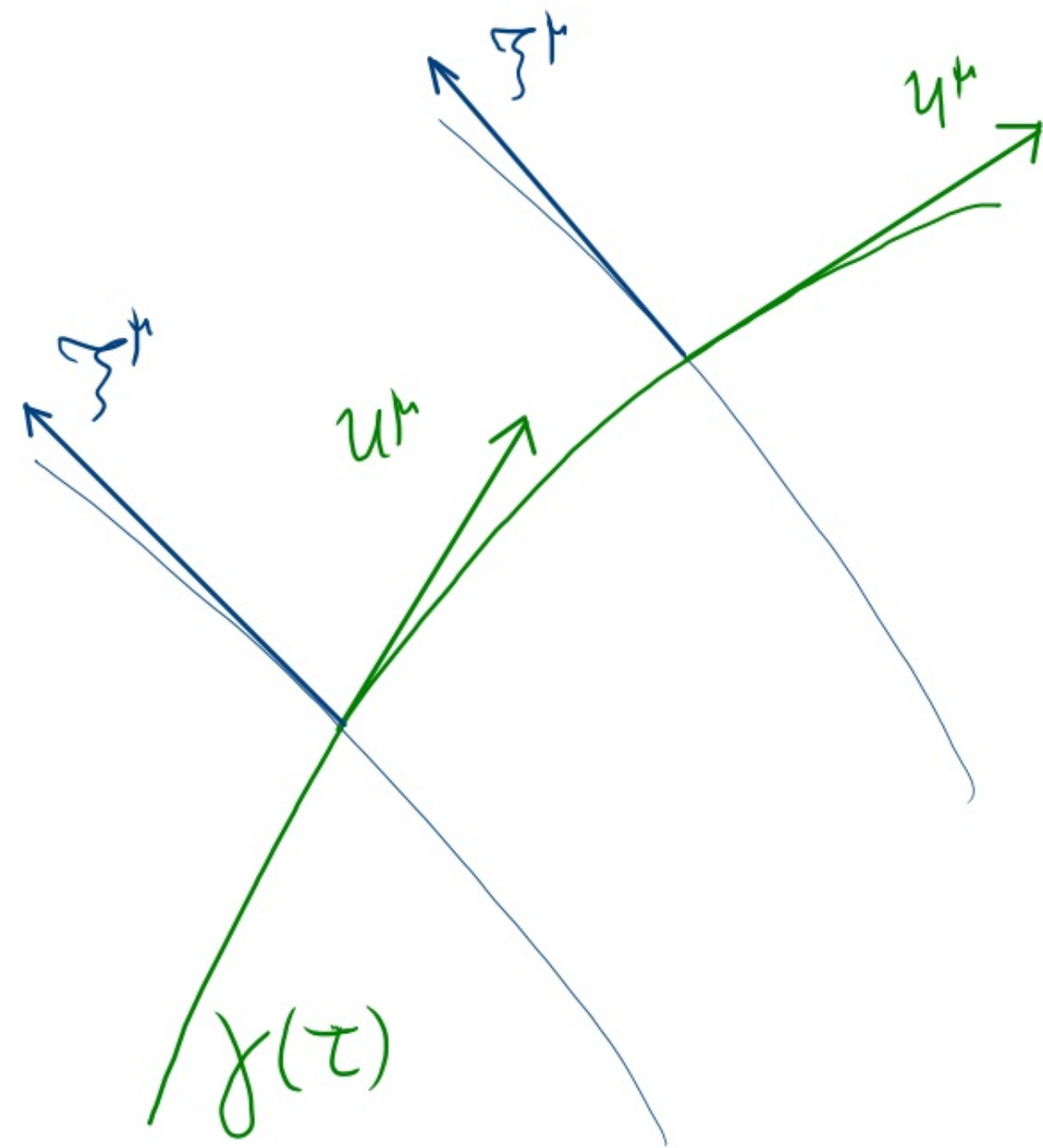
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KVF equation

