Phenomenological Extension to Black Hole Ringdown

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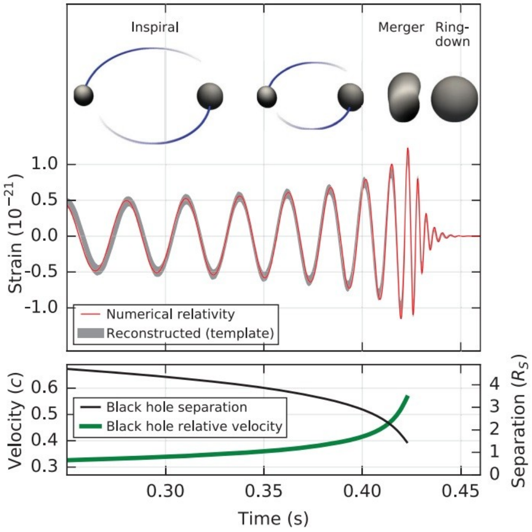
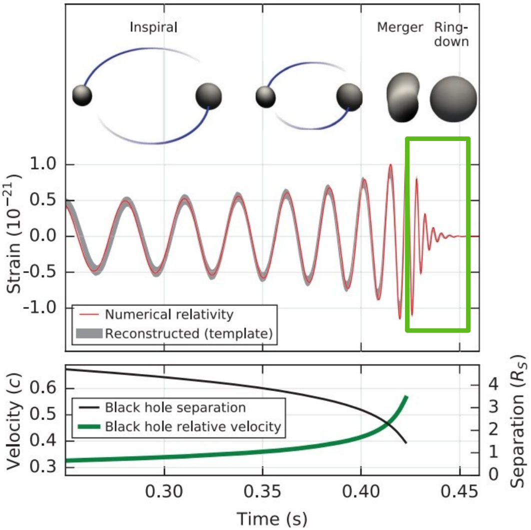
Background



Phys. Rev. Lett. 116, 061102 (2016), arXiv:1602.03837 [gr-qc]



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Importance of Ringdown

- Linear ringdown: adequately described by complex frequencies, the *Quasi-Normal Modes*

$$\Phi(t, r^* = \mathrm{const.}) = \sum_{n=0} A_n e^{-i\omega_n t}, \qquad \omega_n = Re(\omega_n) - i\, Im(\omega_n)$$

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  - *Kerr hypothesis:* Astrophysical BHs
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  - QNMs are a BH property

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- *Test Kerr Hypothesis*
  - Measure 2 ω_n
  - Verify that ω_n correspond to M and J as predicted by GR

Recipe for Linearized Equations

Step 1. Metric for Schwarzschild BH solution

$$\bar{g}_{\mu\nu} = diag\left(-(1-r_H/r), (1-r_H/r)^{-1}, r^2, r^2\sin^2\theta\right), \quad r_H = 2M$$

$$\downarrow$$

Step 2. New metric: $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} + \mathcal{O}(h^2), \quad |h_{\mu\nu}| \ll |\bar{g}_{\mu\nu}|$

$$\downarrow$$

Step 3. Expand perturbation metric, $h_{\mu\nu}$, to *good* basis

$$h_{\mu\nu} = \sum_a \sum_{\ell m} h^{\alpha}_{\ell m}(t,r)\left(\mathbf{t}^{\alpha}_{\ell m}\right)_{\mu\nu}(\theta,\phi)$$

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↓

Step 4. Gauge fix: Regge-Wheeler gauge

↓

Step 5. Plug into Einstein's Equations: $\Delta G_{\mu\nu} = 0$

Step 5: Final Equations

- Plug into Einstein's Equations: $\Delta G_{\mu\nu} = \delta\Gamma^{\kappa}_{\mu\nu;\kappa} - \delta\Gamma^{\kappa}_{\mu\kappa;\nu} = 0$
- Coordinate transformation to *Tortoise*: Send singularity to $-\infty$

$$r^* = r + r_H log\left(\frac{r}{r_H} - 1\right), \quad r > r_H$$

- Wave equations

$$\boxed{\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^{*2}}\right)\Phi^{\pm} - f(r)V_\ell^{\pm}(r)\Phi^{\pm} = 0} \qquad f(r) = 1 - \frac{r_H}{r},$$

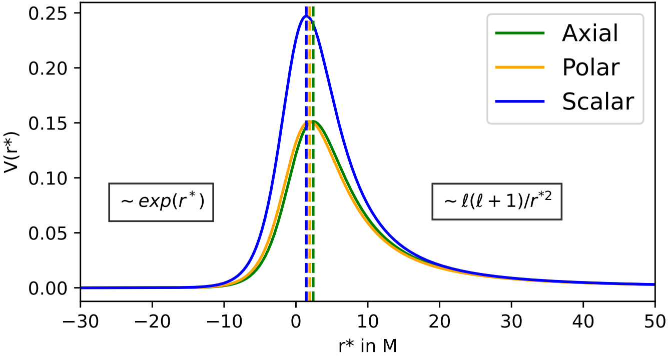
with $\hat{\mathcal{P}}\Phi^+ = (-1)^\ell\Phi^+$ and $\hat{\mathcal{P}}\Phi^- = (-1)^{\ell+1}\Phi^-$

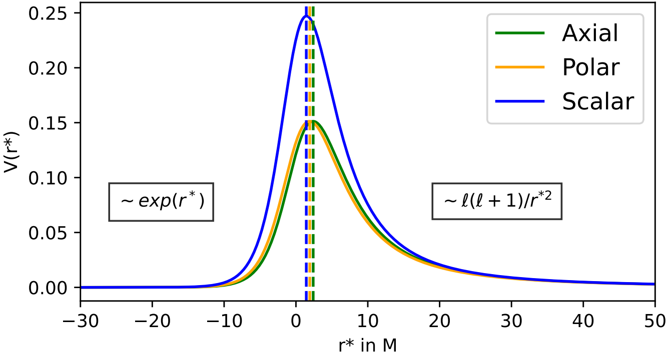
- Perturbation potentials

$$V_\ell^-(r) = \left(\frac{\ell(\ell+1)}{r^2} - \frac{3r_H}{r^3}\right) \qquad \textit{Axial: Regge-Wheeler}$$

$$V_\ell^+(r) = \frac{9\lambda r_H^2 r + 3\lambda^2 r_H r^2 + \lambda^2(\lambda+2)r^3 + 9r_H^3}{r^2(\lambda r + 3r_H)^2} \qquad \textit{Polar: Zerilli}$$

where $\lambda = \ell(\ell+1) - 2$.

