The emergence of spacetime in IIB string theory

A study of the complex Langevin method applied on the IKKT model and variants

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To my beloved Eleni, to whom I owe my connection with timespace.

Contents

Notation

$\mathbb{N} = \{0, 1, 2, \ldots\}$: natural numbers
$\mathbb{N}_n = \{1, 2, \ldots, n\}$: natural numbers up to n, $\forall n \in \mathbb{N}$
$\mathbb{Z} = \{\ldots, -3, -2-1, 0, +1, +2, +3, \ldots\}$: integer numbers
$\mathbb{N}_* = \mathbb{Z}_+ = \mathbb{N} \setminus \{0\}$: natural numbers not including 0
$\mathbb{Z}_* = \mathbb{Z} \setminus \{0\}$: integer number not including 0
$\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} = \{0, 1, \ldots, n-1\}$, $\forall n \in \mathbb{N}_*$

$\mathbb{Q}$: rational numbers
$\mathbb{Q}_* = \mathbb{Q} \setminus \{0\}$: rational numbers not including 0

$\mathbb{R} =]-\infty, +\infty[$: real numbers
$\mathbb{R}_* = \mathbb{R} \setminus \{0\} =]-\infty, 0[\cup]0, \infty[$: non-zero real numbers
$\mathbb{R}_+ =]0, +\infty[$:[1] non–negative real numbers
$\overline{\mathbb{R}} = [-\infty, +\infty] = \mathbb{R} \cup \{\infty\}$:[2] compact real numbers
$\overline{\mathbb{R}}_+ = [0, +\infty] = \{0\} \cup \mathbb{R}_+ \cup \{\infty\}$:[3] compact non–negative real numbers

$\mathbb{C}$: complex numbers with $\imath$ denoting the imaginary unit
$\mathbb{C}_* = \mathbb{C} \setminus \{0 + \imath 0\}$: non-zero complex numbers
$\overline{\mathbb{C}}$: compact complex numbers

$a, b, c, \ldots,$: lowercase Latin letters either denote set elements or functions
$\alpha, \beta, \gamma, \ldots$: lowercase Greek letters either denote set functions or random variables or stochastic processes
$A, B, C, \ldots$: uppercase Latin letters denote sets or random variables or stochastic processes
$\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$: calligraphic Latin letters denote collections of sets
$\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$: Fraktur Latin letters denote families of collections of sets

[1] 0 is not included to form an Abelian group under multiplication, also the range of log.
[2] ∞ is really one topologically, reachable from either positive of negative real numbers.
[3] 0 can now be included under the group multiplication with ∞ its multiplicative inverse.

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Introduction

Superstring theory has been studied intensively as a unified theory that includes quantum gravity. The theory is defined in 10 spacetime dimensions and the connection to the real world — where only 4 dimensions are macroscopic — is realized via compactification of the extra dimensions. How this can actually occur has been investigated perturbatively by using D–brane configurations as a background, leading to tremendously many vacua giving rise to the so–called string landscape. Clearly, it is important to see if this picture remains valid when the issue is addressed in a non–perturbative manner.

The type IIB matrix model, also known as the IKKT model [1], is regarded as one of the most promising candidates for a non–perturbative formulation of superstring theory. The model is defined by dimensionally reducing either 10–dimensional $\mathcal{N} = 1$ super Yang–Mills theory to 0 dimensions or the $\mathcal{N} = 2$ type IIB superstring Green–Schwartz formulation in the Schild gauge [2, 3, 4]. Therefore, spacetime does not exist a priori in this model. The eigenvalues of the bosonic matrices stand as the spacetime coordinates, inferring that spacetime is generated dynamically from the bosonic degrees of freedom of the matrices [5]. Since type IIB superstring theory is defined in 10 dimensions, it is important to understand how our 4–dimensional spacetime emerges by studying this model, hinting that compactifications of extra dynamics is an non–perturbative inherent property of type IIB superstring theory dynamics.

Various attempts have been made to address the emergence of 4–dimensional spacetime from the 10–dimensional background of type IIB superstring theory. In the Lorentzian version of the IKKT matrix model, the indices are contracted by the Minkowski metric

$$\eta = \begin{pmatrix} -1 & & \cdots & \\ & +1 & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ & & \cdots & +1 \end{pmatrix},$$

and the action has the SO_{9+1} Lorentzian symmetry.[1] The bosonic action is unbounded from below, making the (unmodified) study of the Lorentzian model a priory difficult, which is why efforts have been focused on the Euclidean version [6, 7, 8, 9, 10, 11, 12, 13, 14, 15] instead, defined by making a Wick rotation with respect to (analytical continuation to) the temporal direction, and contracting the indices by the Euclidean metric

$$\mathbb{1} = \begin{pmatrix} +1 & & \cdots & \\ & +1 & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ & & \cdots & +1 \end{pmatrix}.$$

The Euclidean version has the SO_{10} rotational symmetry instead of the Lorentzian SO_{9+1}, and it is amenable to numerical simulations because the partition function is finite without any cutoffs [16, 17]. However, it suffers from a severe sign problem, which appears after integrating out the fermions. The complex Pfaffian,[2] stemming from integrating the fermion degrees of freedom out the model into an effective bosonic model, plays a central role in the spontaneous symmetry breaking (SSB) of the SO_{10} rotational symmetry [19, 20]. In models where there are no fermionic degrees of freedom, like the bosonic model, or the Pfaffian is real positive, like in the 4–dimensional supersymmetric toy model, there is no SSB of the rotational symmetry [6, 7, 21]. There is no SSB, either, in the phase–quenched model, which omits the complex phase of the Pfaffian [13], implying that the SSB of the rotational symmetry might be an effect of the model complex phase, warranting its study

[1] Which is a (hyperbolic–rotational) subgroup of the Poincaré group of Minkowski spacetime symmetries.
[2] Which is a determinant in the simplified 4–dimensional [18] or 6–dimensional supersymmetric models [14]).

by re–weighting for instance, in which case the important configurations are generally different between the original model and its phase–quenched counterpart, leading to a severe overlap problem. To reduce this problem, the factorization method [9, 10, 11, 12, 13] simulates a constrained system, in which the expectation value of the phase factor is calculated to determine the true vacuum. The results are consistent with the SSB pattern of $SO_{10} \to SO_3$ predicted using the Gaussian expansion method (GEM) [22, 23]. While this is an interesting dynamic property, its relevance to the emergence of a 4–dimensional spacetime is unclear.[3]

This observation led to Monte Carlo simulations of the Lorentzian IKKT model [24, 25, 26, 27]. The problem of the unbounded bosonic action was solved using separate cutoffs in the temporal and spatial directions [24]. Although the Pfaffian is real, the model has a severe sign problem as well, due to the bosonic part of the action S_{boson} , which appears with a factor $\exp \imath S_{\text{boson}}$ in the partition function. To avoid it, the authors in [24] used an approximation, and they found that 3 out of 9 spatial dimensions start to expand after a critical time. That moment — which results from the dynamics of the model — may be identified as the birth of the universe. Later works [25, 26] computed the expansion rate of the universe numerically, which starts with an exponential–law expansion at early times, followed by a power–law expansion at late times. However, the authors in [27] found that it is simply an effect of the domination of almost 3–dimensional configurations with a singular Pauli–matrix structure, because of the approximation.

Monte Carlo methods have been playing a crucial role in non–perturbative studies of quantum field theories and statistical systems relevant to particle, nuclear and condensed matter physics. However, in many interesting cases, it happens that such methods cannot be applied straightforwardly because the effective Boltzmann weight appearing in expectation values of the form

$$\langle O \rangle = \int O dw$$

can become negative or even complex. A brute–force method would be to use the absolute value $|w|$ of the weight in generating configurations and to treat the phase as an observable,[4]

$$\langle O \rangle = \frac{\langle O \exp \imath \arg w \rangle_0}{\langle \exp \imath \arg w \rangle_0},$$

with

$$\langle O \rangle_0 = \int O d|w|.$$

This re–weighting method indeed works for small systems, but the computational complexity is exponential to the system size due to huge cancellations among configurations, which is commonly referred to as the sign problem.

In recent years there has been major progress in evading the sign problem by complexifying the dynamical variables, which are supposed to be real in the original system, leading to the development of two approaches:

- The generalized Lefschetz–thimble method [28, 29, 30, 31], which amounts to deforming the integration contour in such a way that the complex phase $\arg w$ becomes mild enough to be handled by the re–weighting method.
- The complex Langevin method (CLM) [32, 33], which attempts to define a stochastic process for the complexified variables so that the expectation values with respect to this process are equal to the expectation values defined in the original system, extending the idea of stochastic quantization [34].

In both approaches, holomorphy plays a crucial role. The advantage of the CLM compared to the other one, is that it is computationally less costly, which enables its application to much larger system size. The disadvantage, on the other hand, is that the equivalence to the original system is non–trivial. Progress in this direction was made by clarifying the conditions for the equivalence [35, 36, 37, 38, 39, 40] and by inventing new techniques that made it possible to meet these conditions for a larger space of parameters [18, 41, 42, 43, 44, 45, 46, 47].

In this work [14, 15] we applied the CLM to the Euclidean version of the type IIB matrix model, and reproduced the SSB of the SO_{10} symmetry to SO_3. The application of the CLM to the Lorentzian version [48, 49, 50] may elucidate the spacetime structure that emerges when we exclude the approximation to avoid the sign problem. While this is still an ongoing work, there are some preliminary results [50] that look quite promising.

[3] A priori the 3–dimensional surviving symmetry contradicts the 4–dimensional ansatz, however remembering that this is a Euclidean model result, it becomes unclear if this is only part of the SSB that is sought for. For example, this surviving 3–dimensional symmetry may simply correspond to the spacial part of the expected 4–dimensional spacetime.

[4] $\forall w = \exp z \in \mathbb{C}$ formally, its modulus is $|w| = \exp \Re z$ and its phase is $\arg w = \Im z$.

This thesis is structured as:

Part I includes a pedagogical preview of the mathematical machinery behind stochastic calculus, which is the foundation of the methodology studied. This part should be skipped to chapter 4 for readers familiar with measure and probability theory and or stochastic calculous.

chapter 1 presents the fundamental mathematical structures used throughout this thesis, mainly to establish a notational convention as much as revise core concepts. This includes

- elements of abstract algebra
- elements of analysis and related topics of general topology and metric spaces
- elements of differential geometry and connections to algebra

chapter 2 moves on to develop the core concepts behind probability theory and how they extend to stochastic processes. The starting point is elements of measure theory and how it expands into an applicable theory of probability, which stands as the groundwork for defining stochastic properties.

chapter 3 expands on the topic of stochastic differential equations and technical issues behind solving them into stochastic processes that are "well–behaved". Stochastic calculus stems from merging notions of analysis with that of measure theory. Only elements roughly relevant to the Langevin equation are presented here.

chapter 4 applies aforementioned mathematical background in the context of complex stochastic differential equations and processes, which stand as a precursor of the CLM used in this work. This chapter is based on the concept of stochastic quantization as pioneered by Parisi, Wu and Klauder [32, 33, 34].

Part II focuses on the physics problem addressed in this work and solutions, as well as a preview of the related background behind it.

chapter 5 is a brief revision of core elements of (classical) field theory, provided here for notational conventions mostly. Readers familiar with quantum field theory may skip this chapter.

chapter 6 contains elements of string theory somewhat relevant to the origin of the IKKT model as well as the dynamical compactification of extra dimensions. Readers familiar with string theory will find here how the IKKT model connects to type IIB superstring theory, as well as what issues are addressed by matrix models (dynamical compactification of extra dimensions) and what issues are not (for example the moduli space of string theories).

chapter 7 contains a thorough presentation of the CLM applied in the context of the IKKT model, together with technical aspects that arise when working with field theories (for example exploiting gauge symmetries of fields). This chapter previews the backbone of the methodology used in the research of this doctorate study.

chapter 8 contains the results of the study of the Euclidean IKKT model using the CLM. For completion, the results of the 4–dimensional toy model are included together with the 6–dimensional toy model and the 10–dimensional true model, which are part of this dissertation.

chapter 9 contains the results of the study of the Lorentzian IKKT model using the CLM, with various modifications to the model in exploring the dynamical generation of a Lorentzian spacetime.

Supplementary material for other type IIB superstring theory matrix models can be found in [51, 52, 53, 54, 55, 56]. Further details about the history and evolution of the IKKT matrix model can be found in [4, 57, 58, 59, 60, 61, 62, 63].

A compulsory greek synopsis of the thesis is appended at the end.

Part I.

Background

1. Mathematical foundation

1.1. Algebra

Foundations

Relations

Definition 1.1.1. A binary relation $\sim$ on a set R for which:

- $\forall a \in R$, $a \sim a$ (reflexivity),
- $\forall a, b \in R$, $a \sim b$ if and only if $b \sim a$ (symmetry),
- $\forall a, b, c \in R$ with $a \sim b$ and $b \sim c$, $a \sim c$ (transitivity),

is an equivalence relation on R.

An equivalence relation $\sim$ partitions R into disjoint sets or equivalence classes, defined (and denoted) by

$$\frac{R}{\sim} = \{b \in R | b \sim a\}.$$

Indeed, suppose that $\exists c$ with $c \in [a]$ and $c \in [b]$ and $[a] \neq [b]$. Then, $c \sim a$ and $c \sim b$, hence by symmetry and transitivity $a \sim b$, hence $a \in [b]$ and $b \in [a]$, and since the same holds for all of $[a]$ and all $[b]$, $[a] = [b]$.

Definition 1.1.2. A binary relation $\leq$ on a set R for which:

- $\forall a \in R$, $a \leq a$ (reflexivity),
- $\forall a, b, c \in R$ with $a \leq b$ and $b \leq c$, $a \leq c$ (transitivity),
- $\forall a, b \in R$ with $a \leq b$ and $b \leq a$, $a = b$ (antisymmetry),
- $\forall a, b \in R$, $a \leq b$ or $b \leq a$ (strong connectivity),

is a total ordering on R.

R with a total ordering $\leq$ (implicitly) has a strict total ordering $<$ too:

- $\forall a \in R$, $a \not< a$ (non–reflexivity),
- $\forall a, b, c \in R$, with $a < b$ or $b < c$, $a < c$ (transitivity),
- $\forall a, b \in R$ with $a \neq b$, $a < b$ or $b < a$ (strong connectivity),

which is implied by $\leq$ with the definition:

- $\forall a, b \in R$ with $b \not\leq a$, $a < b$.

If $\exists x \in R$ such that $\forall a \in R$, $a \geq x / a \leq x$, R is bounded from below/above with respect to the total ordering $\leq$. The same definition applies with respect to the strict total ordering $<$. A strict total ordering bound is a total ordering bound and it is unique.

Sets with operation(s)

Definition 1.1.3. A magma is a set G closed under a binary operation $\circ : G \times G \longrightarrow G$.

A monoid is a magma $(G, \circ)$ whose binary operation $\circ$ satisfies:

- $\forall a, b, c \in G$, $a \circ (b \circ c) = (a \circ b) \circ c$ (associativity),
- $\exists \mathbb{1} \in G$ unique, such that $\forall a \in V$, $\mathbb{1} \circ a = a \circ \mathbb{1} = a$ (identity),

$(G, \circ)$ is a group is on top

- $\forall a \in G$, $\exists x \in G$ unique, such that $a \circ x = x \circ a = \mathbb{1}$ (divisibility).

$(G, \circ)$ is an Abelian group if on top

- $\forall a, b \in G$, $a \circ b = b \circ a$ (commutativity),

Definition 1.1.4. A field is a set $\mathbb{K}$ closed under two binary operations $+, \cdot : \mathbb{K} \times \mathbb{K} \longrightarrow \mathbb{K}$ such that:

- $\mathbb{K}$ is an Abelian group with operator $+$, identity 0 and, $x = -a$ inverse $\forall a \in \mathbb{K}$,
- $(\mathbb{K} \setminus \{0\}, \cdot)$ is an Abelian group with identity 1 and $x = a^{-1}$ inverse $\forall a \in \mathbb{K}$,
- $\forall a, b, c \in \mathbb{K}$, $a(b + c) = ab + ac$ or $(a + b)c = ac + bc$ (distributivity).

Excluding 0 from multiplication implies it should either be undefined, or $\forall a \in F$, $a0 = 0a = 0$ (destructibility), which in turn implies that $\forall a, b \in F$ with $ab = 0$, $a = 0$ or $b = 0$.

Definition 1.1.5. A vector space over a field $\mathbb{K}$ is a set V closed under addition $+ : V \times V \longrightarrow V$ and scalar multiplication $\cdot : \mathbb{K} \times V \longrightarrow V$, such that:

- V is an Abelian group with operator $+$, identity 0_V and inverse $-x$, $\forall x \in V$,
- All group properties of field multiplication in $\mathbb{K}$ except for divisibility hold for the scalar multiplication:
  - $\forall a, b \in \mathbb{K}$ and $\forall x \in V$, $a(bx) = (ab)x$ (scalar associativity),
  - $\forall x \in V$, $1x = x$ (scalar identity),
  - $\forall a \in \mathbb{K}$ and $\forall x, y \in V$, $a(x + y) = ax + ay$ (vector distributivity),
  - $\forall a, b \in \mathbb{K}$ and $\forall x \in V$, $(a + b)x = ax + bx$ (scalar distributivity).

  also implying:
  - $\forall a \in \mathbb{K}$ and $\forall x \in V$ with $ax = 0_V$, $a = 0$ or $x = 0_V$.

Thus a field $\mathbb{F}$ is its own vector space by definition 1.1.5.

A linear subspace $U \leq V$ is a subset of V (finitely)[1] closed under the operations of vector space V. $\forall A \subseteq V$, span A is the smallest subspace $U \leq V$ with $A \subseteq V$. $\dim V \leq |A|$ with equality (and thus definition) holding if A is a (not unique) smallest subset for which span $A = V$. These concepts change when relaxing the finiteness constraint.

Definition 1.1.6. An algebra over a field $\mathbb{K}$ is a vector space K, additionally closed under binary multiplication $\circ : K \times K \longrightarrow K$ (magma), such that

- $\forall x, y, z \in K$, $x \circ (y + z) = x \circ y + x \circ z$ and $(x + y) \circ z = x \circ z + y \circ z$ (left and right distributivity),[2]
- $\forall a, b \in \mathbb{K}$ and $\forall x, y \in K$, $(ax) \circ (by) = ab(x \circ y)$ (product compatibility).

Additionally,

[1] An alternative definition of span A is all the (finite) linear combinations of elements in A. This may be different to all the linear combinations of elements in A, if A is infinite.

[2] Notice how "and" is used instead of "or", because unlike in a field, an algebra is generally (and usually) non–commutative.

- If K is a monoid with $\circ$, K is a unital algebra,
- If $\circ$ is associative, K is an associative algebra,
- If $\circ$ is commutative, K is a commutative algebra,
- If $\circ$ is anti–commutative, i.e. $\forall x, y \in K$, $x \circ y + y \circ x = 0$, K is a Grassman algebra.

If K is a group with $\circ$, $\circ = \cdot$ and K simply reduces to a field. Thus a field $\mathbb{F}$ is its own (associative, unital, commutative) algebra.

Definition 1.1.7. A Grassman number over a field $\mathbb{K}$ is defined $\forall a, b \in \mathbb{K}$, as $a + b\ell$ with $\ell^2 = 0$. Grassman numbers form a Grassman algebra U over $\mathbb{K}$ with the algebra operations defined by:

- $\forall \phi = \phi_0 + \phi_1\ell, \psi = \psi_0 + \psi_1\ell \in U$, and $\forall a, b \in \mathbb{K}$, $a\phi + b\psi = a(\phi_0 + \phi_1\ell) + b(\psi_0 + \psi_1\ell) = (a\phi_0 + b\psi_0) + (a\phi_1 + b\psi_1)\ell$,
- $\forall \phi = \phi_0 + \phi_1\ell, \psi = \psi_0 + \psi_1\ell \in U$, $\phi \circ \psi = (\phi_0 + \phi_1\ell)(\psi_0 + \psi_1\ell) = \phi_0\psi_0 + (\phi_0\psi_1 + \phi_1\psi_0)\ell$.

The concept of a base generalizes to that of a generating set $A \subset K$, such that $\forall x \in K$, there exists a combination of elements in A that generate x via any or all of the operations of the algebra K. The usual process is to span a base B out of the generators A via $\circ$, then simply $\operatorname{span} B = K$. $\dim K = |B|$, not $|A|$.

For non–commutative algebras K the commutator operator is defined via

$$[\cdot|\cdot] : K \times K \longrightarrow K : x, y \longmapsto [x|y] = x \circ y - y \circ x,$$

which is identically 0 for commutative algebras.

Operators

Definition 1.1.8. A linear operator $F : V \longrightarrow U : x \longmapsto Fx$ from a vector space V to a vector space U, both over a field $\mathbb{K}$, satisfies

- $\forall a, b \in \mathbb{K}$ and $\forall x, y \in V$, $F(ax + by) = aFx + bFy$ (linearity).

Example 1.1.9. The set $\mathcal{L}(V, U)$ of all linear operators $F : V \longrightarrow U$ is a vector space under function addition and scalar multiplication as

$$\forall a, b \in \mathbb{K},\ \forall x, y \in V \text{ and } \forall F \in \mathcal{L}(V, U),\ F(ax + by) = F(ax + by) = aFx + bFy.$$

Moreover, $\mathcal{L}(V, V) = \mathcal{L}(V) \simeq \mathrm{M}_{\dim V}\mathbb{K}$ is an (associative unital) algebra under function composition as,

$$\forall x \in V \text{ and } \forall F, G \in \mathrm{M}_{\dim V}\mathbb{K},\ (F \circ G)x = F(Gx) = FGx,$$

where $\mathrm{M}_{\dim V}\mathbb{K}$ is the space of all $\dim V \times \dim V$ (square) matrices representing linear operators in $\mathcal{L}(V)$.

Definition 1.1.10. An operator $F : V \longrightarrow U$:

- such that $F(V) \leq U$, is a homomorphism,
- such that $F(V) = U$, f is an epimorphism,
- such that F is injective, F is a monomorphism and $V \lesssim U$,
- F is both a monomorphism and epimorphism (implying it is bijective), F is an isomorphism and $V \simeq U$.

Tensors

$\forall V, U$ vector subspaces with $V \cap U = \emptyset$, their Cartesian product $V \times U$ is identified as their direct sum $V \oplus U$, while

$$V \otimes U = \{x \otimes y \in \mathcal{L}(U, V) | \forall x \in U \text{ and } \forall y \in V\},$$

with $x \otimes y : V \longrightarrow U : a \longmapsto x(y \cdot a)$.[3]

Definition 1.1.11. $\forall \mathcal{V}$ a finite collection of vector spaces,

$$T = \bigotimes \mathcal{V} = \bigotimes_{V \in \mathcal{V}} V$$

is a tensor space of $\operatorname{rank} T = |\mathcal{V}|$. $\forall x \in T$, x is a tensor.

$\forall U, V$ algebraic structures of the same type (groups, fields, vectors spaces, algebras etc) such that $U \subseteq V$, U is a subconstruct (of the corresponding type) of V, denoted as $U \leq V$.

Definition 1.1.12. $\forall V$ a vector space over a field $\mathbb{K}$, and $\forall n \in \mathbb{N}$,

$$\mathrm{T}^n V = \bigotimes_{i=1}^{n} V$$

is the rank n tensor space generated by V.

The direct sum

$$\mathrm{T}V = \bigoplus_{n \in \mathbb{N}} \mathrm{T}^n V = \bigoplus_{n \in \mathbb{N}} \bigotimes_{i=1}^{n} V$$

together with the product, such that $\forall x = (x_n)_{n \in \mathbb{N}}, y = (y_n)_{n \in \mathbb{N}} \in \mathrm{T}V$,[4]

$$xy = (x_n \otimes y_n)_{n \in \mathbb{N}},$$

forms the so called free (graded) tensor algebra generated by V [64].

Linear equations

Equality $=$ in vector spaces and their respective operator algebra can be used to pose questions in the form of (linear) equations.

Definition 1.1.13. $\forall F : V \longrightarrow U$ linear operator between vector spaces V and U, and $\forall y \in F(V) \leq U$, $\exists A \subseteq V$ such that $\forall x \in A$,[5] $Fx = y$.

For the homogeneous case $Fx = 0$,[6] the solution $A = \ker F \leq V$ is a subspace of $V$and called the kernel of F. $\forall x_0$ solution of $Fx = y$, $A = \{x + x_0 | \forall x \in \ker F\}$.

The following are equivalent $\forall F : V \longrightarrow U$:

- $Fx = y$ has an exact solution $\{x_0\}$,
- $\ker F = \{0_V\}$ or $\dim \ker F = 0$,
- $\exists! F^{-1} : F(V) \longrightarrow U$ and thus $\det F \neq 0$, and $x = F^{-1}y$.[7]

[3] See definition 1.2.6 for the definition of an inner product on a vector space in general and definition 1.1.16 specifically for the standard inner product in vector spaces, used here.
[4] $\forall n, m \in \mathbb{N}$, $\mathrm{T}^n V \otimes \mathrm{T}^m V \simeq \mathrm{T}^{n+m} V$.
[5] Note that $y \in F(V)$, and not $y \in U$ generally, is essential for the existence of a solution $A \subseteq V$.
[6] Note that $\forall B \leq U$, $0 \in B$, as a vector (sub)space.
[7] $F : V \longrightarrow U$ injective means its representations are square matrices. This also implies that $V \lesssim U$.

Eigenvalues

Definition 1.1.14. $\forall F : V \longrightarrow V$ operator on a vector space V on a field $\mathbb{K}$, $\forall x \in V$ and $\forall \lambda \in \mathbb{K}$ such that,

$$Fx = \lambda x,$$

x is an eigenvector of F and λ an eigenvalue of F.

$F - \lambda \mathbb{1}$ being singular ($\dim \ker F > 0$) is the definitional requirement for non–zero eigenvectors. Since eigenvectors are non–zero solutions to the homogeneous equation $(F - \lambda \mathbb{1})x = 0$, they form a whole subspace $\ker(F - \lambda \mathbb{1}) \leq V$ corresponding to the eigenvalue λ.

The direct sum of all eigenvector subspaces forms the original vector space V,

$$V = \bigoplus\nolimits_{\lambda \text{ eigenvalue of } F} \ker(F - \lambda \mathbb{1}),$$

therefore F can have at most $\dim V$ eigenvalues if $\dim V < \infty$.

Representations

Definition 1.1.15. $\forall V$ vector space, a smallest subset $A \subset V$ for which $\operatorname{span} A = V$, if it exists,[8] is a base of V.

If $|A| < \infty$, $\dim V = |A|$ and $A = \{a_n\}_{n=1}^{\dim V}$ $\forall x \in V$,

$$x = \sum\nolimits_{n=1}^{\dim X} x_n a_n.$$

Similarly, for $A = \{a_n\}_{n \in \mathbb{N}}$ countable and V complete,

$$x = \sum\nolimits_{n \in \mathbb{N}} x_n a_n,$$

where $(x_n)_{n=1}^{\dim V} \in \mathbb{R}^{\dim V}$. Both cases can be expressed with the generic form

$$x = \sum\nolimits_{a \in A} x_a a. \tag{1.1.1}$$

Definition 1.1.16. $\forall V$ finite dimensional vector space, x_a stands for the base–invariant vector $x \in V$. In the same convention, pair of repeating indices imply summation, for example, $\forall x, y \in V$,

$$x \cdot y = x_a y_a.$$

F_{ab} stands for the base–invariant $F \in \mathcal{M}_{\dim V} \mathbb{K}$ operator on V. If F is hermitian for example,

$$x \cdot F \cdot y = x_a F_{ab} y_b.$$

If a new inner product is defined via a hermitian (and positive definite for Euclidean spaces) operator $g \in \mathcal{M}_{\dim V} \mathbb{K}$,

$$x \cdot_g y = x_a g_{ab} y_b.$$

It also applies on vector level

$$Fx|_a = F_{ab} x_b,$$

which is the familiar matrix product stemming from operator action. In this case, the *free* index a is mandatory and denotes that this is a vector relation. In this notation, a symbol without indices is a scalar. Repeating indices are not free (as they are summed over).

Operator composition is represented by a matrix product, $\forall F, G : V \longrightarrow V$,

$$F \circ G|_{ab} = F_{ac} G_{cb}$$

[8] The most common use cases are finite–dimensional vector spaces V, for which $\dim V = |A|$, and complete infinite–dimensional (see section §1.2. Analysis) vector spaces.

Example 1.1.17. The identity operator $\mathbb{1} : V \longrightarrow V$ has Kronecker's delta as an index form, $\forall x \in V$,[9]

$$\mathbb{1}x = x \text{ or } \mathbb{1}x|_a = \delta_{ab}x_b = x_a.$$

Another example is the antisymmetric tensor ε with $\operatorname{rank}\varepsilon \in \mathbb{N}$ such that

$$\varepsilon_{x_1\dots x_{\operatorname{rank}\varepsilon}} = \begin{cases} +1 & x_1 \dots x_{\operatorname{rank}\varepsilon} \text{ is an even permutation of } \mathbb{N}_{\operatorname{rank}\varepsilon} \\ -1 & x_1 \dots x_{\operatorname{rank}\varepsilon} \text{ is an odd permutation of } \mathbb{N}_{\operatorname{rank}\varepsilon} \\ 0 & \text{otherwise} \end{cases},$$

the simplest non–trivial case being $\operatorname{rank}\varepsilon = 2$ with $\varepsilon_{01} = +1$, $\varepsilon_{10} = -1$ and $\varepsilon_{00} = \varepsilon_{11} = 0$.

$\forall F : V \longrightarrow V$ linear operator, $\det F = \varepsilon_{x_1\dots x_{\dim V}} F_{x_1\dots x_{\dim V}}$.

Index squashing

$\forall T$ tensor with $\operatorname{rank} T \in \mathbb{N}$, its symmetrized version is written as

$$T_{(x_1\dots x_{\operatorname{rank}T})} = \frac{1}{(\operatorname{rank}T)!}\sum\nolimits_{(\sigma:\mathbb{N}_{\operatorname{rank}T}\longrightarrow\mathbb{N}_{\operatorname{rank}T})\in S_{\operatorname{rank}T}} T_{(x_{\sigma(1)}\dots x_{\sigma(\operatorname{rank}T)})},$$

and its antisymmetrized version as

$$T_{[x_1\dots x_{\operatorname{rank}T}]} = \frac{1}{(\operatorname{rank}T)!}\sum\nolimits_{(\sigma:\mathbb{N}_{\operatorname{rank}T}\longrightarrow\mathbb{N}_{\operatorname{rank}T})\in S_{\operatorname{rank}T}} \epsilon_{\sigma(1)\dots\sigma(\operatorname{rank})} T_{(x_{\sigma(1)}\dots x_{\sigma(\operatorname{rank}T)})},$$

where $\forall n \in \mathbb{N}$, S_n is the group of all permutations of n numbers.

For $\operatorname{rank}T = 2$,

$$T_{(xy)} = \frac{1}{2}(T_{xy} + T_{yx}) \text{ and } T_{[xy]} = \frac{1}{2}(T_{xy} - T_{yx}).$$

1.2. Analysis

Topology

Definition 1.2.1. A collection of subsets $\mathcal{T}$ on a set V such that:

- $\emptyset \in \mathcal{T}$ and $V \in \mathcal{T}$,
- $\forall \mathcal{A} \subseteq \mathcal{T}$ subcollection of $\mathcal{T}$,
$$\bigcup \mathcal{A} = \bigcup\nolimits_{A\in\mathcal{A}} A \in \mathcal{T},$$
- $\forall \mathcal{A} \subseteq \mathcal{T}$ finite subcollection of $\mathcal{T}$,
$$\bigcap \mathcal{A} = \bigcap\nolimits_{A\in\mathcal{A}} A \in \mathcal{T}.$$

is a topology (of open subsets) on V. A set V equipped with a topology $\mathcal{T}$ is a topological space $(V, \mathcal{T})$.

$\forall U \subseteq V$, $\mathcal{T}_U = \{A \cap U | A \in \mathcal{T}\}$ is the induced on U topology of V.[10]

[9] Kronecker's δ is the discrete version of Dirac's δ distribution and outright definable as

$$\delta_{xy} = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}.$$

[10] $U \in \mathcal{T}$ is not necessary for $U \in \mathcal{T}_U$; by definition $V \in \mathcal{T}$ therefore $V \cap U = U \in \mathcal{T}_U$. Obviously $\emptyset \cap U = \emptyset \in \mathcal{T}_U$. Union and finite intersection are straightforward to show too.

Definition 1.2.2. A function $f : V \longrightarrow U$ between two topological spaces V and U with topologies $\mathcal{T}_V$ and $\mathcal{T}_U$ respectively such that $\forall A \in \mathcal{T}_U$, $f^{-1}(A) \in \mathcal{T}_V$. A function $f : V \longrightarrow U$ that is bijective, continuous with inverse $f^{-1} : U \longrightarrow V$ continuous as well, is a homeomorphism between V and U, and in such a case V and U are homeomorphic, writing $V \simeq U$.

As in an abstract set, a vector space V may assume a topology $\mathcal{T}$. Topologies on sets can be defined via mappings on its elements, the most common being a metric d on V.

Definition 1.2.3. A limit point $x \in U$ of a subset $U \subseteq V$ of a topological space V with topology $\mathcal{T}$ is such that $\forall U \in \mathcal{T}$ with $x \in U$, $U \setminus \{x\} \neq \emptyset$. The set $\overline{U}$ of all limit points of U is called the closure of U.

Obviously, $\overline{V} \supseteq V$. If $\overline{V} = V$, then V is complete.

Metric

Definition 1.2.4. A metric $d : V \times V \longrightarrow \mathbb{R}$ on a set V satisfies:

- $\forall x, y \in V$, $d(x, y) = 0$ if and only if $x = y$ (identity of indiscernibles),
- $\forall x, y \in V$, $d(x, y) = d(y, x)$ (symmetry),
- $\forall x, y, z \in V$, $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality),

Non–negativity is implied:

- $\forall x, y \in V$, $2d(x, y) = d(x, y) + d(x, y) = d(x, y) + d(y, x) \geq d(x, x) = 0$.

A metric on V implies a topology $\mathcal{T}$ on V by taking unions of ϵ–balls on V defined as $\{y \in V : d(x, y) < \epsilon\}$, $\forall x \in V$.

Norm

Definition 1.2.5. A seminorm $\| \cdot \| : V \longrightarrow \mathbb{R}$ on a vector space V over a field $\mathbb{K}$ satisfies:

- $\forall x, y \in V$, $\|x + y\| \leq \|x\| + \|y\|$ (subadditivity),
- $\forall a \in \mathbb{K}$ and $\forall x \in V$, $\|ax\| = |a|\|x\|$ (absolute homogeneity),

A norm is a seminorm that additionally:

- $\forall x \in V$, if $\|x\| = 0$ then $x = 0_V$ (positive definiteness) which implies $\|0_V\| = \|0x\| = |0|\|x\| = 0\|x\| = 0$.

Non–negativity is implied:

- $\forall x \in V$, $2\|x\| = \|x\| + \|x\| \geq \|x - x\| = \|0_V\| = 0$.

A norm on V implies a metric on V by $d(x, y) = \|y - x\|$, $\forall x, y \in V$. A complete normed space V is a Banach space.

Inner product

Definition 1.2.6. An inner product $\langle \cdot | \cdot \rangle : V \times V \longrightarrow \mathbb{K}$ on a vector space V over a field $\mathbb{K}$ equipped with the extra (conjugate) involution $\cdot^* : V \longmapsto V$, such that,

- $\mathbb{K}$ is an extension of $\mathbb{R}$ and particularly, $\forall a \in \mathbb{K}$ hermitian, i.e. with $a = a^*$, $a \in \mathbb{R}$,
- $\forall a \in \mathbb{K}$, $a^{**} = a$,
- $\forall a, b \in \mathbb{K}$, $(ab)^* = b^* a^*$,

satisfies

- $\forall x, y \in V$, $\langle x | y \rangle = \langle y | x \rangle^*$ (conjugate symmetry) which implies $\langle x | x \rangle = \langle x | x \rangle^* \in \mathbb{R}$, $\forall x \in V$,

- $\forall a, b \in \mathbb{K}$ and $\forall x, y, z \in V$ (linearity),

$$\langle ax + by|z\rangle = a\langle x|z\rangle + b\langle y|z\rangle, \text{ or}$$
$$\langle x|ay + bz\rangle = a\langle x|y\rangle + b\langle x|z\rangle.$$

- $\forall x \in V$, if $x \neq 0_V$ then $\langle x|x\rangle > 0$ (positive definiteness).

An inner product on V implies a norm on V by $\|x\| = \sqrt{\langle x|x\rangle}$. A complete space with an inner product is called a Hilbert space.

If the inner product is real specifically, i.e. $\langle\cdot|\cdot\rangle : V \times V \longrightarrow \mathbb{R}$, the conjugate symmetry reduces to symmetry,

- $\forall x, y \in V, \langle x|y\rangle = \langle y|x\rangle$ (symmetry).

$\forall x, y \in V \setminus \{0_V\}$ with $\langle x|y\rangle = 0$, x and y are orthogonal. If on top $\|x\| = \|y\| = 1$, they are orthonormal.

By definition, a finite–dimensional vector space V allows at most $\dim V$ number of orthogonal vectors, as orthogonality implies linear independence.

Duality

Operators

Definition 1.2.7. $\forall V, U$ normed vector spaces over a field $\mathbb{K}$, a linear operator $F : V \longrightarrow U$ is:

- bounded if $\exists c \in \mathbb{R}$ such that, $\forall x \in V$, $\|Fx\|_U \leq c\|x\|_V$,
  - an isometry (specifically) if $\forall x \in V$, $\|Fx\|_U = \|x\|_V$,
- a functional if $U = \mathbb{K}$.

The set $\mathcal{L}(V, U)$ of all bounded linear operators $F : V \longrightarrow U$ is a vector space, with $\dim \mathcal{L}(V, U) = \dim V \dim U$.

For bounded linear operators $F : V \longrightarrow V$ on the same vector space V, the shorthand notation $\mathcal{L}(V)$ is used. $\mathcal{L}(V)$ still is an algebra.

The set V^* of bounded linear functionals on V is the topological dual set of V. The duality defines a natural pairing map $\langle\cdot|\cdot\rangle_V : V \times V^* \longrightarrow \mathbb{R}$ such that $\forall x \in V$ and $\forall f \in V^*$, $\langle x|f\rangle_V = f(x)$. If V is a real Hilbert space, it is self–dual in the sense that the natural pairing and its inner product coincide (isomorphically).

Definition 1.2.8. $\forall F : V \longrightarrow U$ operator from V to U Hilbert spaces, for the adjoint operator $F^\dagger : U \longrightarrow V$, $\forall x, y \in V$,

$$\langle x|F^\dagger y\rangle_U = \langle Fx|y\rangle_V.$$

The definition of the dual natural pairing map provides a natural (bra–ket) notation (due to Dirac) for (dual or otherwise) vectors x in a vectors space V. In the context of dual spaces, $\langle f| \in V^*$ and $|x\rangle \in V$. An operator $F : V \longrightarrow U$ operates on the left on vectors as $F|x\rangle = Fx$ and on the right for dual vectors respectively.

In the context of a complex inner product, an operator $F : V \longrightarrow V$ on a Hilbert space V is hermitian if and only if $F^\dagger = F$, or equivalently $\forall x \in V$ and $\forall y \in U$,

$$\langle Fx|y\rangle = \langle x|Fy\rangle = \langle x|F|y\rangle,$$

which basically allows the last notation.[11] The same is true for a real inner product, provided F is symmetric.

$\forall \lambda$ eigenvalue of a hermitian operator F, $\lambda \in \mathbb{R}$, and if on top F is positive–definite, $\lambda > 0$.

Finally, the inner product defines a natural projection functional on a Hilbert space V, $\forall x \in V$, as $\langle x| : V \longrightarrow \mathbb{K}$.

[11] Note that in the hermitian notation $\langle x|F|y\rangle$, there is only one inner product, the one on U.

Definition 1.2.9. A hermitian operator $F : V \longrightarrow V$ on a vector space V such that $\forall x \in V$, $\langle x|F|y\rangle > 0$, is positive–definite.

$\forall\lambda$ eigenvalue of a positive–definite operator F, $\lambda > 0$.

Proposition 1.2.10. *$\forall F : V \longrightarrow V$ positive definite operator on a vector space V,*

$$\langle x|F|y\rangle = \langle x|y\rangle_F$$

defines an inner product on V per definition 1.2.6.

Definition 1.2.11. $\forall V$ complete vector space,[12] and base A,

$$|x\rangle = \sum\nolimits_{a\in A} |a\rangle\langle a|x\rangle, \tag{1.2.1}$$

where the operator

$$\sum\nolimits_{a\in A} |a\rangle\langle a| : X \longrightarrow X, \tag{1.2.2}$$

is a projection operator.

The projection operator (1.2.2) does nothing, as is understood by (1.2.1), allowing it to be interjected wherever in operations on V. However, if A is not a base of V, in the sense that $\operatorname{span} A < V$, the operator does act as a projector on $x \in V$ to $\operatorname{span} A$, hence the name. $\forall a \in A$, $\langle a|x\rangle \in \mathbb{K}$ is the component a (projection) of x, and (1.2.1) is identical to (1.1.1).

Definition 1.2.12. By extension, an operator $F : V \longrightarrow U$ from V to U vector spaces assumes a matrix representation given a base A on V,

$$F = \sum\nolimits_{a\in A}\sum\nolimits_{b\in A} |a\rangle\langle a|F|b\rangle\langle b|,$$

whose components are $\langle a|F|b\rangle$, $\forall a, b \in A$.

Combining vector representations 1.1.15 and matrix representations 1.2.11, an expression (calculation) is possible at component level, $\forall x, y \in V$ and $\forall F \in \mathcal{L}(V)$,

$$\langle x|F|y\rangle = \sum\nolimits_{a\in A}\sum\nolimits_{b\in A} \langle x|a\rangle\langle a|F|b\rangle\langle b|y\rangle,$$

simply by injecting projection operators.

At component level, the following shorthand notations will be used throughout:

- $\forall x, y \in V$ vectors, the inner product can be written as

  $$x \cdot y = x^\dagger y = \langle x|y\rangle = x_a y_a \in \mathbb{K}.$$

  where the matrix column–row notation is used for reference. $\forall F, G \in \mathcal{L}(V)$ component matrices, this notation relates to the ordinary matrix product as

  $$F \cdot G = F^\dagger G \in \mathcal{L}(V) \text{ or } F \cdot G|_{ab} = F^*_{ca} G_{cb}.$$

- $\forall x, y \in V$ vectors, the outer product

  $$x \otimes y = xy^\dagger \in \mathcal{L}(V) \text{ or } x \otimes y|_{ab} = x_a y_b.$$

  where the matrix column–row notation is used for reference. $\forall F, G \in \mathcal{L}(V)$ component matrices, this notation relates to the ordinary matrix product as

  $$F \otimes G = FG^\dagger \in \mathcal{L}(V) \text{ or } F \otimes G|_{ab} = F_{ac} G^*_{bc}.$$

[12] A in definition 1.1.15 is at most countable.

