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The Factorization Method for Monte Carlo Simulations of Systems With a Complex Action

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We propose a method for Monte Carlo simulations of systems with a complex action. The method has the advantages of being in principle applicable to any such system and provides a solution to the overlap problem. In some cases, like in the IKKT matrix model, a finite size scaling extrapolation can provide results for systems whose size would make it prohibitive to simulate directly.

1. INTRODUCTION

There exist many interesting systems in high energy physics whose action contains an imaginary part, such as QCD at finite baryon density, Chern-Simons theories, systems with topological terms (like the θ-term in QCD) and systems with chiral fermions. This imposes a severe technical problem in the simulations, requiring an exponentially large amount of data for statistically significant measurements as the system size is increased or the critical point is approached. Furthermore, the overlap problem appears when standard reweighting techniques are applied in such systems and it becomes exponentially hard with system size to visit the relevant part of the configuration space. In [1] it was proposed to take advantage of a factorization property of the distribution functions of the observables one is interested to measure. This approach can in principle be applied to *any* system and it eliminates the overlap problem completely. In some cases it is possible to use finite size scaling to extrapolate successfully to large system sizes where it would have been impossible to measure oscillating factors directly. The method has been applied successfully in matrix models of non perturbative string theory (IKKT) [1], random matrix theory of finite density QCD (RMT) [2] as well as the 2d CP^3 model, the 1d antiferromagnetic model with imaginary B and the 2d compact U(1) with topological charge [3].

In this paper we present our results [1,2] for IKKT and RMT. In the first case we study the space–time dimensionality hoping to *dynamically* recover our 4d space–time and in the second to test the factorization method against known analytical results. In all cases we will be dealing with a system defined by a partition function $Z = \int dA\, \mathrm{e}^{-S_0}\, \mathrm{e}^{i\Gamma}$ and the corresponding phase quenched model $Z_0 = \int dA\, \mathrm{e}^{-S_0}$ where $S = S_0 - i\Gamma$ is the action of the system with its real and imaginary parts. A represents collectively the degrees of freedom of the model and in our case it corresponds to a set of $N \times N$ matrices. In case we are interested in measuring some observable $\mathcal{O}$, we consider the distribution functions $\rho_{\mathcal{O}}(x) = \langle \delta(x - \mathcal{O}) \rangle$ and $\rho^{(0)}_{\mathcal{O}}(x) = \langle \delta(x - \mathcal{O}) \rangle_0$, where $\langle \ldots \rangle_0$ refers to Z_0. Then we define the fiducial system $Z_{\mathcal{O},x} = \int dA\, \mathrm{e}^{-S_0} \delta(x - \mathcal{O})$, the weight factor $w_{\mathcal{O}}(x) = \langle \mathrm{e}^{i\Gamma} \rangle_{\mathcal{O},x}$ and the distribution $\rho_{\mathcal{O}}(x)$ factorizes $\rho_{\mathcal{O}}(x) = \frac{1}{C}\, \rho_{\mathcal{O}}(x)\, w_{\mathcal{O}}(x)$ where $C = \langle \mathrm{e}^{i\Gamma} \rangle_0$. Then $\langle \mathcal{O} \rangle = \frac{1}{C} \int_{-\infty}^{\infty} dx\, x\, \rho^{(0)}_{\mathcal{O}}(x)\, w_{\mathcal{O}}(x)$. The δ–function constraint is implemented in our simulations by considering the system $Z_{\mathcal{O},V} =$

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$\int dA\, e^{-S_0} e^{V(\mathcal{O})}$ where $V(z) = \frac{1}{2}\gamma(z-\xi)^2$ and γ, ξ are parameters which control the constraining of the simulation. The results are insensitive to the choice of γ as long as it is large enough. Then we have that $w_{\mathcal{O}}(x = \langle\mathcal{O}\rangle_{i,V}) = \langle e^{i\Gamma}\rangle_{i,V}$. The distribution of $\mathcal{O}$ in $Z_{i,V}$ has a peak $\bar{x}$ and the quantity $V'(\bar{x})$ is the value of $f_{\mathcal{O}}^{(0)}(x) = \frac{d}{dx}\ln\rho_{\mathcal{O}}^{(0)}(x)$ at $x = \bar{x}$. The function $\rho_{\mathcal{O}}^{(0)}(x)$ can be obtained by integrating an analytic function to which we fit the $f_{\mathcal{O}}^{(0)}(x)$ data points.