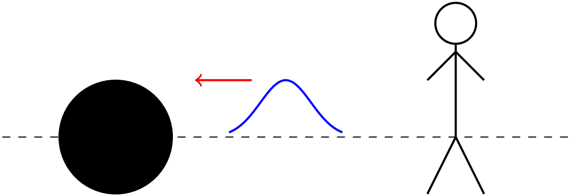

Axial
Polar
Scalar
V(r*)
r* in M
$\sim exp(r^*)$
$\sim \ell(\ell+1)/r^{*2}$
0.25
0.20
0.15
0.10
0.05
0.00
−30
−20
−10
0
10
20
30
40
50

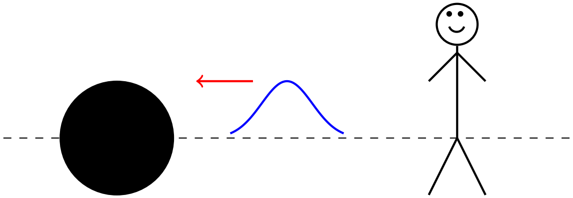


- QNMs eigenvalues to the boundary value problem

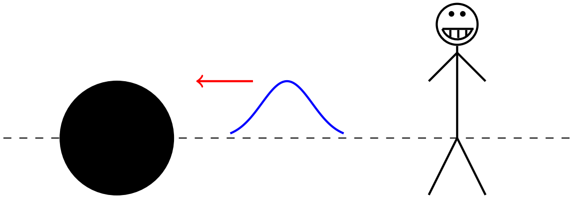
$$\frac{\mathrm{d}^2\Phi_\ell^\pm}{\mathrm{d}r^{*2}} - \left(\omega^2 + f(r)V_\ell^\pm\right)\Phi_\ell^\pm = 0$$

- Outgoing boundary conditions: $\Phi_\ell^\pm \sim e^{\mp i\omega r^*}, \qquad r^* \to \pm\infty$





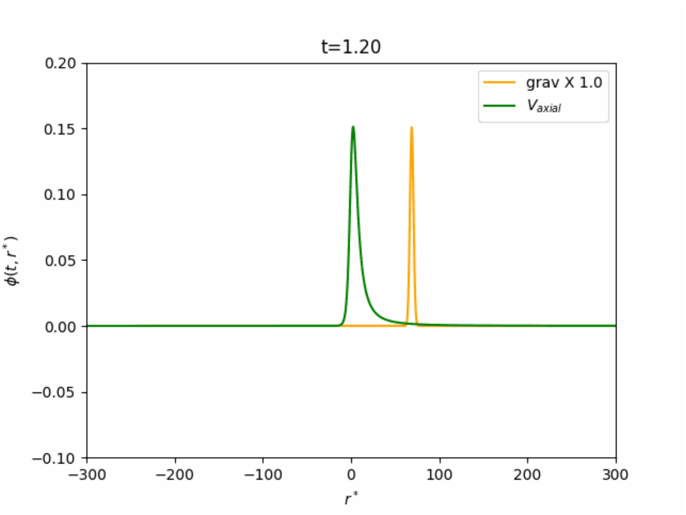
- Contributions to observer's signal:
  - Direct propagation from Initial Conditions
  - QNMs
  - Power-law tail (Price tail)



- Contributions to observer's signal:
  - Direct propagation from Initial Conditions
  - **QNMs**
  - **Power-law tail (Price tail)**
- Initial Condition for derivative $\rightarrow$ Force movement to one direction
  - Field $\Phi(t=0, r^*) = A \cdot \exp\left(-(r^* - \mu)/2\sigma^2\right)$
  - Derivative $\partial_t \Phi(t=0, r^*) = -\partial_* \Phi(t=0, r^*)$

$$\Phi(t, r^* = \text{fixed}) = \sum_{n=0}^{N-1} A_n e^{-Im(\omega_n)t} \sin(Re(\omega_n)t + \phi_n) + A_{tail}(t - t_{tail})^{-(2\ell+3)}$$

Scattering on Axial Potential



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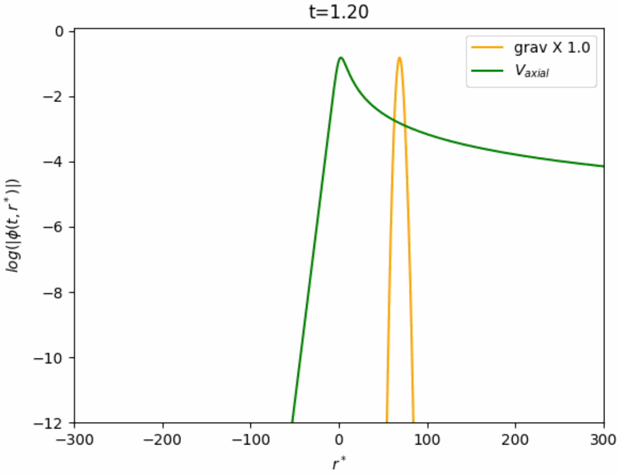

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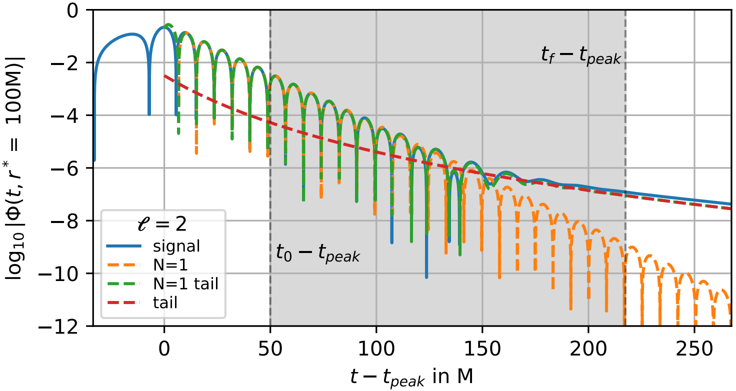
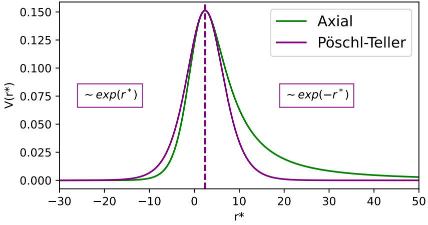


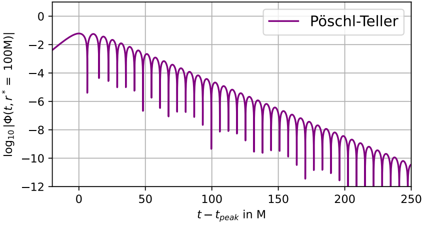
Simple case:
Pöschl-Teller potential

Pöschl-Teller Potential[1]

$$V_{PT}(r^*) = \frac{V_0}{\cosh^2(\alpha(r^* - r_0^*))}$$

- No late-time tail
- Analytical Spectrum: $\omega = (V_0 - \alpha/4)^{1/2} - i\alpha\,(n + 1/2)$
- Match with Regge-Wheeler





[1] Mashhoon, 3rd Marcel Grossmann Meeting, (1982), Ferrari, Mashhoon PRL. 52, 1361 (1984)

- Theory Agnostic: $2N + 2N$ free parameters
  - *No Assumptions* for the frequencies

$$\Phi_N^{\mathrm{TA}} = \sum_{n=0}^{N-1} A_n e^{-Im(\omega_n^{\mathrm{TA}})(t-t_{\mathrm{peak}})} \sin\left(Re(\omega_n^{\mathrm{TA}})(t - t_{\mathrm{peak}}) + \phi_n\right)$$