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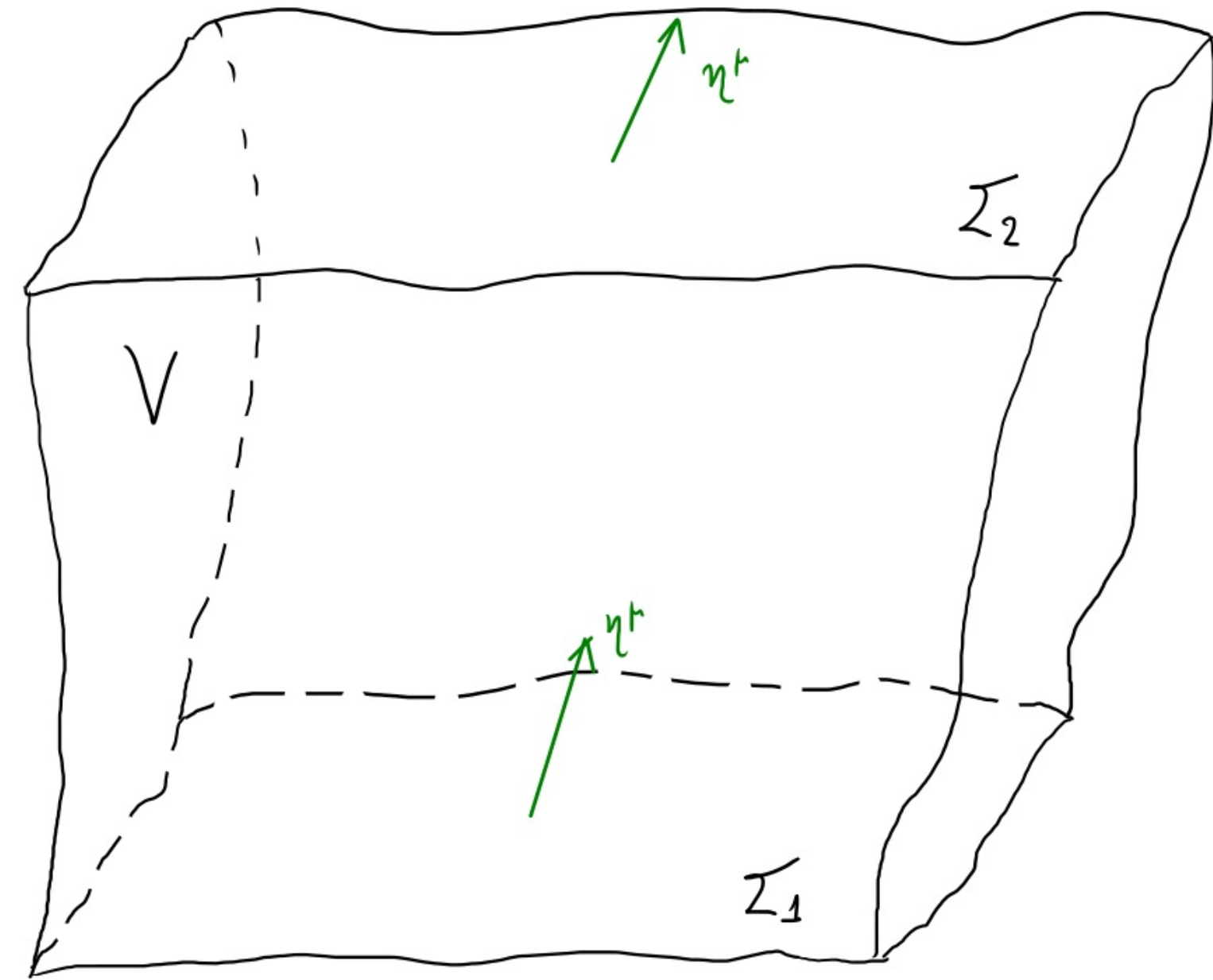
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*KVF equation*



If  $\Sigma^M$  timelike  $\rightarrow$  energy conservation  
" " spacelike momentum "