The inner (dot) product notation will be used as an alternative often to avoid confusion with the inner product induced by a measure on measure spaces.[13] For $\mathbb{K} = \mathbb{R}$, $\dagger = \top$, i.e. the notion of the adjoint becomes that of the transpose. Extended matrix operations in definition 1.2.8 reduce to the ordinary matrix product for hermitian operators. Finally, for a hermitian operator $F : V \longrightarrow U$ and $\forall x, y \in V$, the following symmetric notation makes sense,

$$x \cdot F \cdot y = \langle x|F|y\rangle = x_a F_{ab} y_b \in \mathbb{K}.$$

Infinite–dimension spaces require extra structure to that of a topology, namely a measure and a collection of measurable subsets to define a metric/norm/product, and their completeness as spaces is at risk, even if $\mathbb{K}$ is complete.[14]

Definition 1.2.13. The suprenum $\sup A$ of a subset $A \subseteq V$ of a set V equipped with a topology $\mathcal{T}$ and a total ordering $\leq$ is the lowest upper bound by said ordering of A. For a functional $f : V \longrightarrow \mathbb{K}$, $\sup f = \sup f(V)$.

Spectrum

Definition 1.2.14. $\forall \lambda \in \mathbb{K}$ such that for a bounded linear operator $F : V \longrightarrow V$ of a vector space V over a field $\mathbb{K}$, operator $F - \lambda \mathbb{1} : V \longrightarrow V$ is non–bijective.

Of note is that the spectrum of an (bounded) operator is more than its eigenvalues (definition 1.1.14).

The natural numbers $\mathbb{N}$

Definition 1.2.15. By the axiom of infinity,[15] $\exists \mathfrak{N}$ collection of (inductive) sets such that, $\forall \mathcal{N} \in \mathfrak{N}$:

- $0 = \emptyset = \{\} \in \mathcal{N}$ (empty set),
- $\forall a \in \mathcal{N}$, $\text{next}(a) = a \cup \{a\} \in \mathcal{N}$ (successor),

The intersection (which is straightforwardly also an inductive set),

$$\mathbb{N} = \bigcap \mathfrak{N} = \bigcap_{\mathcal{N} \in \mathfrak{N}} \mathcal{N}$$

is the set of natural numbers.

Informally, the axioms in definition 1.2.15 effectively describes a counting process, which is the essence of natural numbers,

$$\mathbb{N} = \{\{\}, \{\{\}\}, \{\{\{\}\}\}, \{\{\{\{\}\}\}\}, \ldots\} = \{0, 1, 2, 3, \ldots\},$$

and the process is countably infinite.

The successor function defines an addition operation on $\mathbb{N}$ such that

$$\forall a \in \mathbb{N},\ a + 0 = a \text{ and } \forall b \in \mathbb{N},\ a + \text{next}(b) = \text{next}(a + b),$$

which associative and symmetric, and with neutral element 0, making $\mathbb{N}$ a monoid.

The natural numbers are well–ordered by the total ordering $\leq$ such that

$$\forall a, b,\ a \leq b \text{ if and only if } \exists c \in \mathbb{N} \text{ such that } a + c = b.$$

It is unusual to assume a topology on the natural numbers, so it is customary to just assume the power set $2^{\mathbb{N}}$ (maximum topology) if necessary.

[13] See section §2.1. Probability theory for details.
[14] See section §2.1. Probability theory for more details on σ–algebras and measures.
[15] $\exists A$ set with $\emptyset \in A$ such that $\forall a \in A$, $a \cup \{a\} \in A$. Such sets are called inductive, and the axiom allows the existence of (countably) infinity sets.

Finite counting with $\mathbb{N}_n$, $\forall n \in \mathbb{N}_*$

$\mathbb{N}_n \subset \mathbb{N}$ is simply $\{1, 2, 3 \ldots, n\}$, $\forall n \in \mathbb{N}_*$, and is frequently used to enumerate finite ordered sets (vectors) and or indices.

The integers $\mathbb{Z}$

Definition 1.2.16. The equivalence relation $\sim_+$ in $\mathbb{N} \times \mathbb{N}$ such that,

$$\forall (a, b), (c, d) \in \mathbb{N} \times \mathbb{N},\ (a, b) \sim_+ (c, d) \text{ if and only if } a + d = b + c,$$

defines the set of integers as

$$\mathbb{Z} = \frac{\mathbb{N} \times \mathbb{N}}{\sim_+}.$$

Informally, $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, +1, +2, +3, \ldots\} \supset \mathbb{N}$.

$\mathbb{Z}$ is a group under the addition operator defined as,

$$\forall [(a, b)], [(c, d)] \in \mathbb{Z},\ [(a, b)] + [(c, d)] = [(a + c, b + d)],$$

with neutral element $[(0, 0)]$ and opposite $-[(a, b)] = [(b, a)]$, $\forall [(a, b)] \in \mathbb{Z}$.

$\mathbb{Z}$ is a monoid under the multiplication operator defined as,

$$\forall [(a, b)], [(c, d)] \in \mathbb{Z},\ [(a, b)] \cdot [(c, d)] = [(ac + bd, ad + bc)],$$

with neutral element $[(1, 0)]$.

Integers are well–ordered by the total ordering $\leq$ such that

$$\forall [(a, b)], [(c, d)] \in \mathbb{Z},\ [(a, b)] \leq [(c, d)] \text{ if and only if } a + d \leq b + c.$$

$\mathbb{Z}$ can be seen as the minimum group closure of the natural numbers under addition. The ordered pair (a, b) stands for the difference $a - b$ and an equivalence class consists of all such pairs of natural numbers giving the same difference.

The positive integers $\mathbb{Z}_+ = \mathbb{N}_*$

Of interest are the positive integers

$$\mathbb{Z}_+ = \{1, 2, 3, \ldots\} = \mathbb{N} \setminus \{0\} = \mathbb{N}_*,$$

for they make a good case of a magma with respect to multiplication, as they do not include the destructive (for multiplication) 0. Such positive subsets will become more relevant in the grater number sets that follow.

Integers naturally define the operation of absolute value as,

$$\forall a \in \mathbb{Z},\ |a| = \begin{cases} +a & a \geq 0 \\ -a & a \leq 0 \end{cases},$$

bearing in mind that $+0 = -0$ are the neutral element of addition.

The cyclic groups $\mathbb{Z}_n$, $\forall n \in \mathbb{Z}_+$

Of greater interest are the finite mod groups defined by Euclidean division on integers.

Definition 1.2.17. $\forall a, b \in \mathbb{Z}$, $\exists q \in \mathbb{Z}$ and $\exists r + 1 \in \mathbb{N}_b$ unique, such that $a = qb + r$.

Symbolically, $r = a \bmod b$.

$\forall n \in \mathbb{Z}$, the equivalence relation $\sim_n$ matching integers by their Euclidean remainder with n, partitions integers into a finite set of equivalence classes represented by said remainder, defined as

$$\mathbb{Z}_n = \frac{\mathbb{Z}}{\sim_n}.$$

$\mathbb{Z}_n$ is a group under the addiction $+_n$ defined as

$$\forall a, b \in \mathbb{Z}_n,\ a +_n b = (a + b) \bmod n.$$

Informally, $\mathbb{Z}_n = \{0, 1, 2, 3, \ldots, n - 1\}$.

The rational numbers $\mathbb{Q}$

Definition 1.2.18. The equivalence relation $\sim_\cdot$ in $\mathbb{Z} \times \mathbb{Z}_*$ such that,

$$\forall (a, b), (c, d) \in \mathbb{Z} \times \mathbb{Z},\ (a, b) \sim_\cdot (c, d) \text{ if and only if } ad = bc,$$

defines the set of rational numbers as

$$\mathbb{Q} = \frac{\mathbb{Z} \times \mathbb{Z}_*}{\sim_\cdot},$$

where $\mathbb{Z}_* = \mathbb{Z} \setminus \{0\}$.

Rational numbers maintain their integer ordered pair notation in the form of a fraction a/b, $\forall [(a, b)] \in \mathbb{Q}$. The corresponding equivalence class stands for all equivalent fractions.

Informally, $\mathbb{Q} \supset \mathbb{Z}$.

$\mathbb{Q}$ is a group under the addition operator defined as,

$$\forall \frac{a}{b}, \frac{c}{d} \in \mathbb{Q},\ \frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + b \cdot c}{b \cdot d},$$

with neutral element $0/1$ and opposite $-a/b = (-a)/b$, $\forall a/b \in \mathbb{Q}$, where the corresponding equivalence classes are implied.

The positive rational numbers $\mathbb{Q}_+$

Of interest are the positive rational numbers,

$$\mathbb{Q}_+ = \frac{\mathbb{Z}_+ \times \mathbb{Z}_+}{\sim_\cdot},$$

which form a group under the multiplication defined as,

$$\forall \frac{a}{b}, \frac{c}{d} \in \mathbb{Q},\ \frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d},$$

with neutral element $1/1$ and inverse $(a/b)^{-1} = b/a$, $\forall a/b \in \mathbb{Q}$.

The multiplication (and corresponding group structure) of $\mathbb{Q}_+$ extends to $\mathbb{Q}$, and together with the addition operator, $\mathbb{Q}$ is a field (definition 1.1.4).

$\mathbb{Q}$ still is well–ordered by the total ordering $\leq$ such that

$$\forall \frac{a}{b}, \frac{c}{d} \in \mathbb{Q},\ \frac{a}{b} \leq \frac{c}{d} \text{ if and only if } ad \leq bc.$$

Absolute value extends to rational numbers as

$$\forall \frac{a}{b} \in \mathbb{Q},\ \left|\frac{a}{b}\right| = \frac{|a|}{|b|}.$$

$\mathbb{Q}$ can be seen as the minimum field closure of the integers group under multiplication.

The real numbers $\mathbb{R}$

$\forall \{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Q}$ Cauchy sequence of rational numbers, the existence of a limit $a \in \mathbb{Q}$ is not guaranteed. Informally the set of all limit points of $\mathbb{Q}$ (the completion of $\mathbb{Q}$) is the set of real numbers $\mathbb{R}$.

Definition 1.2.19. The set of real numbers is a totally–ordered field, having the least–upper–bound property,

$$\forall A \subseteq \mathbb{R},\ \exists \sup A \in \mathbb{R} \text{ such that } \forall a \in A,\ a \leq \sup A \text{ and } \forall b \in \mathbb{R} \text{ with } a < b,\ \sup A < b.$$

The real numbers is the first set to assume a non–trivial topology $\mathcal{T}$, generated by all the open intervals in $\mathbb{R}$. This topology is also generated by the metric defined $\forall a, b \in \mathbb{R}$ by $d(a, b) = |a - b|$.[16]

Informally, $\mathbb{R} = \overline{\mathbb{Q}} \supset \mathbb{Q}$.

The positive real numbers $\mathbb{R}_+$

The positive real numbers $\mathbb{R}_+$ is of equivalent interest to that of the positive rational numbers $\mathbb{Q}_+$, as a clean group under multiplication with 1 and no destructive element. It is interesting to point out that the functions mapping one set to the other are none other than

$$\log : \mathbb{R}_+ \longrightarrow \mathbb{R} \text{ and } \exp : \mathbb{R} \longrightarrow \mathbb{R}_+,$$

with the corresponding properties of translating one operation into the other,

$$\forall a, b \in \mathbb{R}_+,\ \log(ab) = \log a + \log b, \text{ and } \forall a, b \in \mathbb{R},\ \exp(a + b) = \exp a \exp b.$$

The probability interval $[0, 1]$

The probability interval [0,1] follows a similar pattern, this time with the pair of the logistic function and its inverse sigmoid function:

$$\text{logit} : [0, 1] \longrightarrow \mathbb{R} : x \longmapsto \text{logit}\, x = \log \frac{x}{1 - x} \text{ and } \sigma : \mathbb{R} \longrightarrow [0, 1] : x \longmapsto \sigma(x) = \frac{1}{1 + \exp(-x)},$$

with the corresponding operator in [0,1] now respecting,[17]

$$\forall a, b \in \mathbb{R},\ \sigma(a+b) = \frac{1}{1 + \exp(-(a + b))} = \frac{1}{1 + \exp(-a)\exp(-b)} = \frac{1}{1 + \frac{\sigma(-a)}{\sigma(+a)}\frac{\sigma(-b)}{\sigma(+b)}} = \frac{\sigma(a)\sigma(b)}{\sigma(a)\sigma(b) + \sigma(-a) + \sigma(-b)} = \sigma(a) \circ \sigma(b),$$

so that $\forall a, b \in [0, 1],\ \text{logit}(a \circ b) = \text{logit}\, a + \text{logit}\, b$.

[16] Coincidentally this metric also stands for the Lebesgue measure on the Borel σ–algebra $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{T})$ stemming from the topology $\mathcal{T}$ of $\mathbb{R}$ (see section §2.1. Probability theory for details).

[17] Note that $\sigma(-x) = 1 - \sigma(x)$ and

$$\exp(-x) = \frac{\sigma(-x)}{\sigma(+x)}$$

The complex numbers $\mathbb{C}$

Definition 1.2.20. The set of complex numbers $\mathbb{C}$ is (isomorphic to) $\mathbb{R}^2$ equipped with the corresponding addition and a product stemming from the polynomial notation of complex numbers via an indeterminate $\imath$ (imaginary unit),

$$\forall (x_0, x_1) \in \mathbb{R}^2, x_0 + \imath x_1 \in \mathbb{C},$$

with the condition $\imath^2 + 1 = 0$, making it a field.

The condition $\imath^2 = -1$ implies $\forall n \in \mathbb{Z}$,

$$\imath^{4n+0} = +1,\ \imath^{4n+1} = +\imath,\ \imath^{4n+2} = -1,\ \imath^{4n+3} = -\imath,$$

which means that the group generated by $\imath$ via polynomial multiplication is isomorphic to $\mathbb{Z}_4$, and the cyclicity allows the reduction of all polynomials on $\imath$ back to a linear polynomial of the form $x + \imath y$, making $\mathbb{C}$ closed under polynomial multiplication.

Definition 1.2.21. $\forall z = \Re z + \imath \Im z \in \mathbb{C}$,

- $z^* = \Re z - \imath \Im z$ is the conjugate of z (apparently $z^{**} = z$),
- $\Re z = (z + z^*)/2$ is the real part of z,
- $\Im z = (z - z^*)/2\imath$ is the imaginary part of z,
- $|z| = \sqrt{z^* z} = \sqrt{(\Re z)^2 + (\Im z)^2}$ is the modulus or absolute value of z.

$\mathbb{C}$ is a field under polynomial addition and multiplication, more specifically:

- $\forall z, z' \in \mathbb{C}$, $z + z' = (\Re z + \Re z') + \imath(\Im z + \Im z')$ (addition),
- $\forall z, z' \in \mathbb{C}$, $zz' = (\Re z \Im z - \Re z' \Im z') + \imath(\Re z' \Im z + \Re z \Im z')$ (addition).

It should be noted that functions on $\mathbb{R}$ need care when extending to $\mathbb{C}$.

Polar representation and the $U(1)$ group

The exponential function on exp is defined as

$$\exp : \mathbb{C} \longrightarrow \mathbb{C} : z \longmapsto \exp z = (\cos \Im z + \imath \sin \Im z) \exp \Re z,$$

which implies that exp is periodic in the imaginary direction.

Definition 1.2.22. A complex number $z \in \mathbb{C}$ assumes an alternative (polar) representation by their modulus and their so–called (imaginary) phase:

$$z = \Re z + \imath \Im z = |z| \exp \imath \arg z,$$

where

$$|z| = \sqrt{(\Re z)^2 + (\Im z)^2} \text{ and } \arg z = \arctan \frac{\Im z}{\Re z},$$

with the usual singularity at $z = 0$, where $|z| = 0$ and any phase applies.

The quaternions $\mathbb{H}$

A possible (and common to all further extension possibilities) extension of complex numbers with a richer imaginary unit algebra is the set of quaternions $\mathbb{H}$.

Definition 1.2.23. The set of quaternions $\mathbb{H}$ is (isomorphic to) $\mathbb{R}^4$ equipped with the corresponding addition and a product stemming from the polynomial notation of complex numbers via an indeterminate $\imath$ (imaginary unit),

$$\forall (x_0, x_1, x_2, x_3) \in \mathbb{R}^4, x_0 + \imath x_1 + \jmath x_2 + k x_3 \in \mathbb{C},$$

with the imaginary product algebra

	$+\imath$	$+\jmath$	$+k$
$+\imath$	-1	$+k$	$-\jmath$
$+\jmath$	$-k$	-1	$+\imath$
$+k$	$+\jmath$	$-\imath$	-1

.

Unlike the complex numbers, the quaternion product is not commutative, hence $\mathbb{H}$ is not a field (but a ring).

Clifford Algebras $\mathrm{C}\ell_g V$

Definition 1.2.24. $\forall V$ a vector space over a field $\mathbb{K}$ and $\forall g : V \times V \longrightarrow \mathbb{K}$ non–degenerate symmetric map such that:

- $\forall y \in V$, if $g(x, y) = 0$ then $x = 0$ (non–degenerate),
- $\forall x, y \in V$, $g(x, y) = g(y, x)$ (symmetric),

the Clifford algebra $\mathrm{C}\ell_g V = \mathrm{T}V/g$ with g interpreted as an equivalence relation stemming from $x \otimes y + y \otimes x = 2g(x, y)\mathbb{1} \otimes \mathbb{1}$, meaning the algebra product $\circ$ respects

$$x \circ y + y \circ x = 2g(x, y).$$

$\dim \mathrm{C}\ell_g V = 2^{\dim V}$ [64].

$\forall V$ is equipped with an inner product (symmetric bilinear form) and

$$\forall x, y \in V,\ g(x, y) = \frac{1}{2}(x \circ y + y \circ x) = \frac{1}{2}(xy + yx) = \langle x|y\rangle,$$

the abbreviation $\mathrm{C}\ell V = \mathrm{C}\ell_{\dim V}\mathbb{K}$ is used. If the inner product is not positive definite, the notation splits the positive from the negative eigenvalues of the metric defining said inner product, like $\mathrm{C}\ell V = \mathrm{C}\ell_{p,q}\mathbb{K}$ with $p + q = \dim V$.

Of interest is the special case of exterior algebras $\mathrm{C}\ell_0 V = \bigwedge V$ defined for $g = 0$, resulting in an anticommutative algebra such that,

$$\forall x, y \in V,\ x \circ y + y \circ x = x \wedge y + y \wedge x = 0.$$

1.3. Differential Algebra

Manifolds

Definition 1.3.1. A topological space V, such that $\forall x, y \in V$ with $x \neq y$, $\exists A, B \in \mathcal{T}$ with $x \in A$, $y \in B$ and $A \cap B = \emptyset$, is a Hausdorff (separable) topological space.

The topology $\mathcal{T}$ of a topological space V admits a base $\mathcal{B} \subseteq \mathcal{T}$ such that $\forall U \in \mathcal{T}$, $\exists \mathcal{A} \subseteq \mathcal{B}$ such that

$$U = \bigcup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A$$

Definition 1.3.2. A topological space V that admits a countable base is a second countable (completely separable) one.

A bijective function $f : V \longrightarrow U$ between two topological spaces U and V with topologies $\mathcal{S}$ and $\mathcal{T}$ respectively, that respects said topologies in the sense that $\forall A \in \mathcal{S}$, $f^{-1}(A) \in \mathcal{T}$, is a homeomorphism. U and V are then said to be homeomorphic.

Definition 1.3.3. A topological manifold M modeled after a vector space V on a field $\mathbb{K}$, is a second countable Hausdorff topological space with topology $\mathcal{T}$ that is locally homeomorphic to V with the usual topology coming from $\mathbb{K}^{\dim V}$. This means that $\forall A \in \mathcal{T}$, $\exists \phi : A \longrightarrow V$ homeomorphism, called a chart.

A collection of charts enough to cover M is an atlas of M.

Definition 1.3.4. A differentiable (smooth) manifold M modeled after a vector space V on a field $\mathbb{K}$ is a manifold such that $\forall \mathcal{A} \subseteq \mathcal{T}$ atlas of M such that $\forall A, B \in \mathcal{A}$, $\phi_A^{-1} \circ \phi_B : V \longrightarrow V$ is differentiable on V.

Lie (matrix) groups

Definition 1.3.5. A Lie group G is a group that is also a smooth manifold additionally with smooth charts.

The material presented here applies to the general theory of Lie groups/algebras, but when applied to matrices they assume simpler and more explicit forms [65].

Let $\mathrm{M}_{\dim V}\mathbb{K} \simeq \mathbb{K}^{\dim V \otimes V}$ the vector space of operators (matrices) on a vector space V over a field $\mathbb{K}$, with the usual topology $\mathcal{T}_{\mathrm{M}}$ stemming from $\mathcal{T}_V$ which in turn stems from $\mathcal{T}_{\mathbb{K}}$ (hence the term "usual"), is a monoid under composition (matrix product) $\circ$, [18]because it is closed and associative, and $\exists \mathbb{1} \in \mathrm{M}_{\dim V}\mathbb{K}$ unit element, but $\forall A \in \mathrm{M}_{\dim V}\mathbb{K}$, A^{-1} is not necessarily defined. $\mathrm{M}_{\dim V}\mathbb{K}$ does not only assume a topology but a metric as well, stemming from a definition of a norm, usually the Frobenius norm $\forall A \in \mathrm{M}_{\dim V}\mathbb{K}$, $\|A\| = \sqrt{A_{ab}\overline{A_{ab}}}$.

Definition 1.3.6. The subspace $\mathrm{GL}_{\dim V}\mathbb{K} = \{A \in \mathrm{M}_{\dim V}\mathbb{K} | \det A \neq 0\} \leq \mathrm{M}_{\dim V}\mathbb{K}$ of all invertible matrices on $\mathbb{K}$ is a group under composition $\circ$, called the ($\dim V$–dimensional) *general linear* group over $\mathbb{K}$. Any closed under the relative $\mathcal{T}_{\mathrm{GL}}$ subgroup $G \leq \mathrm{GL}_{\dim V}\mathbb{K}$ is a matrix Lie group.

$\mathrm{GL}_{\dim V}\mathbb{K}$ is a group, because

- $\forall A, B \in \mathrm{GL}_{\dim V}\mathbb{K}$, $\det(AB) = \det A \det B \neq 0$.
- $\det \mathbb{1} \neq 0$,
- $\forall A \in \mathrm{GL}_{\dim V}\mathbb{K}$, $A^{-1} \in \mathrm{GL}_{\dim V}\mathbb{K}$ by definition.

Most frequently used matrix Lie groups are closed under $\mathcal{T}_{\mathrm{M}}$ as well, as subsets of $\mathrm{M}_{\dim V}\mathbb{K}$.

Note that $\mathrm{GL}_{\dim V}\mathbb{K} \in \mathcal{T}_{\mathrm{M}}$, meaning that the set of non–invertible matrices is closed [65]. However $\mathrm{GL}_{\dim V}\mathbb{K}$ is trivially closed under its own (induced) topology $\mathcal{T}_{\mathrm{GL}}$.

Linear groups

Note that $\mathrm{GL}_{\dim V}\mathbb{K} \in \mathcal{T}_{\mathrm{M}}$, meaning that the set of non–invertible matrices is closed [65]. However $\mathrm{GL}_{\dim V}\mathbb{K}$ is trivially closed under its own (induced) topology $\mathcal{T}_{\mathrm{GL}}$, therefore $\mathrm{GL}_{\dim V}\mathbb{K}$, by definition 1.3.6, is a (matrix) Lie group.

Field inclusion transfers, from the defining fields to the corresponding Lie groups. For example, $\mathrm{GL}_{\dim V}\mathbb{R} < \mathrm{GL}_{\dim V}\mathbb{C}$ for $\mathbb{R} < \mathbb{C}$.

The *special linear* group $\mathrm{SL}_{\dim V}\mathbb{K} = \{A \in \mathrm{GL}_{\dim V}\mathbb{K} | \det A = 1\} < \mathrm{GL}_{\dim V}\mathbb{K}$ is a Lie group, as $\det : \mathrm{M}_{\dim V}\mathbb{K} \longrightarrow \mathbb{K}$ is a continuous function on $\mathcal{T}_{\mathrm{M}}$ and all induced topologies, and:

- $\forall A, B \in \mathrm{SL}_{\dim V}\mathbb{K}$, $\det(AB) = \det A \det B = 1$.
- $\det \mathbb{1} = 1$,
- $\forall A \in \mathrm{SL}_{\dim V}\mathbb{K}$, $\det A^{-1} = (\det A)^{-1} = 1$.

[18]Matrix multiplication is denoted as composition for clarity here.

Unitary groups

$\forall \mathbb{K}$ field with a conjugation $^* : \mathbb{K} \longrightarrow \mathbb{K}$ (definition 1.1.4), $\mathrm{M}_{\dim V}\mathbb{K}$ assumes a conjugation

$$^\dagger : \mathrm{M}_{\dim V}\mathbb{K} \longrightarrow \mathrm{M}_{\dim V}\mathbb{K} : A \longmapsto A^\dagger \text{ with } (A^\dagger)_{ab} = (A_{ba})^*,$$

which is nothing more than the adjoint operator defined in 1.2.8.

The *unitary* group $\mathrm{U}_{\dim V}\mathbb{K} = \{A \in \mathrm{GL}_{\dim V}\mathbb{K} | A^\dagger A = AA^\dagger = \mathbb{1} \text{ or } A^{-1} = A^\dagger\} < \mathrm{GL}_{\dim V}\mathbb{K}$ is a Lie group, as:

- $\forall A, B \in \mathrm{SL}_{\dim V}\mathbb{K}$, $(AB)^{-1} = B^{-1}A^{-1} = B^\dagger A^\dagger = (AB)^\dagger$.
- $\mathbb{1}^{-1} = \mathbb{1} = \mathbb{1}^\dagger$,
- $\forall A \in \mathrm{SL}_{\dim V}\mathbb{K}$, $(A^{-1})^{-1} = (A^\dagger)^{-1} = (A^{-1})^\dagger$.

By the definition of a unitary matrix, $|\det A| = 1$, with the modulus function

$$|\cdot| : \mathbb{K} \longrightarrow \mathbb{R}_+ : a \longmapsto |a| = \sqrt{a^* a}$$

induced on $\mathbb{K}$ by its conjugation.[19]

The *special unitary* group $\mathrm{SU}_{\dim V}\mathbb{K} = \mathrm{U}_{\dim V}\mathbb{K} \cap \mathrm{SL}_{\dim V}\mathbb{K} \leq \mathrm{U}_{\dim V}\mathbb{K}$ is a Lie group by construction.

For unitary groups with $\mathbb{K} = \mathbb{C}$, the field symbol is omitted.

Orthogonal subgroups

For $\mathbb{K} = \mathbb{R}$, the (special) orthogonal subgroup $(\mathrm{S})\mathrm{O}_{\dim V} < (\mathrm{S})\mathrm{U}_{\dim V}$ defined by

$$(\mathrm{S})\mathrm{O}_{\dim V} = \{A \in \mathrm{GL}_{\dim V}\mathbb{R} | A^\top A = AA^\top = \mathbb{1} \text{ or } A^{-1} = A^\top \text{ (and } \det A = 1)\}$$

is a Lie group, with the adjoint $\dagger$ replaced by the transpose $\top$.

The definition of unitary/orthogonal matrix groups is based on the definition of the adjoint/transpose of a matrix (operator), which in turn relies on how the operators behaves in relation to the inner product of the model vector space V. According to definition 1.2.6, an inner product shall be positive–definite, in compliance with inducing a norm on V. If that condition is dropped, and the inner product has the general (mixed–signature) form

$$\langle x|y\rangle = \sum\nolimits_{a=1}^{n} x_a y_a - \sum\nolimits_{a=n+1}^{n+k} x_a y_a$$

a new set of (special) orthogonal groups $(\mathrm{S})\mathrm{O}_{n+k}$ is defined with $n + k = \dim V$.[20]

Isometries

An orthogonal group $\mathrm{O}_{\dim V}$ can be extended to a group of isometries $\mathrm{E}_{\dim V}$ of $V \simeq \mathbb{R}^{\dim V}$ as follows:

$$\forall A \in \mathrm{O}_{\dim V} \text{ and } \forall x \in V, \exists! A_x \in \mathrm{E}_{\dim V} \text{ such that } \forall y \in V,\ A_x y = Ay + x,$$

which in fact includes rotations and translations – or isometries overall – in V.

- $\forall A_x, B_y \in \mathrm{E}_{\dim V}$ and $\forall z \in V$, $A_x B_y z = A(Bz + y) + x = ABz + Ay + x = (AB)_{Ay+x} z$ (product closure),
- $\forall A_x, B_y, C_z \in \mathrm{E}_{\dim V}$ and $\forall w \in V$ (associativity),

$$(A_x B_y)C_z w = (AB)_{Ay+x} C_z w = ((AB)C)_{ABz+(Ay+x)} = (A(BC))_{A(Bz+y)+x} = A_x (BC)_{Bz+y} = A_x (B_y C_z) w,$$

- $\exists \mathbb{1}_0$ such that $\mathbb{1}_0 = \mathbb{1}x + 0 = x$ (unit),
- $\forall A_x \in \mathrm{E}_{\dim V}$, $\exists A_x^{-1} = (A^{-1})_{-A^{-1}x}$ (inverse).

It is worth noting that $\mathrm{E}_{\dim V} < \mathrm{GL}_{\dim V+1}\mathbb{R}$ specifically, with the extra degree of freedom necessary to encode translations [65].

[19] For real numbers it is just ± 1, while for complex numbers it is the whole unit circle.
[20] These are labeled with the explicit $n + k$ index to indicate the split of dimensions into positive–definite and negative–definite.

The exponential map

The composition operator (matrix product) of $\mathrm{M}_{\dim V}\mathbb{K}$ may not be commutative, but it is self–commutative, in the sense that $\forall A \in \mathrm{M}_{\dim V}\mathbb{K}$, the exponent notation makes sense, $\forall n \in \mathbb{N}$,

$$A^n = \prod_{i=1}^{n} A \text{ with } A^0 = \mathbb{1}.$$

For $\mathrm{GL}_{\dim V}\mathbb{K}$, $n \in \mathbb{Z}$ more so, as $\forall n \in \mathbb{Z}_+$, $A^{-n} = (A^{-1})^n = (A^n)^{-1}$.

Definition 1.3.7. The exponential map is defined as

$$\exp : \mathrm{M}_{\dim V}\mathbb{K} \longrightarrow \mathrm{GL}_{\dim V}\mathbb{K} : A \longmapsto \exp A = \sum\nolimits_{n\in\mathbb{N}} \frac{1}{n!}A^n.$$

Proposition 1.3.8. *The exponential map is analytic on $M_{\dim V}\mathbb{K}$ on its topology and Frobenius distance.*

Theorem 1.3.9. *$\forall A \in M_{\dim V}\mathbb{K}$, $\det\exp A = \exp\mathrm{tr}\, A$. In addition:*

- $\exp 0 = \mathbb{1}$,
- $(\exp A)^\dagger = \exp A^\dagger$,
- $(\exp A)^{-1} = \exp(-A)$,
- $\forall a, b \in \mathbb{K}$, $\exp((a+b)A) = \exp(aA)\exp(bA)$,
- $\forall B \in M_{\dim V}\mathbb{K}$ *with* $AB = BA$, $\exp(A+B) = \exp A \exp B$,
- $\forall B \in GL_{\dim V}\mathbb{K}$, $\exp(BAB^{-1}) = B\exp AB^{-1}$.

Lie (matrix) algebras

Definition 1.3.10. A Lie algebra $\mathfrak{g}$ is an algebra replacing a composition operator with that of a Lie bracket $[\cdot|\cdot] : \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathfrak{g}$, that satisfies:

- $\forall a, b \in \mathbb{K}$ and $\forall X, Y, Z \in \mathfrak{g}$ (bilinearity or distributivity and scalar product compatibility),

$$\begin{aligned}[aX + bY|Z] &= a[X|Z] + b[Y|Z] \\ [X|aY + bZ] &= a[X|Y] + b[X|Z]\end{aligned},$$

- $\forall X \in \mathfrak{g}$, $[X|X] = 0$ (alternality),
- $\forall X, Y, Z \in \mathfrak{g}$, $[X|[Y|Z]] + [Y|[Z|X]] + [Z|[X|Y]] = 0$ (Jacobi identity).

By alternality, $\forall X, Y \in \mathfrak{g}$, $[X+Y|X+Y] = 0$. By bilinearity, this becomes $[X|Y] + [Y|X] = 0$ (anticommutativity).

All types of moprhisms in definition 1.1.10 extend to Lie groups with the added requirement that the function is continuous with the respective domain topology.

A subalgebra $\mathfrak{h} \leq \mathfrak{g}$ such that $\forall x \in \mathfrak{g}$ and $\forall y \in \mathfrak{h}$, $[X|Y] \in \mathfrak{h}$, is an ideal of $\mathfrak{g}$. The maximal subalgebra $\mathfrak{h} \leq \mathfrak{g}$ such that $\forall X, Y \in \mathfrak{h}$, $[X|Y] = 0$, is the center of $\mathfrak{g}$.

The Lie bracket $[\cdot|\cdot] : \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathfrak{g}$ defines $\forall X \in \mathfrak{g}$ an X–adjoint map

$$\mathfrak{ad}_X : \mathfrak{g} \longrightarrow \mathfrak{g} : Y \longmapsto \mathfrak{ad}_X Y = [X, Y].$$

Definition 1.3.11. $\forall\{X_a\}_{a=1}^{\dim\mathfrak{g}} \subset \mathfrak{g}$ vector base of $\mathfrak{g}$, the structure constants of $\mathfrak{g}$ with respect to said base are defined by

$$[X_a|X_b] = \alpha_{abc}X_c.$$

The alternality of the Lie product implies

$$\alpha_{abc} + \alpha_{bac} = 0,$$

while the Jacobi identity implies

$$\alpha_{abc}\alpha_{cde} + \alpha_{bdc}\alpha_{cae} + \alpha_{dac}\alpha_{cbe} = 0.$$

Definition 1.3.12. $\forall \mathrm{G} \leq \mathrm{GL}_{\dim V}\mathbb{K}$ matrix Lie group, $\exists \mathfrak{g} \leq \mathrm{M}_{\dim V}\mathbb{K}$ Lie matrix algebra, such that $\forall a \in \mathbb{R}$ and $\forall X \in \mathfrak{g}$, $\exp(aX) \in \mathrm{G}$ [65].[21]

Definition 1.3.12 defines subdomains of the exponential map as $\exp : \mathfrak{g} \longrightarrow \mathrm{G}$.

$\forall A \in \mathrm{G}$, the A–adjoint map on $\mathfrak{g}$ is

$$\mathrm{ad}_X : \mathfrak{g} \longrightarrow \mathfrak{g} : Y \longmapsto \mathrm{ad}_X\, Y = XYX^{-1}.$$

Theorem 1.3.13. *$\forall G \leq GL_{\dim V}\mathbb{K}$ $\mathbb{K}$–matrix Lie group, the corresponding $\mathbb{K}$–matrix Lie algebra $\mathfrak{g}$ is an $\mathbb{R}$–matrix Lie algebra too, such that $\forall A \in G$ and $\forall X \in \mathfrak{g}$, for the A–adjoint of X, $\mathrm{ad}_A\, X \in \mathfrak{g}$.*

group	algebra
$\mathrm{GL}_{\dim V}\mathbb{K}$	$\mathfrak{gl}_{\dim V}\mathbb{K} = \mathrm{M}_{\dim V}\mathbb{K}$
$\mathrm{SL}_{\dim V}\mathbb{K}$	$\mathfrak{sl}_{\dim V}\mathbb{K} \leq \mathrm{M}_{\dim V}\mathbb{K}$ such that $\forall X \in \mathfrak{sl}_{\dim V}\mathbb{K}$, $\mathrm{tr}\, X = 0$
$\mathrm{U}_{\dim V}\mathbb{K}$	$\mathfrak{u}_{\dim V}\mathbb{K} \leq \mathrm{M}_{\dim V}\mathbb{K}$ such that $\forall X \in \mathfrak{u}_{\dim V}\mathbb{K}$, $X + X^\dagger = 0$
$\mathrm{SU}_{\dim V}\mathbb{K}$	$\mathfrak{su}_{\dim V}\mathbb{K} \leq \mathrm{M}_{\dim V}\mathbb{K}$ such that $\forall X \in \mathfrak{su}_{\dim V}\mathbb{K}$, $\mathrm{tr}\, X = 0$ and $X + X^\dagger = 0$

Table 1.3.1.: Examples of Lie group/algebra correspondences.

Proposition 1.3.14. *$\forall X, Y \in M_{\dim V}\mathbb{K}$ as a Lie algebra,*

$$\mathrm{ad}_{\exp X}\, Y = \exp \mathfrak{ad}_X\, Y.$$

Complexification of a real matrix Lie algebra

Definition 1.3.15. A vector space V over $\mathbb{R}$ is complexified by the smallest superspace U such that, $\forall A, B \in V$, $A + \imath B \in U$, hence $U \simeq V \oplus V$.

Proposition 1.3.16. *$\forall \mathfrak{g}$ real matrix Lie algebra, $\exists \mathfrak{h}$ vector complexification that is a complex matrix Lie algebra in the sense of the complexified Lie bracket, $\forall X, Y \in \mathfrak{h}$,*

$$[X, Y] = [\Re X + \imath \Im X, \Re Y + \imath \Im Y] = ([\Re X, \Re Y] - [\Im X, \Im Y]) + \imath([\Re X, \Im Y] + \Im X, \Re Y]).$$

algebra	complexified algebra
$\mathfrak{gl}_{\dim V}\mathbb{R}$	$\mathfrak{gl}_{\dim V}\mathbb{C}$
$\mathfrak{sl}_{\dim V}\mathbb{R}$	$\mathfrak{sl}_{\dim V}\mathbb{C}$
$\mathfrak{u}_{\dim V}\mathbb{R}$	$\mathfrak{gl}_{\dim V}\mathbb{C}$
$\mathfrak{su}_{\dim V}\mathbb{R}$	$\mathfrak{sl}_{\dim V}\mathbb{C}$

Table 1.3.2.: Examples of Lie algebra complexifications.

[21] Note that $a \in \mathbb{R}$ suffices even if V is over $\mathbb{K}$.

Physics phase convention

In the context of physics, $\exp \imath : \mathfrak{g} \longrightarrow \mathrm{G} : X \longmapsto \exp(\imath X)$. The result of table 1.3.1 and table 1.3.2 in this context are show in table 1.3.3.

	real		complex
group	algebra	algebra	group
$\mathrm{GL}_{\dim V}\mathbb{K}$	$\mathfrak{gl}_{\dim V}\mathbb{K} = \mathrm{M}_{\dim V}\mathbb{K}$	$\longrightarrow$ $\mathfrak{gl}_{\dim V}\mathbb{K}$	$\mathrm{GL}_{\dim V}\mathbb{K}$
$\mathrm{SL}_{\dim V}\mathbb{K}$	$\mathfrak{sl}_{\dim V}\mathbb{K} \leq \mathrm{M}_{\dim V}\mathbb{K}$ such that $\forall X \in \mathfrak{sl}_{\dim V}\mathbb{K}$, $\operatorname{tr} X = 0$	$\longrightarrow$ $\mathfrak{sl}_{\dim V}\mathbb{K}$	$\mathrm{SL}_{\dim V}\mathbb{K}$
$\mathrm{U}_{\dim V}\mathbb{K}$	$\mathfrak{u}_{\dim V}\mathbb{K} \leq \mathrm{M}_{\dim V}\mathbb{K}$ such that $\forall X \in \mathfrak{u}_{\dim V}\mathbb{K}$, $X = X^\dagger$	$\longrightarrow$ $\mathfrak{gl}_{\dim V}\mathbb{K}$	$\mathrm{GL}_{\dim V}\mathbb{K}$
$\mathrm{SU}_{\dim V}\mathbb{K}$	$\mathfrak{su}_{\dim V}\mathbb{K} \leq \mathrm{M}_{\dim V}\mathbb{K}$ such that $\forall X \in \mathfrak{su}_{\dim V}\mathbb{K}$, $\operatorname{tr} X = 0$ and $X = X^\dagger$	$\longrightarrow$ $\mathfrak{sl}_{\dim V}\mathbb{K}$	$\mathrm{SL}_{\dim V}\mathbb{K}$

Table 1.3.3.: Examples of Lie group/algebra correspondences and their complexified variant.

$\operatorname{tr} X = 0$ stems from $\det \exp X = \exp(\imath \operatorname{tr} X) = 1$ for the special groups.

$X = X^\dagger$ stems from the unitary requirement of the unitary groups.

1.4. Numerical linear algebra methods

The conjugate gradient method

Equation $Fx = y$ for finite dimension admits a linear system of equations representation $F_{ab}x_b = y_a$. To solve an exactly–solvable system like this ($\det F \neq 0$) numerically, there exist several methods, several of which are optimized for specific problems. The conjugate gradient method applies to problems where F is hermitian ($F^\dagger = F$) and positive definite ($\det F > 0$).

The term "conjugation" in this context means "orthogonality". $\forall \{\chi_n\}_{n=1}^{\dim V}$ of F–orthogonal (hence linear independent) vectors, i.e. such that $\forall n, n' \in \mathbb{N}_{\dim V}$, $\langle \chi_{n'}|F|\chi_n\rangle \propto \delta_{nn'}$, $\operatorname{span}\{\chi_n\}_{n=1}^{\dim V} = V$.

Assume x_* is the exact solution to the equation, i.e. $Fx_* = y$ or $x_* = F^{-1}y$. Expressed in the basis $\{\chi_n\}_{n=1}^{\dim V}$ and ignoring the Einstein index notation,

$$\langle \chi_{n'}|y\rangle = \langle \chi_{n'}|F|x_*\rangle = \sum_{n=1}^{\dim V} \langle \chi_{n'}|F|\chi_n\rangle\langle \chi_n|x_*\rangle = \sum_{n=1}^{\dim V} \langle \chi_n|F|\chi_n\rangle\delta_{nn'}\langle \chi_n|x_*\rangle = \langle \chi_{n'}|F|\chi_{n'}\rangle\langle \chi_{n'}|x_*\rangle,$$

so the components of the solution x_* are

$$\langle \chi_n|x_*\rangle = \frac{\langle \chi_n|y\rangle}{\langle \chi_n|F|\chi_n\rangle}. \tag{1.4.1}$$

The foundation of the conjugate gradient method lies in efficiently building an orthogonal basis $\{x\}_{n=1}^{\dim V}$ such that, the corresponding solution can be approximately good enough for a given tolerance. This is extremely useful for considerably large $\dim V$.

Note that the exact solution x_* to $Fx = y$ is also the (unique) minimizer of

$$f : V \longrightarrow \mathbb{R} : \frac{1}{2}\langle x|F|x\rangle - \langle x|y\rangle,$$

whose existence is justified by its second derivative being F. The equivalent equation is then

$$\nabla f(x) = Fx - y = 0,$$

making the conjugate gradient method for F similar to a gradient descent method for a scalar function f, in which $\forall n \in \mathbb{N}$ and $x_n \in V$ a guess of x_*, the search direction is given by the residual $\epsilon_n = -\nabla f(x_n)$.

Algorithm 1.1 Conjugate gradient method in C–like syntax

Let $\epsilon > 0$ be a tolerance for a good approximation.
Let $x_0 \in V$ be an initial guess at x_*.
Let $\epsilon_0 = y - Fx_0$ be the residual to the exact solution.

If $\|\epsilon_0\| < \epsilon$:
Return $x_* = x_0$.

Let $\chi_0 = \epsilon_0$.
Let $n = 0$.

Repeat over $n \in \mathbb{N}_{\dim V}$:
Let

$$\alpha_n = \frac{\langle \epsilon_n | \epsilon_n \rangle}{\langle \chi_n | F | \chi_n \rangle}.$$

Let $x_{n+1} = x_n + \alpha_n \chi_n$ be the better guess.
Let $\epsilon_{n+1} = \epsilon_n - \alpha_n F \chi_n$ be the new residual.