[1] Nee, Völkel, Pfeiffer, PRD 108 (2023) 4, 4

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- Theory Specific: $2N + 1$ free parameters
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  - $\omega_n^{\mathrm{TS}} M^{\mathrm{TS}} = \underbrace{\omega_n^{\mathrm{inj}}}_{\text{assume}} \underbrace{M^{\mathrm{inj}}}_{1}$

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- Vary the fitting window $t \in [t_0 - t_{\mathrm{peak}}, t_f - t_{\mathrm{peak}}]$

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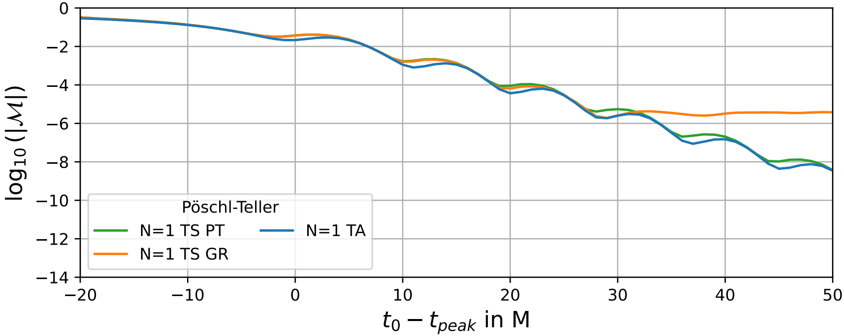
Mismatch Plots

$$\mathcal{M} = 1 - \frac{\langle \Phi_{signal}, \Phi_{fit} \rangle}{\sqrt{\langle \Phi_{signal}, \Phi_{signal} \rangle \langle \Phi_{fit}, \Phi_{fit} \rangle}}, \qquad \langle \Phi_{signal}(t), \Phi_{fit}(t) \rangle = \int_{t_0}^{t_f} \Phi_{signal}(t) \cdot \Phi_{fit}(t) dt$$



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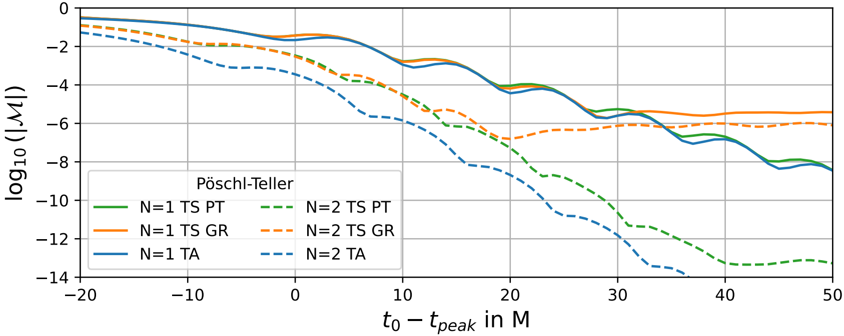
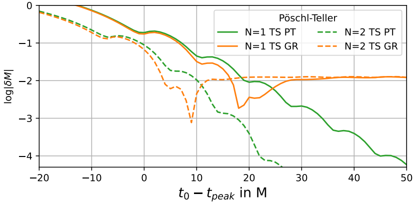


Extraction of parameters

- Fit the same waveform with varying the starting time of the fitting window
- TS correct ω hypothesis (green) & TS wrong ω hypothesis (orange) & TA no ω hypothesis (blue)



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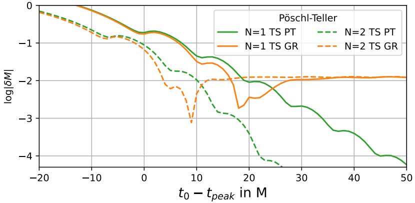

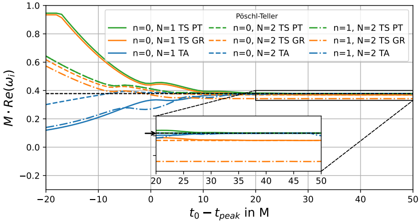

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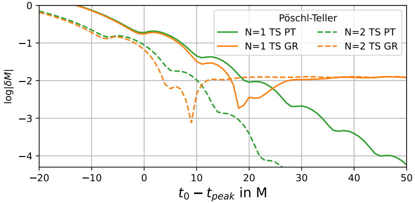

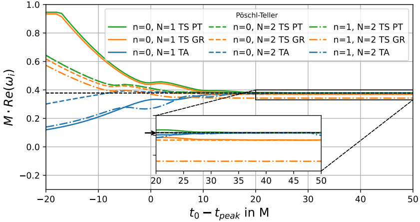

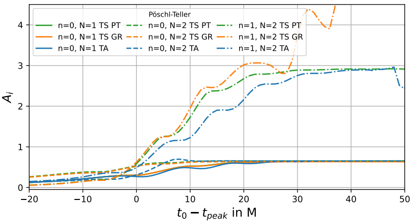

GR case:
Regge-Wheeler Potential

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Model power-law tail: 2 free parameters

$$\Phi_N \rightarrow \Phi_N + A_{tail}\left(t - t_{tail}\right)^{-(2l+3)}$$

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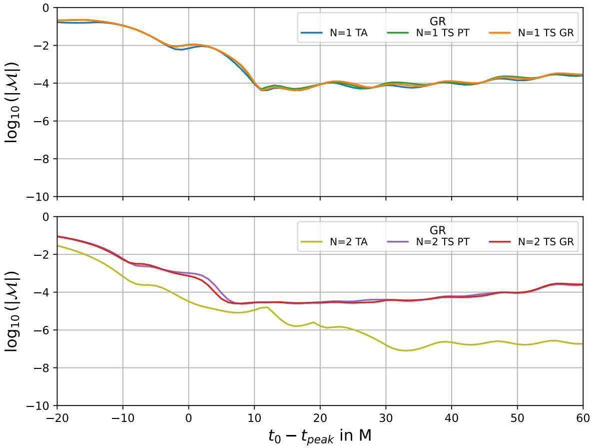

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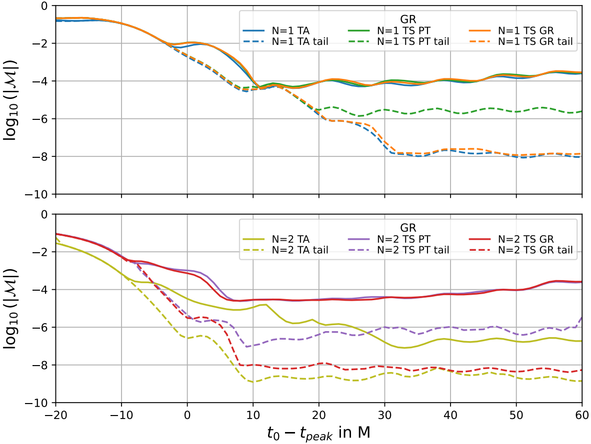
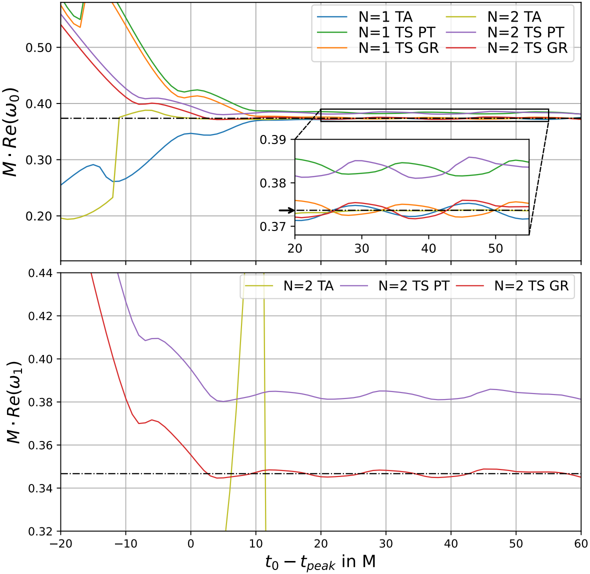