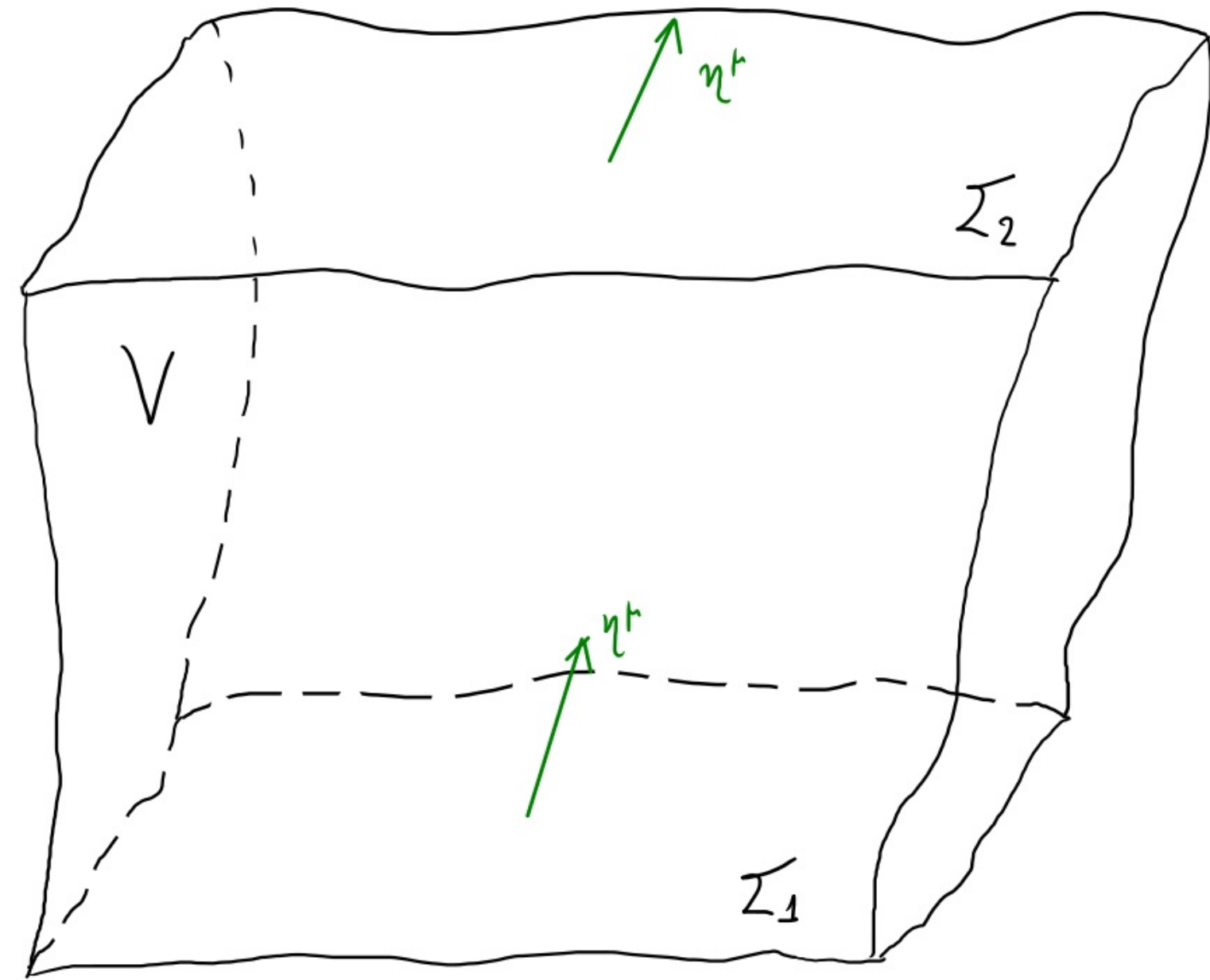


If  $\mathcal{J}^\mu$  timelike  $\rightarrow$  energy conservation

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E.g.  $\mathcal{J}^\mu$  timelike:

Consider  $\left\{ \begin{array}{l} 4\text{-d volume } V \\ \Sigma_1, \Sigma_2 \text{ spacelike} \\ \eta^\mu \text{ timelike} \end{array} \right.$