If $\|\epsilon_{n+1}\| < \epsilon$:
Return $x_* = x_0$.
Break from loop.

Let

$$\beta_n = \frac{\langle \epsilon_{n+1} | \epsilon_{n+1} \rangle}{\langle \epsilon_n | \epsilon_n \rangle}.$$

Let $\chi_{n+1} = \epsilon_{n+1} + \beta_n \chi_n$ be the next orthonormal direction.
Advance $k = k + 1$

Return $x_* = x_{k+1}$.

What makes the conjugate gradient method special is the requirement that the consecutive search directions are orthonormal, therefore, assuming all previous actual search directions are orthonormal, the next one is built per the Gram–Schmidt orthonormalization,

$$\chi_n = \epsilon_n - \sum\nolimits_{i=1}^{n-1} \frac{\langle \chi_i | A | \epsilon_i \rangle}{\langle \chi_i | A | \chi_i \rangle} \chi_i.$$

giving the next best guess

$$x_{n+1} = x_n + \frac{\langle \chi_n | \epsilon_n \rangle}{\langle \chi_n | A | \chi_n \rangle} \chi_n.$$

2. Stochastic processes

2.1. Probability theory

Definition 2.1.1. A collection $\mathcal{F}$ of events on a sample space Ω that satisfies:

- the empty set $\emptyset \in \mathcal{F}$ is an event (a null event),
- $\forall A \in \mathcal{F}$ event, the complement $\neg A \equiv \Omega \setminus A \in \mathcal{F}$ is also an event,
- $\forall \mathcal{A} \subseteq \mathcal{F}$ countable subcollection of events in $\mathcal{F}$, the union

$$\bigcup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A \in \mathcal{F},$$

  is also an event (logical $\vee$),

is an σ–algebra on Ω.

Definition 2.1.1 implies that:

- the sample space $\Omega = \neg\emptyset \in \mathcal{F}$ is an event (the sure event)
- $\forall \mathcal{A} \subseteq \mathcal{F}$ countable subcollection of events in $\mathcal{F}$, the intersection

$$\bigcap \mathcal{A} = \bigcap_{A \in \mathcal{A}} A = \neg \bigcup_{A \in \mathcal{A}} \neg A \in \mathcal{F},$$

  is also an event (logical $\wedge$).

This means that a σ–algebra is closed under countable set operations.

A sample space Ω that admits a σ–algebra $\mathcal{F}$ of events is a measurable space. Ω is implied by the cover of $\mathcal{F}$,

$$\Omega = \bigcup \mathcal{F} = \bigcup_{A \in \mathcal{F}} A.$$

Theorem 2.1.2. *$\forall \mathcal{A}$ collection of events of a set E, $\exists \sigma(\mathcal{A})$ a smallest σ–algebra of events such that $\sigma(\mathcal{A}) \supseteq \mathcal{A}$.*[1]

$\forall \mathfrak{A}$ a family of σ–algebras on a set Ω, their σ–union is

$$\bigvee \mathfrak{A} = \bigvee_{\mathcal{F} \in \mathfrak{A}} \mathcal{F} = \sigma\Big(\bigcup_{\mathcal{F} \in \mathfrak{A}} \mathcal{F}\Big) = \sigma\Big(\bigcup \mathfrak{A}\Big).$$

The arbitrary intersection

$$\bigcap \mathfrak{A} = \bigcap_{\mathcal{F} \in \mathfrak{A}} \mathcal{F}$$

is already a σ–algebra.

Definition 2.1.3. $\forall \mathfrak{A}$ a family of σ–algebras, the product σ–algebra $\otimes \mathfrak{A}$ is defined as the Cartesian product of events in each of the σ–algebras.

$$\bigotimes \mathfrak{A} = \bigotimes_{\mathcal{F} \in \mathfrak{A}} \mathcal{F} = \Big\{ \prod_{\mathcal{F} \in \mathfrak{A}} A_{\mathcal{F}} \Big| A_{\mathcal{F}} \in \mathcal{F}, \forall \mathcal{F} \in \mathfrak{A} \Big\}.$$

[1] The power set 2^{Ω} of Ω is a σ–algebra and also, $\forall \mathcal{A}$ collection of events on Ω, $2^{\Omega} \supseteq \mathcal{A}$ by definition, so such a σ–algebra always exists.

If the sample space Ω admits a topology $\mathcal{T}$, $\mathcal{B}(\Omega) = \sigma(\mathcal{T})$ is the Borel σ–algebra on Ω. The most prominent such example is $\mathbb{R}$ with the usual topology formed from open intervals.

Definition 2.1.4. A finite positive set function $\rho : \mathcal{F} \longrightarrow [0,1] \subset \mathbb{R}$ on an σ–algebra of events $\mathcal{F}$ that satisfies:

- $\rho(\emptyset) = 0$,
- $\forall \mathcal{A} \subseteq \mathcal{F}$ countable disjoint[2] subcollection of events in $\mathcal{F}$,

$$\rho\Big(\bigcup_{A\in\mathcal{A}} A\Big) = \sum\nolimits_{A\in\mathcal{A}} \rho(A)$$

is a probability measure on $\mathcal{F}$.

$\forall A \in \mathcal{F}$ event with $\rho(A) = 0$ is a null event. $\emptyset$ is a null event by definition 2.1.4.

Definition 2.1.5. $\forall \mathfrak{A}$ a family of σ–algebras, and $\forall \{\rho_{\mathcal{F}}\}_{\mathcal{F}\in\mathfrak{A}}$ probability measures defined on them, the product probability measure is defined as the product of measures on the product σ–algebra $\otimes\mathfrak{A}$,[3]

$$\bigotimes\nolimits_{\mathcal{F}\in\mathfrak{A}} \rho_{\mathcal{F}} : \bigotimes \mathfrak{A} \longrightarrow [0,1] : \prod\nolimits_{\mathcal{F}\in\mathfrak{A}} A_{\mathcal{F}} \mapsto \prod\nolimits_{\mathcal{F}\in\mathfrak{A}} \rho_{\mathcal{F}}(A_{\mathcal{F}}).$$

A sample space Ω that admits a σ–algebra of events $\mathcal{F}$ (a measurable space), which in turn admits a probability measure ρ, is a probability space.

A probability space is complete if and on if $\forall A \in \mathcal{F}$ null event, $\forall B \subset A$, $B \in \mathcal{F}$, i.e. $\mathcal{F}$ contains all possible null events as defined by probability ρ.

Any property with respect to the sample space Ω will be said to hold for almost all of Ω if and only if for the subset $A \subseteq \Omega$ on which it does not hold, $\rho(A) = 0$. Henceforth, $\forall \omega \in \Omega$ will mean almost everywhere. For a probability measure in specific, this means that $\rho(\neg A) = 1$, so "almost everywhere" coincides with "almost surely".

Definition 2.1.6. A function $X : \Omega \longrightarrow \mathbb{R}$ from a measurable space Ω with σ–algebra $\mathcal{F}$ for which, $\forall \Delta \in \mathcal{B}(\mathbb{R})$, $X^{-1}(\Delta) \in \mathcal{F}$, is a $\mathcal{F}$–measurable (or just measurable when the σ–algebra is implied) function. If $\mathcal{F}$ additionally admits a probability measure ρ, the measurable function X is a random variable.

Without going into much detail, a (probability) measure ρ defines a (Lebesgue) integral of a measurable function (random variable) on an event $A \in \mathcal{F}$, denoted as

$$\int_A f d\rho.$$

With this definition, for another measure μ on $\mathcal{F}$ with $\mu \ll \rho$ (absolutely continuous to ρ), $\exists f : \Omega \longrightarrow \mathbb{R}$ measurable function (Radon–Nikodym derivative) such that, $\forall A \in \mathcal{F}$

$$\mu(A) = \int_A f d\rho.$$

From definition 2.1.6, X induces a probability measure (a law) ρ_X on $\mathcal{B}(\mathbb{R})$. $\forall \Delta \in \mathcal{B}(\mathbb{R})$,

$$\rho_X(\Delta) = \rho(X^{-1}(\Delta)).$$

Definition 2.1.7. A non–decreasing right continuous function $F : \mathbb{R} \longrightarrow \mathbb{R}$ is a distribution.

[2] $\forall A, B \in \mathcal{A}$ with $A \neq B$, $A \cap B = \emptyset$.

[3] The product measure is a probability one because $\forall A \subseteq [0,1]$,

$$\prod\nolimits_{a\in A} a \in [0,1].$$

In particular, $\forall X : \Omega \longrightarrow \mathbb{R}$ random variable, $\exists F_X : \mathbb{R} \longrightarrow [0,1]$ such that, $\forall x \in \mathbb{R}$

$$F_X(x) = \rho(X^{-1}((-\infty, x))).$$

The correspondence is bilateral, i.e. a distribution F_X can always be associated with a probability measure ρ on $\mathbb{R}$.

Random variables are in essence encoding the event information in $\mathcal{F}$ into quantifiable formats, avoiding the technicality of measure spaces and the corresponding Lebesgue integrals. For example in the target space of a random variable X, together with the distribution F_X it induces on $\mathbb{R}$, the Radon–Nikodym derivative simply reduces to the so called probability density function $f_X : \mathbb{R} \longrightarrow \mathbb{R}$ of the random variable X, such that, $\forall x \in \mathbb{R}$,

$$F_X(x) = \int_{(-\infty, x)} dF_X = \int_{-\infty}^{x} f_X(x')dx',$$

where the latter integration is meant with the Lebesgue measure on $\mathcal{B}(\mathbb{R})$, mapping all subintervals of the form $]a, b[$,[4] $\forall a, b \in \mathbb{R}$, to $|a - b|$.

Definition 2.1.8. For a random variable $X : \Omega \longrightarrow \mathbb{R}$, the integral

$$\mathbb{E}[X] = \langle X \rangle = \int_{\Omega} X d\rho,$$

if defined, is the expectation of X on Ω.[5]

By the distribution F_X associated with X, its expectation is trivially

$$\langle X \rangle = \int_{\mathbb{R}} \mathbb{1} d\rho_X = \int_{\mathbb{R}} x dF_X(x),$$

where $\mathbb{1} : \mathbb{R} \longrightarrow \mathbb{R}$ is the identity function, $\forall x \in \mathbb{R}$, $\mathbb{1}(x) = x$.

Lemma 2.1.9 (Jensen's inequality)**.** $\forall \phi : \mathbb{R} \longrightarrow \mathbb{R}$ *convex and measurable and* $\forall X : \Omega \longrightarrow \mathbb{R}$ *an integrable random variable on a probability space, such that* $\phi \circ X : \Omega \longrightarrow \mathbb{R}$ *is integrable,*

$$\phi(\langle X \rangle) \leq \langle \phi \circ X \rangle = \int_{\mathbb{R}} \phi d\rho_X = \int_{\mathbb{R}} \phi(x) dF_X(x).$$

Definition 2.1.10. For a random variable $X : \Omega \longrightarrow \mathbb{R}$, its variance

$$\text{variance}(X) = \langle (X - \langle X \rangle)^2 \rangle = \langle X^2 \rangle - \langle X \rangle^2,$$

is well–defined as $\phi(x) = x^2$ is both convex and measurable on $\mathcal{B}(\mathbb{R})$.

Similarly for an additional random variable $Y : \Omega \longrightarrow \mathbb{R}$, the covariance of X and Y is

$$\text{covariance}(X, Y) = \langle (X - \langle X \rangle)(Y - \langle Y \rangle) \rangle = \langle XY \rangle - \langle X \rangle \langle Y \rangle.$$

For $\langle X \rangle$ to be unique, $\langle |X| \rangle < \infty$, where the absolute value $|\cdot|$ is both convex and measurable on $\mathcal{B}(\mathbb{R})$.

Example 2.1.11 (Noise)**.** A random variable $\eta : \Omega \longrightarrow \mathbb{R}$ with $\langle \eta \rangle = 0$ and $\exists \sigma \in \mathbb{R}$ such that $\text{variance}(\eta) = \langle \eta^2 \rangle = \sigma^2$, is also called a noise.

[4] All other possible endpoint combinations for the subintervals included.

[5] The alternate (unary) $\langle \cdot \rangle$ notation of the expectation is not to be confused with the (binary) notation $\langle \cdot | \cdot \rangle$ of the inner product in Hilbert space nor its (ternary) counterpart $\langle \cdot | \cdot | \cdot \rangle$ involving hermitian linear bounded operators on such spaces. This notation becomes inconvenient with conditional expectation however.

Definition 2.1.12. A collection $\mathcal{X}$ of random variables, for which

$$\sup_{X\in\mathcal{X}}\langle|X|\rangle<\infty$$

is uniformly integrable.

Lemma 2.1.13. *The law ρ_X of a random variable $X:\Omega\longrightarrow\mathbb{R}$ on a probability space $(\Omega,\mathcal{F},\rho)$ is uniquely determined by its characteristic function defined by*

$$\varphi_X(\alpha)=\langle\exp\imath\alpha X\rangle.$$

If φ_X is absolutely integrable on $\mathbb{R}$, the density f_X correspondent to the law ρ_X of X is the Fourier transform of the characteristic function,

$$f_X(x)=\int_{\mathbb{R}}\exp(-\imath\alpha x)\varphi_X(\alpha)d\alpha.$$

The characteristic function of a random variable also generates its various moments, $\forall n\in\mathbb{N}$,

$$\langle X^n\rangle=(-\imath)^n\frac{d^n\varphi_X}{d\alpha^n}(0).$$

The central moments are respectively $\langle(X-\langle X\rangle)^n\rangle$. The first order central moment is trivially 0. The second order central moment is simply the variance of X,

$$\text{variance}\,X=\langle(X-\langle X\rangle)^2\rangle=\langle X^2\rangle-\langle X\rangle^2=\left(\frac{d\phi_X}{d\alpha}(0)\right)^2-\frac{d^2\phi_X}{d\alpha^2}(0).$$

The characteristic function is what will be addressed as the partition function for an action as a random variable on field configuration space.

Definition 2.1.14. For a probability space:

- A finite subcollection $\mathcal{A}\subseteq\mathcal{F}$ of events is independent if and only if

  $$\rho\Big(\bigcap\nolimits_{A\in\mathcal{A}}A\Big)=\prod\nolimits_{A\in\mathcal{A}}\rho(A),$$

  and an arbitrary subcollection $\mathcal{A}\subseteq\mathcal{F}$ is independent if and only if every finite subcollection of $\mathcal{A}$ is independent.
- A finite family $\mathfrak{A}$ of σ–algebras is independent if and only if every finite selection $\{A_{\mathcal{F}}\in\mathcal{F}|\mathcal{F}\in\mathfrak{A}\}$ is an independent collection of events, and an arbitrary family of $\mathfrak{A}$ of σ– algebras is independent if and only if every finite subfamily of $\mathfrak{A}$ is independent.
- A finite collection of random variables $X=\{X_n:\Omega\longrightarrow\mathbb{R}\}_n$ is independent if and only if every selection

  $$\{X_n(A):A\in\mathcal{A}\}_n$$

  of events in $\mathcal{B}(\mathbb{R})$ is independent.

Proposition 2.1.15. *A finite family of random variables $X=\{X_n:\Omega\longrightarrow\mathbb{R}\}_n$ is independent if and only if $\forall\{\phi_n:\mathbb{R}\longrightarrow\mathbb{R}\}_n$ finite collection of measurable functions,*

$$\Big\langle\prod\nolimits_n\phi_n\circ X_n\Big\rangle=\prod\nolimits_n\langle\phi_n\circ X_n\rangle.$$

$\forall A,B\in\mathcal{F}$ events with $\rho(A)\neq0$, the probability of B occurring given A,

$$\rho(B|A)=\frac{\rho(A\cap B)}{\rho(A)},$$

defines a conditional probability measure $\rho(\cdot|A)$ with $\rho(A|A)=1$. If A and B are independent, $\rho(B|A)=\rho(B)$ and $\rho(A|B)=\rho(A)$ mutually.

By extension, a conditional on $A \in \mathcal{F}$ expectation $\forall X$ random variable is

$$\mathbb{E}[X|A] = \int_{\Omega} X d\rho(\cdot|A).$$

The σ–algebra of a probability space encodes the information said probability space has (observable) access to. Conditioning on a particular event $A \in \mathcal{F}$, results in a sub–σ–algebra $\mathcal{E} \leq \mathcal{F}$,[6]

$$\mathcal{E} = \{A \cap B | B \in \mathcal{F}\},$$

so it makes sense to generalize conditioning on a sub–σ–algebra $\mathcal{E} \leq \mathcal{F}$.

Definition 2.1.16. $\forall \mathcal{E} \leq \mathcal{F}$ sub–σ–algebra of events in a probability space, the conditional on $\mathcal{E}$ expectation $\mathbb{E}[X|\mathcal{E}]$ of a random variable X on Ω is an $\mathcal{E}$–measurable and integrable random variable satisfying $\forall A \in \mathcal{E}$,

$$\mathbb{E}[X \mathbb{1}_A] = \mathbb{E}[\mathbb{E}[X|\mathcal{E}]\mathbb{1}_A].$$

For another random variable Y, the conditional on Y expectation $\mathbb{E}[X|Y]$ of X is simply $\mathbb{E}[X|\sigma(Y)]$.[7]

Theorem 2.1.17. *$\forall(\Omega, \mathcal{F}, \rho)$ probability space and $\forall \mathcal{E} \leq \mathcal{F}$ sub–σ–algebra of events, the conditional on $\mathcal{E}$ expectation $\mathbb{E}[X|\mathcal{E}]$ of an (absolutely–integrable) random variable $X \in L^1(\Omega, \mathcal{F}, \rho)$ is almost surely (ρ–almost) unique.*

For conditioning with another random variable:

Theorem 2.1.18. *$\forall(\Omega, \mathcal{F}, \rho)$ probability space, $\forall(E, \mathcal{E}, \mu)$ measure space, $\forall X : \Omega \longrightarrow \mathbb{R}$ ρ–integrable random variable and $Y : \Omega \longrightarrow \mathbb{R}$ random variable, where a measure μ is induced by Y as $\mu(A) = \rho(Y^{-1}(A))$, $\forall \Delta \in \mathcal{B}(\mathbb{R})$,[8] $\exists \phi : \mathbb{R} \longrightarrow \mathbb{R}$ ρ–unique μ–integrable random variable such that $\forall A \in \mathcal{E}$*

$$\int_{\Delta} \phi d\mu = \int_{Y^{-1}(\Delta)} X d\rho.$$

Therefore $\mathbb{E}[X|Y] = \phi \circ Y : \Omega \longrightarrow \mathbb{R}$, roughly speaking a function of Y as the expectation on X naturally eliminates any information on X. In the usual case that Y is also a random variable (valued on $\mathbb{R}$), ϕ reduces to a measurable function.

The conditional expectation inherits the properties of the Lebesgue integral (expectation) along with some extra properties.

Lemma 2.1.19 (Taking out what is known)**.** *$\forall X : \Omega \longrightarrow \mathbb{R}$ an integrable random variable and $\forall Y : \Omega \longrightarrow \mathbb{R}$ an $\mathcal{E}$–measurable random variable, both on a probability space with $\mathcal{E} \leq \mathcal{F}$, such that XY is integrable,*

$$\mathbb{E}[XY|\mathcal{E}] = Y\mathbb{E}[X|\mathcal{E}].$$

Jensen's inequality (lemma 2.1.9) holds conditionally as well,

$$\phi(E[X|\mathcal{E}]) \leq E[\phi \circ X|\mathcal{E}].$$

For multivariate random variables (ordered collection of random variables) $X = (X_i)_{i=1}^n : \Omega \longrightarrow \mathbb{R}^n$ with the target space assuming the Lebesgue measure on $\mathcal{B}(\mathbb{R}^n)$, the corresponding characteristic function becomes

$$\varphi_X(\alpha) = \langle \exp \imath\alpha \cdot X \rangle,$$

[6] $\mathcal{E}$ is a σ–algebra of Ω from definition 2.1.1.

[7] The $\mathbb{E}$ notation is used for random variable conditioning as it is of the "same kind" as the main argument (also a random variable), unlike conditioning with a sub–σ–algebra. However behind the scenes the forms are equivalent.

[8] This is possible because Y as a random variable (definition 2.1.6) is $\mathcal{F}$–measurable, therefore $Y^{-1}(\Delta) \in \mathcal{F}$, meaning the measure is well defined $\forall \Delta \in \mathcal{B}(\mathbb{R})$. In general $\sigma(Y) \leq \mathcal{F}$ (with equality meaning no conditioning).

where

$$\alpha \cdot X = \sum_{i=1}^{n} \alpha_i X_i$$

is simply a linear combination of the random variables in X expressed in the default inner product of $\mathbb{R}^n$.

Per the definition 2.1.14, a collection of random variables X is independent if and only if the joint distribution F_X and the marginal distributions F_{X_i}, $\forall i \in \mathbb{N}_n$, relate as

$$F_X(x) = \prod_{i=1}^{n} F_{X_i}(x_i),\ \forall x \in \mathbb{R}^n.$$

If the joint f_X and marginal f_{X_i}, $\forall i \in \mathbb{N}_n$, densities exist,

$$f_X(x) = \prod_{i=1}^{n} f_{X_i}(x_i),\ \forall x \in \mathbb{R}^n.$$

For two independent random variables X and Y, the measurable function ϕ in theorem 2.1.18 for $\mathbb{E}[X|Y]$ is in fact the marginal density f_Y, and

$$f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x, y)dx.$$

2.2. Stochastic processes

Definition 2.2.1. A time index set $\mathbb{T}$ is a totally–ordered set with a (unique) strict minimum 0, i.e. $\forall t \in \mathbb{T}$ with $t \neq 0$, $t > 0$ with the induced strict total ordering.[9] A complete time index set $\overline{\mathbb{T}}$ also includes a (unique) strict maximum ∞, labeled as such to represent the end of time.

Both $\mathbb{N}$ and $\mathbb{R}_+ \cup \{0\}$ are time index sets, with their completed variants $\mathbb{N} \cup \{\infty\}$ and $\overline{\mathbb{R}}_+$. However, $\mathbb{N}_n$, $\forall n \in \mathbb{N}$, and $[0, \tau]$, $\forall \tau \in \mathbb{R}_+$ are also complete (finite) time index sets. $\mathbb{N}$ as a time index is discrete, while $\mathbb{R}$ is continuous.

Definition 2.2.2. A collection

$$X = \{X_t : \Omega \longrightarrow \mathbb{R} | t \in \mathbb{T}\} : \mathbb{T} \times \Omega \longrightarrow \mathbb{R}$$

of random variables for a time index $\mathbb{T}$ is a stochastic process.

For a discrete time index, X is also termed a stochastic sequence.

$\forall t \in \Sigma$, $X_t : \Omega \longrightarrow \mathbb{R}$ is a particular state (and a random variable) of the stochastic process X.

$\forall \omega \in \Omega$, $X(\omega) : \mathbb{T} \longrightarrow \mathbb{R}$ is particular sample path of the stochastic process X.

Definition 2.2.3. The σ–algebra of events generated by a stochastic process is

$$\mathcal{X} = \bigvee_{t \in \mathbb{T}} \sigma(X_t).$$

A σ–algebra is generated for subsets of the time index Σ as well. Of particular interest are the subsets $I \subseteq \mathbb{T}$ bounded by two time limits $a, b \in \mathbb{T}$,[10]

- $I = [a, b] = \{t \in \mathbb{T} | a \leq t \leq b\} \subseteq \mathbb{T}$,
- $I = [a, b[= \{t \in \mathbb{T} | a \leq t < b\} \subseteq \mathbb{T}$,
- $I =]a, b] = \{t \in \mathbb{T} | a < t \leq b\} \subseteq \mathbb{T}$,
- $I =]a, b[= \{t \in \mathbb{T} | a < t < b\} \subseteq \mathbb{T}$,

[9] It is indeed unique in the strict ordering sense; assuming $\exists 0' \in \mathbb{T}$ also strict minimum with $0' \neq 0$, both $0' > 0$ and $0' < 0$ from the definition of the strict minimum.

[10] Generalizing the notion of intervals in $\mathbb{R}$, which solely rely on the total ordering of the set.

denoted as

$$\mathcal{X}_I = \bigvee_{t\in I} \sigma(X_t).$$

For a stochastic process X:

- the past of the process is described by $\mathcal{X}_{<t} = \mathcal{X}_{[0,t[}$,
- the complete past of the process is described by $\mathcal{X}_{\leq t} = \mathcal{X}_{[0,t]}$,
- the present of the process is described by $\mathcal{X}_{=t} = \mathcal{X}_{[t,t]}$,
- the future of the process is described by $\mathcal{X}_{>t} = \mathcal{X}_{]t,\infty[}$,
- the complete future of the process is described by $\mathcal{X}_{\geq t} = \mathcal{X}_{[t,\infty[}$,

If the time index $\mathbb{T}$ has a maximum time ∞, the upper time bound of the future of a process X is closed instead.

As a random variable induces a probability measure on $\mathcal{B}(\mathbb{R})$, a stochastic process induces a probability measure on $\mathcal{B}(\mathbb{T}) \otimes \mathcal{B}(\mathbb{R})$, which is a (product) σ–algebra.[11]

A contained in $I \subseteq \mathbb{T}$ stochastic subprocess $X_I : I \times \Omega \longrightarrow \mathbb{R}$ can then be measurable on the $\mathcal{B}(I) \otimes \mathcal{F}$ σ–algebra.

Definition 2.2.4. A filtration $\mathcal{F}$ on a probability space is a family $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$ of sub–σ–algebras of $\mathcal{F}$ such that $\forall t, t' \in \mathbb{T}$ with $t' \leq t$, $\mathcal{F}_{t'} \leq \mathcal{F}_t$.

It is straightforward to see that a filtration $\mathcal{F}$ assumes a total ordering with a global minimum $\mathcal{F}_0$ (and optionally a global maximum $\mathcal{F}_\infty$ if $\mathbb{T}$ is complete).[12] If $\mathbb{T}$ is discrete, the filtration is well–ordered (see also definition 2.2.4).[13]

A filtration models the information growth with time related to an experiment prescribed by the σ–algebra of all events on a probability space, elevating the corresponding probability space to a filtered probability space.

The σ–algebras in definition 2.2.3 form the so–called natural filtration of the process X.

Definition 2.2.5. $\forall t \in \mathbb{T}$,

$$\mathcal{F}_{t+} = \bigcap_{t'>t} \mathcal{F}_t$$

is the immediate future σ–algebra to time t on the filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\in\mathbb{T}}$, while[14]

$$\mathcal{F}_{t-} = \bigvee_{t'<t} \mathcal{F}_t$$

is the past σ–algebra to time t.

A filtration $\mathcal{F}$ is right–continuous if and only if $\forall t \in \mathbb{T}$, $\mathcal{F}_t = \mathcal{F}_{t+}$, i.e. the lower bound of the totally ordered (possibly uncountable) intersection is included in it. A filtration $\mathcal{F}$ is complete if the corresponding probability space is complete and $\forall A \in \mathcal{F}_\infty$ with $\rho(A) = 0$, $A \in \mathcal{F}_{0+}$, i.e. all σ–algebras of the filtration contain all the null events of $\mathcal{F}_\infty$. Both right–continuity and completeness ar labeled as usual conditions.

[11] To include discrete times in this case, a topology has to be devised for $\mathbb{N}$. As a countable set, it makes sense to use the power set $2^{\mathbb{N}}$, which is a trivial topology for all sets. In this case, $\mathcal{B}(\mathbb{N}) = 2^{\mathbb{N}}$ as well. In such cases space delimiters have no meaning and the closed interval will be used.

[12] The notation $\mathcal{F}_\infty$ is used for all the evens in Ω in the sense, that any filtration will maximally contain all events eventually in time $\mathbb{T}$, if at all. This way the symbol $\mathcal{F}$ is reserved for the filtration itself, in par with the notation used for a stochastic process X.

$$\mathcal{F}_\infty \supseteq \bigvee \mathcal{F} = \bigvee_{\tau\in\mathbb{T}} \mathcal{F}_\tau,$$

with equality holding for $\mathbb{T}$ complete ($\exists \max \mathbb{T} = \infty \in \mathbb{T}$).

[13] The difference is that with a partial ordering, not all pairs of elements are comparable but, for those that are, the same properties of the total ordering hold. In total ordering, every pair of elements is comparable.

[14] $\forall \mathfrak{A}$ a family of σ–algebras,

$$\bigcap_{\mathcal{F}\in\mathfrak{A}} \mathcal{F}$$

is also a σ–algebra. Indeed:

- $\forall \mathcal{F} \in \mathfrak{A}$, $\emptyset \in \mathcal{F}$,therefore $\emptyset \in \cap_{\mathcal{F}\in\mathfrak{A}}\mathcal{F}$,
- $\forall A \in \cap_{\mathcal{F}\in\mathfrak{A}}\mathcal{F}$, $A \in \mathcal{F}$ meaning $\neg A \in \mathcal{F}$, $\forall \mathcal{F} \in \mathfrak{A}$,therefore $\neg A \in \cap_{\mathcal{F}\in\mathfrak{A}}\mathcal{F}$,
- $\forall \mathcal{A} \subseteq \cap_{\mathcal{F}\in\mathfrak{A}}\mathcal{F}$ finite subcollection, $\mathcal{A} \subseteq \mathcal{F}$ meaning $\cup_{A\in\mathcal{A}} A \in \mathcal{F}$, $\forall \mathcal{F} \in \mathfrak{A}$, therefore $\cup_{A\in\mathcal{A}} A \in \cap_{\mathcal{F}\in\mathfrak{A}}\mathcal{F}$.

It is labeled as "immediate" because $\cap_{\mathcal{F}\in\mathfrak{A}}\mathcal{F}$, as a lower bound of $\{\mathcal{F}\}_{\mathcal{F}\in\mathfrak{A}}$, is not necessarily included in it (see Definition 2.2.4).

Definition 2.2.6. A stochastic process X such that $\forall t \in \mathbb{T}$, X_t is $\mathcal{F}_t$–measurable, is adapted to the corresponding filtered probability space.

A stochastic process is adapted to its natural filtration $\mathcal{X}$ by definition 2.2.3.

Definition 2.2.7. A stochastic process X such that $\forall t \in \mathbb{T}$, $X_{[0,t]}$ is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$–measurable, is progressively measurable.

A progressively measurable stochastic process is adapted to the filtration of the filtered probability space it is defined on.

A random variable $T : \Omega \longrightarrow \mathbb{T}$ is a random time. T is a stopping time if and only if $\forall t \in \mathbb{T}$, $\{T \leq t\} = \{\omega \in \Omega | T(\omega) \leq t\} \in \mathcal{F}_t$.

The intuition behind a stopping time is that events defined by it only require information from the past alone.

Definition 2.2.8. $\forall T : \Omega \longrightarrow \mathbb{T}$ a stopping time,

$$\mathcal{F}_T = \{A \in \mathcal{F} | A \cap \{T \leq t\} \in \mathcal{F}_t, \forall t \in T\}$$

is the σ–algebra of events occurring by time T. T as a random variable is $\mathcal{F}_T$–measurable.

For T a stopping time and $\forall t \in \mathbb{T}$, t and $T + t$ are stopping times. For a countable collection T of stopping times $\sup T$ and $\inf T$ are stopping times. If $T = \{T_n\}_{n \in \mathbb{N}}$ is a sequence of stopping times,[15] $\limsup_{n \in \mathbb{N}} T$ and $\liminf_{n \in \mathbb{N}} T$ are stopping times.

Proposition 2.2.9. *$\forall X$ progressively measurable stochastic process on a filtration $\mathcal{F}$ of a filtered probability space, and $\forall T$ stopping time, the random variable X_T is $\mathcal{F}_T$–measurable.*

The definition of time–contained natural filtrations (see definition 2.2.3) can also be defined by stopping times as well.

2.3. Martingales

Definition 2.3.1. A progressively measurable stochastic process $X = \{X_t\}_{t \in \mathbb{T}}$ on a filtered probability space with filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t \in \mathbb{T}}$, for which $\forall t \in \mathbb{T}$, $\mathbb{E}[X_t] < \infty$ and $\forall t' < t$:

- $\mathbb{E}[X_t | \mathcal{F}_{t'}] \leq X_{t'}$, is a $\mathcal{F}$–supermartingale,
- $\mathbb{E}[X_t | \mathcal{F}_{t'}] = X_{t'}$, is a $\mathcal{F}$–martingale,
- $\mathbb{E}[X_t | \mathcal{F}_{t'}] \geq X_{t'}$, is a $\mathcal{F}$–submartingale.

Martingales are defined for both discrete and continuous time indices $\mathbb{T}$, however the latter requires some technical assumptions not really relevant to the use of stochastic processes in Monte Carlo simulations, therefore focus will be given in the case of discrete time, as in [66]. This means that stopping times are henceforth also integer–valued random variables, in the range $\mathbb{N}$ (including 0).

Theorem 2.3.2. *$\forall X$ $\mathcal{F}$–martingale and $\forall \mathcal{T}$ non–decreasing sequence of stopping times, [16], $X_{\mathcal{T}}$ is an $\mathcal{F}$–martingale.*

Analogous results hold for supermartingales and submartingales. This result allows replacing the time index with a random time in martingales.

Example 2.3.3. $\forall \{\eta_n\}_{n \in \mathbb{N}}$ countable collection of independent identically distributed noises,[17] the stochastic process X defined $\forall n \in \mathbb{N}$ by

$$X_n = \sum_{i=0}^{n} \eta_i,$$

is an martingale with respect to the filtration defined $\forall n \in \mathbb{N}$ by

$$\bigvee_{i=0}^{n} \sigma(\eta_i).$$

[15] A collection $\{B_a\}_{a \in A}$ of sets is a sequence if and only if A is well ordered.
[16] $\forall \omega \in \Omega$, $\forall n, n' \in \mathbb{N}$ with $n' \leq n$, $T_{n'}(\omega) \leq T_n(\omega)$.
[17] Also a stochastic noise in a sense.

Theorem 2.3.4 (Strong law of large numbers)**.** $\forall X$ *$\mathcal{F}$–super–, $\mathcal{F}$–sub–, $\mathcal{F}$–martingale such that* $\sup_{n\in\mathbb{N}}\langle|X_n|\rangle < \infty$, $\lim_{n\to\infty} X = X_\infty$ *with* $\langle|X_\infty|\rangle < \infty$.

For martingales, the suprenum condition may be replaced with non–negativity of X.

Theorem 2.3.5. $\forall X$ *stochastic process consisting of independent identically distributed random variables* X_n, $\forall n \in \mathbb{N}$, *such that* $\text{variance}(X_0) < \infty$,

$$\lim_{n\to\infty} \sum_{i=0}^{n} X_i = \langle X_0 \rangle, \text{ almost surely.}$$

2.4. Wiener processes

Markov processes have several equivalent definitions, all telling the same thing, that in a present state X_t of such a process X at a time $t \in \mathbb{T}$, the future and the past of the process are independent.

Definition 2.4.1. A stochastic process X such that $\forall t \in \mathbb{T}$, $\forall F_{\text{past}}$ bounded $\mathcal{X}_{\leq t}$–measurable function and $\forall F_{\text{future}}$ bounded $\mathcal{X}_{\geq t}$–measurable function,

$$\mathbb{E}[F_{\text{past}}F_{\text{future}}|\mathcal{X}_{=t}] = \mathbb{E}[F_{\text{past}}|\mathcal{X}_{=t}]\mathbb{E}[F_{\text{future}}|\mathcal{X}_{=t}] \text{ almost surely,}$$

is a Markov process.

An equivalent definition describes a Markov process by its more famous description that its future is not dependent on its past.

Definition 2.4.2. A stochastic process X such that $\forall t \in \mathbb{T}$ and $\forall F_{\text{future}}$ bounded $\mathcal{X}_{\geq t}$–measurable function,

$$\mathbb{E}[F_{\text{future}}|\mathcal{X}_{=t}] = \mathbb{E}[F_{\text{future}}|\mathcal{X}_{\leq t}] \text{ almost surely,}$$

is a Markov process.

$\forall A \in \mathcal{X}_{\leq t}$ with $F_{\text{past}} = \mathbb{1}_A$ and $\forall B \in \mathcal{X}_{\geq t}$ with $F_{\text{future}} = \mathbb{1}_B$, these definitions reduce to probability definitions respectively,

$$\rho(AB|\mathcal{X}_{=t}) = \rho(A|\mathcal{X}_{=t})\rho(B|\mathcal{X}_{=t}) \text{ and } \rho(B|\mathcal{X}_{=t}) = \rho(B|\mathcal{X}_{\leq t}).$$

For a discrete time index $\mathbb{T}$, a Markov process is a Markov chain.

Example 2.4.3. $\forall \eta = \{\eta_n\}_{n\in\mathbb{N}}$ a sequence of independent random variables and $\forall\{f_n : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}\}_{n>0}$ collection of measurable on $\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})$ functions, the stochastic process $X = \{X_n\}_{n\in\mathbb{N}}$ defined $\forall n \in \mathbb{N}$ by,[18]

$$X_n = f_n \circ (X_{n-1} \otimes \eta_n),\ X_0 = \eta_0,$$

is a Markov chain.

The underlying theory of transition probabilities ensures that Markov processes with special conditions can be formed from an initial random condition and a transition function (kernel) that advances the process by induction. This induction depends only on the immediately previous state for the case of Markov chains, as shown in example 2.4.3.

$\forall\mathbb{T}$ a time index which is also a field with addition $+$ and multiplication $\cdot$ (definition 1.1.4), and $\forall\{t_n\}_{n\in\mathbb{N}} \subseteq \mathbb{T}$ sequence of times with $t_0 = 0 \in \mathbb{T}$, $\{\Delta t_n = t_n - t_{n-1}\}_{n>0}$ is a sequence of time increments defined by $\{t_n\}_{n\in\mathbb{N}}$.

Henceforth $\forall X$ a stochastic process, the states of the process at times $\{t_n\}_{n\in\mathbb{N}}$ will be labeled by the new index directly, as $\{X_n\}_{n\in\mathbb{N}}$. In effect, $\mathbb{N}$ constitutes a time discretization of $\mathbb{T}$ (if $\mathbb{T} \simeq \mathbb{R}_+$), and $\{X_n\}_{n\in\mathbb{N}}$ is a countable subprocess of X. If $\mathbb{N}_N$, $\forall N \in \mathbb{N}$ is used instead, a finite subprocess of X is extracted, which is also a vector random variable. The finite–dimensional distributions of a stochastic process refers to the distribution of its finite subprocesses.

$\forall X$ a stochastic process, let ΔX be a process of increments in that $\forall n > 0$, $\Delta X_n = X_n - X_{n-1}$.

[18] $\forall X, Y, Z$ sets and $\forall f : X \longrightarrow Z : x \longmapsto f(x)$ and $\forall g : Y \longrightarrow Z : y \longmapsto f(y)$ functions with the same target set,

$$h = f \otimes g : X \times Y \longrightarrow Z \times Z : (x, y) \mapsto h(x, y) = (f(x), g(y))$$

Definition 2.4.4. A stochastic process X such that $\forall\{t_n\}_{n\in\mathbb{N}} \subseteq \mathbb{T}$ sequence of times, the random variables induced by the corresponding states of the increments process ΔX on said times are independent, is said to be a process of independent increments.

To define such a process, only the initial distribution and those of increment random variables are necessary, as can be seen by the characteristic function of the finite–dimension distributional distributions of finite subselections of states on the times given:

$$\left\langle \exp \imath \sum_{i=0}^{n} \alpha_n X_n \right\rangle = \left\langle \exp \imath \left(\alpha_0 X_0 + \sum_{i=1}^{n} \sum_{j=i}^{n} \alpha_i \Delta X_n \right) \right\rangle = \langle \exp \imath \alpha_0 X_0 \rangle \prod_{i=1}^{n} \left\langle \exp \imath \sum_{j=i}^{n} \alpha_i \Delta X_n \right\rangle$$

Proposition 2.4.5. *A process with independent increments is a Markov process.*

Henceforth for random vectors a generic vector space V on a field of finite dimension with an inner product $\cdot$, and a possible conjugation $\cdot^*$ on a field $\mathbb{K}$ with a topology $\mathcal{T}$,[19] will be assumed as the target space, i.e. $X : \Omega \longrightarrow V$. For random variables, the generic field $\mathbb{K}$ will be used, but in reality either $\mathbb{R}$ or the algebraically closed $\mathbb{C}$ with conjugation will be meant, i.e. $X : \Omega \longrightarrow \mathbb{K}$, with $\mathcal{B}(\mathbb{K})$ defined by the topology $\mathcal{T}$ on $\mathbb{K}$. Lastly, besides being well–ordered, the time index $\mathbb{T}$ will be assumed a field as well.

Definition 2.4.6. A random variable $X : \Omega \longrightarrow \mathbb{K}$ with Gaussian probability density function,

$$f : \mathbb{K} \longrightarrow \mathbb{R}_+ : x \mapsto f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(- \frac{|x-\mu|^2}{2\sigma^2} \right),$$

with $\mu \in \mathbb{R}$ and $\sigma^2 \in \mathbb{R}_+$.

The parameters of the Gaussian density define the first (and all) the moments of X,

$$\langle X \rangle = \mu \text{ and } \text{variance}(X) = \sigma^2$$

The case $\sigma^2 = 0$ is formally definable with Dirac's δ distribution:

Definition 2.4.7. Dirac's δ on a field $\mathbb{K}$ with a topology $\mathcal{T}$, is a measure on $\mathcal{B}(\mathbb{K}) = \sigma(\mathcal{T})$ such that, $\forall A \in \mathcal{B}(\mathbb{K})$,

$$\delta(A) = \begin{cases} 1 & 0 \in A \\ 0 & 0 \notin A \end{cases},$$

where $0 \in \mathbb{K}$ is the unit element of the field addition (and the destructive element of the field product) in $\mathbb{K}$ (definition 1.1.4).

$\forall f : \mathbb{K} \longrightarrow \mathbb{K}$,

$$\int_{\mathbb{K}} f d\delta = f(0).$$

Dirac's δ is extensible to vector spaces with a topology as is.

For a vector random variable $X : \Omega \longrightarrow V$, $\dim V \in \mathbb{N}$, the corresponding (orthonormal Gaussian density becomes

$$f : V \longrightarrow \mathbb{R}_+ : x \mapsto f(x) = \left(\prod_{n=1}^{\dim V} \frac{1}{\sqrt{2\pi\sigma_n^2}} \right) \exp\left(- \sum_{n=1}^{\dim V} \frac{(x_n - \mu_n)^2}{2\sigma_n^2} \right).$$

The base–independent form for a generic vector space V,

$$f : V \longrightarrow \mathbb{R}_+ : x \mapsto f(x) = \sqrt{(2\pi)^{-\dim V} (\det \upsilon)^{-1}} \exp \frac{1}{2} (x-\mu) \cdot \upsilon^{-1} \cdot (x-\mu),$$

[19] See chapter 1. Mathematical foundation for details.

where $\upsilon : V \longrightarrow V$ is a non–singular ($\exists \upsilon^{-1}$) positive definite (and hence hermitian) operator on V and μ and x are vectors on V.

The Gaussian characteristic is

$$\langle \exp \imath\alpha \cdot X \rangle = \exp\left(\imath\alpha \cdot \alpha - \frac{1}{2}\alpha \cdot \upsilon \cdot \alpha\right).$$

Gaussian random variables are uniquely determined by the first two moments μ and υ. If υ is diagonal, the random variables defining the finite random vector X are termed uncorrelated.