Frequency Plots



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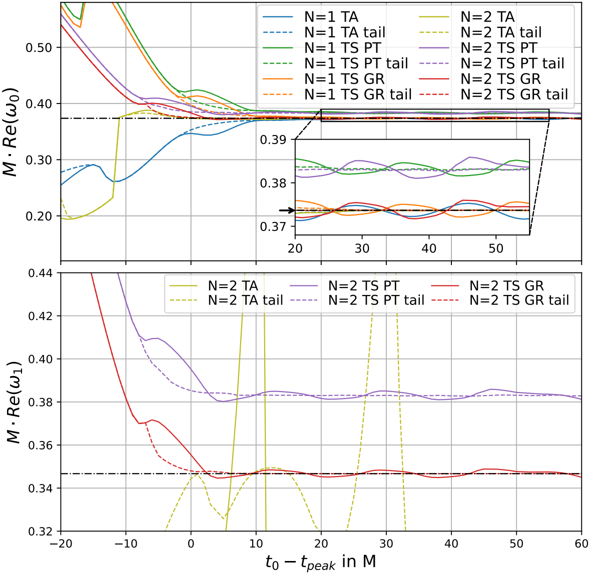
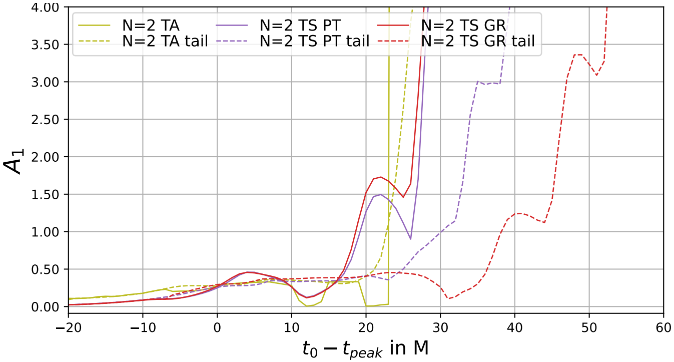
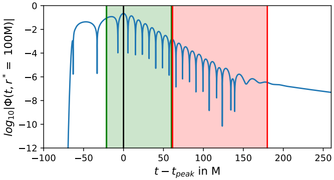


Amplitude Plots

- None of the models find the overtone for late times, like in PT case



VARIATION OF FIT END TIME t_f



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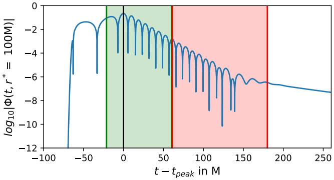
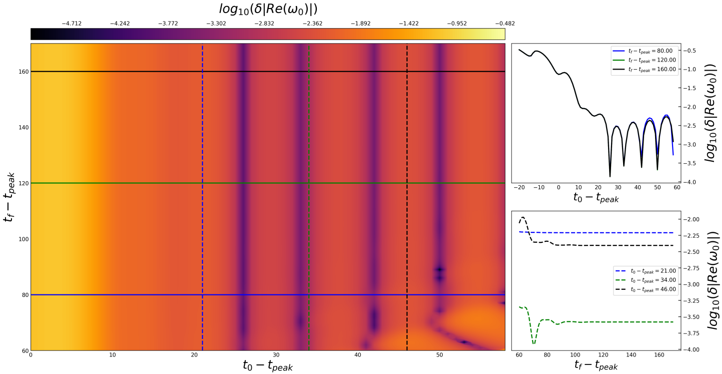


- t_f does not seem to affect frequency's accuracy
- Distorted bottom-right region:
  a small portion of the waveform was included in the fit



Parametrized QNM Framework

Parametrized QNM Framework [1]

- GR : may not be the ultimate theory of gravity
- Modifications : small impact on perturbation potential

[1] V. Cardoso, M. Kimura, A. Maselli, E. Berti, C.F.B. Macedo, R. McManus, PRD 99 (2019) 10,
R. McManus, E. Berti, C.F.B. Macedo, M. Kimura, A. Maselli, V. Cardoso, PRD 100 (2019) 4

PARAMETRIZED QNM FRAMEWORK [1]

- GR : may not be the ultimate theory of gravity
- Modifications : small impact on perturbation potential
- Which modification is correct? $\rightarrow$ ¯_(ツ)_/¯

$\Big\} \Rightarrow$

- Phenomenological Extension to GR ringdown: *Parametrized QNM Framework*
  - Introduce $1/r$ terms to the potentials:
  - Can be mapped to different alternative theories

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- Phenomenological Extension to GR ringdown: *Parametrized QNM Framework*
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$$V^{GR} \longrightarrow V^{PF} = V^{GR} + \sum_{k=0} \underbrace{\frac{1}{r_H^2} a_k(\omega) \left(\frac{r_H}{r}\right)^k}_{\delta V_k}, \quad a_k \ll (1 + 1/k)^k (k+1)$$

  - Shift in QNMs from one δV_k:

$$\omega = \omega_{GR} + \underbrace{a^{(k)} d_{(k)}}_{\text{linear}} + \underbrace{\alpha^{(k)} \partial_\omega \alpha^{(s)} d_{(k)} d_{(s)} + \frac{1}{2} a^{(k)} a^{(s)} e_{(ks)}}_{\text{quadratic}} + \mathcal{O}(a^3)$$

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- Original definition in frequency domain

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}r^{*2}} + \left[\omega^2 - f(r)\left(V^{GR}(r) + \delta V_k\right)\right]\Phi = 0$$

Parametrized QNM Framework

- Original definition in frequency domain

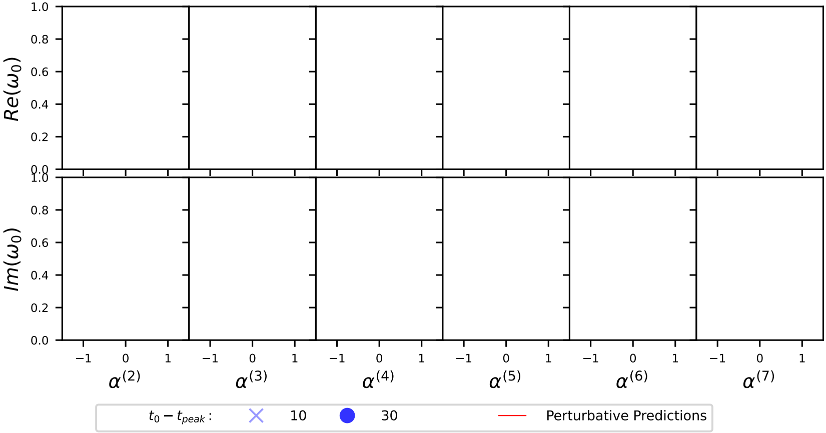
$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}r^{*2}} + \left[\omega^2 - f(r)\left(V^{GR}(r) + \delta V_k\right)\right]\Phi = 0$$