By applying this method we force the system to sample configurations which give the essential contributions to $\langle\mathcal{O}\rangle$, something that would be increasingly difficult with system size in the phase quenched model, eliminating this way the overlap problem. This already allows us get close to the thermodynamic limit with modest computer resources. Furthermore we obtain direct knowledge of $w_{\mathcal{O}}(x)$ and $\rho_{\mathcal{O}}(x)$ which allows us to understand the effect of Γ. This is important for understanding the properties of the system when Γ plays a crucial role. Using the generic scaling properties of the weight factor $w_i(x)$, one may extrapolate the results obtained by direct Monte Carlo evaluations to larger system size. Such an extrapolation is expected to be particularly useful in cases where the distribution function turns out to be positive definite. In those cases we can actually even *avoid* using the reweighting formula by reducing the question of obtaining the expectation value to that of finding the minimum of the free energy, which is (minus) the *log* of the distribution function. Here, the error in obtaining the scaling function propagates to the final result without significant magnifications. Therefore, the extrapolation can be a powerful tool to probe the thermodynamic limit from the accessible system size.

2. NON-PERTURBATIVE STRING THEORY

The so-called IKKT matrix model [4] has been conjectured to be a definition for non-perturbative string theory. A particularly interesting feature of the model is that the eigenvalues of the bosonic matrices A_μ generate space-time *dynamically*. In our case we work with a reduced model where $\mu = 1, \ldots, 6$. The observables that we study are the normalized eigenvalues $\lambda_1/\langle\lambda_1\rangle_0 > \ldots > \lambda_6/\langle\lambda_6\rangle_0$ of the space-time "moment of inertia" $T_{\mu\nu} = \frac{1}{N}\mathrm{Tr}A_\mu A_\nu$. An interesting scenario to investigate would be that the O(6) symmetry is spontaneously broken with some of the eigenvalues (possibly 4) grow large and the rest remain small, providing a mechanism for dynamical compactification of extra dimensions.

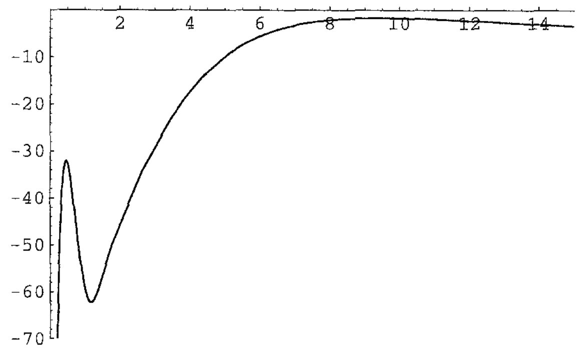

Fig. 1: $\log\rho_5(x)$ *for N=128, n = 16.*

The IKKT model is a good example where finite size scaling for the oscillating factor $w_i(x)$ works well (the index i corresponds to $\lambda_i/\langle\lambda_i\rangle_0$). For $i > 1$ we find fast convergence to a scaling function $\Phi_i(x)$, where $w_i(x) = \exp\{N^2\Phi_i(x)\}$ [1]. We compute $\Phi_i(x)$ for small matrix size $n \le 20$ and the phase quenched distribution function $\rho_i^{(0)}(x)$ for larger size N. Then the factorization formula can be used in order to compute $\rho_i(x)$. Note that in the computation of $\langle\lambda_i\rangle$ the errors do not propagate exponentially with system size since $\langle\lambda_i\rangle$ can be determined from the minimum of the "free energy" $F_i(x) = -\frac{1}{N^2}\log\rho_i(x)$. In Fig. 1 we show our results for $\rho_5(x)$ for $n = 16$ and $N = 128$. A double peak structure of $\rho_5(x)$ is evident and it is expected that the peak for small x will increase with system size [1]. One hopes that this peak will be dominant for $i = 5, 6$ and that the large x peak will be dominant for $i = 4, 3, 2, 1$ and this will realize the SSB scenario. Note the heavy suppression of the $x = 1$

region caused by Γ which is the peak of the phase quenched model distribution of λ_i.