Proposition 2.4.8. *Uncorrelated random variables in a finite–dimensional random vector X are independent.*

Definition 2.4.9. A finite stochastic process $X_{\leq t}$, $\forall t \in \mathbb{T}$, is Gaussian if and only if every discrete finite subprocess $\{X_n\}_{n\in\mathbb{N}_\nu}$, $\forall \nu \in \mathbb{N}$, of X is Gaussian.

Proposition 2.4.10. *$\forall X = \{X_n\}_{n\in\mathbb{N}}$ a Gaussian stochastic sequence and $\forall X$ a random variable with $\lim_{n\to\infty} X_n = X$, X is also Gaussian.*

The proposition 2.4.10 extends to a sequence of stochastic processes.

Definition 2.4.11. A stochastic process X whose finite–dimensional distributions are independent of a time–shift $t \in \mathbb{T}$ are strictly stationary.

All moments of a stochastic process are functions of the form

$$\text{moment}(X) : \mathbb{T} \longrightarrow \mathbb{K} : t \mapsto \text{moment}(X_t).$$

Of particular interest in strictly stationary processes are the first two moments:

- the expectation $\mu(t) = \langle X_t \rangle$ is constant,
- the correlation $\upsilon(t', t) = \text{covariance}(X_{t'}, X_t) = \langle X_{t'} - \mu(t') | X_t - \mu(t) \rangle = \upsilon(t - t')$ depends only on the time difference.

The converse is not true, defining a widely stationary process X, which satisfies these two moment conditions. However a Gaussian wide stationary process is a strictly stationary process.

Let

$$\varphi : V \longrightarrow \mathbb{R}_+ : x \mapsto \varphi(x) = (2\pi)^{-\dim V/2} \exp \frac{1}{2}(x - \mu) \cdot (x - \mu)$$

be the standard Gaussian density with mean 0 and variance $1_{\dim V}$.

Definition 2.4.12. $\forall x \in \mathbb{K}$, a Wiener process (or Brownian motion) $W = \{W_t\}_{t\in\mathbb{T}}$ with $W_0 = x$ (a deterministic initial condition) is a stationary standard Gaussian process. The Wiener process W with $W_0 = 0$ is a standard Wiener process.

As such, a Wiener process is a process of independent increments, and more specifically:

- $\forall t \in \mathbb{T}$, $\langle \Delta W_t \rangle = 0$,
- $\forall \Delta t \in \mathbb{T}$ on t, $\text{variance}(\Delta W_t) = \langle \Delta W_t | \Delta W_t \rangle = \Delta t$,

For the Wiener process W itself:

- $\forall t \in \mathbb{T}$, $\langle W_t \rangle = x$,
- $\forall t', t \in \mathbb{T}$, $\text{covariance}(W_{t'}, W_t) = \langle W_{t'} - x | W_t - x \rangle = \min\{t', t\}$.

A Brownian motion has a continuous modification with continuous sample paths [66], therefore it will henceforth be assumed as such directly.

Theorem 2.4.13. *$\forall x \in \mathbb{R}$ and a finite stochastic process $X_{\leq t}$ with $X_0 = x$ adapted to a filtration $\mathcal{F}$ such that, $\forall t \in \mathbb{T}$ $\forall \Delta t \in \mathbb{T}$ on t:*

- *$\mathbb{E}[\Delta X_t | \mathcal{F}_t] = 0$ almost surely or X is a martingale,*
- *$\mathbb{E}[(\Delta X_t)^2 | \mathcal{F}_t] = \Delta t$ almost surely or $\{X_t^2 - t\}_{t\in\mathbb{T}}$ is a martingale,*

is a Wiener process.

Properties of a Wiener process W with $W_0 = x$

- (stochastic exponent) $\forall \alpha \in \mathbb{K}$, the process

$$\left\{ M_t = \exp\left(\alpha W_t - \frac{1}{2}\alpha^2 t^2 \right) \right\}$$

  is a martingale with respect to the natural filtration $\mathcal{W}$ of W,[20] i.e.

$$\mathbb{E}[\exp \alpha \Delta W_t | \mathcal{W}_t] = \exp \frac{1}{2}\alpha^2 \Delta t \text{ almost surely.}$$

- (strong Markov property) $\forall T$ stopping time, $\{\Delta W_t = W_{t+T} - W_t\}_{t \in \mathbb{T}}$ is a Wiener process independent of $\mathcal{W}$.

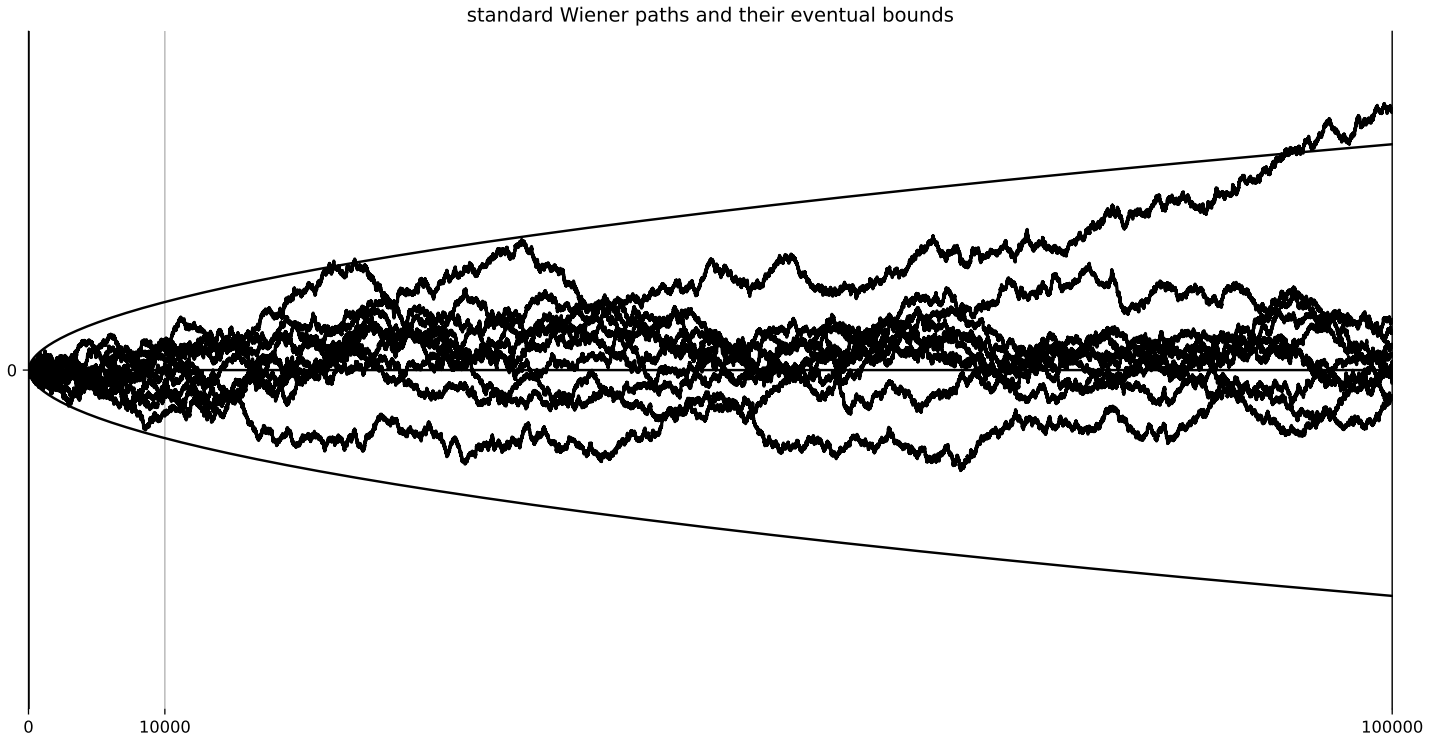


Figure 2.4.1.: A collection of Wiener process realization with their eventual bounds. One of the sample paths appears to require more thermalization time, as it still breaches the bounds.

Properties of a standard Wiener process W

- (spatial homogeneity) $\forall x \in \mathbb{K}$, $\{W_t + x\}_{t \in \mathbb{T}}$ is a Wiener process starting on x,
- (symmetry) $-W$ is a (standard) Wiener process,
- (scaling) $\forall \alpha \in \mathbb{T}$, $\{\sqrt{\alpha} W_{t/\alpha}\}_{t \in \mathbb{T}}$ is a (standard) Wiener process,
- (time reversibility) $\forall \alpha \in \mathbb{T}$, $\{W_t\}_{t \leq \alpha}$ and $\{W_\alpha - W_{\alpha - t}\}_{t \leq \alpha}$ are identical in probability,
- (strong law of large numbers) $\lim_{t \to \infty} tW_t = 0$ almost surely.
- (law of iterated logarithm) A standard Wiener process W is eventually bound by $\pm\sqrt{2t \log\log t}$ (figure 2.4.1):

$$\limsup_{t \to \infty} \frac{W_t}{\sqrt{2t \log\log t}} = +1 \text{ and } \liminf_{t \to \infty} \frac{W_t}{\sqrt{2t \log\log t}} = -1$$

[20] See definitions 2.2.3 and 2.2.4.

3. Stochastic calculus

3.1. The Itô integral

Initially, time integrals of the type

$$\int_0^\tau f_t dW_t$$

for (possible random) functions f satisfying specific conditions and a measure induced by the Wiener process W need to be defined.[1] Assume a probability space Ω with events $\mathcal{F}_\infty$, a probability measure ρ on them, and a filtration $\mathcal{F}$ on a time index $\mathbb{T}$, satisfying all usual conditions listed after definition 2.2.5.

Definition 3.1.1. $\forall p \in \mathbb{N}$, let $L^p(\rho)$ be all the almost–everywhere p–norm ρ–integrable random functions, i.e. $\forall X : \Omega \longrightarrow V \in L^p(\rho)$

$$\int \|X\|_p^p d\rho < \infty,$$

where the p–norm is defined $\forall x \in V$ by

$$\|x\|_p^p = \sum\nolimits_{i=1}^{\dim V} |x_i|^p,$$

with the special case $\|x\|_2^2 = x \cdot x$ for the inner product of V.

$L^p(\rho)$ is a Hilbert (infinite–dimensional vector) space with inner product defined $\forall X, Y \in L^2(\rho)$ by

$$\langle X|Y\rangle = \int X \cdot Y d\rho$$

For $V = \mathbb{R}$, all p–norms reduce to an absolute value defined by the addition on $\mathbb{R}$ and the inner product to the product on $\mathbb{R}$.[2] For simplicity, X is assumed real.

$\forall \tau \in \mathbb{T}$, $\mathcal{H}_2[0,\tau]$ is the space of all progressively measurable stochastic processes $X : \mathbb{T} \times \Omega \longrightarrow \mathbb{R}$ that satisfy

$$\int_0^\tau \langle X_t|X_t\rangle dt < \infty. \tag{3.1.1}$$

The inner product of $L^2(\rho)$ appears to naturally extend to $\mathcal{H}_2[0,\tau]$ via the correspondence

$$X \longleftrightarrow \int_0^\tau X_t dW_t,$$

defining an isometry between the two spaces, however special conditions apply [66], namely $\forall X \in \mathcal{H}_2[0,\tau]$

$$\left\langle \int_0^\tau X_t dW_t \right\rangle = 0 \text{ and } \left\langle \int_0^\tau X_t dW_t \middle| \int_0^\tau X_t dW_t \right\rangle = \int_0^\tau \langle X_t|X_t\rangle dt, \tag{3.1.2}$$

where it becomes apparent that the integral

$$\int_0^\tau X_t dW_t$$

is (naturally) a random variable on $\Omega \longrightarrow \mathbb{R}$. If however $\tau \in \mathbb{T}$ is assumed variable, the aforementioned integral becomes a stochastic process itself. In what follows, τ may be ∞, therefore time index $\mathbb{T}$ itself is simply assumed, whether finite or infinite.

[1] One can naively expect that the process $W : \mathbb{T} \times \Omega \longrightarrow \mathbb{R}$ induces a measure (law) ρ_W on $\mathcal{B}(\mathbb{R})$ by the product measure defined on $\mathcal{B}(\mathbb{T}) \otimes \mathcal{F}_\infty$, however stochastic integration involving time requires some extra conditions, which are explored in this chapter.

[2] See definition 1.1.4 for details.

Theorem 3.1.2. $\forall X \in \mathcal{H}_2(\mathbb{T})$, *the stochastic process*

$$\left\{ \int_0^\tau X_t dW_t \right\}_{\tau \in \mathbb{T}} \tag{3.1.3}$$

is an almost surely continuous martingale, such that

$$\left\langle \sup_{\tau \in \mathbb{T}} \left\langle \int_0^\tau X_t dW_t \middle| \int_0^\tau X_t dW_t \right\rangle \right\rangle \leq 4 \int_0^\tau \langle X_t | X_t \rangle dt.$$

Example 3.1.3. The stochastic process $\overline{W}$ defined $\forall x \in \mathbb{R}$, $\forall f, g : \mathbb{T} \longrightarrow \mathbb{R}$, non–random functions, such that $\forall \tau \in \mathbb{T}$,

$$\overline{W}_\tau = x + f(\tau) + \int_0^\tau g(t) dW_t,$$

is a Gaussian process with independent increments, such that

$$\langle \overline{W}_\tau \rangle = x + f(\tau).$$

Condition (3.1.1) states that random variables in $\mathcal{H}_2(\mathbb{T})$ are square–integrable in mean. If this condition is relaxed to almost certain square–integrability, $\mathcal{H}_2(\mathbb{T})$ is expanded to $\mathcal{L}_2(\mathbb{T})$, which allows the definition of a process integral like (3.1.3) but with a random time $T : \Omega \longrightarrow \mathbb{T}$ [66],

Example 3.1.4 (a Langevin process)**.** $\forall \alpha, \beta$ progressively measurable with respect to the filtration $\mathfrak{F}$ stochastic processes such that $\sqrt{\alpha}, \beta \in \mathcal{L}_2(\mathbb{T})$, and W a standard Wiener process with $\mathfrak{F}$–independent increments,[3], the stochastic process X such that $X(0)$ is $\mathcal{F}_0$–measurable and $\forall \tau \in \mathbb{T}$,

$$X_\tau - X_0 = \int_0^\tau \alpha_t dt + \int_0^\tau \beta_t dW_t, \tag{3.1.4}$$

defines a Brownian motion with drift α and noise β.[4]

Definition 3.1.5. The integral form (3.1.3) defined an equivalent differential form,

$$dX_t = \alpha_t dt + \beta_t dW_t, \tag{3.1.5}$$

(3.1.5) is the precursor for stochastic differential equations, whose solutions are stochastic processes.

Definition 3.1.6 (Itô's formula)**.** $\forall f : \mathbb{R} \longrightarrow \mathbb{R}$ twice differentiable on $\mathbb{R}$ and $\forall t \in \mathbb{T}$,

$$df \circ W_t = \partial_x f \circ W_t dW_t + \frac{1}{2} \partial_x^2 \circ W_t dt,\ \partial_x = \frac{\partial}{\partial x}.$$

A weaker condition applies to Itô's formula; it is sufficient for f to be once–differentiable only, but $\exists g : \mathbb{R} \longrightarrow \mathbb{R}$ measurable on $\mathcal{B}(\mathbb{R})$ such that $\forall x \in \mathbb{R}$,

$$\partial_x f(x) - \partial_x f(0) = \int_0^x g(y) dy.$$

Then, $\forall t \in \mathbb{T}$,

$$df \circ W_t = \partial_x f \circ W_t dW_t + \frac{1}{2} g \circ W_t dt.$$

Theorem 3.1.7 (Itô's formula)**.** $\forall f : \mathbb{T} \times \mathbb{R} \longrightarrow \mathbb{R}$, *piecewise–twice continuously differentiable on* $\mathbb{T} \times \mathbb{R}$ *(assuming* $\mathbb{T} = \mathbb{R}$*),* $\forall t \in \mathbb{T}$ *and* $\forall \omega \in \Omega$,[5]

$$df_t \circ W_t = \partial_t f_t \circ W_t dt + \partial_x f_t \circ W_t dW_t + \frac{1}{2} \partial_x^2 f \circ W_t dt.$$

[3] $\forall t \in \mathbb{T}$ and $\forall \Delta t \in \mathbb{T}$ on t, ΔW_t is independent of the events $\mathcal{F}_t \in \mathfrak{F}$.

[4] These functions assume for consistency a notation similar to stochastic processes.

[5] Such formulas should normally be written $\forall \omega \in \Omega$, which is omitted for brevity. $\forall f : \mathbb{R} \longrightarrow \mathbb{R}$ function and $\forall X : \mathbb{T} \times \Omega \longrightarrow \mathbb{R}$ stochastic process, $\forall t \in \mathbb{T}$, $X_t : \Omega \longrightarrow \mathbb{R}$ is a random variable, so only $f \circ X_t$ makes sense if ω–dependence is to be omitted. Such formulations become difficult when multiple variables get involved, and some textbooks write $f(X_t)$ when they actually mean $\forall \omega \in \Omega$, $f(X_t(\omega))$, in effect treating a random variable as a variable and not the function on the sample space Ω that it actually is.

$\forall X$ a random variable of the form

$$dX_t = \alpha_t dt + \beta_t dW_t$$

$\forall \alpha, \beta$ as in example 3.1.4, Itô's formula becomes $\forall t \in \mathbb{T}$ and $\forall \omega \in \Omega$,

$$df_t \circ X_t = \partial_t f_t \circ X_t dt + \alpha_t \partial_x f_t \circ X_t dt + \beta_t \partial_x f_t \circ X_t dW_t + \frac{1}{2}\beta_t^2 \partial_x^2 f_t \circ X_t dt, \tag{3.1.6}$$

In what follows, explicit dependence on time will be omitted.[6] In such notation, (3.1.6) becomes

$$df \circ X = \partial_t f \circ X dt + \alpha \partial_x f \circ X dt + \beta \partial_x f \circ X dW + \frac{1}{2}\beta^2 \partial_x^2 f \circ X dt. \tag{3.1.7}$$

Definition 3.1.8. $\forall f : \mathbb{T} \times V \longrightarrow \mathbb{R}$ twice continuously differentiable functional for a real vector space $V \simeq \mathbb{R}^{\dim V}$, $\nabla_x f : V \longrightarrow V$ stands for the vector of partial derivatives of f on V (excluding time $\mathbb{T}$), and $\nabla_x \otimes \nabla_x f : V \longrightarrow \mathcal{L}(V)$ stands for the (Jacobian) matrix of second–order partial derivatives of f. In this notation, the derivative with respect to time is still abbreviated as $\partial_t f$.

By definition 3.1.8, $\forall f : \mathbb{T} \times V \longrightarrow \mathbb{R}$, Itô's formula generalizes $\forall t \in \mathbb{T}$ to

$$df \circ X = \partial_t f \circ X dt + \alpha \cdot \nabla_x f \circ X dt + \beta \cdot \nabla_x f \circ X dW + \frac{1}{2}\beta \cdot \nabla_x \otimes \nabla_x f \circ X \cdot \beta dt,$$

where $(\cdot|\cdot)$ is the inner product of V. [7]

There is an alternative definition of the stochastic integral due to Stratonovich, denoted with $\bullet$ instead of $\cdot$,[8] which relates to Itô's formula by the differential notation

$$f \circ W \bullet dW = f \circ W dW + \frac{1}{2}\partial_x f \circ W dt.$$

Stratonovich's integral changes Itô's formula (for $f : \mathbb{R} \longrightarrow \mathbb{R}$) into a form that resembles its deterministic counterpart,

$$df \circ W = \partial_x f \circ W \bullet dW = \partial_x f \circ W dW + \frac{1}{2}\partial_x^2 f \circ W dt.$$

The intuitive difference between the two formulations is the logic by which time differences $\Delta\tau \in \mathbb{T}$ and their effect on stochastic processes are treated. By Itô's logic, the reference point for the corresponding ΔX is τ, whereas by Stratonovich's logic, it is $(\tau + \Delta\tau)/2$.

The intuition behind Itô's formula is describing the total differential $df \circ W$ of the stochastic process $f \circ W : \mathbb{T} \times \Omega \longrightarrow \mathbb{R}$ induced by a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ and the Wiener process W. This can be illustrated by deducing Itô's formula from the (naive) Taylor expansion in computing the total differential $df \circ W$. Aside from eliminating dt higher orders, $\langle (dW_t)^2 \rangle \propto dt$, $\forall t \in \mathbb{T}$, which is why the second order derivative remains in Itô's formula.

3.2. Stochastic differential equations

Definition 3.2.1. $\forall \beta \in \mathcal{L}_2(\mathbb{T})$ and $\forall X$ stochastic process, such that $\forall t \in \mathbb{T}$,

$$dX_t = \beta_t dW_t - \frac{1}{2}\beta_t^2 dt,$$

[6] The function composition $\circ$ takes precedence over function operations. Operators on functions (as vectors, like ∇) take precedence over function composition.

[7] By chapter 1. Mathematical foundation, the notation $\beta \cdot \nabla_x \otimes \nabla_x f \circ X \cdot \beta$ is possible because V is real and the Jacobian $\nabla_x \otimes \nabla_x f$ is symmetric (hermitian) for f twice continuously differentiable on V.

[8] The usual notation for Stratonovich's calculus is $\circ$, but since the function composition is explicitly used here in consistency with the functional nature of random variables, $\bullet$ is used instead to avoid confusion.

the stochastic process $P = \exp X$ is a stochastic exponent. Applying Itô's formula (3.1.7) for $f = \exp$ and no time dependence (i.e. $P_t = f \circ X_t$, $\forall t \in \mathbb{T}$),

$$dP_t = \beta_t P_t dW_t - \frac{1}{2}\beta_t^2 P_t dt + \frac{1}{2}\beta_t^2 P_t dt,$$

or

$$dP_t = \beta_t P_t dW_t,\ P_0 = 1,$$

which is a first instance of a stochastic differential equation, that is not directly integrable.

For the sake of defining a Cauchy problem for stochastic differential equations, all used symbols and assumptions are re–iterated.

Definition 3.2.2. Let Ω be a probability space of events $\mathcal{F}$ with probability (measure) ρ, filtered by $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$ for a (continuous) time index $\mathbb{T}$. Let $W : \mathbb{T} \times \Omega \longrightarrow \mathbb{R}$ be a Wiener process with $W_0 = x \in \mathbb{R}$ and $\xi : \Omega \longrightarrow \mathbb{R}$ a random variable independent of W_τ, $\forall \tau \in \mathbb{T}$. Let the filtration of Ω be generated by the processes W and ξ, as in $\forall \tau \in \mathbb{T}$

$$\mathcal{F}_\tau = \sigma(\xi) \vee \bigvee_{t=0}^{\tau} \sigma(W_t).$$

Finally let $\alpha, \beta : \mathbb{T} \times \mathbb{R} \longrightarrow \mathbb{R}$ be measurable on $\mathcal{B}(\mathbb{T}) \times \mathcal{B}(\mathbb{R})$ functions.

The continuous $\mathfrak{F}$–adapted stochastic process $X : \mathbb{T} \times \Omega \longrightarrow \mathbb{R}$ with initial condition $X_0 = \xi$ is a strong solution to the stochastic differential equation

$$dX_t = \alpha_t \circ X_t dt + \beta_t \circ X_t dW_t,$$

if $\forall \tau \in \mathbb{T}$,

$$\int_0^\tau (|\alpha_t \circ X_t| + |\beta_t \circ X_t|^2) dt < \infty,$$

and

$$X_\tau = \xi + \int_0^\tau \alpha_t \circ X_t dt + \int_0^\tau \beta_t \circ X_t dW_t.$$

Theorem 3.2.3. *If α, β satisfy the Lipchitz condition, meaning $\exists \ell > 0$ such that, $\forall \tau \in \mathbb{T}$ and $\forall x, y \in \mathbb{R}$,*

$$|\alpha_\tau(x) - \alpha_\tau(y)| + |\beta_\tau(x) - \beta_\tau(y)| \leq \ell |x - y|,$$

and the corresponding linear growth condition, meaning

$$|\alpha_\tau(x)| + |\beta_\tau(x)| \leq \ell(1 + |x|),$$

and $\langle |\xi|^2 \rangle < \infty$, $\exists X$ unique strong solution with $X_0 = \xi$ to the stochastic differential equation

$$dX_t = \alpha_t \circ X_t dt + \beta_t \circ X_t dW_t, \tag{3.2.1}$$

satisfying $\sup_{t\in\mathbb{T}} \langle |X_t|^2 \rangle < \infty$.

(3.2.1) is the most generic form of a time–dependent first–order linear stochastic differential equation, consisting of a drift function α and a noise function β. The drift is basically the deterministic part of the equation, and the noise is what makes it stochastic.

The linear variant of (3.2.1),

$$dX_t = \alpha_t X_t dt + \beta_t X_t dW_t,$$

has general solution, $\forall \tau \in \mathbb{T}$,

$$X_\tau = \exp\left(\int_0^\tau \beta_t dW_t + \int_0^\tau \left(\alpha_t - \frac{1}{2}\beta_t^2 \right) dt \right).$$

Definition 3.2.4. Let $V \simeq \mathbb{R}^{\dim V}$ be a vector space of finite dimension and an inner product

$$(\cdot|\cdot) : V \times V \longrightarrow \mathbb{R} : x, y \mapsto x \cdot y = \sum_{i=1}^{\dim V} x_i y_i,$$

and consequent 2–norm

$$\|\cdot\| : V \longrightarrow \mathbb{R}_+ : x \mapsto \|x\| = \sqrt{x \cdot x},$$

$x \in V$ and $W : \mathbb{T} \times \Omega \longrightarrow V$ a multidimensional Wiener process with independent components and $W_0 = x$. Likewise in definition (**??**), let $\xi : \Omega \longrightarrow V$ be a stochastic process independent of W_τ, $\forall \tau \in \mathbb{T}$. Let the filtration $\mathcal{F}$ be generated by the initial ξ and W as in

$$\mathcal{F}_\tau = \bigotimes \sigma(\xi) \vee \bigvee_{t=0}^{\tau} \bigotimes \sigma(W_t),$$

where the product σ–algebras of the corresponding (independent) components appear. Let $\alpha : \mathbb{T} \times V \longrightarrow V$ be a measurable on $\mathcal{B}(\mathbb{T}) \times \mathcal{B}(V)$ vector function, and $\beta : \mathbb{T} \times V \longrightarrow \mathcal{L}(V)$ a measurable on $\mathcal{B}(\mathbb{T}) \times \mathcal{B}(\mathcal{L}(V))$ matrix function, assuming $\mathcal{L}(V)$ has the Frobenius norm, defined by the inner product

$$\cdot : \mathcal{L}(V) \times \mathcal{L}(V) \longrightarrow \mathbb{R} : F, G \mapsto F \cdot G = \operatorname{tr} F^\top G = \sum_{i=1}^{\dim V} \sum_{j=1}^{\dim V} F_{ij} G_{ij},$$

as

$$\|\cdot\| : \mathcal{L}(V) \longrightarrow \mathbb{R}_+ : F \mapsto \|F\| = \sqrt{F \cdot F}.$$

The continuous in the topology of V induced by its inner product $\mathcal{F}$–adapted vector stochastic process $X : \mathbb{T} \times \Omega \longrightarrow V$ with initial condition $X_0 = \xi$ is a strong solution to the vector stochastic differential equation

$$dX_t = \alpha_t \circ X_t dt + \beta_t \circ X_t dW_t,$$

if $\forall \tau \in \mathbb{T}$,[9]

$$\int_0^\tau (\|\alpha_t \circ X_t\| + \|\beta_t \circ X_t\|^2) dt < \infty,$$

and

$$X_\tau = \xi + \int_0^\tau \alpha_t \circ X_t dt + \int_0^\tau \beta_t \circ X_t dW_t.$$

Theorem 3.2.5. *If α, β (uniformly in components) satisfy the Lipchitz condition, meaning $\exists \ell > 0$ such that, $\forall \tau \in \mathbb{T}$ and $\forall x, y \in \mathbb{R}$,*

$$\|\alpha_\tau(x) - \alpha_\tau(y)\| + \|\beta_\tau(x) - \beta_\tau(y)\| \leq \ell \|x - y\|,$$

and the corresponding linear growth condition, meaning

$$\|\alpha_\tau(x)\| + \|\beta_\tau(x)\| \leq \ell(1 + \|x\|),$$

and $\langle \|\xi\|^2 \rangle < \infty$, $\exists X$ unique strong solution with $X_0 = \xi$ to the stochastic differential equation

$$dX_t = \alpha_t \circ X_t dt + \beta_t \circ X_t dW_t, \tag{3.2.2}$$

satisfying $\sup_{t \in \mathbb{T}} \langle \|X_t\|^2 \rangle < \infty$.

The vector stochastic differential equation (3.2.2) is in effect a (linear) set of stochastic differential equations.

$\forall W' : \mathbb{T} \times \Omega \longrightarrow V$ dependent vector Wiener process, $\exists M \in \mathcal{L}(V)$ transformation matrix such that $W = MW'$ is an independent vector Wiener process, therefore all linear stochastic Cauchy problems reduce to ones with independent noise, as in definition 3.2.4.

[9] Note that $\forall t \in \mathbb{T}$, $\beta_t \circ X_t : \Omega \longrightarrow \mathcal{L}(V)$ is an "operator" random variable, thus $\forall \omega \in \Omega$, the result may act on the image of another random vector, namely $dW_t : \Omega \longrightarrow V$ in this case. Hence the shorthand notation $\beta_t \circ X_t dW_t : \Omega \longrightarrow V$ is in fact a random vector in itself as a matrix–vector product.

Definition 3.2.6 (Langevin equation)**.** Another simplification of (3.2.1), is with time–independent drift $\sigma : V \longrightarrow \mathbb{R}$ and noise $\mu : V \longrightarrow \mathbb{R}$ and deterministic initial condition $x \in V$,

$$dX_t = \sigma \circ X_t dt + \mu \circ X_t dW_t,\ X_0 = x. \tag{3.2.3}$$

Adapting the conditions of theorem 3.2.5, namely the Lipchitz condition, $\exists \ell \in \mathbb{R}_+$ such that, $\forall x, \Delta x \in V$,

$$\|\Delta\sigma(x)\| + \|\Delta\mu(x)\| \leq \ell\|\Delta x\|,$$

and the linear growth condition

$$\|\sigma(x)\| + \|\mu(x)\| \leq \ell(1 + \|x\|),$$

assert by theorem 3.2.5 the existence and uniqueness of a solution to (3.2.2).

$\langle\cdot\rangle_x$ denotes the expectation with respect to the probability measure ρ_x induced on $\mathcal{B}(V)$ by the stochastic process $X : \mathbb{T}\times\Omega \longrightarrow V$ (solution of (3.2.3)) starting at $X_0 = x \in \mathbb{R}$,[10] a notation which is possible thanks to the well placed Cauchy problem for stochastic differential equations (existence and uniqueness given initial condition).

3.3. Diffusion processes

Definition 3.3.1. $\forall X : \mathbb{T} \times \Omega \longrightarrow V$ a Markov process, a function $\varrho : (\mathbb{T} \times V)^2 \longrightarrow \mathbb{R}_+$ such that:

- $\forall t, t' \in \mathbb{T}$ and $\forall x \in V$, $\varrho(t, x, t', \cdot) : V \longrightarrow \mathbb{R}_+$ and $\varrho(t, x, t', \cdot) : V \longrightarrow \mathbb{R}_+$ are measurable on $\mathcal{B}(V)$,[11]
- $\forall t \in \mathbb{T}$ and $\forall x, x' \in V$, $\varrho(t, x, t, x') = \delta(x - x')$, where δ is the Dirac distribution stemming from the Dirac measure $\delta : \mathcal{B}(V) \longrightarrow \overline{\mathbb{R}}_+$ such that $\forall A \in \mathcal{B}(V)$ and $\forall F : V \longrightarrow V$,

$$\delta(A) = \begin{cases} 1 & 0 \in A \\ 0 & 0 \notin A \end{cases} \text{ and } \int_V F d\delta = F(0).$$

- $\forall t, t', t'' \in \mathbb{T}$ with $t'' < t' < t$ and $\forall x, x', x'' \in V$, the Chapman–Kolmogorov equation holds [66],

$$\varrho(t'', x'', t, x) = \int_V \varrho(t'', x'', t', x')\varrho(t', x', t, x)d\mu(x'),$$

  where integration is meant with respect to the Lebesgue measure μ giving parallelogram volume on all parallelogram subsets of V.[12]

Proposition 3.3.2. *The density law $\upsilon_X : \mathbb{T} \otimes V \longrightarrow \mathbb{R}_+$ of a Markov process X is uniquely defined by an initial law $\upsilon_0 : V \longrightarrow \mathbb{R}_+ : x \longmapsto f_X(0, x)$ and it transition function $\varrho_X : (\mathbb{T} \times V)^2 \longrightarrow \mathbb{R}_+$.*[13] *Conversely, $\forall \upsilon$ transition function, $\exists X$ Markov process corresponding to it.*[14]

In that sense, the transition function progresses a Markov process, so all is needed is a starting point, in the distribution space. Namely $\forall t \in \mathbb{T}$ and $\forall x \in V$,

$$\upsilon_X(t, x) = \int_V \upsilon_0(y)\varrho(0, y, t, x)d\mu(y).$$

Definition 3.3.3. A Markov process X such that $\forall t, \Delta t \in \mathbb{T}$, $\exists \mu : \mathbb{T} \times V \longrightarrow V$ drift and $\exists \sigma : \mathbb{T} \times V \longrightarrow \mathcal{L}(V)$ noise (or diffusion [66]), such that:

[10] According to definition 2.1.6, this is only applicable to random variables, unless the product measure of $\mathcal{B}(\mathbb{T})\otimes\mathcal{F}_\infty$ is also a probability on $\mathbb{T}\times\Omega$.
[11] In other words they are random variables on V with the Borel σ–algebra of events and therefore have a law defined on V. In some textbooks, like [66], alternative definitions involving corresponding functions as measures are often given.
[12] $\forall \{x_n\}_{n=1}^{\dim V} \subset V$ base of V, and the change–base operator $B \in \mathcal{L}(V)$ it defines, the volume of the parallelogram with the base vector as edges has volume $\mu(B) = |\det B|$.
[13] Here the notation of the probability measure $\rho_X : \mathcal{B}(\mathbb{T}) \otimes \mathcal{B}(V) \longrightarrow \mathbb{R}_+$ is abused to stand for the corresponding probability density function $f_X : \mathbb{T} \times V \longrightarrow \mathbb{R}_+$.
[14] In fact, there exists a whole family of Markov processes, distinct (but not necessarily uniquely) by an initial distribution.

- $\lim_{\Delta t\to 0}(\Delta t)^{-1}\mathbb{E}[\Delta X_t|X_t=x]=\mu(t,x)$,
- $\lim_{\Delta t\to 0}(\Delta \tau)^{-1}\mathbb{E}[\Delta X_t\cdot\Delta X_t|X_t=x]=\sigma(t,x)\cdot\sigma(t,x)$, or more generally,
- $\lim_{\Delta t\to 0}(\Delta t)^{-1}\mathbb{E}[\Delta X_t\otimes\Delta X_t|X_t=x]=\sigma(t,x)\otimes\sigma(t,x)$, [citation needed]

is a diffusion.

Theorem 3.3.4 (Kolmogorov backward equation)**.** *$\forall X$ diffusion with continuous on V drift μ and noise σ, and probability density $\forall\upsilon_0:V\longrightarrow\mathbb{R}_+$ and $\forall\tau\in\mathbb{T}$ such that*

$$\upsilon:[0,\tau]\otimes V\longrightarrow\mathbb{R}_+:t,x\mapsto\int_V\upsilon_0(\chi)\varrho(t,x,\tau,\chi)d\mu(\chi)$$

is continuously twice differentiable on V, the latter satisfies the following Cauchy problem:

$$-\partial_t\upsilon=\mu\cdot\nabla_x\upsilon+\frac{1}{2}\operatorname{tr}((\sigma\otimes\sigma)\cdot(\nabla_x\otimes\nabla_x\upsilon)),\ \lim_{t\to\tau}\upsilon(t,x)=\upsilon_0(x),$$

The transition function ϱ_X of the diffusion X is uniquely defined by the drift μ and the noise σ.

Theorem 3.3.5 (Kolmogorov forward equation (Fokker–Planck))**.** *$\forall X$ diffusion with continuous on V drift μ and noise σ, probability density $\forall\upsilon_0:V\longrightarrow\mathbb{R}_+$ and $\forall\tau\in\mathbb{T}$ such that*

$$\upsilon:\mathbb{T}\otimes V\longrightarrow\mathbb{R}_+:t,x\mapsto\int_V\upsilon_0(\chi)\varrho(\tau,\chi,t,x)d\mu(\chi)$$

is continuously twice differentiable on V, the latter satisfies the following Cauchy problem:

$$\partial_t\upsilon=\frac{1}{2}\operatorname{tr}((\nabla_x\otimes\nabla_x)\cdot(\sigma\otimes\sigma\upsilon))-\nabla_x\cdot(\mu\upsilon),\ \upsilon(0,x)=\upsilon_0(x).$$

The Fokker–Planck equation essentially describes the evolution of the probability distribution $\upsilon(t,\cdot)$ of the state random variable X_t with time $t\in\mathbb{T}$.

Theorem 3.3.6. *$\forall X$ solution to a well–placed (stochastic) Cauchy problem as in definition 3.2.4,*

$$dX_t=\mu_t\circ X_t dt+\sigma_t\circ X_t dW_t,\ X_0=\xi,$$

X is a diffusion process with drift μ and noise σ.

For the special case of time–independent drift μ and noise σ,

$$dX_t=\mu\circ X_t dt+\sigma\circ X_t dW_t,\ X_0=\xi,$$

the corresponding to X transition function ϱ_X is time–translation–invariant, $\forall t,t',\Delta t\in\mathbb{T}$ and $\forall x,x'\in V$,

$$\varrho_X(t'+\Delta t,x',t+\Delta t,x)=\varrho_X(t',x',t,x),$$

meaning that the transition function depends only on time differences $\Delta t\in\mathbb{T}$. X is then a homogeneous diffusion.

4. Stochastic Quantization

In this chapter, the machinery of stochastic processes calculus is summarized and reformulated in physics terms.

4.1. Assumptions

Stochastic differential equations

Let Ω be a probability space with a probability measure ρ (definition 2.1.4) on events $\mathcal{F}_\infty$ (2.1.1) filtered by $\mathcal{F}$ (definition 2.2.4). Let $V \simeq \mathbb{R}^{\dim V}$ be a finite–dimensional real vector space (definition 1.1.5).

Let $\mathbb{T} = \mathbb{R}_+$ be a real time index (definition 2.2.1) with the usual topology $\mathcal{T}$ (definition 1.2.1) of open intervals, total ordering $<$ (definition 1.1.2), field operations $+$ and $\cdot$ (definition 1.1.4), and the Lebesgue measure on the Borel σ–algebra $\mathcal{B}(\mathbb{T}) = \sigma(\mathcal{T})$, and $\tau : \Omega \longrightarrow \mathbb{T}$ a stopping time (definition 2.2.8).

Let $\Phi : \mathbb{T} \times \Omega \longrightarrow V$ be a stochastic process (usually with the usual conditions, see for example definitions 2.2.2, 2.2.8, 2.2.8, and usually a diffusion), whose notation is changed hereon to stand for a (physical) field Φ.[1] For $\mathbb{T} \times \Omega$ assume the product measure of the Lebesgue time measure and the probability measure ρ on the product σ–algebra $\mathcal{B}(\mathbb{T}) \otimes \mathcal{F}_\infty$ (definition 2.1.5).

Let time–independent drift $\mu : V \longrightarrow V$ and noise $\sigma : V \longrightarrow \mathcal{L}(V)$, that, together with a Wiener process $W : \mathbb{T} \times \Omega \longrightarrow V$, define a pair of a stochastic equation for the field ϕ,

$$d\Phi(\tau) = \mu \circ \Phi(\tau) d\tau + \sigma \circ \Phi(\tau) dW(\tau),\ \Phi(0) = \Phi_0 : \Omega \longrightarrow V,$$

and a Fokker–Planck equation for its probability transition density $\varrho : \mathbb{T} \times V \longrightarrow \mathbb{R}_+$,

$$\frac{\partial}{\partial \tau} \varrho = \frac{1}{2} \sigma \cdot (\nabla \otimes \nabla \varrho) \cdot \sigma - \nabla \cdot (\mu \varrho),\ \varrho(0) = \varrho_0 : V \longrightarrow \mathbb{R}_+, \tag{4.1.1}$$

with initial time $\tau_0 = 0$ and initial configuration $\phi_0 = \langle \Phi_0 \rangle$. A typical initial distribution corresponds to said deterministic field initial condition, which is represented by Dirac's δ distribution on V (definition 2.4.7),

$$\varrho_0(\phi) = \delta(\phi - \phi_0),$$

although this is only formulaic, as Dirac's δ measure has no corresponding density function (Radon–Nikodym derivative).

Rewriting (4.1.1) as

$$\frac{\partial}{\partial \tau} \varrho = -\mathcal{A}^\top \varrho,$$

defines a (differential) Hamiltonian operator $\mathcal{A}$ that prescribes the time evolution of an observable $O : \mathbb{T} \times V \longrightarrow \mathbb{R}$,

$$\frac{\partial}{\partial \tau} \langle O \circ \Phi \rangle = \langle \mathcal{A} O \circ \Phi \rangle.$$

[1] Not to be confused with an algebraic field as in definition 1.1.4.

Expectation

$\forall \Phi, \Psi : \Omega \longrightarrow V \in L^2(\rho)$ square integrable random variables, the following inner product is well defined,

$$\langle \Phi | \Psi \rangle = \int_\Omega \Phi \cdot \Psi d\rho, \tag{4.1.2}$$

where $\cdot$ is the inner product of V. In consistency with the Dirac notation for vector spaces, $\langle \Phi | \Psi \rangle$ is the projection of Ψ on Φ. However, looking for a base may not be straightforward if $\dim L^2(\rho)$ is uncountable. In such a case, a measure is required for $L^2(\rho)$, which in turn requires a σ–algebra on $L^2(\rho)$, which in turn if it is to be the Borel σ–algebra $\mathcal{B}(L^2(\rho))$, a topology is needed on $L^2(\rho)$. The inner product (4.1.2) guarantees a topology on $L^2(\rho)$ and as such a Borel σ–algebra $\mathcal{B}(L^2(\rho))$. Recall that $L^p(\rho)$ spaces in particular group random variables by version.[2] Thus $L^2(\rho)$ is a measurable space. Defining a measure on such a space however is highly non–trivial and will be postponed when simplified assumptions will enable its good definition. Formally however, one define a projection operator (in Riemannian notation) as

$$\int_{L^2(\rho)} |\Phi\rangle\langle\Phi| \mathcal{D}\Phi,$$

where $\mathcal{D}\Phi$ represents integration with respect to whatever well–defined measure $L^2(\rho)$ may have. In the context of physics, this is known as a (Feynman) path integral [67].

The inner product (4.1.2), as it is built on the probability measure ρ of Ω, is in fact a correlator between Ψ and Φ.