- Pass to time domain: *replace* ω^2 with $-\partial_{tt}$

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^{*2}}\right)\Phi - f(r)\left(V^{GR}(r) + \delta V_k\right)\Phi = 0$$

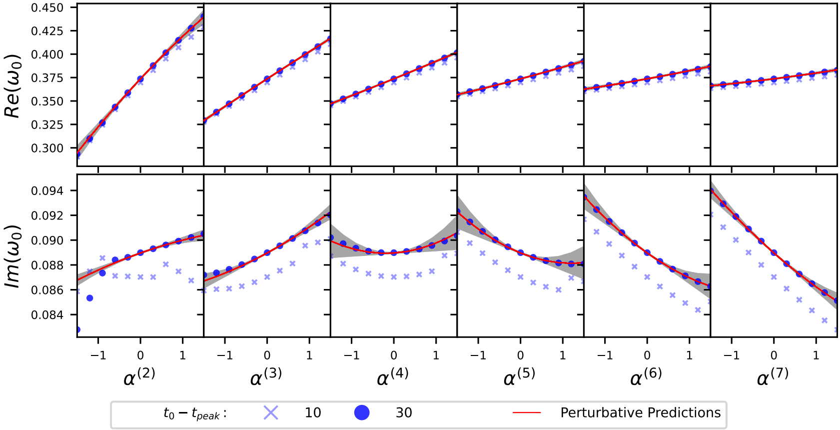
Frequency Results

- Fundmanetal mode & tail



Frequency Results

- Fundmanetal mode & tail
- Robust extraction of $N = 0$ mode frequencies
- Agreement with perturbative prediction



Conclusion

- Fit starting time t_0 impacts the parameters' extraction
- Potential with tail-less spectrum does not induce oscillation on the extracted parameters

- Fit end time t_f does not seem to impact the parameters' extraction
- Lack of modeling of the power-law tail $\Rightarrow$ unstable extraction of ringdown parameters
- Modeling of the tail $\Rightarrow$ stabilization of parameter extraction

- Time evolution results recover QNMs in agreement with theoretical predictions

THANKS FOR YOUR ATTENTION :)

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BACKUP

First Approach: Scalar Field Evolution Around a Black Hole

- Obeys Wave Equation

$$D_\mu D^\mu \psi = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \right) = 0$$

- Spherical Symmetry $\rightarrow$ decompose in spherical harmonics

$$\psi(t, r, \theta, \phi) = \sum_{\ell, m} \frac{u_l(t, r)}{r} Y_{\ell m}(\theta, \phi)$$

- Even Parity Waves evolving under the equation

$$\Box u_\ell(t, r) - V_{scalar} u_\ell(t, r) = 0$$

where the scalar potential is

$$V_{scalar} = \underbrace{\left(1 - \frac{r_H}{r}\right)}_{f(r)} \left(\frac{\ell(\ell + 1)}{r^2} + \frac{r_H}{r^3} \right)$$

Step 3: Expand Perturbation Metric $h_{\mu\nu}$

- Expand the perturbation in a convenient basis
- Goal: Compute the 'coefficients' of the basis $\rightarrow$ solving PDEs.
- Basis: *Zerilli Tensor Harmonics*
  1. Spherical symmetry $\rightarrow$ should include spherical harmonics
  2. Every element should be orthogonal to every other
- Symmetric Metric $\rightarrow$ 10 independent components

$$h_{\mu\nu}(t, r, \theta, \phi) = \sum_{\alpha} \sum_{\ell m} h^{\alpha}_{\ell m}(t, r) \left(\boldsymbol{t}^{\alpha}_{\ell m}\right)_{\mu\nu} (\theta, \phi) = \left(\begin{array}{c|c|c} S_{1x1} & S_{1x1} & V_{1x2} \\ \hline S_{1x1} & S_{1x1} & V_{1x2} \\ \hline & & \\ V_{2x1} & V_{2x1} & T_{2x2} \end{array}\right)$$

Step 3: Expand Perturbation Metric $h_{\mu\nu}$

$$h_{\mu\nu}(t,r,\theta,\phi) = \sum_{\alpha}\sum_{lm} h^{\alpha}_{lm}(t,r)\left(\mathbf{t}^{\alpha}_{lm}\right)_{\mu\nu}(\theta,\phi) = \left(\begin{array}{c|c|c} S_{1x1} & S_{1x1} & V_{1x2} \\ \hline S_{1x1} & S_{1x1} & V_{1x2} \\ \hline V_{2x1} & V_{2x1} & T_{2x2} \end{array}\right)$$

Polar (Even)

- 7 components
- Transform under partiy: $(-1)^{\ell}$

Axial (Odd)

- 3 components
- Transform under parity: $(-1)^{\ell+1}$

$$h_{\mu\nu} = h^{axial}_{\mu\nu} + h^{polar}_{\mu\nu}$$

Step 4: Fix Perturbation Gauge

- Infinitesimal Coordinate Transformation $x'^{\mu} = x^{\mu} + \xi^{\mu}$
- Equivalent spacetimes are given by metric perturbations that obey

$$h'_{\mu\nu} = h_{\mu\nu} - (\bar{D}_{\mu}\xi_{\nu} - \bar{D}_{\nu}\xi_{\mu})$$

- Split ξ^{μ} into axial-polar
- Fixing Gauge $\equiv$ Conditions on ξ^{μ}: Simplify Equations

$$\left.\begin{array}{ll} \text{1 condition} & \text{Axial} \\ \text{3 conditions} & \text{Polar} \end{array}\right\} \Rightarrow \text{Reduce DoF} \quad 10 \rightarrow 6$$

-

$$\begin{cases} \partial_{\theta}(\sin\theta h_{02}) = -\partial_{\phi}(h_{03}/\sin\theta), & \partial_{\theta}(\sin\theta h_{12}) = -\partial_{\phi}(h_{13}/\sin\theta) \\ h_{23} = 0 & , h_{33} = \sin^2\theta\, h_{22} \end{cases}$$