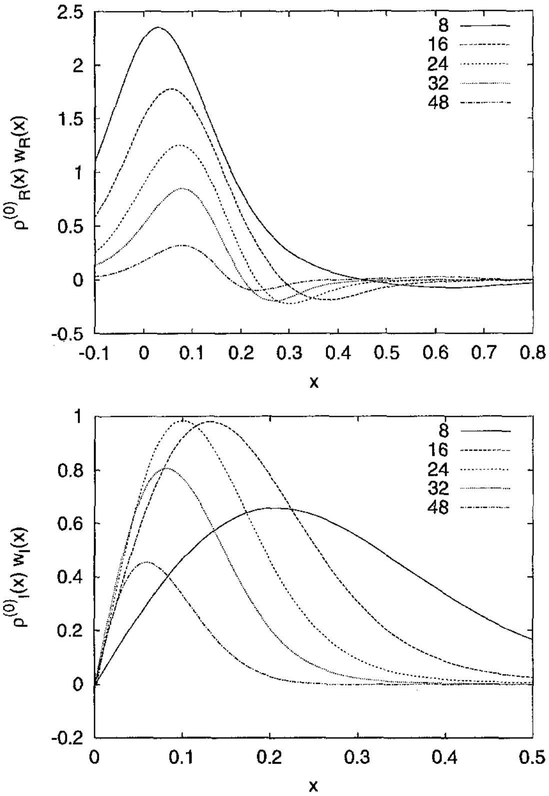

Fig. 2: *$\rho_R(x)$ and $\rho_I(x)$ for $\mu = 0.2$.*

3. RMT OF FINITE DENSITY QCD

We consider RMT with one quark flavour and zero quark mass [5]. The model is chosen in order to study the correctness and effectiveness of the factorization method, since one can compare results with known analytical solutions even for finite N. The observable we measure is the "quark number density" ν as a function of the chemical potential μ, and we consider the distribution functions $\rho_i(x)$, where $i = R, I$ corresponds to the real and imaginary parts of ν respectively. Notice that the effect of Γ is dramatic, causing a discontinuous transition in ν. In [2] our results nicely reproduce the exact results known for finite N and we are able to achieve large enough values of $N \leq 48$ to obtain the thermodynamic limit. Unfortunately, the function $w_R(x)$ is not positive definite and the important contributions come from the region where it changes sign. As expected, we find that finite size scaling does not work as well as in the case of the IKKT model (although we obtain agreement up to order of magnitude for the values of $N \leq 96$ that we explored). We also find it very difficult to explore the crossover region near the phase transition point $\mu_c = 0.527\ldots$ for $N > 8$ since $|w_i(x)|$ becomes very small. Since RMT is a schematic model of finite density QCD, we expect that the factorization method will be useful to explore the phase diagram of QCD.

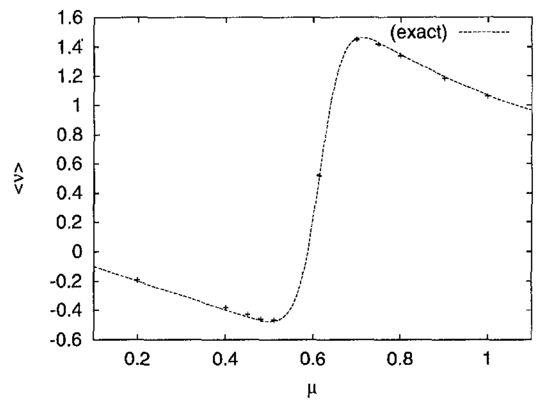

Fig. 3: *$\langle \nu \rangle$ for $N = 8$.*

REFERENCES

1. K. N. Anagnostopoulos and J. Nishimura, Phys. Rev. D **66** (2002) 106008.
2. J. Ambjørn et al., JHEP **0210** (2002) 062.
3. V. Azcoiti, G. Di Carlo, A. Galante and V. Laliena, Phys. Rev. Lett. **89** (2002) 141601; hep-lat/0210004.
4. N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, Nucl. Phys. B **498** (1997) 467; H. Aoki, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, Prog. Theor. Phys. **99** (1998) 713.
5. E. V. Shuryak and J. J. Verbaarschot, Nucl. Phys. A **560** (1993) 306; J. J. Verbaarschot and T. Wettig, Ann. Rev. Nucl. Part. Sci. **50** (2000) 343.