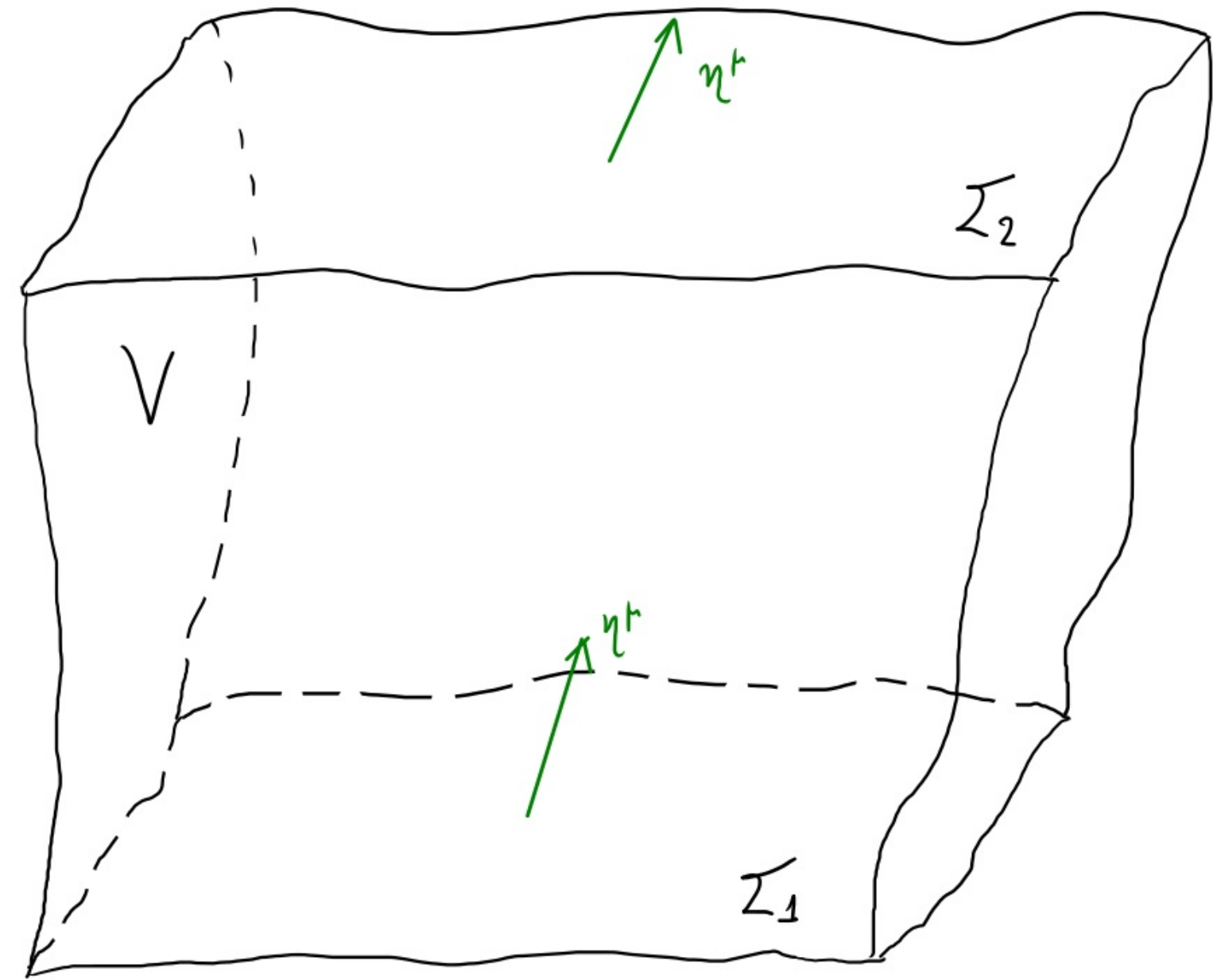


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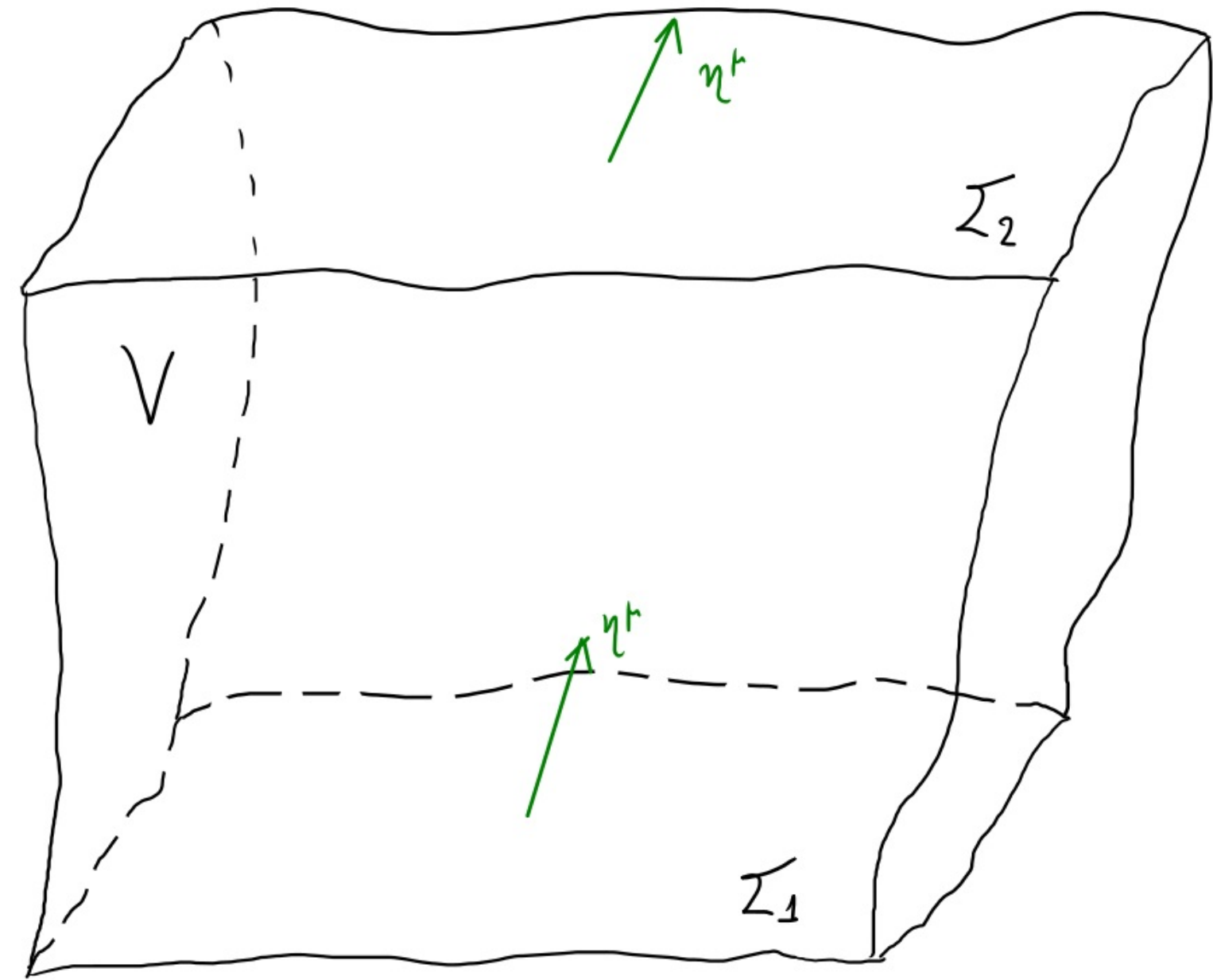
$$\int_V \nabla_\mu J^\mu \sqrt{|g|} d^4x = \int_{\Sigma_2} J_\mu \eta^\mu \sqrt{\gamma} d^3x - \int_{\Sigma_1} J_\mu \eta^\mu \sqrt{\gamma} d^3x + \int_\infty J_\mu \eta^\mu$$