Numerical Method

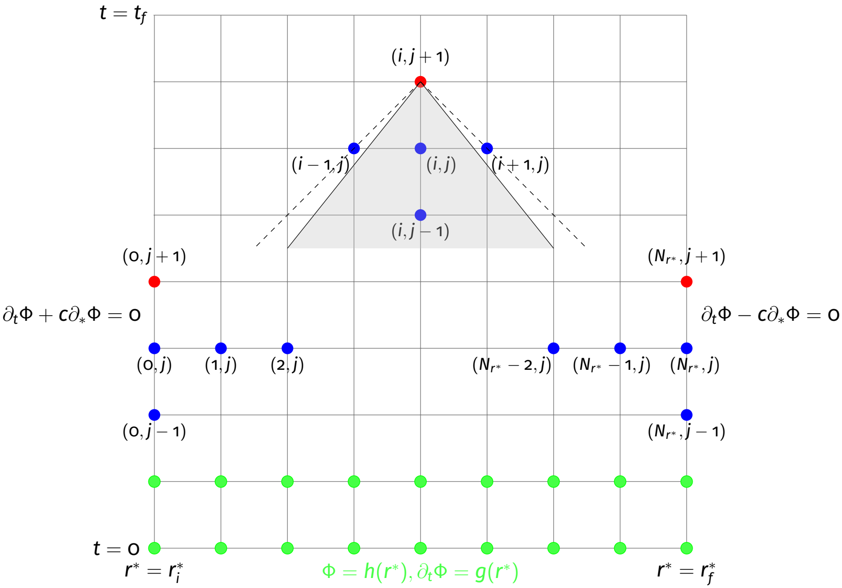
- Staggered Leapfrog

$$\Phi_i^{j+1} = \left(2\Phi_i^j - \Phi_i^{j-1}\right) - CFL^2 \left(\Phi_{i+1}^j - 2\Phi_i^j + \Phi_{i-1}^j\right) - \Delta t^2 V \cdot \Phi_i^j$$

- CFL criterion: $CFL = c\Delta t / \Delta r^* \leq c_{\max}$
- Outgoing Boundary Conditions
  - $$c\frac{\partial u}{\partial n} \pm \frac{\partial u}{\partial t} = 0$$
  - 2nd order upwind discretization

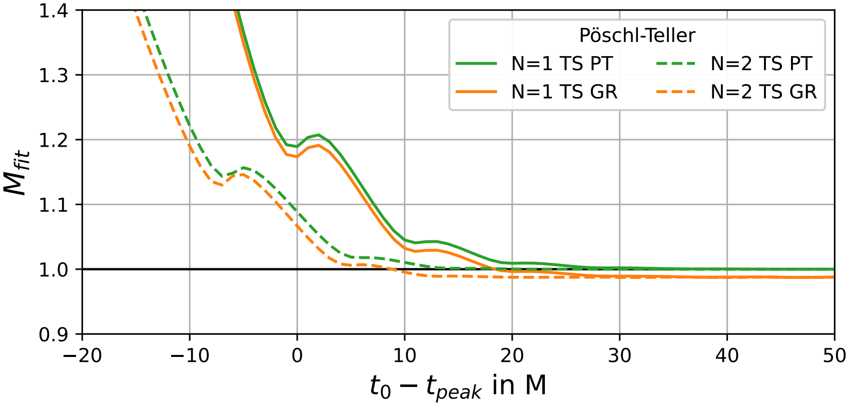
$$\text{left:} \quad \Phi_0^{j+1} = \Phi_0^j + \frac{CFL}{2}\left(-\Phi_2^j + 4\Phi_1^j - 3\Phi_0^j\right)$$

$$\text{right:} \quad \Phi_N^{j+1} = \Phi_N^j - \frac{CFL}{2}\left(3\Phi_N^j - 4\Phi_{N-1}^j + \Phi_{N-2}^j\right)$$



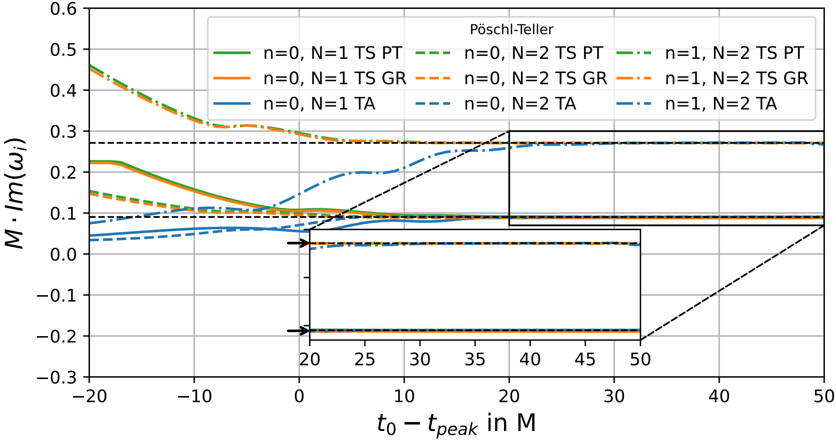
Staggered Leapfrog Stencil (Center), with outgoing BC (left and right), initial conditions determining the first two steps (lower green) and CFL criterion (triangle in shadow).

PT: Mass

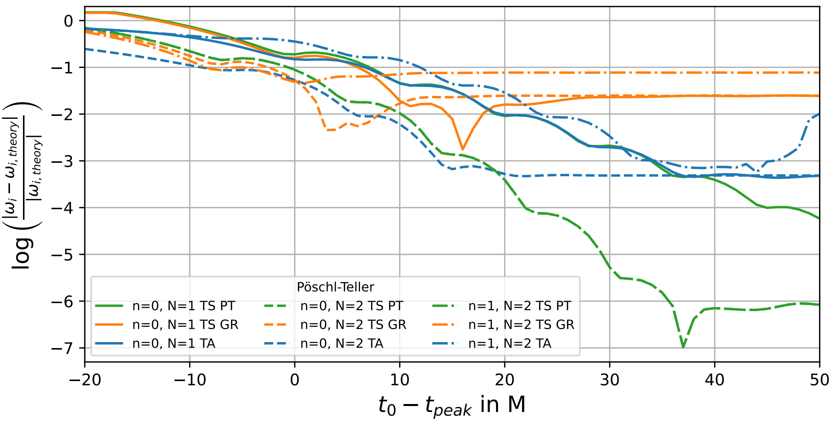


Pöschl-Teller
N=1 TS PT
N=2 TS PT
N=1 TS GR
N=2 TS GR
1.4
1.3
1.2
1.1
1.0
0.9
M_{fit}
−20
−10
0
10
20
30
40
50
$t_0 - t_{peak}$ in M