$\mathcal{H}_2$,[3] the vector space generated by stochastic processes Φ satisfying (3.1.2), also formally admits an inner product (and a corresponding isometry to L^2 at that), $\forall \Phi, \Psi : \mathbb{T} \times \Omega \longrightarrow V \in \mathcal{H}_2(\mathbb{T})$,

$$\langle \Phi | \Psi \rangle = \int_{\mathbb{T}} \langle \Phi(\tau) | \Psi(\tau) \rangle d\tau = \left\langle \int_{\mathbb{T}} \Phi(\tau) dW(\tau) \middle| \int_{\mathbb{T}} \Psi(\tau) dW(\tau) \right\rangle,$$

where integration is meant with the measure induced on $\mathbb{T}$ by the stopping time τ.

Realized sampling ($\omega \in \Omega$)

Mathematically speaking, random variables $\Omega \longrightarrow V$, and by extension, stochastic processes $\mathbb{T} \times \Omega \longrightarrow V$ are tied to a probability space Ω. Fixing $\omega \in \Omega$ however yields a value in V out of a random variable or a sample path $\mathbb{T} \longrightarrow V$ out of a stochastic process.

Hereon all random variables and stochastic processes will be assumed *realized* for a $\omega \in \Omega$. In this context, the stochastic differential equation appears to be an ordinary one,

$$d\phi(\tau) = \mu(\phi(\tau))d\tau + \sigma(\phi(\tau))dW(\tau),\ \phi(0) = \phi_0 \in V,$$

and the corresponding Fokker–Plank equation remains the same.

Note that, formulating problems in realized sample paths does not eliminate their stochasticity; every realization happens randomly and the notion of expectation is relevant. Because diffusion processes have all the usual (good) properties with respect to the filtration $\mathcal{F}$ of the corresponding probability space Ω, namely they are adapted and progressively measurable, the filtration conditional may be omitted for brevity from expectations.

Discretized time

It is worth noting that path realization affect the stopping time $\tau : \Omega \longrightarrow \mathbb{T}$. The original assumption for the time index is $\mathbb{T} = \mathbb{R}_+$. A straightforward discretization would be to switch to $\mathbb{T} = \mathbb{N}$. An alternate approach is to assume a sequence of

[2] Meaning they only differ on subsets of $A \subseteq \Omega$ with $\rho(A) = 0$.
[3] See section §3.1. The Itô integral.

stopping times $\tau : \mathbb{N} \times \Omega \longrightarrow \mathbb{R}_+$ which, as it being stopping times, is (safely) realized to a sequence of times $\tau : \mathbb{N} \longrightarrow \mathbb{R}_+$ for fixed $\omega \in \Omega$.

In this context, the (ordinary) stochastic differential equation becomes

$$\Delta\phi_n = \mu(\phi_n)\Delta\tau_n + \sigma(\phi_n)\Delta W_n,\ \Delta\cdot_n = \cdot_n - \cdot_{n-1},\ \forall n \in \mathbb{Z}_+. \tag{4.1.3}$$

To generate a process ϕ, ϱ_0 with known x_0 needs to be known to realize an initial condition ϕ_0. $\forall n \in \mathbb{Z}_+$, $\Delta\tau_n > 0$. Recalling the definition 2.4.12 of a Wiener process,

$$\langle \Delta W_n \rangle = 0 \text{ and } \langle \Delta W_n | \Delta W_n \rangle = \Delta\tau_n,\ \forall n \in \mathbb{Z}_+,$$

so all is required is a distribution with these two moments, for example a Gaussian distribution with $\mu = 0$ and $\sigma = \sqrt{\Delta\tau_n}$, to generate the noise ΔW_n. If a random variable η satisfying the standard Gaussian distribution ($\mu = 0$ and $\sigma = 1$), equation (4.1.3) becomes

$$\Delta\phi_n = \mu(\phi_n)\Delta\tau_n + \sigma(\phi_n)\eta\sqrt{\Delta\tau_n},\ \forall n \in \mathbb{Z}_+. \tag{4.1.4}$$

Given ϕ_0, simply $\phi_n = \phi_{n-1} + \Delta\phi_n$ $\forall n \in \mathbb{Z}_+$.

To avoid unnecessarily referring to a (non–definable) formal derivative of a Wiener process, it is this well–defined discretized version of the stochastic differential equation that will be referred to upon as the Langevin equation (and the corresponding process as a Langevin process).

It is worth noting that the L^2 function space now realizes a well–defined projection operator for a countable collection $\{\phi_n\}_{n\in\mathbb{N}} \subset L^2$,

$$\sum\nolimits_{n\in\mathbb{N}} |\phi_n\rangle\langle\phi_n| = \sum\nolimits_{n\in\mathbb{N}} |n\rangle\langle n|,$$

where in the latter part, the specifics of base selection are ignored for the sake of generality. L^2 as a vector space has its own vector space of bounded operators $\mathcal{L}(L^2)$. $\forall A \in \mathcal{L}(L^2)$ hermitian (symmetric if $\mathbb{K} = \mathbb{R}$), $\forall \phi, \psi$ and for a particular (countable) base of L^2,

$$\langle\phi|A|\psi\rangle = \sum\nolimits_{n\in\mathbb{N}} \sum\nolimits_{m\in\mathbb{N}} \langle\phi|n\rangle\langle n|A|n\rangle\langle n|\psi\rangle = \sum\nolimits_{n\in\mathbb{N}} \sum\nolimits_{m\in\mathbb{N}} \phi_n A_{nm} \psi_m. \tag{4.1.5}$$

In this discretized time context, the Dirac notation may be abused for stochastic processes as well. $\forall \phi, \psi : \mathbb{N} \times \Omega \longrightarrow V$,

$$\langle\phi|\psi\rangle = \sum\nolimits_{n\in\mathbb{N}} \langle\phi_n|\psi_n\rangle,$$

assuming the sum converges.[4]This simply means that the identity operator on stochastic processes is not bounded, which is not unexpected. Correlators like (4.1.5), may however converge under the right conditions for the operator A and the differential equation generating ϕ and ψ.

Einstein indexing

Per definition 1.1.16, assume that $\forall \phi \in V$, $x \in \mathbb{N}_{\dim V}$ is the integer index running through coordinates ϕ_x. In such a case,

$$\Delta\phi = \mu(\phi)\Delta\tau + \sigma(\phi)\eta\sqrt{\Delta\tau} \text{ is written as } \Delta\phi_x = \mu_x(\phi)\Delta\tau + \sigma_x(\phi)\eta\sqrt{\Delta\tau},$$

with the free index x depicting the vector nature of the equation, and equivalently, the fact that it is in fact a system of equations.

In this form it becomes apparent how this equation may be generalized with a vector Wiener process as

$$\Delta\phi_x = \mu_x(\phi)\Delta\tau + \sigma_{xy}(\phi)\eta_y\sqrt{\Delta\tau} = \Delta\phi_x = \mu_x(\phi)\Delta\tau + \sum\nolimits_y \sigma_{xy}(\phi)\eta_y\sqrt{\Delta\tau},$$

[4]For finite time index, it does, however, a finite time index is in fact a variable one, assuming it simulates the corresponding infinite–time (and continuous) stochastic process, therefore concerns of convergence are relevant. As a counter example,

$$\|\Delta W\|^2 = \sum\nolimits_{n\in\mathbb{N}} \|\Delta W_n\|^2 = \sum\nolimits_{n\in\mathbb{N}} \Delta\tau_n = \infty,$$

for stationary stopping time τ or with a fixed time step.

where the noise $\sigma : V \longrightarrow \mathcal{L}(V)$ now maps to a linear operator instead of a vector.

The Fokker–Planck equation becomes

$$\frac{\partial}{\partial\tau}\varrho = \frac{1}{2}\partial_x\partial_y(\sigma_{xz}\sigma_{yz}\varrho) - \partial_x(\mu_x\varrho) = \frac{1}{2}\sum\nolimits_x \partial_x \sum\nolimits_y \partial_y \sum\nolimits_z (\sigma_{xz}\sigma_{yz}\varrho) - \sum\nolimits_x \partial_x(\mu_x\varrho).$$

The repeating index summation convention (definition 1.1.16) is emphasized here for clarity.

Action and stationary noise

The generic noise function $\sigma : V \longrightarrow V$ is replaced by a constant $\sigma \in \mathbb{R}_+$. In consistency with the fact that (realized) $\phi \in V$, (realized) $\eta \in V$ instead of $\mathbb{R}$, and the noise term of equation (4.1.4) becomes

$$\sigma(\phi)\eta\sqrt{\Delta\tau} \longrightarrow \eta\sigma\sqrt{\Delta\tau} \longrightarrow \eta\sqrt{\Delta\tau},$$

or $\sigma_{xy} = \delta_{xy}$, where in the latter change, the constant σ is absorbed as standard deviation of the distribution of η,

$$\langle\eta|\eta\rangle = \sigma^2\Delta\tau.$$

Furthermore, the drift function $\mu : V \longrightarrow V$ is assumed to be a derivative of a bounded from below functional $f : V \longrightarrow \mathbb{R}$, i.e. $\mu = \nabla f$.

Langevin equation (4.1.4) becomes

$$\Delta\phi = \nabla f(\phi)\Delta\tau + \eta\sqrt{\Delta\tau} \text{ or } \Delta\phi_x = \partial_x f(\phi)\Delta\tau + \eta_x\sqrt{\Delta\tau} \tag{4.1.6}$$

where the defining trait is now the functional f, to be known as *action* in a physics context. (4.1.6) is the template for what is known as *stochastic quantization* in field theory [68].

The corresponding Fokker–Planck equation simplifies to

$$\frac{\partial}{\partial\tau}\varrho = \frac{1}{2}\sigma^2\partial_x\partial_x\varrho - \partial_x(\varrho\partial_x f) = \frac{1}{2}\sigma^2\partial_x\partial_x\varrho - \delta_{xy}\partial_x(\varrho\partial_y f) = \frac{1}{2}\sigma^2\partial_x\partial_x\varrho - \partial_x\varrho\partial_x f - \varrho\partial_x\partial_x f,$$

or

$$\frac{\partial}{\partial\tau}\varrho = \frac{1}{2}\sigma^2\nabla^2\varrho - \nabla\cdot(\varrho\nabla f) = \frac{1}{2}\sigma^2\nabla^2\varrho - \nabla f\cdot\nabla\varrho - \varrho\nabla^2 f.$$

Symmetry fixing

Assume the functional f has a symmetry prescribed by a Lie group G with a corresponding Lie algebra $\mathfrak{g}$,[5] namely $\forall\phi \in V$ and $\forall g \in \mathrm{G}$, that f remains unchanged if ϕ is replaced by $g\phi$.

$\lambda \in \mathfrak{g}$ such that $g = \exp \imath\lambda$, defined the infinitesimal change $\Delta\phi = \imath\lambda\phi$.

Assume G if of finite dimension and has K generators $(\lambda_a)_{a=1}^K \subset \mathfrak{g}$, such that $\forall g \in \mathrm{G}$, $\exists!(g_a)_{a=1}^K \in \mathbb{R}^K$ such that

$$g = \exp \imath g_a\lambda_a,$$

giving

$$\Delta\phi = \imath g_a\lambda_a\phi,$$

thus the symmetry of f under G translates to parametric freedom of ϕ.

A usual approach in fixing $(g_a)_{a=1}^K$ is by minimizing a squared norm–like function $\mathcal{N}$ on ϕ.[6] The simplest example

$$\mathcal{N}(\phi) \propto \|\phi\|^2 = \phi\cdot\phi,$$

[5] See section §1.3. Differential Algebra for a brief reference.
[6] See definition 1.2.5 and 1.2.6.

where the inner product of V was used for the default norm on V. The differential of the norm becomes

$$\Delta\mathcal{N} \propto \phi \cdot \Delta\phi = \imath\phi \cdot g_a \lambda_a \phi,$$

which is nothing but a modified inner product on V.

Symmetry fixing of ϕ leads to a modified discretized Langevin process (and equation),

$$\phi'(\tau) = g(\tau)\phi(\tau) \text{ and } \phi(\tau + \Delta\tau) = \phi'(\tau) + \mu(\phi'(\tau))\Delta\tau + \sigma(\phi'(\tau))\eta\sqrt{\Delta\tau}.$$

4.2. Complexification

ϕ scalar

$\phi \in \mathbb{R}$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$

The Langevin equation becomes

$$\Delta\phi = \frac{\partial}{\partial\phi} f(\phi)\Delta\tau + \eta\sqrt{\Delta\tau}, \tag{4.2.1}$$

with the corresponding Fokker–Planck equation

$$\frac{\partial}{\partial\tau}\varrho = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial\phi^2}\varrho - \frac{\partial}{\partial\phi}\left(\varrho\frac{\partial}{\partial\phi}f\right) = \frac{1}{2}\sigma^2\frac{\partial^2}{\partial\phi^2}\varrho - \frac{\partial}{\partial\phi}\varrho\frac{\partial}{\partial\phi}f - \varrho\frac{\partial^2}{\partial\phi^2}f.$$

$\phi \in \mathbb{C}$ and $f : \mathbb{C} \longrightarrow \mathbb{R}$

This case may seem peculiar but is in fact well–defined, so long as the operator ∇ is properly understood. In the simplest case of $V = \mathbb{C}$, we get the corresponding Wirtinger derivatives of $f : \mathbb{C} \longrightarrow \mathbb{R}$, for $\phi \in \mathbb{C}$,

$$\frac{\partial}{\partial\phi} = \frac{1}{2}\left(\frac{\partial}{\partial\Re\phi} - \imath\frac{\partial}{\partial\Im\phi}\right) \text{ and } \frac{\partial}{\partial\phi^*} = \frac{1}{2}\left(\frac{\partial}{\partial\Re\phi} + \imath\frac{\partial}{\partial\Im\phi}\right).$$

Recall that functionals of the form $f : V \longrightarrow \mathbb{R}$ are not holomorphic, but in such cases holomorphism is irrelevant. In effect the process breaks down to two processes, per $\mathbb{C} \simeq \mathbb{R}^2$. The crucial detail here is that for the variable $\phi \in \mathbb{C}$ the corresponding Wirtinger derivative is the conjugate one.

$\phi \in \mathbb{R}$ and $f : \mathbb{R} \longrightarrow \mathbb{C}$

This may appear as a non–conventional case, it appears however in several applications, where the action f is manifestly complex. Once again, assume the simplest case $V = \mathbb{R}$, and therefore $f = \Re f + \imath\Im f : \mathbb{R} \longrightarrow \mathbb{C}$, i.e. a trajectory in $\mathbb{C}$. $\forall\phi \in \mathbb{R}$

$$\frac{\partial}{\partial\phi} f = \frac{\partial}{\partial\phi}\Re f + \imath\frac{\partial}{\partial\phi}\Im f,$$

which creates an incompatibility in the Langevin equation,

$$\Delta\phi = \frac{\partial}{\partial\phi} f(\phi)\Delta\tau + \eta\sqrt{\Delta\tau}$$

whose consistency requires the complexification of $\phi \in \mathbb{R}$.

The complexification of $\phi \in \mathbb{C}$ can happen in many ways, the most straightforward one being to assume an extra imaginary part as in

$$\phi \longrightarrow \phi + \imath\psi.$$

This replacement affects the definition of the functional f (for scalar $\phi \in \mathbb{C}$ it becomes $f : \mathbb{C} \longrightarrow \mathbb{C}$), which – aside from the definition of f – affects how f is derivated.

$$\frac{\partial}{\partial \phi} f = \frac{1}{2}\left(\frac{\partial}{\partial \Re\phi}(\Re f + \imath \Im f) - \imath \frac{\partial}{\partial \Im\phi}(\Re f + \imath \Im f)\right) = \frac{1}{2}\left(\frac{\partial}{\partial \Re\phi}\Re f + \frac{\partial}{\partial \Im\phi}\Im f\right) + \imath\frac{1}{2}\left(\frac{\partial}{\partial \Re\phi}\Im f - \frac{\partial}{\partial \Im\phi}\Re f\right),$$
$$\frac{\partial}{\partial \phi^*} f = \frac{1}{2}\left(\frac{\partial}{\partial \Re\phi}(\Re f + \imath \Im f) + \imath \frac{\partial}{\partial \Im\phi}(\Re f + \imath \Im f)\right) = \frac{1}{2}\left(\frac{\partial}{\partial \Re\phi}\Re f - \frac{\partial}{\partial \Im\phi}\Im f\right) + \imath\frac{1}{2}\left(\frac{\partial}{\partial \Re\phi}\Im f + \frac{\partial}{\partial \Im\phi}\Re f\right).$$

Example 4.2.1. $\forall \phi \in \mathbb{C}$, $f = \phi$,

$$\frac{\partial}{\partial \phi^*}\phi = \frac{1}{2}\left(\frac{\partial}{\partial \Re\phi^*}\Re\phi - \frac{\partial}{\partial \Im\phi^*}\Im\phi\right) + \imath\frac{1}{2}\left(\frac{\partial}{\partial \Re\phi^*}\Im\phi + \frac{\partial}{\partial \Im\phi^*}\Re\phi\right)$$
$$= \frac{1}{2}\left(\frac{\partial}{\partial \Re\phi^*}\Re\phi^* + \frac{\partial}{\partial \Im\phi^*}\Im\phi^*\right) + \imath\frac{1}{2}\left(\frac{\partial}{\partial \Im\phi^*}\Re\phi^* - \frac{\partial}{\partial \Re\phi^*}\Im\phi^*\right) = 1,$$

or $f = \phi^*$,

$$\frac{\partial}{\partial \phi^*}\phi^* = \frac{1}{2}\left(\frac{\partial}{\partial \Re\phi^*}\Re\phi^* - \frac{\partial}{\partial \Im\phi^*}\Im\phi^*\right) + \imath\frac{1}{2}\left(\frac{\partial}{\partial \Re\phi^*}\Im\phi^* + \frac{\partial}{\partial \Im\phi^*}\Re\phi^*\right) = 0.$$

$\phi \in \mathbb{C}$ and $f : \mathbb{C} \longrightarrow \mathbb{C}$

Before exploring the general case of a complex Langevin equation, it is important to analyze it with Einstein indexing based on $\mathbb{C} \simeq \mathbb{R}^2$.[7]

$$\forall \phi \in \mathbb{C},\ \phi = \phi_0 + \imath\phi_1,$$

where by convention, the real part is indexed by 0.

Vector operations on $\mathbb{R}^2$ carry over as is on $\mathbb{C}$,

$$\forall x, y \in \mathbb{R} \text{ and } \forall \phi, \psi \in \mathbb{C},\ (x\phi + y\psi)_a = x\phi_a + y\psi_a.$$

The complex product requires a rank 3 tensor $\bigcirc : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$ to define,

$$\forall \phi, \psi \in \mathbb{C},\ (\phi\psi)_a = \bigcirc_{abc}\phi_b\psi_c,\ \bigcirc = \left(\begin{array}{cc|cc} +1 & & & +1 \\ & -1 & +1 & \end{array}\right).$$

Multiplication with unities becomes

$$(1\phi)_a = \delta_{ab}\phi_b \text{ and } (\imath\phi)_a = \epsilon_{ab}\phi_b,$$

where ϵ is the anti–symmetric tensor depicting the sign of the permutation of its indices.

Conjugation is defined via a matrix (linear operation) $\sigma : \mathbb{C} \longrightarrow \mathbb{C}$,

$$\forall \phi \in \mathbb{C},\ (\phi^*)_a = \sigma_{ab}\phi_b.$$

The analog of an inner product $\bullet : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$ in $\mathbb{C}$ becomes,

$$\forall \phi, \psi \in \mathbb{C},\ (\phi^*\psi)_a = \bigcirc_{abc}\sigma_{bc}\phi_c\psi_d = \bullet_{abc}\phi_b\psi_c,\ \bullet = \left(\begin{array}{cc|cc} +1 & & & +1 \\ & +1 & -1 & \end{array}\right).$$

Note how the fact that $|\phi|^2 = \phi^*\phi = \phi_a\phi_a$, $\forall \phi \in \mathbb{C}$, cannot be expresses in terms of $\bullet$. This is an important aspect when examining a fully complex Langevin equation.

[7] This analysis is contained here and is not to be confused with complex vector and matrix variables to be introduced later.

The notion of complexification applies to complex numbers as well, in the sense of extending $\mathbb{C}$ to $\mathbb{C}\otimes\mathbb{C}$, which is equivalent to complexifying the real components of a complex number. With the double complex set $\mathbb{C}\otimes\mathbb{C}$, a second imaginary unit $\jmath$ is introduced, and a third one $\imath\jmath$ stems from the complex Cartesian product,[8]

$$\mathbb{C}\ni\phi=\phi_0+\imath\phi_1\longrightarrow(\phi_{00}+\jmath\phi_{01})+\imath(\phi_{10}+\jmath\phi_{11})=\phi_{00}+\jmath\phi_{01}+\imath\phi_{10}+\imath\jmath\phi_{11}\in\mathbb{C}\times\mathbb{C}.$$

In index notation ϕ assumes an extra free index, as in $\phi_a\longrightarrow\phi_{ab}$.

Conjugation extends naturally too,

$$\phi^*=\phi_{00}-\jmath\phi_{01}-\imath\phi_{10}+\imath\jmath\phi_{11}\text{ or }(\phi^*)_{ab}=\sigma_{ac}\sigma_{bd}\phi_{cd},$$

thus $\phi^*\in\mathbb{C}\otimes\mathbb{C}$ is not the one $\phi^*\in\mathbb{C}$ extends too, which is $\sigma_{ac}\phi_{cb}$ instead, so care is needed when extending complex derivatives.

$\forall f:\mathbb{C}\longrightarrow\mathbb{C}$, the complex derivative of f, shorthand–denoted as ∂f is indexed and complexified as

$$\partial_a\longrightarrow\sigma_{ac}\partial_{cb}\text{ hence }(\partial f)_a=\bigcirc_{acd}\partial_c f_d\longrightarrow\bigcirc_{acd}\sigma_{ce}\partial_{eb}f_d=\bullet_{acd}\partial_{cb}f_d=(\partial f)_{ab},$$

where the correction stems from the disagreement between conjugation in $\mathbb{C}\otimes\mathbb{C}$ and the extension of a conjugate in $\mathbb{C}$. After complexification, $f:\mathbb{C}\otimes\mathbb{C}\longrightarrow\mathbb{C}$, but $\partial f:\mathbb{C}\otimes\mathbb{C}\longrightarrow\mathbb{C}\otimes\mathbb{C}$, because the derivative is affected by the complexification.

Example 4.2.2. Let $\mu,\lambda\in\mathbb{R}$ and

$$f:\mathbb{C}\longrightarrow\mathbb{C}:\phi\longmapsto f(\phi)=\mu|\phi|^2-\imath\lambda|\phi|^4,$$

a typical scalar (complex) action functional. As a complex functional it should assume a free index, but the image form it takes cannot be written in index form using the standard operations.[9]

The modulus of ϕ becomes

$$|\phi|^2=\phi_a\phi_a\longrightarrow(\phi_a\phi_a)_b=\bigcirc_{bcd}\phi_{ac}\phi_{ad}\in\mathbb{C},$$

and

$$|\phi|^4=\phi_a\phi_a\phi_b\phi_b\longrightarrow(\phi_a\phi_a\phi_b\phi_b)_c=\bigcirc_{cdg}\bigcirc_{def}\bigcirc_{ghi}\phi_{ae}\phi_{af}\phi_{bh}\phi_{bi},$$

allowing the indexed expression of f,

$$f_a=\mu\bigcirc_{abc}\phi_{db}\phi_{dc}-\lambda\epsilon_{ab}\bigcirc_{bcd}\bigcirc_{ceg}\bigcirc_{dfh}\phi_{ie}\phi_{ig}\phi_{jf}\phi_{jh}.$$

This internal complex index notation was used here to exhibit the need of complexifying

ϕ vector

The aforementioned concepts generalize straightforwardly to a (finite–dimensional) vector complexification $V\simeq\mathbb{R}^{\dim X}\longrightarrow\mathbb{C}^{\dim X}$, simply on a component level. The arguments per the derivation, apply to the partial derivatives (components) of the functional f.

[8] Note that $\mathbb{C}\otimes\mathbb{C}$ with $k=\imath\jmath$ is not isomorphic to $\mathbb{H}$, the imaginary units algebra differ,

$\mathbb{C}\otimes\mathbb{C}$:

	$+1$	$+\imath$	$+\jmath$	$+k$
$+1$	$+1$	$+\imath$	$+\jmath$	$+k$
$+\imath$	$+\imath$	-1	$+k$	$-\jmath$
$+\jmath$	$+\jmath$	$-k$	-1	$+\imath$
$+k$	$+k$	$-\jmath$	$-\imath$	$+1$

vs $\mathbb{H}$:

	$+1$	$+\imath$	$+\jmath$	$+k$
$+1$	$+1$	$+\imath$	$+\jmath$	$+k$
$+\imath$	$+\imath$	-1	$+k$	$-\jmath$
$+\jmath$	$+\jmath$	$-k$	-1	$+\imath$
$+k$	$+k$	$+\jmath$	$-\imath$	-1

[9] More qualitatively, think about being unable to "hide non–symbolically" the imaginary unit representation of the complex $\mu|\phi|^2-\imath\lambda|\phi|^4$ by using index notation. There is no way to write this as $f_a(\phi)=\ldots$.

ϕ matrix

$\mathcal{L}(V)$ as an algebra (definition 1.1.6, example 1.1.9), is field–like – the only difference being that the set $\mathcal{L}(V)$ is not a group under the binary product $\circ$ – meaning it can be equipped with a conjugation when $V \simeq \mathbb{C}^{\dim V}$,

$$^\dagger : \mathcal{L}(V) \longrightarrow \mathcal{L}(V) : \phi \longmapsto \phi^\dagger \text{ with } \phi^\dagger|_{ab} = \phi^*_{ba},$$

and the corresponding product,

$$\cdot : \mathcal{L}(V) \times \mathcal{L}(V) \longrightarrow \mathcal{L}(V) : \phi, \psi \longmapsto \phi\psi = \phi \circ \psi \text{ with } \phi\psi|_{ab} = \phi_{ac}\psi_{cb}.$$

The corresponding real axis of the conjugation is the set of positive–definite hermitian operators ϕ, for which $\phi = \phi^\dagger$. Indeed, $\forall \psi \in \mathcal{L}(V)$,

$$(\psi^\dagger\psi)^\dagger = \psi^\dagger\psi^{\dagger\dagger} = \psi^\dagger\psi \text{ since } (\psi^*_{cb}\psi_{ca})^* = \psi^*_{ca}\psi_{cb}.$$

The corresponding conjugate derivative of a functional $f : \mathcal{L}(V) \longrightarrow \mathbb{C}$ becomes

$$\frac{\partial}{\partial\phi^*} f\bigg|_{ab} = \frac{\partial}{\partial\phi^*_{ab}} f.$$

$\forall \phi \in \mathcal{L}(V)$ hermitian, $\phi^\dagger = \phi$ or $\phi^*_{ab} = \phi_{ba}$ and

$$\frac{\partial}{\partial\phi^*} f\bigg|_{ab} = \frac{\partial}{\partial\phi_{ba}} f.$$

Example 4.2.3. Let $f(\phi) = \|\phi\|^2 = \operatorname{tr}\phi^\dagger\phi$,

$$\frac{\partial}{\partial\phi^*} \operatorname{tr}\phi^\dagger\phi\bigg|_{ab} = \frac{\partial}{\partial\phi^*_{ab}}\phi^*_{dc}\phi_{dc} = \phi^*_{dc}\frac{\partial}{\partial\phi^*_{ab}}\phi_{dc} = \phi^*_{dc}\delta_{ad}\delta_{bc} = \phi^*_{ab} \text{ or } \frac{\partial}{\partial\phi^*}\operatorname{tr}\phi^\dagger\phi = \phi^\dagger,$$

or $f(\phi) = (\operatorname{tr}\phi^\dagger\phi)^2$,

$$\frac{\partial}{\partial\phi^*}(\operatorname{tr}\phi^\dagger\phi)^2 = 2\operatorname{tr}\phi^\dagger\phi\frac{\partial}{\partial\phi^*}\operatorname{tr}\phi^\dagger\phi = 2(\operatorname{tr}\phi^\dagger\phi)\phi^\dagger,$$

since $\operatorname{tr}\phi^\dagger\phi \in \mathbb{R}_+$, or $f(\phi) = \|\phi\|^4 = \operatorname{tr}\phi^\dagger\phi\phi^\dagger\phi$,

$$\begin{aligned}\frac{\partial}{\partial\phi^*}\operatorname{tr}\phi^\dagger\phi\phi^\dagger\phi\bigg|_{ab} &= \frac{\partial}{\partial\phi^*_{ab}}\phi^*_{dc}\phi_{de}\phi^*_{fe}\phi_{fc} = \phi^*_{dc}\frac{\partial}{\partial\phi^*_{ab}}\phi_{de}\phi^*_{fe}\phi_{fc} + \phi^*_{dc}\phi_{de}\phi^*_{fe}\frac{\partial}{\partial\phi^*_{ab}}\phi_{fc}\\ &= \phi^*_{dc}\delta_{ad}\delta_{be}\phi^*_{fe}\phi_{fc} + \phi^*_{dc}\phi_{de}\phi^*_{fe}\delta_{af}\delta_{bc} = \phi^*_{ac}\phi^*_{fb}\phi_{fc} + \phi^*_{db}\phi_{de}\phi^*_{ae} = \phi^\dagger_{bf}\phi_{fc}\phi^\dagger_{ca} + \phi^\dagger_{bd}\phi_{de}\phi^\dagger_{ea},\end{aligned}$$

or

$$\frac{\partial}{\partial\phi^*}\operatorname{tr}\phi^\dagger\phi\phi^\dagger\phi = 2\phi^\dagger\phi\phi^\dagger.$$

The case corresponding to the real process in that of the matrix ϕ being hermitian. In such case $\phi^\dagger = \phi$ or $\phi^* = \phi^\top$, so the corresponding to above derivatives are with respect to $\phi^\top$, not ϕ.

4.3. Stochastic Quantization

The stochastic setting for a scalar field theory consists of

- a time index $\mathbb{T}$,
- a field configuration space V with the Borel σ–algebra $\mathcal{B}(V)$ and $\ell : \mathcal{B}(V) \longrightarrow \mathbb{R}_+$ the corresponding Lebesgue measure,
- a time–independent drift $\mu : V \longrightarrow V$ that stems from a time–independent action functional $S : V \longrightarrow \mathbb{R}$ as $\mu = -\nabla S$, and

- a vector Wiener process $\eta : \mathbb{T} \times \Omega \longrightarrow V$,
- a consistency noise setting $\sigma = \sqrt{2}$,

such that $\forall \phi : \mathbb{T} \times \Omega \longrightarrow V$,

$$\Delta\phi(\tau) = \sigma\Delta\eta(\tau) - \nabla S \circ \phi(\tau)\Delta\tau,\ \phi(0) = \phi_0 : \Omega \longrightarrow V, \tag{4.3.1}$$

and a Fokker–Planck equation for its probability transition density $\varrho : \mathbb{T} \times V \longrightarrow \mathbb{R}_+$ (with $\Sigma = \mathbb{1}$),

$$\frac{\partial}{\partial\tau}\varrho = \nabla \cdot (\nabla\varrho + \rho\nabla S) = \nabla^2\varrho + \varrho\nabla^2 S + \nabla\varrho \cdot \nabla S,\ \varrho(0) = \mathbb{1} : V \longrightarrow \mathbb{R}_+ : \phi \longmapsto \delta(\phi - \phi_0). \tag{4.3.2}$$

Note that the expectation form of the Langevin equation (4.3.1), with the assumption of time–independence of ϕ yields

$$\langle \nabla S \circ \phi \rangle = 0,$$

which basically stands for the (classical) equation of motion for ϕ.

The main argument of stochastic quantization is that the corresponding field theory stems as the stationary limit of a stochastic process defined by the stochastic differential equation (4.3.1) [68].

Thus, of interest are the stationary solutions of the corresponding deterministic Fokker–Planck equation (4.3.2). If the vector field ϕ is indexed by x, the stationary Fokker–Planck equation is written as

$$\partial_x(\partial_x\varrho + \varrho\partial_x S) = \partial_x\partial_x\varrho + \varrho\partial_x\partial_x S + \partial_x\varrho\partial_x S = 0.$$

with apparent solution

$$\varrho_\infty(\phi) \propto \exp(-S(\phi)).$$

Indeed, $\partial_x\varrho_\infty = -\varrho_\infty\partial_x S$ and $\partial_x\partial_x\varrho_\infty = -\partial_x(\varrho_\infty\partial_x S) = \varrho_\infty\partial_x S\partial_x S - \varrho_\infty\partial_x\partial_x S$, the stationary part of (4.3.2) becomes,

$$\partial_x\partial_x\varrho_\infty + \varrho_\infty\partial_x\partial_x S + \partial_x\varrho_\infty\partial_x S = \varrho_\infty\partial_x S\partial_x S - \varrho_\infty\partial_x\partial_x S + \varrho_\infty\partial_x\partial_x S - \varrho_\infty\partial_x S\partial_x S = 0.$$

Monte Carlo methods for evaluating stochastic integrals $\langle O \circ \Phi \rangle$ of functionals $O : V \longrightarrow \mathbb{R}$, rely on stochastic processes to generate (realized) Markov chains for efficient sampling of the configuration space for the Monte Carlo calculation.

The stochastic differential equation of $O \circ \Phi$ is

$$\Delta(O \circ \phi) = \nabla O \circ \phi \cdot \Delta\phi = \nabla O \circ \phi \cdot \nabla S \circ \phi(\tau)\Delta\tau + \sigma\nabla O \circ \phi \cdot \Delta W(\tau),\ O \circ \phi(0) = O \circ \phi_0 : \Omega \longrightarrow V,$$

which is a more generic stochastic differential equation that still relies on a vector Wiener process, even if O (and thus $O \circ \Phi$) is scalar. The distribution of functionals of stochastic processes is not a trivial matter, and will not be expanded here [66]. The corresponding evolution equation of the expectation $\langle O \circ \Phi \rangle$ is [68],

$$\begin{aligned}\frac{\partial}{\partial\tau}\langle O \circ \phi \rangle &= \nabla \cdot (\nabla\langle O \circ \phi \rangle - \langle O \circ \phi \rangle\nabla S) = \nabla^2\langle O \circ \phi \rangle - \langle O \circ \phi \rangle\nabla^2 S - \nabla\langle O \circ \phi \rangle \cdot \nabla S \\ &= \nabla \cdot (\nabla\langle O \circ \phi \rangle - \langle O \circ \phi \rangle\mu) = \nabla^2\langle O \circ \phi \rangle - \langle O \circ \phi \rangle\nabla \cdot \mu - \nabla\langle O \circ \phi \rangle \cdot \mu,\ \langle O \circ \phi \rangle(0) = \langle O \circ \phi_0 \rangle.\end{aligned} \tag{4.3.3}$$

4.4. The complex action problem

Any time–evolving process (even deterministic) can be simulated by their defining differential equation, provided a corresponding discretization scheme, and stochastic processes are no different.

A simulation of a stochastic process $\phi : \Omega \times \mathbb{T} \longrightarrow \mathbb{K}$, is a sample path for $\omega \in \Omega$ discretized on either $\mathbb{N}$ or a discrete stopping time τ.

Monte Carlo stands for a range of techniques by which expectation values $\langle O \circ \phi \rangle$ of functionals $O : V \longrightarrow \mathbb{K}$ are estimated efficiently.[10] The calculation relies on obtaining a representative (pseudo)random sample path on the configuration space V.

[10] Note that $O \circ \phi : \Omega \longrightarrow \mathbb{K}$ is in fact evaluated, O simply the formula of the observable set on V. For example

$$O(x) = x^* x$$

is a functional on V, but the expectation value that is sought is $\langle O \circ \phi \rangle = \langle \phi^* \phi \rangle$.

Monte Carlo methods improve the sampling process, by sampling in a way that maximizes contributions to the functional integral $\langle O \circ \phi \rangle$.

Of note is the fact that a theory or quantity is of interest at equilibrium of ϕ, i.e. at the limit $\tau \to \infty$. In practice, there exists a thermalization time τ_0 by which ϕ has practically achieved equilibrium and depends on the speed of (stochastic) convergence of ϕ, and is usually determined empirically, by examining the time history of the functional $O \circ \phi$.

A customary Monte Carlo method (important sampling) for estimating observables on a (Euclidean) theory with action functional $S : V \longrightarrow \mathbb{K}$ relies on generating instances of ϕ with a transition probability that depends on the partition value

$$Z = \langle \exp(-S \circ \phi) \rangle$$

which gives a good probability only for $\mathbb{K} = \mathbb{R}$. The logic behind importance sampling is to lead a Markov chain of samples in to domain of high contribution to observable integrals $\langle O \circ \phi \rangle$ for Said Monte Carlo method does not break if $\mathbb{K} = \mathbb{C}$, however its effect in optimizing the simulation does.

Reweighting and the overlap problem

An obvious solution to the complex action problem would be to split the action functional parts as $S = S_0 - \imath\Gamma$, and make evaluations on the S_0 functional of both the observable functional O and the phase factor $\exp \imath\Gamma$,

$$\langle O \circ \phi \rangle = \frac{\langle O \exp \imath\Gamma \circ \phi \rangle_0}{\langle \exp \imath\Gamma \rangle_0},$$

where $\langle \cdot \rangle_0$ is estimated on sample path(s) generated by S_0. This approach generates a new overlap problem, by which the two phase–quenched observables do not become important in the same domain of V.

The Complex Langevin method

The Langevin process (4.3.1) is defined for complex S provided Φ is complexified accordingly, i.e. the domain $\mathbb{C}$ for random variables defined on V becomes $\mathbb{C} \times \mathbb{C}$ if necessary, altering the definitions of S and all functionals O. Generally speaking ϕ assumes an analytic continuation. This process has a well defined Fokker–Planck equation, and thus probability, therefore it is conjectured that it generates sample paths compatible with Monte Carlo simulations. Keeping the corresponding Wiener process η "uncomplexified", under some special assumptions, $\langle \Im O \circ \phi \rangle = 0$, making the complexified Langevin process compatible with the original functionals. A slightly more generic treatment with a complex noise can be found in [35, 36].

To skip the cumbersome index notation of section §4.2. Complexification, hereon complexified objects will be denoted with an overline. Also random variables will be lower–cased as if they are realized (even if they are not).[11] The first object that requires explicit definition is the complexified vector space $\overline{V}$ over $\mathbb{C}$, assuming V is a vector space over $\mathbb{R}$, in which vector representations are allowed to have values from the new field.

A compact form of the (real) Langevin equation with drift $-\nabla S$ and stationary noise η,

$$\Delta\phi = \Delta\eta - \nabla S \circ \phi \Delta\tau,\ \phi(0) = \phi_0 \in V, \tag{4.4.1}$$

and the corresponding Fokker–Planck Hamiltonian operator

$$\mathcal{A} = \nabla \cdot \nabla - \nabla S \cdot \nabla$$

which has 0 as a non–degenerate eigenvalue and non–negative spectrum overall, implying stability of the stationary Fokker–Planck solution

$$\varrho_\infty \circ \phi \propto \exp(-S \circ \phi),$$

where $\nabla_\Re$ is the derivative with respect to a/the real ϕ.[12]

[11] See section §4.1. Assumptions for details on the assumption of realized stochastic processes in context of numerical calculations.
[12] Upon complexification of ϕ, this real ϕ becomes $\Re\overline{\phi}$.

If the action functional is complex, the minimal edit to the Langevin equation is to complexify ϕ, which will in turn edit S as well, however the derivative operator $\nabla_\Re$ with respect now to $\phi = \Re\overline{\phi}$ remains as is in the definition of $\mathcal{A}$. This minimal edit allows leaving the Wiener process η real, leading to the complex Langevin equation

$$\Delta\overline{\phi} = \Delta\eta - \overline{\nabla S} \circ \overline{\phi}\Delta\tau,$$

which can be decomposed as a pair of real Langevin equations,[13]

$$\begin{aligned}\Delta\Re\overline{\phi} &= \Delta\eta - \Re\overline{\nabla S} \circ \overline{\phi}\Delta\tau \\ \Delta\Im\overline{\phi} &= \quad\quad - \Im\overline{\nabla S} \circ \overline{\phi}\Delta\tau\end{aligned},$$

which in turn has a real corresponding Fokker–Planck Hamiltonian

$$\begin{aligned}\Re\mathcal{A} &= \overline{\nabla} \cdot \overline{\nabla} - \Re\overline{\nabla S} \cdot \overline{\nabla} \\ \Im\mathcal{A} &= \overline{\nabla} \cdot \overline{\nabla} - \Im\overline{\nabla S} \cdot \overline{\nabla}\end{aligned}.$$

$\overline{S} : \overline{V} \longrightarrow \mathbb{C}$ is the analytic continuation of the action S, i.e. the complete form it takes after complexification of ϕ. The corresponding Fokker–Planck probability density $\varrho : \mathbb{T} \times \overline{V} \longrightarrow \mathbb{R}_+$ is a distribution over the complexified $\overline{V}$.

Recall from section §4.2. Complexification that,

$$\overline{\nabla} = \frac{1}{2}(\nabla_\Re - \imath\nabla_\Im),\ \nabla_\Re = \nabla,$$

allowing the definition of a "complexified" Hamiltonian,

$$\overline{\mathcal{A}} = \overline{\nabla} \cdot \overline{\nabla} - \overline{\nabla S} \cdot \overline{\nabla},$$

Applied on holomorphic observable functionals O, by the Cauchy–Riemann equations, $\nabla_\Im = \imath\nabla_\Re$, implying $\nabla = \nabla_\Re$ and in turn, $\overline{\mathcal{A}} = \mathcal{A}$. Henceforth thus, observables will be assumed to be holomorphic functions of ϕ.

Correspondingly, let $\overline{\varrho} : \mathbb{T} \times V \longrightarrow \mathbb{C}$ be a complex distribution that satisfies

$$\frac{\partial}{\partial\tau}\overline{\varrho} = \nabla \cdot \nabla\overline{\varrho} + \nabla \cdot (\overline{\varrho}\nabla S),$$

in contrast to the proper Fokker–Planck density ϱ satisfying

$$\frac{\partial}{\partial\tau}\varrho = \overline{\nabla} \cdot \overline{\nabla}\varrho + \overline{\nabla} \cdot (\varrho\overline{\nabla S}). \tag{4.4.2}$$

The main conjecture of the complex Langevin method is that $\forall O : V \longrightarrow \mathbb{R}$ observable functional with $\overline{O} : \overline{V} \longrightarrow \mathbb{C}$ analytic continuation and $\forall\tau \in \mathbb{T}$,

$$\langle\overline{O} \circ \overline{\phi}\rangle_{\varrho(\tau)} = \langle O \circ \phi\rangle_{\overline{\varrho}(\tau)} \text{ given } \langle\overline{O} \circ \overline{\phi}\rangle_{\varrho(0)} = \langle O \circ \phi\rangle_{\overline{\varrho}(0)}, \tag{4.4.3}$$

and $\exists!\overline{\varrho}(\infty) = \lim_{\tau\to\infty}\overline{\varrho(\tau)}$,[14] such that

$$\lim_{\tau\to\infty}\langle O \circ \phi\rangle_{\overline{\varrho}(\tau)} = \langle O \circ \phi\rangle_{\overline{\varrho}(\infty)}. \tag{4.4.4}$$

What the first complex Langevin method conjecture (4.4.3) states in words is that the complex "probability" measure defined by the complex action functional of the corresponding theory is "equivalent" to the proper (Fokker–Planck) probability measure implied by its (complex) Langevin formulation. The correspondence is:

original (complex) theory $\longrightarrow$ (complex) Langevin formulation

complex probability $\overline{\varrho} : \mathbb{T} \times V \longrightarrow \mathbb{C}$ $\longrightarrow$ proper probability $\varrho : \mathbb{T} \times \overline{V} \longrightarrow \mathbb{R}_+$

original O realized on original ϕ as $O \circ \phi$ $\longrightarrow$ holomorphic $\overline{O}$ realized on complexified $\overline{\phi}$ as $\overline{O} \circ \overline{\phi}$

What the second complex Langevin conjecture (4.4.4) states in words is that the (complex) spectrum (definition 1.2.14) of the corresponding Fokker–Planck Hamiltonian $\mathcal{A}^\top$ (which applies to the complex $\overline{\varrho}$), lies in the positive real half of $\mathbb{C}$, i.e. $\forall\alpha \in \mathbb{C}$ eigenvalue of $\mathcal{A}^\top$, $\Re\alpha \geq 0$ [35, 36]. The defining element of $\mathcal{A}$ is the process drift $-\nabla S$.