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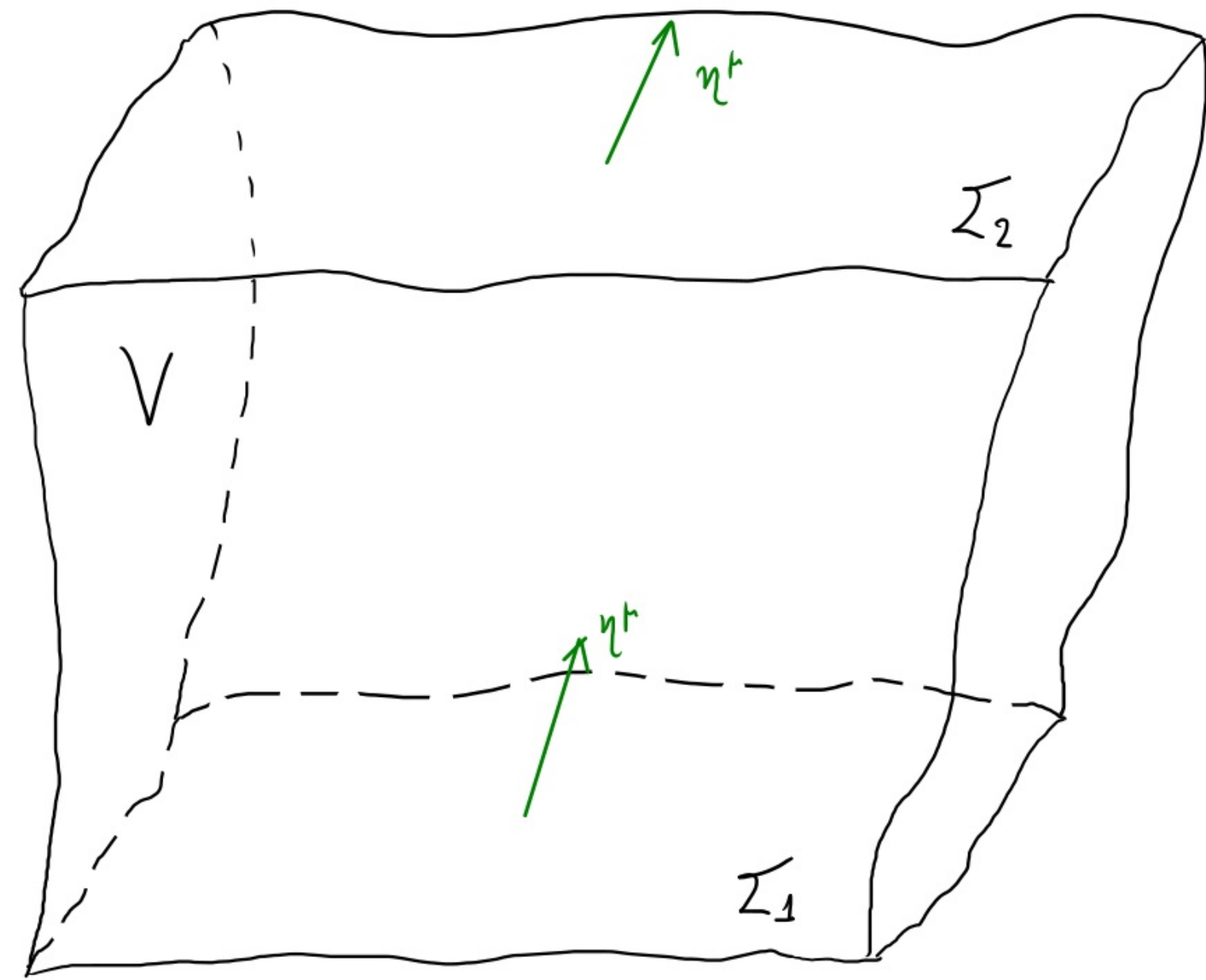
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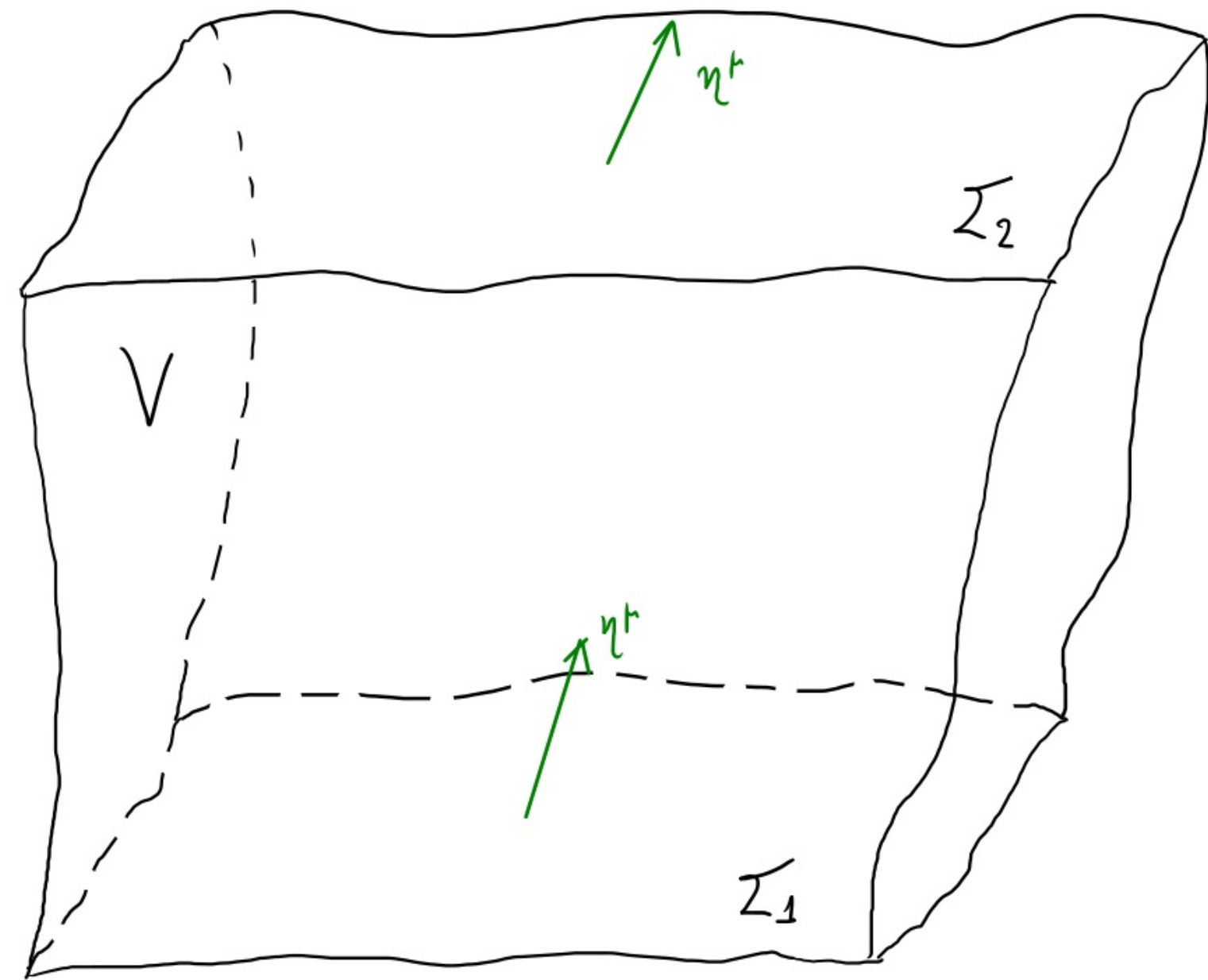
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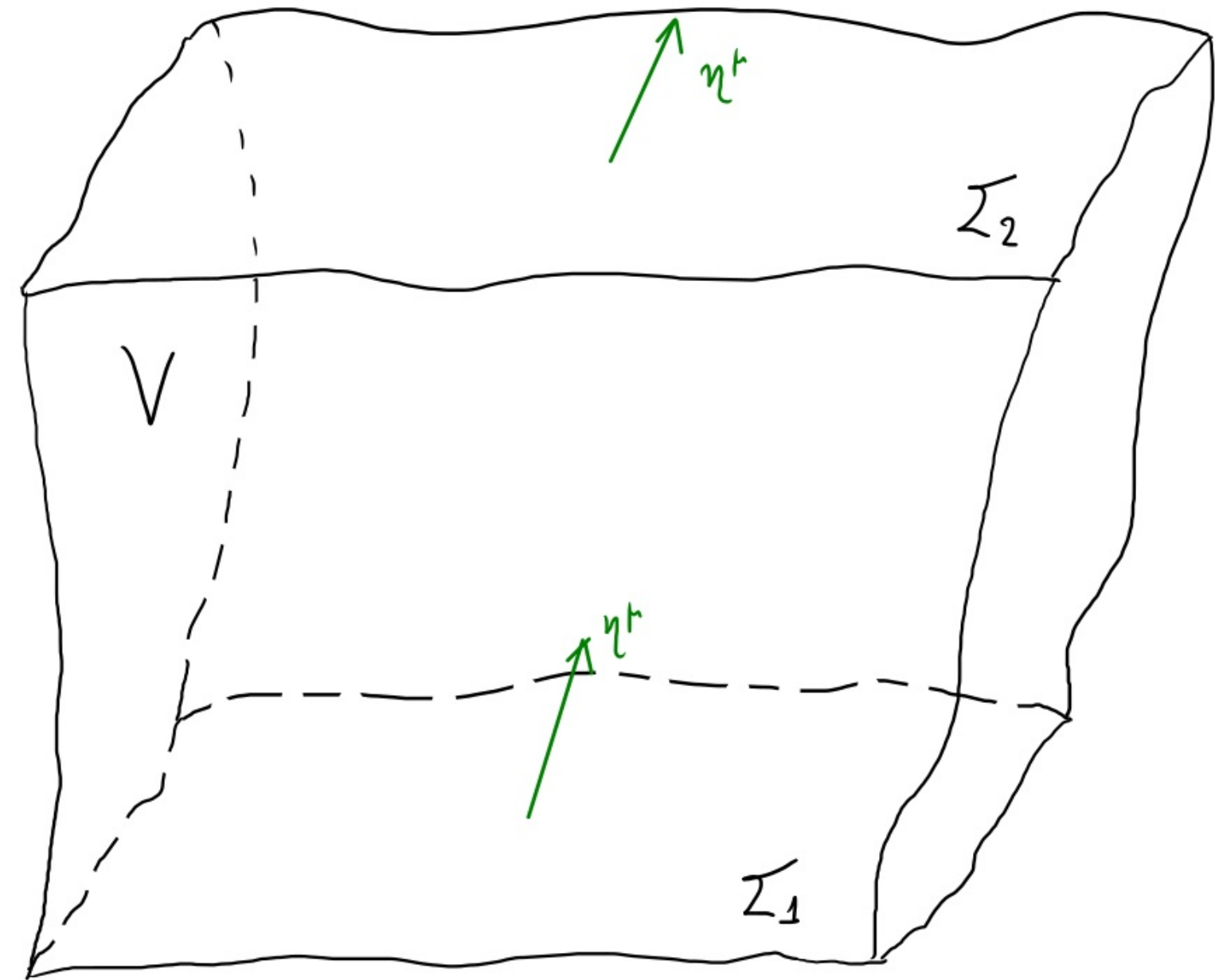
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Choose  $\Sigma_1, \Sigma_2$  s.t.  $\xi^\mu = \eta^\mu = (\partial_0)^\mu$   
 $g_{00} = -1$

$$\int_{\Sigma} J_\mu \eta^\mu \sqrt{\gamma} d^3x = \int_{\Sigma} T^{00} \sqrt{\gamma} d^3x = E = \text{const}$$



Example: Schwarzschild metric

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2)$$

$$g_{\mu\nu} = \text{diag} \left( - \left(1 - \frac{2M}{r}\right), \frac{1}{\left(1 - \frac{2M}{r}\right)}, r^2, r^2 \sin^2\theta \right)$$



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$K \perp F_s$

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$K^\mu$  time translations  $\rightarrow$  energy conservation

$R^\mu$   $\varphi$  - translations  $\rightarrow$  angular momentum conservation

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Conserved quantities along geodesics:

$$E = -K_\mu u^\mu = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau}$$

$$L = R_\mu u^\mu = r^2 \sin^2\theta \frac{d\varphi}{d\tau}$$