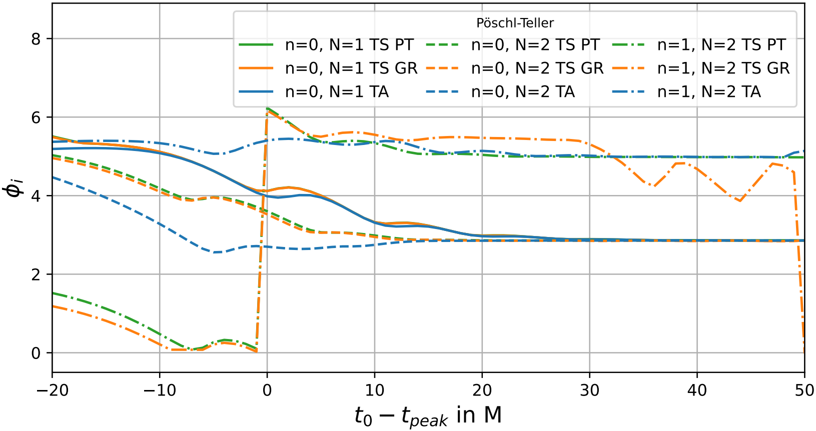
PT: Imaginary



PT: $\delta\omega$

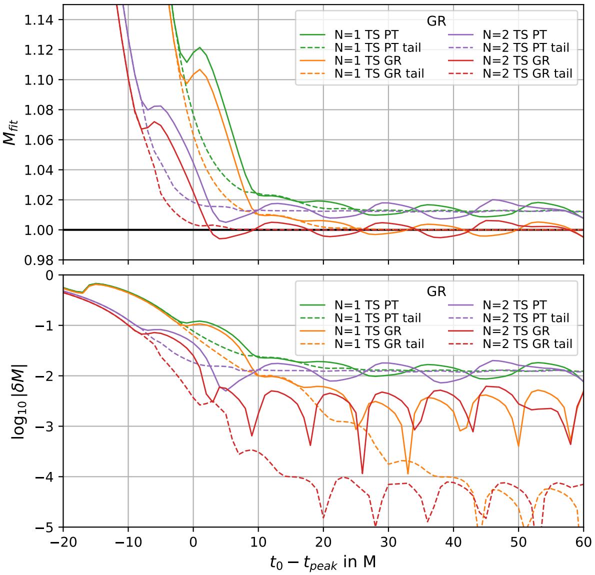


PT: Phase



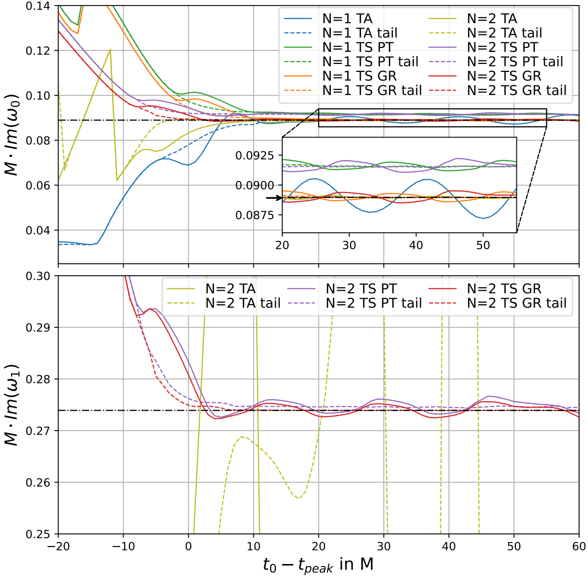
Pöschl-Teller
n=0, N=1 TS PT
n=0, N=1 TS GR
n=0, N=1 TA
n=0, N=2 TS PT
n=0, N=2 TS GR
n=0, N=2 TA
n=1, N=2 TS PT
n=1, N=2 TS GR
n=1, N=2 TA
ϕ_i
$t_0 - t_{peak}$ in M

GR: Mass

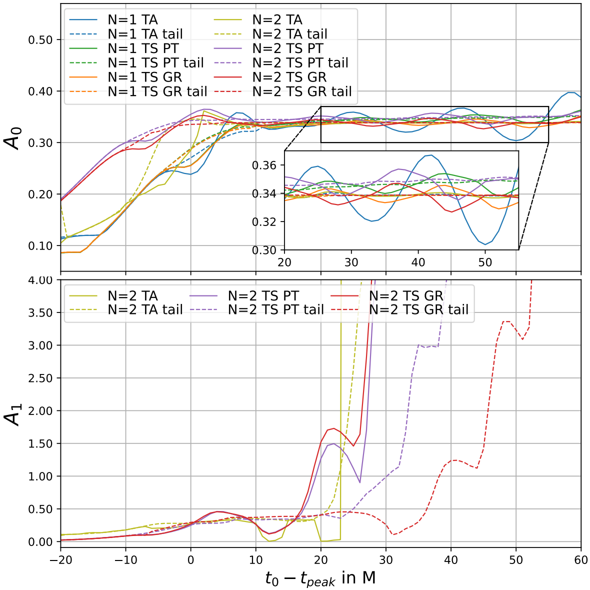


GR
N=1 TS PT
N=1 TS PT tail
N=1 TS GR
N=1 TS GR tail
N=2 TS PT
N=2 TS PT tail
N=2 TS GR
N=2 TS GR tail
M_{fit}
1.14
1.12
1.10
1.08
1.06
1.04
1.02
1.00
0.98
$\log_{10}|\delta M|$
0
−1
−2
−3
−4
−5
−20
−10
0
10
20
30
40
50
60
$t_0 - t_{peak}$ in M

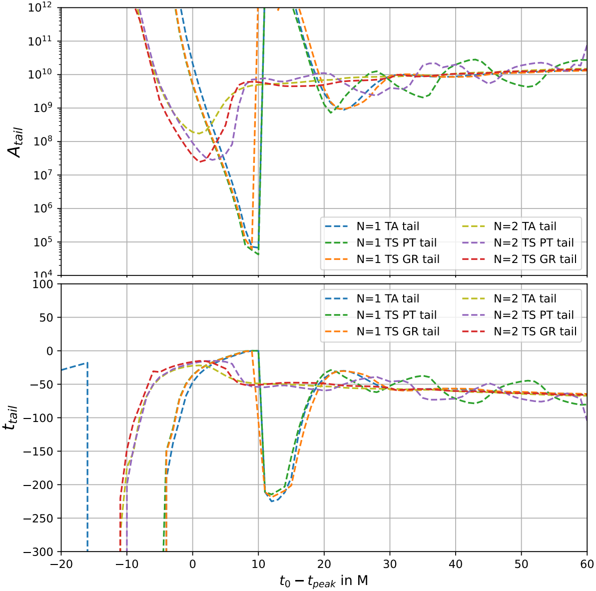
GR: Imaginary



GR: Amplitude

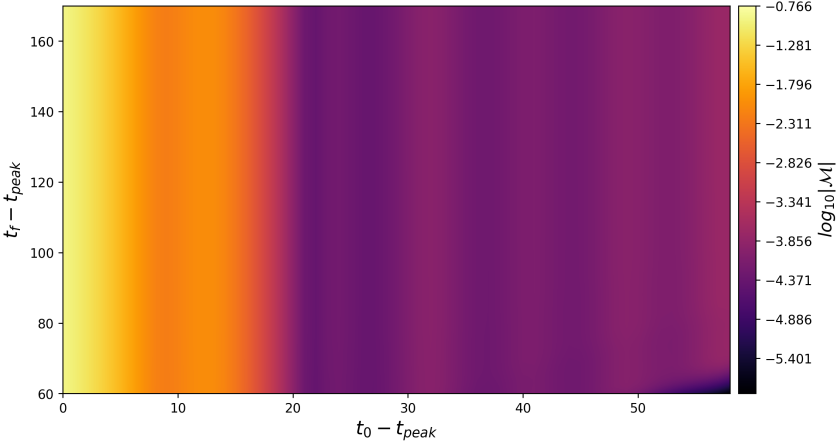


GR: Tail



A_{tail}
t_{tail}
$t_0 - t_{peak}$ in M
N=1 TA tail
N=1 TS PT tail
N=1 TS GR tail
N=2 TA tail
N=2 TS PT tail
N=2 TS GR tail