[13] It may appear that only one of them is stochastic, but in fact the equations are coupled so the system is stochastic.
[14] Notably such that $\overline{\varrho}(\infty) \circ \phi = \exp(-S \circ \phi)$.

Complex action functional decomposition and requirements

A partition function more generally (assuming the real case)

$$Z : (V \longrightarrow \mathbb{R}) \longrightarrow \mathbb{R} : S \longmapsto Z(S) = \langle \exp(-S \circ \phi) \rangle$$

maps an action functional $S : V \longrightarrow \mathbb{R}$ to an integral over the entire configuration space that is used to evaluate expectation values of observables. Even when the action functional is real, it must also be positive definite for the partition integral to converge, i.e. $\forall \phi \in V,\ S(\phi) \geq 0$.

In case of a complex (and/or complexified) action functional $\overline{S}$, the condition becomes $\forall \phi \in V,\ \Re \overline{S}(\phi) \geq 0$, as

$$\langle \exp(-S \circ \phi) \rangle = \langle \exp(-\Re S \circ \phi) \exp(-\imath \Im S \circ \phi) \rangle,$$

where $\exp(-\Re S(\phi))$ decides the convergence of the partition integral, as the imaginary phase is periodic.

A complex action functional can be expressed in two ways,

$$S = \Re S + \imath \Im S = |S| \exp \imath \arg S = |S| \cos \arg S + \imath |S| \sin \arg S. \tag{4.4.5}$$

In the latter case, it is often convenient to explore $\Re \exp \imath \arg S = \cos \arg S$ instead.

Part II.

IKKT matrix model

5. Field theory

The approach to string theory in this text is to ascend from the fundamental tools in building quantum field theories rather than a physics–oriented approach.

Before proceeding to the main material some conventions need to be established.

Einstein indexing

In index notation, a tensor field ϕ has as many free indices as its rank. For example:

- a scalar ϕ
- a vector ϕ_a
- a matrix ϕ_{ab}
- a tensor of rank n, $\phi_{a_1 \dots a_n}$

Euclidean versus Lorentzian metric signature

For a real finite–dimensional vector space $\mathcal{X}$ with an inner product

$$\cdot : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R} : x, y \longmapsto x \cdot y = x_a y_a = \delta_{ab} x_a y_b,$$

the positive definite symmetric (hermitian for complex) defining matrix $\mathbb{1}$ is implied in the contraction. $\forall A$ such matrix defines a new Euclidean inner product,

$$\cdot_A : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R} : x, y \longmapsto x \cdot_A y = x \cdot A \cdot y = A_{ab} x_a y_b.$$

Such matrices belong to the general linear group $\mathrm{GL}_{\dim \mathcal{X}}(\mathbb{R})$ of invertible (diagonalizable matrices), so there exists a base on which A is diagonal. As positive–definite, all eigenvalues of A are positive.

The positive–definiteness of A is required by the triangular inequality condition of the metric the corresponding inner product induces on $\mathcal{X}$. If that condition is dropped, A is allowed to have negative eigenvalues as well. The most popular (diagonalized) non–positive–definite metric is the Lorentzian one defined by $\eta_{ij} = \delta_{ij}$, $\forall i, j{>}0$ and $\eta_{00} = -1$, or

$$\eta = \begin{pmatrix} -1 & & \cdots & \\ & +1 & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ & & \cdots & +1 \end{pmatrix} \text{ versus } \mathbb{1} = \begin{pmatrix} +1 & & \cdots & \\ & +1 & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ & & \cdots & +1 \end{pmatrix}.$$

Even in real vector spaces, a non–trivial metric implies a conjugation involution in the sense

$$x^\dagger = A \cdot x = x \cdot A,$$

which offers an alternative encoding of a custom metric as a conjugation using the trivial inner product defined by $\mathbb{1}$. In index notation, such a conjugate is usually represented by an upper index instead of a lower one, with the defining matrix implied.

$\forall x \in \mathcal{X}$ with Lorentzian metric g,

$$x^a = \eta^{ab} x_b \text{ or } x_a = \eta_{ab} x^b \text{ and } x_a x^a = \eta_{ab} x^a x^b$$

Qualitatively speaking, the role of g is to raise or lower indices, and contraction is no longer represented by plain pairs of repeating indices, the pair has to be one of upper and lower index. This effect extends straightforwardly to tensors as well. Also by convention, the standard vector components are henceforth indexed above.

By convention indexing in Euclidean signature ranges from 1 to $\dim \mathcal{X}$ while in Lorentzian signature from 0 to $\dim \mathcal{X} - 1$.

Wick rotation

A point of interest is that one can connect the Euclidean inner–product and the Lorentzian one via the correspondence

$$x^0 = \imath x^{\dim \mathcal{X}} \text{ or } x^{\dim \mathcal{X}} = \imath x^0,$$

due to $\imath^2 = -1$, called a Wick rotation or else analytic continuation, as the multiplicative role of $\imath = \exp(\imath\pi/2)$ in $\mathbb{C}$ is to rotate by $\pi/2$ counter–clockwise. Wick rotation is a mechanism to translate a theory with Lorentzian metric signature to one with Euclidean signature or vice–versa.

5.1. Classical Field Theory

Lagrangian

The first and most fundamental asset in quantum field theory is classical field theory. The key elements of a field theory are:

- A background space $\mathcal{X}$, which can be any of:
  - a (real finite–dimensional) vector space,
  - any of its local variants, like (most generally) a manifold.
- A degrees of freedom target space $\mathbb{F}$, which can be any of:
  - an algebraic field $\mathbb{K}$,
  - a $\mathbb{K}$–vector space,
  - a $\mathbb{K}$–tensor space
    $$\bigotimes \mathcal{F} = \bigotimes_{\mathbb{F}\in\mathcal{F}} \mathbb{F},$$
    over a collection $\mathcal{F}$ of $\mathbb{K}$–vector spaces,[1]
- A symmetry Lie group G with a corresponding Lie algebra, usually
  - a rotation group, like $\mathrm{SO}_{\dim \mathbb{F}}\mathbb{K}$ or its Poincare–equivalent, characteristic of bosons
  - a spin group (that is not an aforementioned rotation group), characteristic of fermions,
- A resulting configuration space $\mathbb{F}^{\mathcal{X}}$, which is:
  - a tensor space if $\mathcal{X}$ is a vector space,
  - a tensor bundle if $\mathcal{X}$ is a manifold.
- A Lagrangian function,
  $$\mathcal{L} : \mathbb{F}^{\mathcal{X}} \longrightarrow \mathbb{K}^{\mathcal{X}} : \phi \longmapsto \mathcal{L} \circ \phi.$$
- An action functional ($\mathcal{X}$–integral of Lagrangian),[2]
  $$S : \mathbb{F}^{\mathcal{X}} \longrightarrow \mathbb{K} : \phi \longmapsto \langle \mathcal{L} \circ \phi \rangle_{\mathcal{X}} = \int_{\mathcal{X}} \mathcal{L} \circ \phi(x) dx.$$

$\mathcal{L} \circ \phi$ will be often abbreviated by $\mathcal{L}$ when an explicit formula on ϕ is given on it, hence there is no confusion. Technically speaking the Lagrangian also depends on $\nabla\phi$ which has been omitted above for clarity.

Integration in $\mathcal{X}$ stems from the Lebesgue measure on $\mathcal{B}(\mathcal{X})$, which is always defined, as both vector spaces and manifolds assume a topology. The bra–ket notation will be used throughout with the corresponding integration space understood from the content of the bra–ket or the context.

[1] It is understood that rank 0 tensors are scalars in $\mathbb{K}$ and rank 1 tensors are vectors in F.
[2] Indexing over $\mathcal{X}$ is done with functional notation, i.e. $\forall x \in \mathcal{X}$ and $\forall \phi \in \mathbb{F}^{\mathcal{X}}$, $\phi(x) \in \mathbb{F}$.

Euler–Lagrange equations

The equations of motion for a particular field theory stems from the principle of least action.

$$\delta S = \left\langle \frac{\partial}{\partial\phi}\mathcal{L}\delta\phi + \frac{\partial}{\partial\partial_a\phi}\mathcal{L}\partial_a\delta\phi \right\rangle_{\mathcal{X}} = \left\langle \frac{\partial}{\partial\phi}\mathcal{L}\delta\phi + \partial_a\left(\frac{\partial}{\partial\partial_a\phi}\mathcal{L}\delta\phi\right) - \partial_a\frac{\partial}{\partial\partial_a\phi}\mathcal{L}\delta\phi \right\rangle_{\mathcal{X}} = \left\langle \left(\frac{\partial}{\partial\phi}\mathcal{L} - \partial_a\frac{\partial}{\partial\partial_a\phi}\mathcal{L}\right)\delta\phi \right\rangle_{\mathcal{X}},$$

where the total derivative term

$$\left\langle \partial_a\left(\frac{\partial}{\partial\partial_a\phi(x)}\mathcal{L}\delta\phi(x)\right) \right\rangle_{\mathcal{X}} = 0$$

was dropped as dependent on the boundary of integration $\partial\mathcal{X}$, which is zero as said boundary lies at infinity and all functionals are implied square–integrable on $\mathcal{X}$, meaning the fall–off fast enough at infinity, and thus definitely 0 on $\partial\mathcal{X}$.

The least action principle $\delta S = 0$, $\forall\Delta\phi \in \mathbb{F}^{\mathcal{X}}$, yields the Euler–Lagrange equations (of motion) of the theory

$$\frac{\partial}{\partial\phi}\mathcal{L} = \partial_a\frac{\partial}{\partial\partial_a\phi}\mathcal{L}$$

Noether's theorem

A Lagrangian $\mathcal{L}$ may be invariant under transformations/operators that act on vectors in $\mathbb{F}^{\mathcal{X}}$. Such symmetries are closed under composition, $\mathbb{1}$ is a symmetry for any Lagrangian, and for every symmetry, the inverse operator is also a symmetry. Therefore the set of symmetries of a Lagrangian forms a group. If said group is also a smooth manifold, it is a Lie group.[3]

The symmetries of the Lagrangian pass on to the corresponding action functional. However, the action functional is additionally immune to total derivatives (divergences), $\forall\alpha \in \mathbb{K}$,

$$\delta\mathcal{L} = \alpha\partial_a\Lambda^a$$

of the Lagrangian due to the diminishing boundary conditions in the integral.

Theorem 5.1.1 (Noether's)**.** *$\forall G$ symmetry Lie group of an action functional S, such that $\forall A \in G$ with $\phi \longrightarrow \exp(\imath A)\cdot\phi$ or equivalently (infinitesimally) $\forall A \in \mathfrak{g}$ (the tangent to G Lie algebra) with $\delta\phi = \imath A\cdot\phi$ and $\exists\Lambda$ vector field such that for a Lagrangian corresponding to action S,*

$$\delta\mathcal{L} = \partial_a\Lambda^a,$$

$\exists J$ vector field (Noether current) such that

$$J^a = \frac{\partial\mathcal{L}}{\partial\partial_a\phi}\cdot A\cdot\phi - \Lambda^a,$$

which is conserved, $\partial_a J^a = 0$ on shell (the subspace of $\mathbb{F}^{\mathcal{X}}$ satisfying the equations of motion of $\mathcal{L}$).

Noether's theorem holds for local (gauge) symmetries too [69], which assume the form $\forall x \in \mathcal{X}$,

$$\delta\phi(x) = \imath A(x)\cdot\phi(x),$$

meaning that symmetries are now part of orbits of $\mathcal{X}$ in G.

Currents stemming from (gauge) symmetries of the action are not the only ones that can appear in a Lagrangian. Their presence can be manifest as interacting with the fields terms, putting the free theory off shell.

[3] See section §1.3. Differential Algebra for details on matrix Lie groups and corresponding algebras.

The energy–momentum tensor

The spacetime translations are a special set of symmetries that all action functionals satisfy and are infinitesimally modeled by a set of Killing vectors ξ as

$$\delta\phi = \xi^a \partial_a \phi.$$

Since this is a vector of symmetries, the corresponding Noether current (energy–momentum tensor assumes another free index,[4]

$$T^a{}_b = \frac{\partial \mathcal{L}}{\partial \partial_a \phi} \cdot \partial_b \phi + \delta^a{}_b \mathcal{L},$$

where the dot product is meant with respect to $\mathbb{F}$, where as the contraction of indices is meant wit respect to $\mathcal{X}$. On shell $\partial_a T^{ab} = 0^b$.

Spinor fields are such that $\mathcal{X}$ is a Grassman algebra instead of a (in a sense classical) scalar/vector/tensor space.

Wick rotation

Wick rotation is a special case of analytic continuation of a field theory on complex spacetime. The transformation itself replaces the time coordinate x^0 with a coordinate

$$x_{\dim \mathcal{X}} = \imath x^0,$$

which transforms bilinear contractions with Lorentzian signature into ones with euclidean (positive definite) one, namely $\forall x \in \mathcal{X}$,

$$\sum_{a=1}^{\dim \mathcal{X}} \sum_{b=1}^{\dim \mathcal{X}} \delta_{ab} x_a x_b = \sum_{a=0}^{\dim \mathcal{X}-1} \sum_{b=0}^{\dim \mathcal{X}-1} \eta_{ab} x^a x^b.$$

With a euclidean metric signature, index leveling becomes obsolete, and by convention, lower indices are used.

5.2. Quantum Field Theory

The path integral

The path integral represent a measure–theoretic approach to second quantization [67, 68].[5] The configuration space $\mathbb{F}^{\mathcal{X}}$ as a vector space is infinite–dimensional and concerns about a possible Lebesgue measure leave the definition of the Feynman path integral formal,

$$\langle \dots \rangle_{\mathbb{F}^{\mathcal{X}}} = \int \dots \mathcal{D}\phi,$$

where here the bracket is understood as integrating on $\mathbb{F}^{\mathcal{X}}$, defining the Lebesgue measure on $\mathcal{B}(\mathbb{F}^{\mathcal{X}})$, but it becomes possible on the lattice–regularized theory (finitely discretized $\mathcal{X}$).

A much as the action functional $S : \mathbb{F}^{\mathcal{X}} \longrightarrow \mathbb{K}$ is characteristic of a classical field theory, by extension, the generating functional, also labeled the partition function,

$$Z : \mathbb{F}^{\mathcal{X}} \longrightarrow \mathbb{K} : J \longmapsto \langle \exp(\imath S + \imath \langle J\phi \rangle_{\mathcal{X}}) \rangle_{\mathbb{F}^{\mathcal{X}}}. \tag{5.2.1}$$

$J = 0$ corresponds to the free theory.

[4] Note that

$$\delta^a{}_b = \eta^{ac} \eta_{cb},$$

so g appears in the definition of a similar–index energy–momentum T^{ab}.

[5] Similarly to stochastic quantization, they are compatible.

Connection with stochastic quantization

The generating functional is the characteristic understood in probability theory (lemma 2.1.13), and as such it generates the probability law of the theory

$$\rho = \frac{\exp \imath S_{\text{lorentzian}}}{\langle \exp \imath S_{\text{lorentzian}} \rangle_{\mathbb{F}^{\mathcal{X}}}},$$

and all expectations of (observable) functions $O : \mathbb{F}^{\mathcal{X}} \longrightarrow \mathbb{K}$,

$$\langle O \rangle_{\mathbb{F}^{\mathcal{X}}} = \frac{\langle O \exp \imath S_{\text{lorentzian}} \rangle_{\mathbb{F}^{\mathcal{X}}}}{\langle \exp \imath S_{\text{lorentzian}} \rangle_{\mathbb{F}^{\mathcal{X}}}}.$$

This is formal apparently, as the imaginary unit in the Boltzmann factor $\exp \imath S_{\text{lorentzian}}$ invalidates any proper definition of a probability with it.[6]

Wick–rotating $\mathcal{X}$ as in $x_{\dim \mathcal{X}} = \imath x^0$, leads to the so–called Euclidean version $S_{\text{euclidean}}$ of a theory with a positive definite and diminishing entropic factor $\exp(-S_{\text{euclidean}})$. The connection of stochastic quantization to that of path integration is one tying a stochastic process to the probability law generating path integral (5.2.1) defines.

The conjecture of the complex stochastic quantization is that, as a valid stochastic process it admits a probability law anyway, even if the corresponding theory has a probabilistically ill–defined generating path integral Z. Setting up such a stochastic process is not trivial, complexification may be necessary, which in turn complexifies observables.[7]

Examples

In what follows, a 4–dimensional spacetime $\mathcal{X}$ is assumed, i.e. $\dim \mathcal{X} = 4$

A massive scalar theory $\mathbb{F} = \mathbb{C}$ with Higgs potential

A massive scalar field theory in a Higgs potential sports the Lagrangian

$$\mathcal{L} = \eta^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi - m^2 \phi^* \phi - \lambda (\phi^* \phi)^2,$$

with equation of motion

$$\Box \phi - m^2 \phi - \lambda |\phi|^2 \phi = 0, \; \Box = \eta^{\mu\nu} \partial_\mu \partial_\nu.$$

This theory has $\mathrm{U}_1\mathbb{C}$ symmetry group. Indeed, $\forall a \in \mathbb{R}$, substituting $\phi \to \phi \exp \imath a$ in the Lagrangian leaves it invariant.

A massless boson (vector) theory $\mathbb{F} = \mathbb{R}^{\dim \mathcal{X}}$

A typical massless vector field Lagrangian is that of photons

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \; F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

where J_μ is an external current (source), with equations of motion

$$\partial_\mu F^{\mu\nu} = 0,$$

and $\mathrm{U}_1\mathbb{C}$ symmetry. This is a special case of a Yang–Mills theory.

[6] It is for this reason, the Lorentzian signature of the theory is stressed here.
[7] See section §4.2. Complexification for a technical presentation of the concept.

A massless boson (vector) theory with fermions (Weyl spinors) $\mathbb{F} = \mathbb{R}^{\dim \mathcal{X}}$

The vector Lagrangian becomes

$$-\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \imath\alpha\overline{\psi}\gamma^{\mu} D_{\mu}\psi,\ D_{\mu} = \partial_{\mu} - \imath\alpha A_{\mu},$$

where γ are generators of a Clifford algebra, satisfying

$$\{\gamma^{\mu}|\gamma^{\nu}\} = 2\gamma^{(\mu} \circ \gamma^{\nu)} = 2\eta^{\mu\nu},$$

leading to the equations of motion

$$\partial_{\mu} F^{\mu\nu} + \alpha\overline{\psi}\gamma^{\nu}\psi = 0,$$

with $\alpha > 0$ being a coupling constant between ψ or $\overline{\psi}$ and A.

Yang–Mills theories

Bosonic

A (purely) bosonic classical generic Yang–Mills Lagrangian has the form

$$\mathcal{L} = \frac{1}{4} \operatorname{tr}_{\mathcal{X}}(F_a \cdot F_a),\ F_{a\mu\nu} = \partial_{\mu} A_{a\nu} - \partial_{\nu} A_{a\mu} + \alpha f_{abc} A_{b\mu} A_{c\nu} \tag{5.2.2}$$

with a (generally non–Abelian) gauge symmetry group G with a corresponding algebra $\mathfrak{g}$,[8] whose generators λ satisfy $\forall a \in \mathbb{N}_{\dim \mathfrak{g}}$,

$$\operatorname{tr}_{\mathfrak{g}}(\lambda_a \lambda_b) = \frac{1}{2}\delta_{ab} \text{ and } \lambda_a \circ \lambda_b = [\lambda_a|\lambda_b] = 2\lambda_{[a}\lambda_{b]} = \imath f_{abc}\lambda_c,$$

f being the structure constants characteristic to the algebra $\mathfrak{g}$.[9]

The field strength in general is defined via the commutator of the covariant derivative ($A_{\mu} = \lambda_a A_{a\mu}$)

$$D_{\mu} = \partial_{\mu} - \imath\alpha A_{\mu},\ A_{\mu} = \lambda_a A_{a\mu},$$

as ($F_{\mu\nu} = \lambda_a F_{a\mu\nu}$),

$$\begin{aligned} F_{\mu\nu} = 2\imath\alpha^{-1} D_{[\mu} D_{\nu]} = \imath\alpha^{-1}[D_{\mu}|D_{\nu}] = \imath\alpha^{-1}[\partial_{\mu} - \imath\alpha A_{\mu}|\partial_{\nu} - \imath\alpha A_{\nu}] = \imath\alpha^{-1}([\partial_{\mu}|\partial_{\nu}] - \imath\alpha[\partial_{\mu}|A_{\nu}] - \imath\alpha[A_{\mu}|\partial_{\nu}] - \alpha^2[A_{\mu}|A_{\nu}]) \\ 2\imath\alpha^{-1}(\partial_{[\mu}\partial_{\nu]} - \imath\alpha\partial_{[\mu}A_{\nu]} - \imath\alpha A_{[\mu}\partial_{\nu]} - \alpha^2 A_{[\mu}A_{\nu]}) = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} - \imath\alpha[A_{\mu}|A_{\nu}], \end{aligned}$$

where $\partial_{\mu}\partial_{\nu} - \partial_{\nu}\partial_{\mu} = 0$ and

$$A_{\mu}\partial_{\nu} - A_{\nu}\partial_{\mu} = \lambda_a A_{a\mu}\partial_{\nu} - \lambda_a A_{a\nu}\partial_{\mu} = 0,$$

as total derivative terms. Expanding on $\mathfrak{g}$–components,

$$\begin{aligned} \lambda_a F_{a\mu\nu} = \partial_{\mu}\lambda_a A_{a\nu} - \partial_{\nu}\lambda_a A_{a\mu} - \imath\alpha[\lambda_b A_{b\mu}|\lambda_c A_{c\nu}] = \lambda_a\partial_{\mu}A_{a\nu} - \lambda_a\partial_{\nu}A_{a\mu} - \imath\alpha[\lambda_b|\lambda_c]A_{b\mu}A_{c\nu} \\ = \partial_{\mu}\lambda_a A_{a\nu} - \partial_{\nu}\lambda_a A_{a\mu} + \alpha\lambda_a f_{abc} A_{b\mu} A_{c\nu}. \end{aligned}$$

Note that the non–commutativity of A in Yang–Mills theories stems from the non–commutativity of its gauge group G (and corresponding algebra $\mathfrak{g}$), i.e. from the internal degrees of freedom of A.

The commutator Jacobi identity becomes a Bianchi identity, as

$$[D_{\rho}|[D_{\mu}|D_{\nu}]] + [D_{\nu}|[D_{\rho}|D_{\mu}]] + [D_{\mu}|[D_{\nu}|D_{\rho}]] = 6D_{[\rho}D_{\nu}D_{\mu]} = 6D_{[\rho}F_{\mu\nu]} = 2(D_{\rho}F_{\mu\nu} + D_{\nu}F_{\rho\mu} + D_{\mu}F_{\nu\rho}) = 0,$$

The equation of motion of A in a Yang–Mills Lagrangian (5.2.2),

$$D^{\mu}F_{\mu\nu} = 0 \text{ or } D^{\mu}F_{a\mu\nu} = \partial^{\mu}F_{a\mu\nu} + \alpha f_{abc}A_{b}{}^{\mu}F_{c\mu\nu} = 0. \tag{5.2.3}$$

[8] See section §1.3. Differential Algebra for some details.

[9] Note how each trace has a subscript indicating the space tracing is happening. For example, $\forall X \in \mathcal{X}$, $\operatorname{tr}_{\mathcal{X}} X = \eta_{\mu\nu}X^{\mu\nu}$.

Bosonic with fermions

Editing (5.2.2) to include fermions,

$$\mathcal{L} = \frac{1}{4}\operatorname{tr}_{\mathcal{X}}(F_a \cdot F_a) + \frac{1}{2}\imath\alpha \operatorname{tr}_{\mathcal{U}}(\overline{\psi}_a \gamma \cdot D\psi_a),\ D_\mu = \partial_\mu - \imath\alpha\lambda_a A_{a\mu} \tag{5.2.4}$$

where $\mathcal{U}$ is the vector space on Grassman numbers the fermion field reside.

The equation of motion (5.2.3) acquires a source current,

$$D^\mu F_{\mu\nu} + \alpha\overline{\psi}\gamma_\nu\psi = \lambda_a D^\mu F_{a\mu\nu} + \alpha\overline{\psi}_a\gamma_\nu\psi_a = 0.$$

$\mathcal{N} = 1$ super Yang–Mills theories

$\mathcal{N} = 1$ super Yang–Mills theories have a similar form to (5.2.4). In super Yang–Mills theories, the covariant derivative is defined as

$$D_\mu \cdot = \partial_\mu \cdot + \imath\alpha[A_\mu|\cdot].$$

The supersymmetries of a super Yang–Mills theory are

$$\delta_\epsilon A_\mu = \overline{\epsilon}\gamma_\mu\psi \text{ and } \delta_\epsilon\psi = -\frac{1}{2}F \cdot \gamma\epsilon,\ \gamma_{\mu\nu} = \gamma_{[\mu}\gamma_{\nu]}.$$

Of interest are the zero–volume super Yang–Mills theories. By taking $\dim \mathcal{X} = 0$, all spacetime derivatives vanish, yielding

$$F_{\mu\nu} = -\imath\alpha[A_\mu|A_\nu] \text{ and } D_\mu \cdot = \imath\alpha[A_\mu|\cdot].$$

Substituting in (5.2.4),

$$\mathcal{L} = \frac{1}{4}\operatorname{tr}_{\mathcal{X}}(F \cdot F) - \frac{1}{2}\imath\alpha \operatorname{tr}_{\mathcal{U}}(\overline{\psi}\gamma \cdot D\psi) = -\alpha^2\left(\frac{1}{4}\operatorname{tr}_{\mathcal{X}}([A|A] \cdot [A|A]) + \frac{1}{2}\operatorname{tr}_{\mathcal{U}}(\overline{\psi}\gamma \cdot [A|\psi])\right). \tag{5.2.5}$$

6. String theory

String theories rose out of the necessity for incorporating the gravitational interaction into a unified framework enticing the Standard model. The dynamic geometric nature of the classical general relativity theory of gravity, in which matter and the spacetime background affect each other dynamically, makes it impossible to directly produce a renormalizable quantum field variant. String theories offer not only a way to quantize gravity, but to incorporate the Standard Model with it into a unified framework for describing all fundamental aspects of the universe as observed to this day [2, 4].

6.1. The bosonic string

Assuming a fixed for now background spacetime $\mathcal{X}$ and a metric g with Lorentzian signature $(-+\ldots+)$, the life cycle of a single (point) particle is represented by a monoparametric (world) line

$$X : T \longrightarrow \mathcal{X} : \tau \longmapsto X(\tau).$$

The world line of a particle is expected to be at least continuous, and world line intersections represent particle interaction events.

Even if it describes events, the parameter τ of a world line can be thought of as an internal time. String theory assumes a second "modal" parameter σ, as in

$$X : T \times \Sigma \longrightarrow \mathcal{X} : \tau, \sigma \longmapsto X(\tau, \sigma),$$

a degree of freedom whose modes of "vibration" serve for particle type distinction. The added advantage is that world lines now become world sheets, and interactions are no longer singular events in space time, but topological properties of the world sheets involved [2].

Strings belong to a more general category of p–dimensional p–branes, embedded in a spacetime $\mathcal{X}$, $\forall p \in \mathbb{N}_{\dim \mathcal{X}}$, such that $p = 0$ corresponds to particles and $p = 1$ corresponds to strings,

$$X : T \times \prod\nolimits_i \Sigma \longrightarrow \mathcal{X} : (\sigma_i)_i \longmapsto X(\sigma),$$

where $T \times \prod_i \Sigma$ is a $(p+1)$–dimensional parametric vector space.

The string action

0–branes

The motion of a free point particle ($p = 0$) in a curved spacetime $\mathcal{X}$, is along geodesics (straight lines for a curved spacetime), therefore the action is proportional to the invariant worldling length,

$$S_0 = -m_0 \int d\mu_0,$$

where

$$d\mu_0 = \sqrt{-g_{\mu\nu}(X) dX^\mu dX^\nu} d\tau = \sqrt{-\|dX\|^2} d\tau = \imath \|dX\| d\tau$$

and m_0 a mass scale in inverse length units ($c = \hbar = 1$). This action is linked to a mass scale that cannot be zero, therefore an equivalent action with an auxiliary field h is used to describe massless point particles as well,

$$S_0 \propto \langle h^{-1} \|X\|^2 - m_0^2 h \rangle_T, \tag{6.1.1}$$

which is also easier to quantize.

This action is equivalent to the invariant length action. The equation of motion of h

$$-\frac{\delta S}{\delta h} = h^{-2}\|X\|^2 + m_0 = 0$$

yields $h = \imath m_0^{-1}\|X\|$. Replacing the solution for h in the action (6.1.1), yield the original invariant length action.

p–branes

This process generalizes to the p–brane action

$$S_p = -m_p \int d\mu_p,$$

with the inverse length mass scale replaced by the p–brane inverse volume tension m_p, and the $(p+1)$–dimensional p–brane hyper–volume

$$d\mu_p = \sqrt{-\det G} d^{p+1}\sigma,$$

where the induced metric on $(p+1)$–dimensional $T \times \Sigma$ is

$$G_{\alpha\beta} = g_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu,$$

where $p = \dim \Sigma$.

S_p extremizes the $(p+1)$–dimensional world hyper–volume, in the same sense that S_0 extremizes the length of its world line.

The string action in flat spacetime

In the case of a flat spacetime $\mathcal{X}$, the induced metric becomes,

$$G_{\alpha\beta} = \eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu = \partial_\alpha X \cdot \partial_\beta X,$$

where $\cdot$ stands for the Lorentzian–signature inner product defined by η.[1]

For strings ($p = 1$), it becomes

$$\det G = \|\partial_0 X\|^2\|\partial_1 X\|^2 - |\partial_0 X \cdot \partial_1 X|^2 = (\partial_0 X \cdot \partial_0 X)(\partial_1 X \cdot \partial_1 X) - (\partial_0 X \cdot \partial_1 X)(\partial_1 X \cdot \partial_0 X)$$

leading to the Nambu–Goto action

$$S_1 = -m_1\langle\sqrt{-\det G}\rangle_{T\times\Sigma},$$

which extremizes the area of the world sheet that a propagating string generates in spacetime $\mathcal{X}$.

The string tension m_1 defines a natural string length scale as

$$\ell_1 = \frac{1}{\sqrt{\pi m_1}}. \tag{6.1.2}$$

The string sigma model action

The sigma model action is

$$S_1 = -\frac{1}{2}m_1\langle\sqrt{-\det h}h^{\alpha\beta}G_{\alpha\beta}\rangle_{T\times\Sigma} = -\frac{1}{2}m_1\langle\sqrt{-\det h}h^{\alpha\beta}\partial_\alpha X \cdot \partial_\beta X\rangle_{T\times\Sigma}, \tag{6.1.3}$$

where h is an auxiliary metric on the world sheet Σ with measure–theoretic volume $\sqrt{-\det h}$ with

$$\delta \det h = -\det h h_{\alpha\beta}\delta h^{\alpha\beta} \text{ or } \delta\sqrt{-\det h} = -\frac{1}{2}\sqrt{-\det h}h_{\alpha\beta}\delta h^{\alpha\beta}, \tag{6.1.4}$$

[1] Recall that a Lorentzian–signature metric breaks the positive definiteness of a metric, induced norm and in–turn induced inner product. See

which action is classically equivalent to the Nambu–Goto action. The equation of motion for the auxiliary metric h stems from the variation of the action $\delta S_{p(\sigma)}$ with respect to δh, also encoded in the vanishing of the respective energy–momentum tensor T, with

$$T_{\alpha\beta} = -\frac{2}{m_1\sqrt{-\det h}}\frac{\delta S_p}{\delta h^{\alpha\beta}} = 0,$$

which, together with (6.1.4), gives

$$T_{\alpha\beta} = G_{\alpha\beta} - \frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}G_{\gamma\delta} = 0, \tag{6.1.5}$$

which equates the auxiliary metric h to the world sheet induced metric G.

The corresponding equation of motion in non–index form (let $\operatorname{tr} G = h^{\alpha\beta}G_{\alpha\beta}$) is

$$G = \frac{1}{2}h\operatorname{tr} G \text{ leading to } \det G = \frac{1}{2}\det(h\operatorname{tr} G) = \frac{1}{2}\det h(\operatorname{tr} G)^2.$$

For strings ($p = 1$),[2]

$$\det G = \frac{1}{2}\det h(\operatorname{tr} G)^2 \text{ or } \sqrt{-\det G} = \frac{1}{2}\sqrt{-\det h}\operatorname{tr} G \text{ or } \sqrt{-\det G} = \frac{1}{2}\sqrt{-\det h}h^{\alpha\beta}G_{\alpha\beta}, \tag{6.1.6}$$

which proves the equivalence of the string sigma model action to the Nambu–Goto action for strings ($p = 1$).

The string sigma model action has three distinct symmetries:

Poincaré symmetry: Lorentz infinitesimal rotations and spacetime translations $X^\mu \longrightarrow a^\mu{}_\nu X^\nu + b^\mu$, with the auxiliary world metric satisfying $\delta h_{\alpha\beta} = 0$, as the world sheet maintains its geometry with Lorentz transformations a and spacetime translations b.

Diffeomorphism invariance: $\sigma^\alpha \longrightarrow f^\alpha(\sigma)$ with $h_{\alpha\beta}(\sigma) = \partial_\alpha f^\gamma \partial_\beta f^\delta h_{\gamma\delta}(f(\sigma))$,

Weyl invariance: $h_{\alpha\beta} \longrightarrow \exp(-\phi(\tau,\sigma))h_{\alpha\beta}$ and $\delta X^\mu = 0$.

Diffeomorphism and Weyl invariance are local transformations, which allow a gauge selection on the auxiliary metric field. For $p = 1$, h has 4 components,

$$h = \begin{pmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{pmatrix},$$

but is symmetric ($h_{01} = h_{10}$), therefore it has 3 independent components, 2 of which are fixed by reparametrization (diffeomorphism) invariance gauge selection and one by scale (Weyl) invariance; the flat metric can thus be chosen,

$$h = \eta = \begin{pmatrix} -1 & \\ & +1 \end{pmatrix},$$

resulting in a simpler string sigma model action,

$$S_1 \propto -\langle \eta^{\alpha\beta}\eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu \rangle_{T\times\Sigma} = \langle \partial_0 X \cdot \partial_0 X - \partial_1 X \cdot \partial_1 X \rangle_{T\times\Sigma}. \tag{6.1.7}$$

Gauge fixing the auxiliary world sheet metric h to the flat metric η requires that the worlds sheet topology actually allows such a metric.[3] It is worth noting that this is not a complete gauge fixing, in the sense that there exist reparametrizations that are also Weyl rescalings, satisfying

$$\partial^{(\alpha}\xi^{\beta)} = \Lambda\eta^{\alpha\beta},$$

where ξ is a parameter vector for infinitesimal parametrizations and Λ a corresponding infinitesimal Weyl rescaling parameter [2].

Strings can topologically be classified as either closed or (finitely) open. All world sheets topologies:

- of a closed propagating string are homeomorphic to an infinite cylinder (figure 6.1.1)
- of an open propagating string are homeomorphic to an infinite strip (figure 6.1.1)

[2] $\operatorname{tr} G = h^{\alpha\beta}G_{\alpha\beta} \geq 0$?

[3] The world sheet as a manifold may have a metric defined in an atlas covering it. Assuming a global flat metric means the topology of the manifold has a single chart atlas, i.e. be diffeomophic to a Euclidean space.

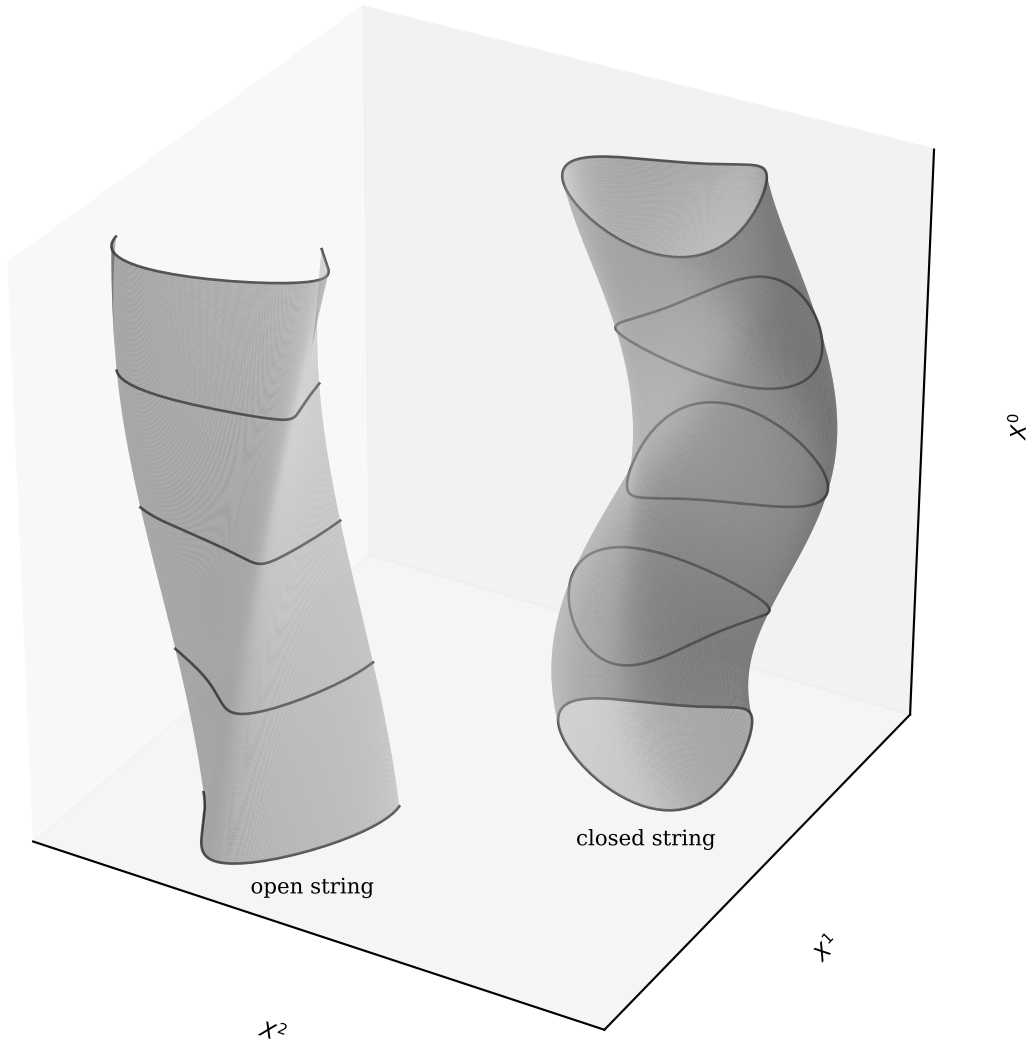


Figure 6.1.1.: The world sheets of a closed (homeomorphic to a cylinder) and an open (homeomorphic to a strip) string respectively.

The string equations of motion

Extremizing the string sigma model action leads to a wave equation of motion,

$$\Box X = 0, \Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta, \tag{6.1.8}$$

which holds for arbitrary world–sheet dimension $p+1$.

By eliminating the equation of motion of the auxiliary metric by gauge fixing it, the vanishing of its corresponded energy–momentum tensor (6.1.5) becomes a (manual) constraint,

$$T_{\alpha\beta} = G_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta}G_{\gamma\delta} = G_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\operatorname{tr} G,$$

which implies the vanishing of its trace, $\operatorname{tr} T = \eta^{\alpha\beta}T_{\alpha\beta} = 0$ [2].

Assuming, without loss of generality, that $\sigma \in [0,\pi]$ is finite, variation of the string sigma model action yields the boundary term

$$\partial S_1 \propto -\langle \partial_0 X \cdot \delta X^\mu|_{\sigma=\pi} - \partial_0 X \cdot \delta X|_{\sigma=0}\rangle_T$$

the vanishing of which defines the string topology, classified as:

closed strings: $X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+\pi)$ (periodic boundary conditions)

open strings:

- **with Neumann boundary conditions:** $\partial_0 X^\mu(\tau,0) = \partial_0 X^\mu(\tau,\pi) = 0$, which respects $\dim\mathcal{X}$–Poincaré invariance,
- **with Dirichlet boundary conditions:** $\forall\mu < \dim\mathcal{X} - p$ with $\mu \neq 0$, $X^\mu(\tau,0) = X_0^\mu$ and $X^\mu(\tau,\pi) = X_\pi^\mu$ and Neumann boundary conditions for $\forall\mu \geq \dim\mathcal{X} - p$ and $\mu = 0$, which breaks $\dim\mathcal{X}$–Poincaré invariance. The modern interpretation of Dirichlet open string boundary condition embeddings are as Dp–branes ("D" for "Dirichlet"), which, if space filling ($p+1 = \dim\mathcal{X}$), respect Poincaré–invariance [2].

Classical solutions

Reparametrizing to lightcone world sheet coordinates,

$$\sigma^\pm = \tau \pm \sigma \text{ with } \partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1) \text{ and } \eta = \begin{pmatrix} \eta_{++} & \eta_{+-} \\ \eta_{-+} & \eta_{--} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix},$$

the equation of motion (6.1.8) becomes (suppressing free indices)

$$\Box X = 0, \Box = \partial_+\partial_-, \tag{6.1.9}$$

while the energy–momentum tensor becomes

$$T = \begin{pmatrix} T_{++} & T_{+-} \\ T_{-+} & T_{--} \end{pmatrix} = \begin{pmatrix} \partial_+ X \cdot \partial_+ X & \\ & \partial_- X \cdot \partial_- X \end{pmatrix},$$

where the vanishing of the off–diagonal elements is tautological from the tracelessness identity for a flat world sheet metric η.

The general solution of the equations of motion can then be expressed are left–propagating X^+ and right–propagating X^- on σ strings as

$$X(\sigma,\tau) = X^-(\sigma^-) + X^+(\sigma^+). \tag{6.1.10}$$

The vanishing of the energy–momentum tensor as part of the world sheet metric equation of motion implies

$$\partial_- X^- \cdot \partial_- X^- = 0 \text{ and } \partial_+ X^+ \cdot \partial_+ X^+ = 0.$$

The general solution of (6.1.9) for the open string (with Neumann boundary conditions)

$$X(\tau,\sigma) = x + \ell_1\alpha_0\tau + \imath\ell_1 \sum\nolimits_{m\in\mathbb{Z}_*} m^{-1}\alpha_m \exp(-\imath m\tau)\cos(m\sigma),\ \alpha_0 = \ell_1 p, \tag{6.1.11}$$

where x denotes the center of mass of the string and p its free momentum. The closed string allows for propagating wave modes, while the modes of the open string are stationary.

In what follows, we will refer to the open string only, for keeping the presentation brief. For the closed string similar results hold with double the algebra, and level matching, further details can be found in [2].

The requirement that the solution is real, yields $\alpha_{-n} = \alpha_n^*$.

Quantization

The non–zero Poisson brackets of X with its canonical momentum $P = m_1\partial_0 X$,

$$[P^\mu(\tau,\sigma)|X^\mu(\tau,\sigma')]_{\text{poisson}} = \eta^{\mu\nu}\delta(\sigma-\sigma'),$$

can be used to quantize the string by replacing Poisson brackets with commutators $[\cdot|\cdot] \to \imath[\cdot|\cdot]$ and the solution (6.1.11),

$$[\alpha_m^\mu|\alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n|0}.$$

The Hamiltonian stemming from the Lagrangian $\mathcal{L}_1$ of the string sigma model action (6.1.7) (hence forth denoted S_1 simply) becomes

$$H = L_0 = \langle\partial_0 X\cdot P - \mathcal{L}_1\rangle_\Sigma = \frac{1}{2}m_1\langle\partial_0 X\cdot\partial_0 X + \partial_1 X\cdot\partial_1 X\rangle_\Sigma = \frac{1}{2}\sum\nolimits_{n\in\mathbb{Z}}\alpha_{-n}\cdot\alpha_n,$$

where $\forall m\in\mathbb{Z}$,

$$L_m = \frac{1}{2}\sum\nolimits_{n\in\mathbb{Z}}\alpha_{m-n}\cdot\alpha_n \tag{6.1.12}$$

are the Virasoro generators appearing in the mode expansion of the energy–momentum tensor. The first derivatives of the string embedding are

$$\partial_\pm X = \frac{1}{2}(\partial_0 X \pm \partial_1 X) = \frac{1}{2}\ell_1\sum\nolimits_{m\in\mathbb{Z}}\alpha_m\exp(-\imath m\sigma^\pm)$$

yielding,[4]

$$T_{\pm\pm} = \partial_\pm X\cdot\partial_\pm X = \frac{1}{2}\ell_1^2\sum\nolimits_{m\in\mathbb{Z}}L_m\exp(-\imath m\sigma^\pm)$$

The Virasoro algebra

The classical Virasoro algebra generators, with the quantum commutator notation satisfy $\forall m,n\in\mathbb{Z}$,

$$[L_m|L_n] = (m-n)L_{m+n}.$$

However the quantized Virasoro algebra generators shall have normal ordering, which is relevant only for

$$L_0 = \frac{1}{2}\sum\nolimits_{n\in\mathbb{Z}} :\alpha_{-n}\cdot\alpha_n: = \frac{1}{2}\alpha_0^2 + \sum\nolimits_{n\in\mathbb{N}}\alpha_{-n}\cdot\alpha_n,$$

[4] The square of partial sums of a sequence $\alpha:\mathbb{N}\longrightarrow\mathbb{R}$,

$$\sum\nolimits_{n\in\mathbb{N}}\alpha_n,$$

becomes

$$\sum\nolimits_{m\in\mathbb{N}}\sum\nolimits_{n\in\mathbb{N}}\alpha_m\cdot\alpha_n = \sum\nolimits_{m\in\mathbb{N}}\sum\nolimits_{n=0}^{m}\alpha_{m-n}\cdot\alpha_n.$$

This formula extended over $\mathbb{Z}$ has unbounded inner sums,

$$\sum\nolimits_{m\in\mathbb{Z}}\sum\nolimits_{n\in\mathbb{Z}}\alpha_m\cdot\alpha_n = \sum\nolimits_{m\in\mathbb{Z}}\sum\nolimits_{n\in\mathbb{Z}}\alpha_{m-n}\cdot\alpha_n,$$

which roughly translates to summing diagonal slices over the infinite $\mathbb{Z}$ grid instead of horizontal/vertical.

as it is the only generator containing matching conjugate pairs of α modes, which in turn extends the classical Virasoro algebra as

$$[L_m|L_n] = (m-n)L_{m+n} + \frac{1}{12}\dim\mathcal{X} m(m^2-1)\delta_{m+n|0} - a\delta_{m0},$$

where a is a constant stemming from the normal ordering arbitrariness of L_0 [2]. The central extension term added to the classic Virasoro algebra stands for a conformal anomaly.

Closed strings

The extraction of the Virasoro algebra was done on open strings (stationary solutions) for the sake of simplicity. However closed strings are of interest as they are the foundation of Type II (super)string theories which are of interest in this work.

Analogous to the open string solution (6.1.11) of the string sigma model equation of motion (6.1.9), the closed string (periodic) boundary condition $X(\tau,\sigma+\pi) = X(\tau,\sigma)$ does not require left–movers X^+ and right–movers X^- to form standing waves,

$$2X^{\pm}(\sigma^{\pm}) = x + \ell_1\alpha_0\sigma^{\pm} + \imath\ell_1\sum\nolimits_{m\in\mathbb{Z}_*} m^{-1}\alpha_m^{\pm}\exp(-2\imath m\sigma^{\pm}),$$

where now we get two sets of string modes α^+ and α^- respective to each propagation directions, resulting in two sets of Virasoro modes L^+ and L^- of the corresponding energy–momentum tensor, each set independently satisfying the Virasoro algebra, while each set commutes with the other. Classically:

$$[L_m^{\pm}|L_n^{\pm}] = (m-n)L_{m+n}\delta_{\pm\pm}.$$

The conformal anomaly

Without going into details (see [2, 4] for further details), the Faddeev–Popov gauge fixing produces extra terms, the ghost action

$$S_1^{\text{ghost}} \propto \langle c^-\partial_+ b_{--} + c^+\partial_- b_{++}\rangle_\Sigma$$

where c is a vector ghost field and b a traceless symmetric antighost tensor field, with corresponding energy–momentum tensor

$$T_{\pm\pm} = -\imath\left(\partial_\pm c^\pm b_{\pm\pm} + \frac{1}{2}c^\pm\partial_\pm b_{\pm\pm}\right).$$

The Virasoro modes satisfy $\forall m,n\in\mathbb{Z}$ the extended algebra

$$[L_m^{\text{ghost}}|L_n^{\text{ghost}}] = (m-n)L_{m+n}^{\text{ghost}} - \frac{1}{6}m(13m^2-1)\delta_{m+n|0}.$$

The total Virasoro modes $\forall m\in\mathbb{Z}$

$$L_m^{\text{total}} = L_m + L_m^{\text{ghost}} - a\delta_{m|0}$$

thus satisfy the algebra

$$[L_m^{\text{total}}|L_n^{\text{total}}] = (m-n)L_{m+n}^{\text{total}} + \frac{1}{12}m((\dim\mathcal{X}-26)m^2 + (2+24a-\dim\mathcal{X}))\delta_{m+n|0},$$

where a is a constant correction on L_0 stemming from its normal ordering arbitrariness [2]. The conformal anomaly (algebra extension) vanishes for $\dim\mathcal{X} = 26$ and $a = 1$ for a bosonic string theory.

The aforementioned critical dimension for bosonic strings can be derived by studying the conformal anomaly of fermion ghosts arising from the Faddeev–Popov gauge fixing of the string action.The general form of the conformal anomaly stemming from a conformal analysis of a Euclidean version is

$$c(\varepsilon,\lambda) = -2\varepsilon(6\lambda^2 - 6\lambda + 1),$$

with ε and λ being the tensor ranks of each of the two ghost fields stemming from the gauge fixing.

For bosonic strings, the two fermion ghosts have $\varepsilon = 1$ and $\lambda = 2$ respectively, leading to

$$c_{\text{bosonic}} = -2 \cdot 1 \cdot (6 \cdot 2^2 - 6 \cdot 2 + 1) = -2(24 - 12 + 1) = -2 \cdot 13 = -26,$$

which can be countered by the contribution of $\dim \mathcal{X}$ bosonic ($\lambda_{\text{bosonic}} = 1$) dimensions,

$$\lambda_{\text{bosonic}} \dim \mathcal{X} + c_{\text{bosonic}} = 0 \text{ or } \dim \mathcal{X} = 26.$$

A similar argument for string theory with fermions fixed to a superconformal gauge leads to $\dim \mathcal{X} = 10$. In superstring theories two bosonic ghosts appear with $\varepsilon = -1$ and $\lambda = 3/2$, giving

$$c_{\text{fermionic}} = -2 \cdot (-1) \cdot \left(6 \cdot \frac{3^2}{2^2} - 6 \cdot \frac{3}{2} + 1\right) = 2\left(\frac{27}{2} - 9 + 1\right) = 2 \cdot \frac{11}{2} = 11,$$

thus the total conformal anomaly is

$$c_{\text{super}} = c_{\text{fermionic}} + c_{\text{bosonic}} = 11 - 26 = -15,$$

that assuming contributions from both bosonic degrees of freedom ($\lambda_{\text{bosonic}} = 1$) and their superpartners ($\lambda_{\text{fermionic}} = 1/2$),

$$(\lambda_{\text{bosonic}} + \lambda_{\text{fermionic}}) \dim \mathcal{X} + c_{\text{super}} = 0 \text{ or } \dim \mathcal{X} = 10.$$

6.2. Fermions and superstrings

For string theory to be able to account for the standard model, a description and modeling of fermions is necessary. In string theory, fermions manifest as a supersymmetry [2], and the two most common approaches to adding fermions are:

- the Ramond–Neveu–Schwarz (RNS) formalism which is world sheet supersymmetric, and
- the Green–Schwarz (GS) formalism which is spacetime supersymmetric.

In this section, the latter formalism will be presented which leads to related type II superstring theories. The foundation is extending spacetime $\mathcal{X}$ to a superspace $\mathcal{X} \oplus \mathcal{U}^{\oplus \mathcal{N}}$, where $\mathcal{U}$ is informally a vector space over a 1–dimensional Grassman algebra instead of a field (definition 1.1.7), with $\dim \mathcal{U} = 2^{\dim \mathcal{X}/2}$, and $\mathcal{N}$ is the global factor to counting the number of supersymmetric degrees of freedom (supercharges).

A first candidate

The $(p+1)$–dimensional sigma model action

$$S_p = -\frac{1}{2} m_p \langle \sqrt{-\det h} h^{\alpha\beta} \Pi_\alpha \cdot \Pi_\beta \rangle_{T \times \Sigma}, \tag{6.2.1}$$

also models the dynamics of a Dp–brane, where $\forall \mu \in \mathbb{Z}_{\dim \mathcal{X}}$, and $\forall \alpha \in \mathbb{Z}_{\dim \Sigma + 1}$,

$$\Pi^\mu{}_\alpha = \partial_\alpha X^\mu$$

is the conjugate momentum for a bosonic string. Assume $\mathcal{N}$ supersymmetries, thus requiring $\mathcal{N}$ d–dimensional Majorana spinor embeddings, indexed as Θ^{Aa}, $\forall A \in \mathbb{Z}_{\mathcal{N}}$, and $\forall a \in \mathbb{Z}_{\dim \mathcal{U}}$. Type II string theory exhibits $\mathcal{N} = 2$ supersymmetry, so henceforth the supersymmetric component groups of Θ will be henceforth labeled Θ^+ and Θ^-.[5]

Relevant to spinors is the Grassman (anti–commuting) algebra whose ($\dim \mathcal{X}$ total number of) generators Γ satisfy (in matrix representation notation) the (general) Dirac algebra.

$$\{\Gamma^\mu | \Gamma^\nu\} = 2\eta^{\mu\nu} \mathbb{1}, \text{ or } \{\Gamma^\mu | \Gamma^\nu\}_{ab} = \{\Gamma^\mu{}_{ac} | \Gamma^\nu{}_{cb}\} = 2\eta^{\mu\nu} \delta_{ab}.$$

[5] These embeddings have opposite chirality in type IIA string theory and the same chirality in type IIB string theory.

In such a case, $\forall\psi$ a Majorana spinor, $\overline{\psi} = \psi^\dagger \Gamma_0$ and $\forall\phi$ another Majorana spinor and $\forall\mu \in \mathbb{Z}_{\dim \mathcal{X}}$,

$$\overline{\phi}\Gamma^\mu\psi + \overline{\psi}\Gamma^\mu\phi = 0.$$

Of importance in a Dirac algebra with $\dim \mathcal{X}$ generators is the chiral matrix

$$\Gamma^* = \prod\nolimits_{\mu\in\mathbb{Z}_{\dim\mathcal{X}}} \Gamma^\mu,$$

for which, $\forall\mu \in \mathbb{Z}_{\dim \mathcal{X}}$, $\{\Gamma^*|\Gamma^\mu\} = 0$ and $\Gamma^*\Gamma^* = \mathbb{1}$.

Supersymmetry in the sense of transforming the super world hyper volume takes the form (fermionic vector index suppressed) [59]

$$\delta_\epsilon \Theta^\pm = \epsilon^\pm \text{ and } \delta_\epsilon X^\mu = \imath(\overline{\epsilon}^- \Gamma^\mu \Theta^- - \overline{\epsilon}^+ \Gamma^\mu \Theta^+), \tag{6.2.2}$$

where ϵ is the (here $\dim\mathcal{U}$–dimensional) antisymmetric tensor (here matrix) $\forall A \in \mathbb{Z}_{\mathcal{N}=2}$.

The supergroup of transformations generated from Poincaré transformations and supersymmetric transformations (6.2.2)is the super–Poincaré group.

The simplest supersymmetric extension of Π, for which $\delta_{\text{supersymmetry}}\Pi = 0$ is [2, 59],[6]

$$\Pi^\mu{}_\alpha = \partial_\alpha X^\mu - \imath(\overline{\Theta}^- \Gamma^\mu \partial_\alpha \Theta^- - \overline{\Theta}^+ \Gamma^\mu \partial_\alpha \Theta^+).$$

It is prudent at this point to assume the $\mathcal{N} = 2$ supersymmetry that arises from type IIB string theory in particular. In this notation, the two super world hypervolume embeddings (Majorana–Weyl spinors) Θ^+ and Θ^- are of equal chirality and one can set $\Theta = \Theta^+ = \Theta^-$, which reduces the fermionic degrees of freedom by half. This comes down to $\Pi^\mu{}_\alpha = \partial_\alpha X^\mu$, indicating that additional terms may be needed to represent supersymmetry in type IIB superstrings.

Kappa (κ) symmetry

Notably the supersymmetric part $\mathcal{U}$ has only half of its degrees of freedom independent ($\dim\mathcal{U}/2 = 2^{\dim\mathcal{X}/2-1}$) [2].

This is modeled after the κ–symmetry, defined from the transformations [2, 59],

- $\forall\mu \in \mathbb{Z}_{\dim \mathcal{X}}$, $\delta_\kappa X^\mu = \imath(\overline{\Theta}^- \Gamma^\mu \delta_\kappa \Theta^- - \overline{\Theta}^+ \Gamma^\mu \delta_\kappa \Theta^+)$,
- $\delta_\kappa \Theta^\pm = 2\Gamma^\mu \Pi_\mu{}^\alpha \kappa^{\pm\alpha}$,
- $\forall\alpha, \beta \in \mathbb{Z}_{\dim \Sigma+1}$, $\delta_\kappa(\sqrt{-\det h}h^{\alpha\beta}) = -8\imath(\partial_\gamma \overline{\Theta}^- \kappa^{-\beta} P^{+\alpha\gamma} - \partial_\gamma \overline{\Theta}^+ \kappa^{+\beta} P^{+\alpha\gamma})$

where $\forall\alpha, \beta \in \mathbb{Z}_{\dim \Sigma+1}$, $P^{\pm\alpha\beta} = \sqrt{-\det h}h^{\alpha\beta} \pm \varepsilon^{\alpha\beta}$.

As a reminder, ε is the antisymmetric tensor, such that $\forall A \in \mathrm{M}_{\dim V}\mathbb{K}$,

$$\det A = \varepsilon^{\alpha_1\dots\alpha_{\dim V}} A_{1\alpha_1} \dots A_{\dim V \alpha_{\dim V}}.$$

For $\dim V = 2$,

$$\varepsilon = \begin{pmatrix} & +1 \\ -1 & \end{pmatrix} \text{ and } \det A = \varepsilon^{\alpha\beta} A_{0\alpha} A_{1\beta}.$$

Mind that $\varepsilon_{\alpha\beta} = -\varepsilon^{\alpha\beta}$ and $\varepsilon^{\alpha\gamma}\varepsilon_{\gamma\beta} = \delta^\alpha{}_\beta$.

Corrections to the proposed Dp–brane action candidate (6.2.1) result is the full Green–Schwarz action, which consists of the string sigma model equivalent action (that does not have κ symmetry)

$$S_p = -\frac{1}{2} m_p \langle \sqrt{-\det h} h^{\alpha\beta} \Pi_\alpha \cdot \Pi_\beta \rangle_{T\times\Sigma},$$

[6] Indeed, bar boundary terms,

$$\delta_\epsilon(\overline{\Theta}^A \Gamma^\mu \partial_\alpha \Theta^A) = \delta_\epsilon \overline{\Theta}^A \Gamma^\mu \partial_\alpha \Theta^A + \overline{\Theta}^A \Gamma^\mu \partial_\alpha \delta_\epsilon \Theta^A = \overline{\epsilon}^A \Gamma^\mu \partial_\alpha \Theta^A + \overline{\Theta}^A \Gamma^\mu \partial_\alpha \epsilon^A = \partial_\alpha(\overline{\epsilon}^A \Gamma^\mu \Theta^A) = \partial_\alpha \delta_\epsilon X^\mu = \delta_\epsilon \partial_\alpha X^\mu.$$

and a separate part

$$\Delta S_1 = -m_1\epsilon^{\alpha\beta}\langle \imath\partial_\alpha X\cdot(\overline{\Theta}^+\Gamma\partial_\beta\Theta^+ + \overline{\Theta}^-\Gamma\partial_\beta\Theta^-) + \overline{\Theta}^+\Gamma\partial_\alpha\Theta^+\cdot\overline{\Theta}^-\Gamma\partial_\beta\Theta^-\rangle_{T\times\Sigma},$$

which is the missing part to the modification of the bosonic string action together with respecting the extra κ symmetry.

Assuming type IIB strings ($\Theta^+ = \Theta^- = \Theta$), the κ symmetry simplifies to [59]:

- $\forall\mu\in\mathbb{Z}_{\dim\mathcal{X}}$, $\delta_\kappa X^\mu = 0$,
- $\delta_\kappa\Theta = 2\Gamma^\mu\Pi_\mu{}^\alpha\kappa^{\pm\alpha}$,
- $\forall\alpha,\beta\in\mathbb{Z}_{\dim\Sigma+1}$, $\delta_\kappa(\sqrt{-\det h}h^{\alpha\beta}) = 0$,

and the full Green–Schwarz action becomes ($\Pi_\alpha = \partial_\alpha X$)[7]

$$S_{\text{Green–Schwarz}} = S_{\text{boson}} + S_{\text{fermion}} = -\frac{1}{2}m\langle\sqrt{-\det h}h^{\alpha\beta}G_{\alpha\beta} + 4\imath\epsilon^{\alpha\beta}\partial_\alpha X\cdot\overline{\Theta}\Gamma\partial_\beta\Theta\rangle_{T\times\Sigma}, \tag{6.2.3}$$

where $\overline{\Theta}\Gamma\partial_\alpha\Theta\cdot\overline{\Theta}\Gamma\partial_\beta\Theta = 0$ as a Grassman quartic term.

Compactification of extra dimensions

As explained in section §6.1. The bosonic string, superstring theory generally requires $\dim\mathcal{X} = 10$,[8] which is different from the macroscopically observed and expected spacetime dimension of 4. In analytical string theory, the subject is approached by studying the topology of the spacetime background, and looking for topologies that are compactified in the 6 extra dimensions, in the sense that the characteristic scale of said dimensions becomes irrelevant at low enough energy (or large enough length) scales, with the most prominent example that of Calabi–Yao n–folds.

The topic of string theory background topology goes beyond the scope of this work,[9] however, the motivation — compactification of extra dimensions — remains, and is the primary focus.

6.3. Matrix models

The Schild gauge

Integrating out the world–sheet metric by using its equation of motion (6.1.5) (see (6.1.6) as well), the original Nambu–Goto action for the bosonic strings emerges,

$$S_{\text{boson}} = -m\langle\sqrt{-\det G}\rangle_{T\times\Sigma}.$$

For strings in particular ($p = 1$),

$$\begin{aligned}\det G = \varepsilon^{\gamma\delta}G_{0\gamma}G_{1\delta} = 2\varepsilon^{\gamma\delta}(\partial_0 X\cdot\partial_\gamma X)(\partial_1 X\cdot\partial_\delta X) = \\ = \varepsilon^{\gamma\delta}\eta_{\mu\nu}\eta_{\kappa\lambda}\partial_0 X^\mu\partial_\gamma X^\nu\partial_1 X^\kappa\partial_\delta X^\lambda = \eta_{\mu\nu}\eta_{\kappa\lambda}\partial_0 X^\mu\partial_1 X^\kappa\varepsilon^{\gamma\delta}\partial_\gamma X^\nu\partial_\delta X^\lambda.\end{aligned} \tag{6.3.1}$$

Note that

$$\varepsilon^{\gamma\delta}\partial_\gamma X^\nu\partial_\delta X^\lambda = -\varepsilon^{\delta\gamma}\partial_\gamma X^\nu\partial_\delta X^\lambda = -\varepsilon^{\gamma\delta}\partial_\delta X^\nu\partial_\gamma X^\lambda = -\varepsilon^{\gamma\delta}\partial_\gamma X^\lambda\partial_\delta X^\nu,$$

therefore flipping μ and ν with κ and λ,

$$\eta_{\mu\nu}\eta_{\kappa\lambda}\partial_0 X^\mu\partial_1 X^\kappa\varepsilon^{\gamma\delta}\partial_\gamma X^\nu\partial_\delta X^\lambda = \eta_{\kappa\lambda}\eta_{\mu\nu}\partial_0 X^\kappa\partial_1 X^\mu\varepsilon^{\gamma\delta}\partial_\gamma X^\lambda\partial_\delta X^\nu = -\eta_{\mu\nu}\eta_{\kappa\lambda}\partial_1 X^\mu\partial_0 X^\kappa\varepsilon^{\gamma\delta}\partial_\gamma X^\nu\partial_\delta X^\lambda.$$

Splitting the term in (6.3.1) leads to[1, 59],

$$2\det G = \eta_{\mu\nu}\eta_{\kappa\lambda}H^{\mu\kappa}H^{\nu\lambda},\ H^{\mu\nu} = \varepsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu. \tag{6.3.2}$$

[7] Abbreviating the string tension m_1 to m.
[8] M–theory, which is not presented in this text, closely requires $\dim\mathcal{X} = 11$, however the argument about compactification remains the same [2].
[9] See [2] for a good discussion of the matter.

Recall that the Poisson bracket for two scalar functions f and g is given by [4, 59],

$$\sqrt{-\det h}[f|g]_{\text{Poisson}} = \varepsilon^{\alpha\beta}\partial_\alpha f \partial_\beta g, \tag{6.3.3}$$

therefore

$$2\det G = \det h \eta_{\mu\nu}\eta_{\kappa\lambda}[X^\mu|X^\kappa]_{\text{Poisson}}[X^\nu|X^\lambda]_{\text{Poisson}} = \det h \operatorname{tr}([X|X]_{\text{Poisson}} \cdot [X|X]_{\text{Poisson}}),$$

and the bosonic action becomes

$$\sqrt{2}S_{\text{boson}} = -m\langle\sqrt{-\det h}\operatorname{tr}_{\mathcal{X}}([X|X]_{\text{Poisson}} \cdot [X|X]_{\text{Poisson}})\rangle_{T\times\Sigma}.$$

In a similar fashion, the fermionic action becomes

$$S_{\text{fermion}} = -2\imath m\langle\sqrt{-\det h}\overline{\Theta}\Gamma \cdot [X|\Theta]_{\text{Poisson}}\rangle_{T\times\Sigma}.$$

The Schild action, defined on a different world sheet auxiliary metric gauge h_{Schild},

$$\begin{aligned} 4S_{\text{Schild}} &= \langle\sqrt{-\det h_{\text{Schild}}}(a(\operatorname{tr}_{\mathcal{X}}([X|X]_{\text{Poisson}} \cdot [X|X]_{\text{Poisson}}) - 2\imath\overline{\psi}\operatorname{tr}(\Gamma \cdot [X|\psi]_{\text{Poisson}})) + b)\rangle_{T\times\Sigma} \\ &= \langle(\sqrt{-\det h_{\text{Schild}}})^{-1}a\eta_{\mu\nu}\eta_{\kappa\lambda}\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}\partial_\alpha X^\mu\partial_\beta X^\kappa\partial_\gamma X^\nu\partial_\delta X^\lambda - 2\imath\eta_{\mu\nu}\varepsilon^{\alpha\beta}\partial_\alpha X^\mu\overline{\psi}\operatorname{tr}\partial_\beta\Gamma^\nu\psi + \sqrt{-\det h_{\text{Schild}}}b\rangle_{T\times\Sigma}, \end{aligned} \tag{6.3.4}$$

is equivalent to the Green–Schwarz action [1, 59].

Indeed, by integrating the auxiliary metric volume $\sqrt{-\det h_{\text{Schild}}}$,

$$2\sqrt{-\det h_{\text{Schild}}} = \sqrt{-ab^{-1}\eta_{\mu\nu}\eta_{\kappa\lambda}\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}\partial_\alpha X^\mu\partial_\beta X^\kappa\partial_\gamma X^\nu\partial_\delta X^\lambda} \text{ or } \sqrt{-2\det h_{\text{Schild}}} = \sqrt{-ab^{-1}\det G}, \tag{6.3.5}$$

and substituting the Schild world sheet auxiliary metric volume solution (6.3.5) to the Schild action (6.3.4), the Green–Schwarz action (6.2.3) is retrieved for suitable a and b.

The $\mathcal{N} = 2$ supersymmetry of the Green–Schwarz action manifests in the Schild gauge in two forms [1]:

- Homogeneous transformations (redefining ϵ),

$$\delta_\epsilon\psi = -\frac{1}{2}[X^\mu|X^\nu]_{\text{Poisson}}\Gamma^{\mu\nu}\epsilon \text{ and } \delta_\epsilon X = \imath\bar{\epsilon}\Gamma\psi \text{ and } \epsilon = \frac{1}{2}(\epsilon^1 - \epsilon^2),$$

  that vanish for vanishing X.

- Inhomogeneous transformations (ξ being complementary to ϵ),

$$\delta_\xi\psi = \xi \text{ and } \delta_\xi X = 0,\ \xi = \frac{1}{2}(\epsilon^1 + \epsilon^2),$$

  that for vanishing X.

Note that $\forall\mu,\nu \in \mathbb{Z}_{\dim\mathcal{X}}$,

$$\Gamma^{\mu\nu} = \Gamma^{[\mu}\Gamma^{\nu]} = \frac{1}{2}[\Gamma^\mu|\Gamma^\nu].$$

The corresponding bosonic and fermionic parts of the Schild action are (dropping the b term and redefining the coupling constraint $a = g^{-2}$),

$$S = S_{\text{boson}} + S_{\text{fermion}} = \frac{1}{4}g^{-2}\langle\sqrt{-\det h}\operatorname{tr}_{\mathcal{X}}([X|X]_{\text{Poisson}} \cdot [X|X]_{\text{Poisson}})\rangle_{T\times\Sigma} - \frac{1}{2}g^{-2}\langle\sqrt{-\det h}\overline{\psi}\operatorname{tr}_{\mathcal{X}}(\Gamma \cdot [X|\psi]_{\text{Poisson}})\rangle_{T\times\Sigma} \tag{6.3.6}$$

Varying the action of the bosonic model S_{boson} ($S_{\text{fermion}} = 0$ and no supersymmetry),

$$\begin{aligned}
4\sqrt{-\det h}g^2\delta S_{\text{boson}} &= \delta\langle\eta_{\mu\nu}\eta_{\kappa\lambda}\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}\partial_\alpha X^\mu\partial_\beta X^\kappa\partial_\gamma X^\nu\partial_\delta X^\lambda\rangle_{T\times\Sigma}\\
&= \eta_{\mu\nu}\eta_{\kappa\lambda}\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}\langle\delta\partial_\alpha X^\mu\partial_\beta X^\kappa\partial_\gamma X^\nu\partial_\delta X^\lambda\\
&+ \partial_\alpha X^\mu\delta\partial_\beta X^\kappa\partial_\gamma X^\nu\partial_\delta X^\lambda\\
&+ \partial_\alpha X^\mu\partial_\beta X^\kappa\delta\partial_\gamma X^\nu\partial_\delta X^\lambda\\
&+ \partial_\alpha X^\mu\partial_\beta X^\kappa\partial_\gamma X^\nu\delta\partial_\delta X^\lambda\rangle_{T\times\Sigma}\\
&= \eta_{\mu\nu}\eta_{\kappa\lambda}\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}\langle(\delta^\mu{}_\rho\delta_\alpha{}^\epsilon\partial_\beta X^\kappa\partial_\gamma X^\nu\partial_\delta X^\lambda\\
&+ \partial_\alpha X^\mu\delta^\kappa{}_\rho\delta_\beta{}^\epsilon\partial_\gamma X^\nu\partial_\delta X^\lambda\\
&+ \partial_\alpha X^\mu\partial_\beta X^\kappa\delta^\nu{}_\rho\delta_\gamma{}^\epsilon\partial_\delta X^\lambda\\
&+ \partial_\alpha X^\mu\partial_\beta X^\kappa\partial_\gamma X^\nu\delta^\lambda{}_\rho\delta_\delta{}^\epsilon)\delta\partial_\epsilon X^\rho\rangle_{T\times\Sigma}\\
&= \langle(\eta_{\rho\nu}\eta_{\kappa\lambda}\varepsilon^{\epsilon\beta}\varepsilon^{\gamma\delta}\partial_\beta X^\kappa\partial_\gamma X^\nu\partial_\delta X^\lambda\\
&+ \eta_{\mu\nu}\eta_{\rho\lambda}\varepsilon^{\alpha\epsilon}\varepsilon^{\gamma\delta}\partial_\alpha X^\mu\partial_\gamma X^\nu\partial_\delta X^\lambda\\
&+ \eta_{\mu\rho}\eta_{\kappa\lambda}\varepsilon^{\alpha\beta}\varepsilon^{\epsilon\delta}\partial_\alpha X^\mu\partial_\beta X^\kappa\partial_\delta X^\lambda\\
&+ \eta_{\mu\nu}\eta_{\kappa\rho}\varepsilon^{\alpha\beta}\varepsilon^{\gamma\epsilon}\partial_\alpha X^\mu\partial_\beta X^\kappa\partial_\gamma X^\nu)\delta\partial_\epsilon X^\rho\rangle_{T\times\Sigma}\\
&= \langle((\eta_{\rho\nu}\eta_{\kappa\lambda}\varepsilon^{\epsilon\beta}\partial_\beta X^\kappa + \eta_{\mu\nu}\eta_{\rho\lambda}\varepsilon^{\alpha\epsilon}\partial_\alpha X^\mu)\varepsilon^{\gamma\delta}\partial_\gamma X^\nu\partial_\delta X^\lambda\\
&+ \varepsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\kappa(\eta_{\mu\rho}\eta_{\kappa\lambda}\varepsilon^{\epsilon\delta}\partial_\delta X^\lambda + \eta_{\mu\nu}\eta_{\kappa\rho}\varepsilon^{\gamma\epsilon}\partial_\gamma X^\nu))\delta\partial_\epsilon X^\rho\rangle_{T\times\Sigma}\\
&= \langle((\eta_{\rho\nu}\eta_{\kappa\lambda}\varepsilon^{\epsilon\beta}\partial_\beta X^\kappa + \eta_{\mu\nu}\eta_{\rho\lambda}\varepsilon^{\alpha\epsilon}\partial_\alpha X^\mu)H^{\nu\lambda}\\
&+ H^{\mu\kappa}(\eta_{\mu\rho}\eta_{\kappa\lambda}\varepsilon^{\epsilon\delta}\partial_\delta X^\lambda + \eta_{\mu\nu}\eta_{\kappa\rho}\varepsilon^{\gamma\epsilon}\partial_\gamma X^\nu))\delta\partial_\epsilon X^\rho\rangle_{T\times\Sigma}\\
&= \langle(H_{\rho\kappa}\varepsilon^{\epsilon\beta}\partial_\beta X^\kappa + H_{\mu\rho}\varepsilon^{\alpha\epsilon}\partial_\alpha X^\mu\\
&+ H_{\rho\lambda}\varepsilon^{\epsilon\delta}\partial_\delta X^\lambda + H_{\nu\rho}\varepsilon^{\gamma\epsilon}\partial_\gamma X^\nu)\delta\partial_\epsilon X^\rho\rangle_{T\times\Sigma}\\
&= 4\langle H_{\rho\mu}\varepsilon^{\epsilon\alpha}\partial_\alpha X^\mu\delta\partial_\epsilon X^\rho\rangle_{T\times\Sigma},
\end{aligned}$$

where the matrix H is the Poisson bracket defined in (6.3.2). Integration by parts and discarding total derivative integrants (turning to vanishing boundary terms)

$$\sqrt{-\det h}\delta S_{\text{boson}} = g^{-2}\langle(\partial_\epsilon H_{\rho\mu}\varepsilon^{\epsilon\alpha}\partial_\alpha X^\mu + H_{\rho\mu}\varepsilon^{\epsilon\alpha}\partial_\epsilon\partial_\alpha X^\mu)\delta X^\rho\rangle_{T\times\Sigma}.$$

Taking into account that $\varepsilon^{\epsilon\alpha}\partial_\epsilon\partial_\alpha = 0$,

$$\sqrt{-\det h}\delta S_{\text{boson}} = -g^{-2}\langle\varepsilon^{\epsilon\alpha}\partial_\epsilon H_{\rho\mu}\partial_\alpha X^\mu\delta X^\rho\rangle_{T\times\Sigma} = -g^{-2}\langle[H_{\rho\mu}|X^\mu]_{\text{Poisson}}\delta X^\rho\rangle_{T\times\Sigma} = -g^{-2}\langle[H_{\rho\mu}|X^\mu]_{\text{Poisson}}\delta X^\rho\rangle_{T\times\Sigma}$$

or

$$\delta S_{\text{boson}} = -g^{-2}\langle[[X_\rho|X_\mu]_{\text{Poisson}}|X^\mu]_{\text{Poisson}}\delta X^\rho\rangle_{T\times\Sigma} = 0,$$

leading to the bosonic equation of motion (free indices suppressed),

$$\eta_{\mu\nu}[X^\mu|[X^\nu|X]_{\text{Poisson}}]_{\text{Poisson}} = 0.$$

The Lorentzian IKKT matrix model

The Green–Schwarz action (6.3.4) in the Schild gauge constitutes a hint on a model that can describe strings non–perturbatively. In 1996, Ishibashi, Kawai, Kitazawa and Tsuchiya proposed a matrix model (IKKT model) as a regularization of type IIB string theory as described by the Green–Schwarz action (6.3.4) in the Schild gauge [1].

The precise correspondence between the models is [1, 4, 59],

Type IIB string theory		IKKT N–size matrix model
∂	$\longrightarrow$	0

$$X \in \mathcal{X}^{T\times\Sigma} \longrightarrow A \in \mathrm{M}_N\mathbb{C}^{\dim\mathcal{X}} \text{ traceless hermitian}$$
$$\psi \in \mathcal{U}^{T\times\Sigma} \longrightarrow \psi \in \mathrm{M}_N\mathbb{C}^{\dim\mathcal{U}} \text{ traceless hermitian}$$
$$g^{-2} \longrightarrow N$$
$$\imath[\cdot|\cdot]_{\text{poisson}} \longrightarrow [\cdot|\cdot]$$
$$\langle\sqrt{-\det h}(\cdot)\rangle_{T\times\Sigma} \longrightarrow \mathrm{tr}_{T\times\Sigma}(\cdot)$$

Recall that in type IIB string theory with κ–symmetry, eliminating half the fermionic degrees of freedom gives

$$\dim\mathcal{U} = 2^{\dim\mathcal{X}/2-1}.$$

The key points in this correspondence are:

- The dimensional reduction of the background to a point (effectively eliminating $\mathcal{X}$–derivatives).
- The discretization of the world sheet objects using hermitian matrices.

Applying these modifications to the modified Schild action (6.3.6),

$$S_{\text{boson}} = \frac{1}{4}g^{-2}\langle\sqrt{-\det h}\,\mathrm{tr}([X|X]_{\text{Poisson}}\cdot[X|X]_{\text{Poisson}})\rangle_{T\times\Sigma} - \frac{1}{2}g^{-2}\langle\sqrt{-\det h}\overline{\psi}\,\mathrm{tr}(\Gamma\cdot[X|\psi]_{\text{Poisson}})\rangle_{T\times\Sigma},$$

the (Lorentzian) IKKT action becomes

$$\begin{aligned} S &= -N\,\mathrm{tr}_{T\times\Sigma}\left(\frac{1}{4}\,\mathrm{tr}_{\mathcal{X}}([A|A]\cdot[A|A]) + \frac{1}{2}\overline{\psi}\Gamma\cdot[A|\psi]\right) \\ &= -N\,\mathrm{tr}_{T\times\Sigma}\left(\frac{1}{4}\eta_{\mu\nu}\eta_{\kappa\lambda}[A^\mu|A^\kappa][A^\nu|A^\lambda] + \frac{1}{2}\eta_{\mu\nu}\overline{\psi}_\alpha\Gamma^\nu{}_{\alpha\beta}[A^\mu|\psi_\beta]\right) \end{aligned} \tag{6.3.7}$$

where the A bosons are 10 $N\times N$ hermitian and traceless matrices, and Γ are the 10 Majorana–Weyl representation 16×16 matrices. The fermion vector indices are exposed for clarity.

$T\times\Sigma$ hermiticity

The worldsheet embedding X is discretized by hermitian matrices by design of the IKKT matrix model so that the spacetime generated is real in lieu of the real eigenvalues of the bosonic matrices A.

Relation to super Yang–Mills theories

The Lorentzian IKKT action at zero spacetime volume resembles the zero–volume $\mathcal{N}=1$ super Yang–Mills action (5.2.5), where there is a direct correspondence between the $N\times N$ matrix structure of the $\mathcal{N}=2$ type IIB superstring theory and the $\mathcal{N}=1$ super Yang Mills internal SU_N gauge structure.[10]

$\mathcal{X}$ dimensional reduction

The IKKT matrix model encodes the world sheet geometry in a new fuzzy object (the bosonic matrices A), being in line with Connes' approach to geometry via operators (matrices) [4, 70]. But there is more to the correspondence between the IKKT bosonic matrices A and the worldsheet geometry X in type IIB superstring theory [5, 71].

[10] As briefly presented in section §5.2. Quantum Field Theory.

Symmetries of the IKKT model

With the type IIB / IKKT correspondence introduced in [1], $\mathcal{N} = 2$ the supersymmetry transformations of the Schild action become in the IKKT model (in full index notation)

$$\delta_1 \psi_\alpha = \frac{1}{2} \imath [A^\mu | A^\nu] \Gamma^{\mu\nu}{}_{\alpha\beta} \epsilon_\beta \text{ and } \delta_1 A^\mu = \imath \bar{\epsilon}_\alpha \Gamma^\mu{}_{\alpha\beta} \psi_\beta,$$

and

$$\delta_2 \psi_\alpha = \xi_\alpha \text{ and } \delta_2 A^\mu = 0^\mu,$$

respectively.

Finally the gauge symmetry of the action is $\mathrm{SU}_N\mathbb{C}$ for both A and ψ. The infinitesimal transformation $\forall \imath U \in \mathfrak{su}_N\mathbb{C}$ (hermitian) is

$$\delta_{\text{gauge}} \psi_\alpha = [\imath U | \psi_\alpha] \text{ and } \delta_{\text{gauge}} A^\mu = [\imath U | A^\mu]. \tag{6.3.8}$$

Finally, the model is translationally–symmetric,

$$\delta_{\text{translation}} \psi_\alpha = 0_\alpha \text{ and } \delta_{\text{translation}} A^\mu = \alpha^\mu \mathbb{1}.$$

Let Q_1 and Q_2 be the corresponding supersymmetric generators, and P the translational generators, with

$$\overline{Q}_1 = Q_1 + Q_2 = \text{ and } \overline{Q}_2 = \imath(Q_1 - Q_2).$$

By the equation of motion of ψ,

$$\Gamma^\mu{}_{\alpha\beta} [A_\mu | \psi_\beta] = 0_\alpha,$$

and up to gauge symmetry (6.3.8), it was shown in [1] and reviewed in [71] that,

$$[\bar{\epsilon}_\alpha \overline{Q}_i | \bar{\xi}_\alpha \overline{Q}_j] = -2\delta_{ij} \bar{\epsilon}_\alpha \Gamma^\mu{}_{\alpha\beta} \xi_\beta P_\mu, \tag{6.3.9}$$

which stands for the full (on–shell) supersymmetry $\mathcal{N} = 2$ algebra of the IKKT matrix model, further hinting at the interpretation of the eigenvalues of A a spacetime. From the fact that $\mathcal{N} = 2$ supersymmetry is maximal in $\dim \mathcal{X} = 10$ dimensions, any theory with supersymmetry (6.3.9) must include gravity, provided it is unitary and has a massless spectrum [5, 71].