GR: Variation of fit end time t_f



Map to specific theory: Uncoupled case

- Master equation for Reissner-Nordström BH

$$f\frac{\mathrm{d}}{\mathrm{d}r}\left(f\frac{\mathrm{d}\Psi}{\mathrm{d}r}\right)+\left[\left(1-\frac{r_-}{r_H}\right)^{-2}\omega^2-f\left(V_-+\delta V\right)\right]\Psi=0,$$

with modification from spherically symmetric BH

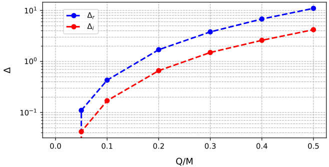
$$\delta V=2\frac{r_-}{r_H}\omega_0^2-\frac{1}{r_H^2}\left(\frac{\lambda+6}{3}\frac{r_-}{r_H}\right)\left(\frac{r_H}{r}\right)^3+\frac{1}{r_H^2}\left(\frac{5}{2}\frac{r_-}{r_H}\right)\left(\frac{r_H}{r}\right)^4,$$

- The amplitudes for each additional power of $1/r$ are

$$\alpha^{(0)}=2\omega_0^2\frac{r_-}{r_H},\qquad \alpha^{(3)}=-\frac{\lambda+6}{3}\frac{r_-}{r_H},\qquad \alpha^{(4)}=\frac{5}{2}\frac{r_-}{r_H}.$$

- If $|Q|\ll M$

$$\omega_{RN-PF}=\left(1-\frac{r_-}{r_H}\right)\left(\frac{2\omega_0}{r_H}+d_0a^{(0)}+d_3a^{(3)}+d_4a^{(4)}\right).$$



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Map to specific theory: Coupled case

- Dynamical Chern-Simons gravity: scalar couples with gravitational field
- The corresponding potentials are

$$
\begin{aligned}
V_{11} =& V^{-}, \\
V_{22} =& V_{scalar} + \frac{1}{r_H^2}\frac{144\pi\ell(\ell+1)}{\beta r_H^4}\left(\frac{r_H}{r}\right)^8, \\
V_{12} =& V_{21} = \frac{1}{r_H^2}\frac{12}{\sqrt{\beta} r_H^2}\sqrt{\pi\frac{(\ell+2)!}{(\ell-2)!}}\left(\frac{r_H}{r}\right)^5,
\end{aligned}
$$

- The amplitudes for each additional power of $1/r$ are

$$
\begin{aligned}
a_{22}^{(8)} =& \bar{\gamma}^2 144\pi\ell(\ell+1), \\
a_{12}^{(5)} =& a_{21}^{(5)} = 12\bar{\gamma}\sqrt{\pi\frac{(\ell+2)!}{(\ell-2)!}},
\end{aligned}
$$

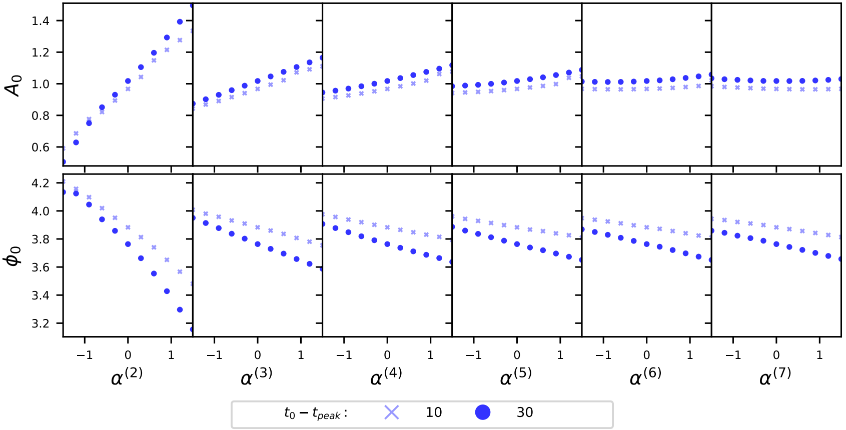
  with $\bar{\gamma} = \beta^{-1/2} r_H^{-2}$.
- Tensor-led modes are

$$
\omega = \omega_0 + e_{(55)}^{1221} a_{12}^{(5)},
$$

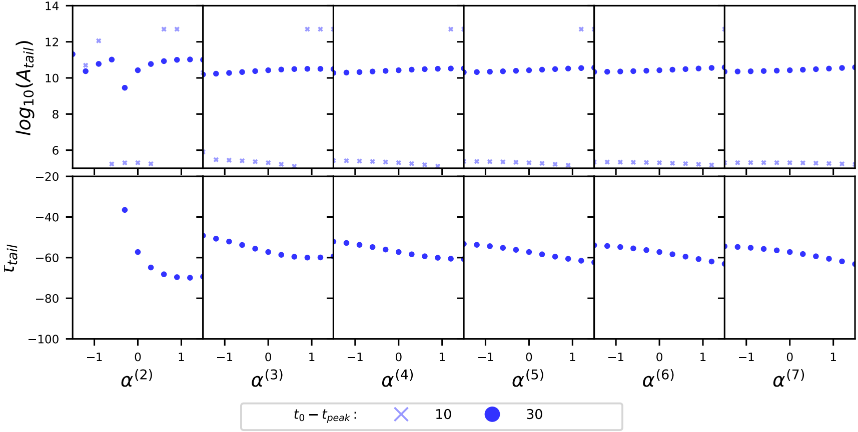
- Scalar-led modes are

$$
\omega = \omega_0 + 2d_{(8)} a_{22}^{(8)} + e_{(88)}\left(a_{22}^{(8)}\right)^2 + e_{(55)}^{1221} a_{21}^{(5)}.
$$

Amplitude & Phase Results

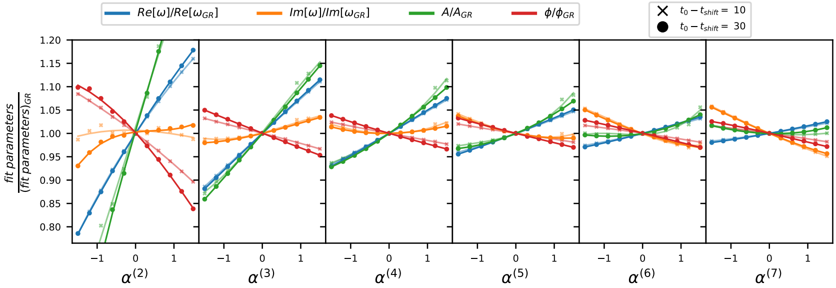


PF: Tail parameters



Comparison with GR values

- Dependance on $t_0 - t_{\rm peak}$
- Robust extraction of $N = 0$ mode frequencies
- Agreement with perturbative prediction



PF: Comparison with GR values