The Euclidean IKKT matrix model

The Wick rotation of the Lorentzian IKKT action (6.3.7) involves the following modifications:[11]

Lorentzian		Euclidean
A_0	$\longrightarrow$	$\imath A_{10}$
Γ_0	$\longrightarrow$	$\imath \Gamma_{10}$
η	$\longrightarrow$ metric signature change $\longrightarrow$	$\mathbb{1}$
index in $\mathbb{Z}_{\dim \mathcal{X}}$	$\longrightarrow$ index $0 \to \dim \mathcal{X}$ $\longrightarrow$	index in $\mathbb{N}_{\dim \mathcal{X}}$

For the Pauli matrices

$$\sigma_0 = \mathbb{1},\ \sigma_1 = \begin{pmatrix} & +1 \\ +1 & \end{pmatrix},\ \sigma_2 = \imath \begin{pmatrix} & +1 \\ -1 & \end{pmatrix} \text{ and } \sigma_3 = \begin{pmatrix} +1 & \\ & -1 \end{pmatrix},$$

[11] Mind the use of lower indices for the Lorentzian analogs.

the representation for the gamma matrices Γ of the Euclidean model is chosen as (Γ_0 shown for completeness)

$$\begin{aligned}
\Gamma_0 &= \imath\sigma_0\otimes\sigma_0\otimes\sigma_0\otimes\sigma_0. \quad \Gamma_1 = \imath\sigma_2\otimes\sigma_2\otimes\sigma_2\otimes\sigma_2,\\
\Gamma_2 &= \imath\sigma_2\otimes\sigma_2\otimes\sigma_0\otimes\sigma_1, \quad \Gamma_3 = \imath\sigma_2\otimes\sigma_2\otimes\sigma_0\otimes\sigma_3,\\
\Gamma_4 &= \imath\sigma_2\otimes\sigma_1\otimes\sigma_2\otimes\sigma_0, \quad \Gamma_5 = \imath\sigma_2\otimes\sigma_3\otimes\sigma_2\otimes\sigma_0,\\
\Gamma_6 &= \imath\sigma_2\otimes\sigma_0\otimes\sigma_1\otimes\sigma_2, \quad \Gamma_7 = \imath\sigma_2\otimes\sigma_0\otimes\sigma_3\otimes\sigma_2,\\
\Gamma_8 &= \imath\sigma_1\otimes\sigma_0\otimes\sigma_0\otimes\sigma_0, \quad \Gamma_9 = \imath\sigma_3\otimes\sigma_0\otimes\sigma_0\otimes\sigma_0,\\
&\Gamma_{10} = \sigma_0\otimes\sigma_0\otimes\sigma_0\otimes\sigma_0.
\end{aligned}$$

The Euclidean IKKT action then becomes (all lower indices now)

$$\begin{aligned}
S &= S_{\text{boson}} + S_{\text{fermion}}\\
&= -N\delta_{\mu\nu}\,\mathrm{tr}_{T\times\Sigma}\left(\frac{1}{4}\delta_{\kappa\lambda}[A_\mu|A_\kappa][A_\nu|A_\lambda] - \frac{1}{2}\overline{\psi}_\alpha\Gamma_{\mu\alpha\beta}[A_\nu|\psi_\beta]\right) = -N\,\mathrm{tr}_{T\times\Sigma}\left(\frac{1}{4}[A_\mu|A_\nu][A_\mu|A_\nu] - \frac{1}{2}\overline{\psi}_\alpha\mathcal{M}_{\alpha\beta}\psi_\beta\right),
\end{aligned} \tag{6.3.10}$$

where the linear antisymmetric fermion operator $\mathcal{M}$ is defined $\forall\psi$ by

$$(\mathcal{M}\psi)_\alpha = \Gamma_{\mu\alpha\beta}[A_\mu|\psi_\beta]. \tag{6.3.11}$$

$\mathcal{M}$ is an $N^2\dim\mathcal{U}\times N^2\dim\mathcal{U}$ matrix, traceless on each of its $N\times N$ submatrices indexed by its non–spinor indices,

$$\mathcal{M}_{\alpha aa'\beta bb'} = \overline{\mathcal{M}}_{\alpha aa'\beta bb'} - \delta_{aa'}\overline{\mathcal{M}}_{\alpha NN\beta bb'} - \delta_{bb'}\overline{\mathcal{M}}_{\alpha aa'\beta NN} + \overline{\mathcal{M}}_{\alpha NN\beta NN}\delta_{aa'}\delta_{bb'},$$

where

$$\overline{\mathcal{M}}_{\alpha aa'\beta bb'} = \Gamma_{\mu\alpha\beta}(A_{\mu a'b}\delta_{ab'} - A_{\mu b'a}\delta_{ba'}). \tag{6.3.12}$$

Integrating fermions out of the corresponding Euclidean partition function,

$$Z = \int \mathcal{D}A\mathcal{D}\overline{\psi}\mathcal{D}\psi\exp(-S) = \int \mathcal{D}A\exp(-S_{\text{boson}})\,\mathrm{pf}\,\mathcal{M} = \int \mathcal{D}A\exp(-S_{\text{effective}}),$$

where for an antisymmetric matrix like $\mathcal{M}$, $\mathrm{pf}_{(\mathcal{U}\times T\times\Sigma)^2}\,\mathcal{M} = \sqrt{\det_{(\mathcal{U}\times T\times\Sigma)^2}\,\mathcal{M}}$, and

$$S_{\text{effective}} = S_{\text{boson}} - \log\mathrm{pf}_{(\mathcal{U}\times T\times\Sigma)^2}\,\mathcal{M} = S_0 - \imath\arg\mathrm{pf}_{(\mathcal{U}\times T\times\Sigma)^2}\,\mathcal{M},$$

with $S_0 = \Re S_{\text{effective}} = S_{\text{boson}} - \log|\,\mathrm{pf}_{\mathcal{U}\times T\times\Sigma}\,\mathcal{M}|$.

Dynamical compactification of extra dimensions

The advantage of studying type IIB string theory is that it admits regularizations (prominently the IKKT matrix model) that can in turn be studied non–perturbatively via finite–size–approximation simulations. One powerful aspect of the IKKT matrix model in the zero–volume limit is the dynamic emergence of spacetime, and as such it is expected that the compactification of the extra dimensions will also occur dynamically.

The bosonic matrix *A*

Each of the bosonic matrices in A stands for each of the 10 spacetime coordinates, so each array of eigenvalues shall represent an event in such spacetime. However, as is evident from the IKKT action, the induced geometry is non–commutative (a fuzzy one), which means that said bosonic matrices cannot be concurrently diagonalized to yield a clear cut event in spacetime. In fact, non–commutativity translates to an uncertainty in the coordinates of an event, hence called a fuzzy event.

$\forall A\in\mathrm{M}_N\mathbb{C}$, and $\forall g\in\mathrm{GL}_N\mathbb{C}$,

$$\mathrm{tr}_{T\times\Sigma}\,A = \mathrm{tr}_{T\times\Sigma}\,\mathrm{ad}_g\,A = \mathrm{tr}_{T\times\Sigma}(gAg^{-1}).$$

However $A \in \mathfrak{su}_N\mathbb{C}$ as hermitian and traceless, thus the requirement that $\mathrm{ad}_g A$ remains hermitian,

$$(\mathrm{ad}_g A)^\dagger = (gAg^{-1})^\dagger = (g^{-1})^\dagger Ag^\dagger = gAg^{-1} = \mathrm{ad}_g A,$$

requires that $\forall g \in \mathrm{GL}_N\mathbb{C}$, $g^{-1} = g^\dagger$, therefore $g \in \mathrm{U}_N\mathbb{C}$, which defines the internal matrix symmetry for the bosonic field A. Recall that $\dim \mathrm{U}_N\mathbb{C} = \dim \mathfrak{u}_N\mathbb{C} = N^2$ and for a base ℓ in $\mathfrak{u}_N\mathbb{C}$, and $\forall g \in \mathrm{U}_N\mathbb{C}$,

$$g = \exp \imath g_a \ell_a.$$

The spacetime symmetry

A Euclidean spacetime is manifestly rotationally symmetric, hence a solid indicator that the 6 extra dimensions in type IIB superstring theory — and by extension in the IKKT matrix model — are compactified, is the spontaneous rotational symmetry breaking $\mathrm{SO}_{\dim \mathcal{X}} \to \mathrm{SO}_D$, with $D < \dim \mathcal{X}$.[12]

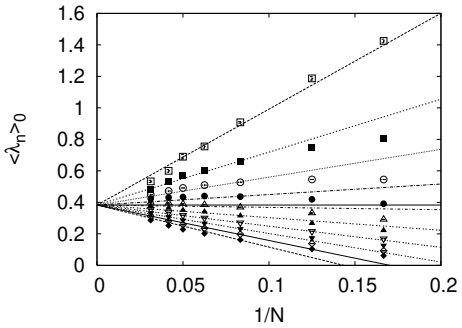


Figure 6.3.1.: The expectations $\langle\lambda\rangle_0$ with respect to the partition function of phase–quenched model S_0, for various matrix sizes N and their extrapolation to $N \to \infty$, at which point all eigenvalue expectations converge to a single value [13].

The eigenvalues of the (symmetric) moment of inertia matrix,

$$\Lambda_{\mu\nu} = N^{-1} \operatorname{tr}_{T\times\Sigma}(A_\mu A_\nu), \tag{6.3.13}$$

provide a length scale for each of the $\dim \mathcal{X}$ extends of spacetime. It is expected that for an isotropic $\dim \mathcal{X}$–dimensional spacetime all $\dim \mathcal{X}$ eigenvalues are identical. Note however that an anisotropy in specific spacetime directions has no preference in direction become different, therefore when averaging said eigenvalues, in a Monte Carlo simulation for instance, the broken symmetry may disappear as the differences among the eigenvalues cancel out when averaged with random orderings. To circumvent this, the ordered vector of eigenvalues is measured in expectation instead, so that differences have an added effect, if any or at all. This is possible for hermitian A, because in such a case Λ is hermitian too (and positive definite at that), thus having real positive eigenvalues.

In the case of generally complex Λ,[13]ordering has to be done by criteria matching that of the expected physical measurement. This most of the times means ordering eigenvalues of Λ by their real part, but using their modulus is not uncommon.[14]

Isotropy in the phase–quenched Euclidean IKKT model

As described in chapter 5. Field theory, a complex action is a source of problems when studying the corresponding system via Monte Carlo methods, that rely on the action to define sampling probabilities. One approach is to use the phase–quenched model, named as such because the imaginary part of the action produces an imaginary phase in the partition integral Z.

It has been shown in [13] that the phase–quenched model has no symmetry breaking, i.e. after reducing the breaking order parameters added to the model initially, the full SO_{10} symmetry is restored, at finite scale $\ell^2 \approx 0.4$, which is consistent to the analytic result $\ell^2 = 0.383$ of the IKKT model studies wither the Gaussian Expansion Method [16, 17], as shown in figure 6.3.1 [13].

Therefore the imaginary part of the action may be responsible for the spontaneous anisotropy by which 4 large dimensions out of 10 shall emerge. The first milestone of this research is to explore the full Euclidean IKKT model, in search of such anisotropy, as a numerical indication that the IKKT model is indeed a good candidate for type IIB superstring theory.

[12] See section §1.3. Differential Algebra for details on symmetries as matrix Lie groups.
[13] For example in the complexification of the Langevin process, see section §4.2. Complexification for a generic treatment of complexification.
[14] In the context of the complex Langevin method, the real part is preferred, because it is expected that the imaginary part eventually vanishes at thermalization of the process.

Euclidean IKKT variants

The Euclidean IKKT model may be defined for $\dim \mathcal{X} < 10$, which reduce to simpler toy models, which are a priori expected to exhibit a similar spontaneous rotational symmetry breaking. The ones with a convergent partition functional are the 4–dimensional [19] and the 6–dimensional [14] toy models. For these two models it is $\det_{T\times\Sigma} \mathcal{M}$ that appears after integrating the fermions out instead of $\mathrm{pf}_{T\times\Sigma} \mathcal{M}$.

The fermion dimension for even spacetime dimension $\dim \mathcal{X}$ is $2^{\dim \mathcal{X}/2-1}$. As shown in [7, 19, 21], the $\dim \mathcal{X} = 4$ model exhibits no spontaneous rotational symmetry breaking, leaving the $\dim \mathcal{X} = 6$ toy model as the only viable candidate for testing the dynamical compactification hypothesis.

For $\dim \mathcal{X} = 4$, the corresponding gamma matrices are $\Gamma_\mu = \imath\sigma_\mu$, $\forall \mu \in \{0, 1, 2, 3\}$, where $\Gamma_4 = -\imath\Gamma_0$, or

$$\begin{aligned} \Gamma_0 = \imath\sigma_0, \quad \Gamma_1 = \imath\sigma_1, \\ \Gamma_2 = \imath\sigma_2, \quad \Gamma_3 = \imath\sigma_3, \\ \Gamma_4 = \sigma_0. \end{aligned}$$

The corresponding fermion determinant $\det_{T\times\Sigma} \mathcal{M}$ is real for this model.

For $\dim \mathcal{X} = 6$, the corresponding gamma matrices are

$$\begin{aligned} \Gamma_0 = \imath\sigma_0 \otimes \sigma_0, \quad \Gamma_1 = \imath\sigma_1 \otimes \sigma_2, \\ \Gamma_2 = \imath\sigma_2 \otimes \sigma_2, \quad \Gamma_3 = \imath\sigma_3 \otimes \sigma_2, \\ \Gamma_4 = \imath\sigma_0 \otimes \sigma_1, \quad \Gamma_4 = \imath\sigma_0 \otimes \sigma_3, \\ \Gamma_6 = \sigma_0 \otimes \sigma_0. \end{aligned}$$

The corresponding fermion determinant $\det_{T\times\Sigma} \mathcal{M}$ is complex for this model.

7. Methodology

7.1. The Gaussian Expansion Method (GEM)

Concept

Hereon the Euclidean IKKT model will be used for reference as is developed in [22, 23]. The full Euclidean IKKT action (6.3.10), written in $\dim \mathcal{X}$–agnostic but N–dependent form,[1] is

$$S = S_{\text{boson}} + S_{\text{fermion}},$$

with

$$S_{\text{boson}} = -\frac{1}{4} N \operatorname{tr}_{T\times\Sigma}[A_\mu|A_\nu][A_\mu|A_\nu], \tag{7.1.1}$$

and

$$S_{\text{fermion}} = -\frac{1}{2} N \operatorname{tr}_{T\times\Sigma} \overline{\psi}_\alpha \mathcal{M}_{\alpha\beta}\psi_\beta = \frac{1}{2} N \operatorname{tr}_{T\times\Sigma} \overline{\psi}_\alpha \Gamma_{\mu\alpha\beta}[A_\mu|\psi_\beta]. \tag{7.1.2}$$

with a moment of inertia tensor (6.3.13)

$$\Lambda_{\mu\nu} = N^{-1} \operatorname{tr}(A_\mu A_\nu), \tag{7.1.3}$$

generating the ordered eigenvalues as order parameters for the spacetime extends.

In terms of bosonic degrees of freedom, the theory has manifest $\mathrm{SO}_{\dim \mathcal{X}}$ symmetry, while in terms of internal string (here finite size $N \times N$ matrix) degrees of freedom, SU_N is manifest.

A perturbative analysis of a (quantum) field theory[2] relies on the action of the theory having a known or analytically solvable part, which is usually quadratic in terms of the fields, generating a Gaussian term in the partition function which is analytically integrable. Mass terms in the action of most common quantum field theories are such.

The particular (Euclidean IKKT) model does not contain such a term on either S_{boson} or S_{fermion}. The main idea of the Gaussian Expansion Method (GEM) is to introduce such a Gaussian term S_0 in the IKKT action,

$$S = S_{\text{GEM}} - S_0 \text{ or } S_{\text{GEM}} = S + S_0,$$

treating S_{GEM} perturbatively and $-S$ as the one–loop counter–term.[3] The requirement that S is Gaussian leaves a lot of parametric freedom on which results of the corresponding theory are expected to depend. The generic ansatz of GEM (effectively) is that such a dependence has plateaus, which are interpreted as effective parametric independence [22], as results are expected to be independent from the parameters defining the artificial Gaussian part.[4]

Gaussian action

Bosonic

The most general bosonic Gaussian action term in the context of an N–size A–bosonic matrix model that respects both the matrix internal SU_N and the bosonic rotational $\mathrm{SO}_{\dim \mathcal{X}}$ symmetries is

$$S_0|_{\text{boson}} = \frac{1}{2} N^2 \operatorname{tr}_{\mathcal{X}}(\mathbb{1} \cdot \Lambda) = \frac{1}{2} N^2 \delta_{\mu\nu} \Lambda_{\mu\nu}.$$

[1] Recall that possible values for $\dim \mathcal{X} \leq 10$ with a convergent IKKT partition function are 4, 6 and 10.
[2] Not presented in this text, see [69] for details on perturbative analysis of quantum field theories
[3] For details on renormalization in quantum field theories, see [69].
[4] See chapter 8. The Euclidean IKKT matrix model for specifics on each model explored.

As explained earlier in section §6.3. Matrix models, and in context with the fact that $\dim \mathcal{X} = 10$ in type IIB superstring theory, the necessity to compactify the extra 6 dimensions to yield a 4–dimensional (euclidean in this case) spacetime as part of superstring phenomenology, requires to manifestly break $\mathrm{SO}_{\dim \mathcal{X}}$ in this model, adding parametric freedom to the bosonic Gaussian part (by replacing $\mathbb{1} \to m_{\text{boson}}$),

$$S_0|_{\text{boson}} = \frac{1}{2} N^2 \operatorname{tr}_{\mathcal{X}}(m_{\text{boson}} \cdot \Lambda) = \frac{1}{2} N^2 m_{\text{boson}}|_{\mu\nu} \Lambda_{\mu\nu} = \frac{1}{2} N m_{\text{boson}}|_{\mu\nu} \operatorname{tr}_{T\times\Sigma}(A_\mu A_\nu),$$

where m_{boson} is a (bosonic mass) parameter matrix imposing anisotropy in the bosonic degrees of freedom. Assuming the moment of inertia matrix λ is always diagonalizable as a real and symmetric matrix,[5] the mass parameter matrix can be reduced to a mass vector corresponding to the moment of inertia eigenvalues hereon labeled with a single (bosonic) index, yielding a bosonic Gaussian part

$$S_0|_{\text{boson}} = \frac{1}{2} N^2 m_{\text{boson}}|_\mu \lambda_\mu. \tag{7.1.4}$$

To simplify, the following replacement can be made

$$\lambda_\mu = N^{-1} \operatorname{tr}_{T\times\Sigma} A_\mu^2, \tag{7.1.5}$$

effectively using a manifestly diagonal bosonic Gaussian action term independent of bosonic representation, without loss of generality, as it is also Gaussian and has the same parametric freedom as (7.1.4).

Fermionic

The fermionic Gaussian part assumes a similar form,

$$S_0|_{\text{fermion}} = N m_{\text{fermion}}|_{\alpha\beta} \operatorname{tr}_{T\times\Sigma}(\overline{\psi}_\alpha \psi_\beta).$$

Note how m_{fermion} only corresponds to $\dim \mathcal{U}$ degrees of freedom, in contrast to $\mathcal{M}$ corresponding to $\dim \mathcal{U} \times N \times N$ degrees of freedom. As expected m_{fermion} shall depend on Γ matrices, though it depends largely on $\dim \mathcal{X}$ and how it affects $\dim \mathcal{U}$ and fermion representations.

Observables and the free energy

The path integral Z can be renormalized in terms of the Gaussian path integral Z_0 as

$$Z = Z_0 \langle \exp(-(S - S_0)) \rangle_0,\ Z_0 = \langle \exp(-S_0) \rangle_0,$$

in a logic similar to that of reweighting, effectively treating $\exp(-S)$ as an observable (functional).[6]

The Boltzmann factor expands as

$$\langle \exp(-(S - S_0)) \rangle_0 = \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n \langle (S - S_0)^n \rangle = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} (-1)^n \langle (S - S_0)^n \rangle$$

with

$$\langle (S - S_0)^n \rangle = \langle (S_{\text{boson}} - S_0 + S_{\text{fermion}})^n \rangle = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} \langle (S_{\text{boson}} - S_0)^{n-k} S_{\text{fermion}}^k \rangle_0$$

In the free energy ($\log 1 = 0$),

$$-F = \log Z = \log Z_0 + \log \left(\sum_{n=1}^{\infty} (-1)^n \sum_{k=0}^{n} \frac{1}{(n-k)!k!} \langle (S_{\text{boson}} - S_0)^{n-k} S_{\text{fermion}}^k \rangle_0 \right) \tag{7.1.6}$$

[5] It is real because $\forall \mu \in \mathbb{N}_{\dim \mathcal{X}}$, A_μ is hermitian thus $\operatorname{tr}_{T\times\Sigma} A_\mu \in \mathbb{R}$ (not to mention traceless, thus $\operatorname{tr}_{T\times\Sigma} A_\mu = 0$), and thus $\forall \nu \in \mathbb{N}_{\dim \mathcal{X}}$, $\operatorname{tr}_{T\times\Sigma}(A_\mu A_\nu) \in \mathbb{R}$ also. $\forall A, B \in \mathbb{M}_N\mathbb{C}$ hermitian matrices,

$$\operatorname{tr}(AB) = A_{ij} B_{ji} = A_{ij} B_{ij}^* = (A_{ij}^* B_{ij})^* = (A_{ji} B_{ij})^* = \operatorname{tr}(AB)^*.$$

[6] See section §4.4. The complex action problem on reweighting.

the effect of the logarithm is to reduce the sum expansion of all correlators $\langle\cdot\rangle_0$ (higher order expectations) to connected correlators $\langle\cdot\rangle_0|_{\text{connected}}$, corresponding to connected Feynman diagrams [22, 23], leading to an expression of the form

$$-F = \log Z_0 + \sum_{n=1}^{\infty}\sum_{k=0}^{n} C_k \langle (S_{\text{boson}} - S_0)^{n-k} S_{\text{fermion}}^{2k}\rangle_0|_{\text{connected}},\ C_k \propto \frac{1}{(n+k)!}(-1)^{n-k}.$$

Observables assume a similar expansion,

$$O = \langle O\rangle_0 + \sum_{n=1}^{\infty}\sum_{k=0}^{n} C_k \langle O(S_{\text{boson}} - S_0)^{n-k} S_{\text{fermion}}^{2k}\rangle_0|_{\text{connected}}.$$

Any finite order $n \in \mathbb{N}_*$ truncation of (7.1.6), is dependent on S_0 and thus is a parametric calculation of m and $\mathcal{A}$. By the generic GEM ansatz, stationary points satisfying

$$\frac{\partial}{\partial m_{\text{boson}}|_\mu} F = 0,\ \forall \mu \in \mathbb{N}_{\dim \mathcal{X}} \text{ and } \frac{\partial}{\partial m_{\text{fermion}}|_{\alpha\beta}} F = 0,\ \forall \alpha, \beta \in \mathbb{Z}_{\dim \mathcal{U}},$$

that are also forming qualitatively dense regions in the combined m_{boson} and m_{fermion} parameter space is sought.

GEM broken symmetry ansatz

The dynamical compactification of extra dimensions as a broken rotational symmetry of the bosonic matrices A, $\mathrm{SO}_{\dim \mathcal{X}} \to \mathrm{SO}_D$, $D < \dim \mathcal{X}$, can be simulated in GEM context by proper parametric adjustment of S_0. The most trivial example is setting

$$m_{\text{boson}}|_\mu = m\ \forall \mu \in \mathbb{N}_{\dim \mathcal{X}} \text{ and } m_{\text{fermion}} = 0,$$

leaving $\mathrm{SO}_{\dim \mathcal{X}}$ unbroken.

In general, reducing $\mathrm{SO}_{\dim \mathcal{X}}$ may leave residual symmetries in the remnant directions, usually in the form of either

$$\mathrm{SO}_D \times \mathrm{SO}_C \text{ with } C < \dim \mathcal{X} - D \text{ or } \mathrm{SO}_D \times \mathbb{Z}_C \text{ with } C \leq \dim \mathcal{X} - D, \text{ and } D < \dim \mathcal{X},$$

where SO_C corresponds to reminiscent rotational symmetry on the compactified dimensions, and $\mathbb{Z}_C$ corresponds to exchange symmetry on said dimensions. These specific symmetry partitions have been studied extensively in [22, 23].

It is worth noting that in all models, the parametric freedom of S_0 is restricted by descending ordering of the bosonic masses $m_{\text{boson}}|_\mu$.

A free energy corresponds to each ansatz, and the one (in a parametric plateau) showing the smallest free energy can be the true vacuum, assuming the ansatz search space is exhaustive.chapter 8. The Euclidean IKKT matrix modelcontains specific results on the vacuum with the lowest free energy of the models under study.

7.2. Complex Langevin Method (CLM)

The main conjecture for the Euclidean IKKT model is that the imaginary part of the action due to (integrating out) the fermions is responsible for the spontaneous rotational symmetry breaking standing for the compactification of the extra dimensions in string theory; specifically $\mathrm{SO}_{10} \to \mathrm{SO}_4$ for the original type IIB string theory model, standing for the emergence of the macroscopically observed 4–dimensional spacetime.

In this section the application of the complex Langevin stochastic calculus is applied to type IIB string theory in the context of its IKKT finite matrix regularization.

The Euclidean IKKT effective action reads

$$S_{\text{effective}} = S_{\text{boson}} - \log \mathrm{pf}_{(\mathcal{U}\times T\times \Sigma)^2} \mathcal{M} \text{ with } S_{\text{boson}} = -\frac{1}{4} N \,\mathrm{tr}_{T\times\Sigma}([A_\mu|A_\nu][A_\mu|A_\nu]), \tag{7.2.1}$$

The bosonic part, becomes

$$\begin{aligned}S_{\text{boson}} &= -\frac{1}{4}N\,\mathrm{tr}_{T\times\Sigma}([A_\mu|A_\nu][A_\mu|A_\nu]) = -\frac{1}{4}N\,\mathrm{tr}_{T\times\Sigma}((A_\mu A_\nu - A_\nu A_\mu)(A_\mu A_\nu - A_\nu A_\mu))\\ &= -\frac{1}{4}N\,\mathrm{tr}_{T\times\Sigma}(A_\nu A_\mu A_\mu A_\nu - A_\nu A_\mu A_\nu A_\mu - A_\mu A_\nu A_\mu A_\nu + A_\mu A_\nu A_\nu A_\mu)\\ &= -\frac{1}{2}N\,\mathrm{tr}_{T\times\Sigma}(A_\mu A_\mu A_\nu A_\nu - A_\mu A_\nu A_\mu A_\nu) = -\frac{1}{2}N\,\mathrm{tr}_{T\times\Sigma}(A_\mu[A_\mu|A_\nu]A_\nu),\end{aligned}$$

whose (complex) derivative is,[7]

$$\begin{aligned}\frac{\partial}{\partial A_o^\top}S_{\text{boson}} &= -\frac{1}{2}N\,\mathrm{tr}_{T\times\Sigma}\frac{\partial}{\partial A_o^\top}(A_\mu A_\mu A_\nu A_\nu - A_\mu A_\nu A_\mu A_\nu)\\ &= -\frac{1}{2}N\,\mathrm{tr}_{T\times\Sigma}(2(A_o A_\nu A_\nu + A_\mu A_\mu A_o) - A_\nu A_o A_\nu - A_\mu A_\mu A_o - A_o A_\nu A_\nu - A_\mu A_o A_\mu)\\ &= -\frac{1}{2}N\,\mathrm{tr}_{T\times\Sigma}(A_o A_\mu A_\mu - 2A_\mu A_o A_\mu + A_\mu A_\mu A_o)\\ &= -\frac{1}{2}N\,\mathrm{tr}_{T\times\Sigma}([A_o|A_\mu]A_\mu - A_\mu[A_o|A_\mu]) = -\frac{1}{2}N\,\mathrm{tr}_{T\times\Sigma}[[A_o|A_\mu]|A_\mu] = -\frac{1}{2}N\,\mathrm{tr}_{T\times\Sigma}[A_\mu|[A_\mu|A_o]],\end{aligned}$$

where A_μ are hermitian and traceless $N\times N$ matrices, for which $A_\mu^\dagger = A_\mu$ or $A_\mu^* = A_\mu^\top$, $\forall\mu$.

Similarly, the derivative of the fermionic part is

$$\begin{aligned}\frac{\partial}{\partial A_\mu^\top}S_{\text{fermion}} &= -\frac{\partial}{\partial A_\mu^\top}\log\mathrm{pf}_{(\mathcal{U}\times T\times\Sigma)^2}\,\mathcal{M} = -\frac{1}{2}\frac{\partial}{\partial A_\mu^\top}\log\det{}_{(\mathcal{U}\times T\times\Sigma)^2}\,\mathcal{M} = -\frac{1}{2}\frac{\partial}{\partial A_\mu^\top}\,\mathrm{tr}_{(\mathcal{U}\times T\times\Sigma)^2}\log\mathcal{M}\\ &= -\frac{1}{2}\,\mathrm{tr}_{(\mathcal{U}\times T\times\Sigma)^2}\left(\frac{\partial}{\partial A_\mu^\top}\mathcal{M}\mathcal{M}^{-1}\right).\end{aligned}$$

which defines a (complex) Langevin equation,[8]

$$\Delta A_\mu = \upsilon_\mu\Delta\tau + \eta_\mu\sqrt{\Delta\tau}\text{ with drift }\upsilon_\mu = -\frac{\partial}{\partial A_\mu^\top}S_{\text{effective}} = \frac{1}{2}\,\mathrm{tr}_{T\times\Sigma}\left(N[A_\nu|[A_\nu|A_\mu]] + \mathrm{tr}_{\mathcal{U}}\frac{\partial}{\partial A_\mu^\top}\mathcal{M}\mathcal{M}^{-1}\right),\tag{7.2.2}$$

and a Wiener process generated discretely by hermitian η_μ with $\sigma = \sqrt{2}$.

Consistency in the complex Langevin process (7.2.2) demands that $A\in\mathrm{M}_N\mathbb{C}^{\dim\mathcal{X}}$, i.e. hermiticity is broken, and A is complexified.[9] Derivation with A^* and $A^\top$ is the same when A is hermitian, so whichever is chosen when complexifying A leads to a valid extension of said Langevin process. In this study, $A^\top$ was chosen.

Observables are evaluated as estimated means on (thermalized) sample paths of the stochastic process generated by (7.2.2),

$$\langle O\rangle = \frac{\sum_{n>n_0}O_n\Delta\tau_n}{\sum_{n>n_0}\Delta\tau_n}.\tag{7.2.3}$$

The reason for reaching thermalization first is the fact that a field theory is the stochastic limit at equilibrium of a corresponding stochastic process as approached by stochastic quantization.[10] Note that summation is performed after thermalization time $n_0\in\mathbb{N}$, which stands as an approximation of the $\tau\to\infty$ equilibrium limit.

The continuum limit $\Delta\tau\to 0$ corresponds to the expectation values averaged with the noise probability density η, which correspond to the expectations defined by the Fokker–Planck probability density given by (4.4.2). Assuming a total duration of T, one can formally write

$$\langle O\rangle = \lim_{T\to\infty}\left(T^{-1}\lim_{\tau_0\to\infty}\int_0^T d\tau O(\tau+\tau_0)\right).\tag{7.2.4}$$

[7] See section §4.2. Complexification for details.
[8] Note that tracing A is different to tracing $\mathcal{M}$.
[9] Note that tracelessness of A is maintained nonetheless.
[10] See chapter 4. Stochastic Quantization for details.

Holomorphicity of O is crucial in the validity of estimator (7.2.4) [35, 36, 37] in the context of the complex Langevin equation, a results that otherwise stems from stochastic quantization (real Langevin) [16, 32, 68].[11]

Stepping

For the transient phase of the Langevin stochastic process, an adaptive step size scheme is adopted to reduce the probability of instabilities to flourish under discretization of the process. The drift term is the only one responsible for such divergences therefore its amplitude

$$\|\upsilon\|^2 \propto= \sum\nolimits_{\mu} \left\| \frac{\partial}{\partial A_{\mu}^{\top}} S_{\text{effective}} \right\|^2 = \sum\nolimits_{\mu} \sum\nolimits_{a} \sum\nolimits_{b} \left| \frac{\partial}{\partial A_{\mu ba}} S_{\text{effective}} \right|^2 \tag{7.2.5}$$

is expected to guide the time stepping routine $\forall n \in \mathbb{N}$ as,

$$\Delta\tau_n = \begin{cases} \Delta\tau_0 & \|\upsilon_n\| \leq \|\upsilon_0\| \vee n \leq n_0, \\ \Delta\tau_0 \dfrac{\|\upsilon_0\|}{\|\upsilon_n\|} & \|\upsilon_n\| > \|\upsilon_0\| \wedge n > n_0, \end{cases}$$

where n_0 is the number of steps after (adequate) thermalization of the process. Estimators (7.2.3) are computed over the thermalized time history only ($n > n_0$). $\Delta\tau_0$ is taken as small as possible, but it is cut off by requirement of thermalization, as too small a time step requires too many iterations per simulation to thermalize.

Validity of the complexified Langevin method

A necessary condition according to [38] for (complexified) estimators which are basically functionals of the (complexified) Langevin process A, to converge to the corresponding observables at equilibrium, is for the drift norm (7.2.5) to vanish asymptotically faster than exponentially, hence the drift norm is an extra quantity in need of monitoring, to validate or discard a run.

As assumed in section §4.4. The complex action problem, an observable O shall admit a holomorphic extension $\overline{O}$ as part of complexification of ϕ into $\overline{\phi}$. While this also includes an extended Hamiltonian $\overline{\mathcal{A}}$, the expectation of $\overline{O}$ obeys the real version of the time evolution (4.3.3)

$$\frac{\partial}{\partial\tau}\langle\overline{O}\rangle = \mathcal{A}\langle\overline{O}\rangle \text{ with } \langle\overline{O}_0\rangle = \langle\overline{O}(\phi_0)\rangle \text{ for } \phi_0 = \phi(\tau_0) \text{ and } \frac{\partial}{\partial\tau}\varrho = -\overline{\mathcal{A}}^{\top}\varrho,$$

with formal solution

$$\langle\overline{O}\rangle = :\exp(\tau\mathcal{A}):\langle\overline{O}_0\rangle,$$

where expectation is assumed over the real probability Fokker–Planck distribution ϱ (and not the complex one $\overline{\varrho}$), and

$$\exp(\tau\mathcal{A}) = \sum\nolimits_{n\in\mathbb{N}} \frac{\tau^n \mathcal{A}^n}{n!},$$

with notation $:\mathcal{A}^n:$, $\forall n \in \mathbb{N}$, denoting the reordering of all ∇ operators in $\mathcal{A}^n$ to the right.[12]

For $\overline{O}$ holomorphic, $\mathcal{A}\overline{O} = \overline{\mathcal{A}O}$ [38], allowing the redefinition

$$\langle\overline{O}\rangle = :\exp(\tau\overline{\mathcal{A}}):\langle\overline{O}_0\rangle = \sum\nolimits_{n\in\mathbb{N}} \frac{\tau^n}{n!}\langle:\overline{\mathcal{A}}^n:\overline{O}_0\rangle, \tag{7.2.6}$$

where placing the expectation inside the exponents, reduces the expectation of $\overline{O}$ to a power series.

[11] Averaging is properly done with the noise density η, however η is stationary (and a Wiener process behind it at that) therefore noise sampling is equivalent to sampling one noise process over stochastic time τ.

[12] Per the example in [38],

$$(f+\partial)^2 = f^2 + f\partial + \partial f + \partial^2 \text{ versus } :(f+\partial)^2: = f^2 + 2f\partial + \partial^2.$$

$\forall n \in \mathbb{N}$, the integral $\langle :\overline{\mathcal{A}}^n : \overline{O}_0 \rangle_\varrho$ has maximum contributions from the dependence of $\overline{\mathcal{A}}$ from the drift υ of the corresponding Langevin process. The drift υ lives in the configuration space of ϕ, and as such, it may have a norm $\|\upsilon\|$.

Let

$$p(u,\tau) = \langle \delta(\|\upsilon\| - u) \rangle$$

be the probability distribution of the drift norm. Assume

$$\exists \kappa \in \mathbb{R}_+ \text{ and } \exists n_0 \in \mathbb{N} \text{ such that } \forall n > n_0,\ p(u,\tau) \sim \exp(-\kappa n) = \sum\nolimits_{n\in\mathbb{N}} \frac{(-\kappa)^n}{n!}. \tag{7.2.7}$$

For such a drift norm distribution, contributions to (7.2.6) become (expectations over p this time)

$$\langle \|\upsilon\|^n \rangle \sim \frac{n!}{\kappa^{n+1}},$$

yielding an approximate radius of convergence κ for the power series (7.2.6) [38], making (7.2.7) a sufficient condition for validity of the complex Langevin method.[13]

In figure 8.1.1 and figure 8.2.1 in chapter 8. The Euclidean IKKT matrix model, examples of valid and invalid simulation runs can be seen when applied to toy models.

Fermion action and noisy estimators

The fermionic drift can be evaluated using any complex stationary stochastic process (noise), whose instances $\eta \in V$ satisfy $\langle \eta^* \eta \rangle = \mathbb{1}$, as

$$\operatorname{tr} \mathcal{M} = \mathcal{M}_{Aaa'Aaa'} = \delta_{Aaa'Bbb'}\mathcal{M}_{Aaa'Bbb'} = \langle \eta^*_{Aaa'}\eta_{Bbb'}\rangle_{\mathbb{F}}\mathcal{M}_{Aaa'Bbb'} = \langle \eta^*_{Aaa'}\mathcal{M}_{Aaa'Bbb'}\eta_{Bbb'}\rangle = \langle \eta^*\mathcal{M}\eta\rangle.$$

The fermionic drift term becomes

$$\operatorname{tr}\left(\frac{\partial}{\partial A^\top_\mu}\mathcal{M}\mathcal{M}^{-1}\right) = \left\langle \eta^* \frac{\partial}{\partial A^\top_\mu}\mathcal{M}\chi \right\rangle \text{ with } \chi = \mathcal{M}^{-1}\eta.$$

$\mathcal{M}^\dagger\mathcal{M}$ is hermitian and positive definite, therefore — assuming invertibility of $\mathcal{M}^\dagger\mathcal{M}$ — equation

$$\mathcal{M}^\dagger\mathcal{M}\chi = \mathcal{M}^\dagger\eta,$$

is computationally solvable using the conjugate gradient method (also see section §1.4. Numerical linear algebra methods).[14] Operations of $\mathcal{M}$ or $\mathcal{M}^\dagger$ on fermions are optimized by its definition (6.3.11) [7], as we did in [14, 15].

Bosonic field hermiticity and gauge cooling

The original theory assumes $\forall \mu,\ A^\dagger_\mu = A_\mu$ (hermitian traceless matrices), however $S_{\text{effective}}$ being complex breaks consistency in the Langevin equation (7.2.2), and A bosons require complexification, becoming generic traceless matrices in $\mathrm{M}_N\mathbb{C}$.

Upon complexifying a hermitian traceless matrix A, its eigenvalues turn from real to complex, however, (by choice) the matrix can be (and is in this work) kept traceless. $\forall A, B \in \mathbb{M}_N\mathbb{C}$ with $\operatorname{tr} A = \operatorname{tr} B = 0$ (based on the previous assumption), $\operatorname{tr}[A|B] = \operatorname{tr}(AB) - \operatorname{tr}(BA) = 0$, due to the cyclic property of the trace, hence the bosonic drift is traceless as well. By a similar argument about the original tracelessness of A, every $N \times N$ submatrix of $\mathcal{M}$ remains traceless as well after complexification.

The symmetry group of $\operatorname{tr}_{T\times\Sigma}$ on $\mathrm{M}_N\mathbb{C}$ is $\mathrm{GL}_N\mathbb{C}$, in consistency with the complexification of $\mathrm{U}_N\mathbb{C}$ (table 1.3.2 on page 27).

$\dim \mathrm{GL}_N\mathbb{C} = \dim \mathfrak{gl}_N\mathbb{C} = N^2$ still, and for a base ℓ in $\mathfrak{gl}_N\mathbb{C}$, and $\forall g \in \mathrm{GL}_N\mathbb{C}$,

$$g = \exp g_a\ell_a = \exp(\Re g_a + \imath\Im g_a)\ell_a.$$

[13] The argument presented here is a qualitative non–rigorous short version of the argument developed in [38, 42].
[14] When expectation values accumulate near 0, the conjugate gradient method may be slow to converge.

The Frobenius matrix norm extended to tensor field A as

$$\|A - A^\dagger\|^2 \propto N^{-1} \operatorname{tr}_{T\times\Sigma}(A_\mu - A_\mu^\dagger)^\dagger(A_\mu - A_\mu^\dagger) = -N^{-1} \operatorname{tr}_{T\times\Sigma}(A_\mu - A_\mu^\dagger)^2,$$

is a good measure of hermiticity of A.

To avoid unnecessary excursions of the Langevin stochastic process in the imaginary direction, the gauge symmetry of the effective action is utilized, to minimize said hermiticity norm.

$\forall A \in \mathrm{M}_N\mathbb{C}$ traceless, $(A_\mu - A_\mu^\dagger)^\dagger(A_\mu - A_\mu^\dagger) \in \mathfrak{su}_N\mathbb{C}$, i.e. is always hermitian and traceless. The trace of such matrix is invariant only under $\arg g = \exp \imath\Im g_a\ell_a \in \mathrm{U}_N\mathbb{C}$, therefore $|g| = \exp \Re g_a\ell_a$ generates a variation. For a boson A,

$$\Delta A = |g|A|g|^{-1} - A = \exp(+\Re g_a\ell_a)A\exp(-\Re g_a\ell_a) - A \to \Re g_a\ell_a A - A\Re g_a\ell_a = \Re g_a[\ell_a|A],$$

and for its adjoint,

$$\Delta A^\dagger = (|g|A|g|^{-1})^\dagger - A^\dagger = \exp(-\Re g_a\ell_a)A^\dagger\exp(+\Re g_a\ell_a) - A^\dagger \to A^\dagger\Re g_a\ell_a - \Re g_a\ell_a A^\dagger = -\Re g_a[\ell_a|A^\dagger],$$

leading to a variation in the norm

$$\Delta\|A - A^\dagger\|^2 \propto -N^{-1} \operatorname{tr}_{T\times\Sigma}((A_\mu - A_\mu^\dagger)\Delta(A_\mu - A_\mu^\dagger)) = -N^{-1} \operatorname{tr}_{T\times\Sigma}((A_\mu - A_\mu^\dagger)(\Re g_a[\ell_a|A_\mu] + \Re g_a[\ell_a|A_\mu^\dagger]))$$
$$= -N^{-1} \operatorname{tr}_{T\times\Sigma}((A_\mu - A_\mu^\dagger)(\Re g_a[\ell_a|A_\mu + A_\mu^\dagger])) = -N^{-1}\Re g_a \operatorname{tr}_{T\times\Sigma} \ell_a[A_\mu - A_\mu^\dagger|A_\mu + A_\mu^\dagger] \propto N^{-1}\Re g_a \operatorname{tr}_{T\times\Sigma} \ell_a[A_\mu|A_\mu^\dagger] = \Re g_a G_a.$$

For $\mathrm{U}_N\mathbb{C}$,[15] the base matrices satisfy $\ell_{abc} \otimes \ell_{ade} = \delta_{be}\delta_{cd}$.

$H \propto -\ell_a G_a$ sets the direction of steepest descent along which a minimum is sought,

$$H_{ab} \propto -\ell_{cab}G_c = -N^{-1}\ell_{cab}\ell_{cde}[A_\mu|A_\mu^\dagger]_{ed} = -N^{-1}\delta_{ae}\delta_{bd}[A_\mu|A_\mu^\dagger]_{ed} = -N^{-1}[A_\mu|A_\mu^\dagger]_{ab},$$

so a new $B = gAg^{-1}$ with $g = \exp \gamma H$ and γ that minimizes $\|B - B^\dagger\|^2$, which for fixed A is a function of γ.

Gauge cooling as it is called [41, 42, 44] modifies the discretized complex Langevin process by adding an intermediate step as,[16]

$$B(\tau) = g(\tau)A(\tau)g^{-1}(\tau) \text{ and } A(\tau + \Delta\tau) = B(\tau) + \Delta\tau\upsilon(\tau) + \sqrt{\Delta\tau}\eta(\tau) \text{ with } g(\tau) = \exp(-\gamma N^{-1}[A(\tau)|A^\dagger(\tau)]). \tag{7.2.8}$$

Spontaneous $\mathrm{SO}_{\dim\mathcal{X}}$ symmetry breaking mechanism

In introducing a symmetry breaking mechanism to the model, the diagonal elements of the moment of inertia matrix λ as in (6.3.13) are used, indexed only once, to avoid confusion with the Einstein index summation convention (definition 1.1.16). The eigenvalues of the moment of inertia matrix Λ are non–holomorphic functions in the complexified A domain, while on the other hand the diagonal elements of Λ (denoted with λ) can equivalently be used for breaking the rotational SO_{10} symmetry.

The symmetry breaking term is controlled by a global parameter $\varepsilon \in \mathbb{R}_+$ as [14, 15, 18]

$$\Delta S_{\text{boson}} = \frac{1}{2}N^2\varepsilon m_{\text{boson}}|_\mu\lambda_\mu = \frac{1}{2}N\varepsilon m_{\text{boson}}|_\mu \operatorname{tr}_{T\times\Sigma} A_\mu^2,$$

where the bosonic masses in m_{boson} define specific symmetry breaking parameters. Note that if $m_\mu = m_\nu\ \forall\mu,\nu \in \mathbb{N}_{\dim\mathcal{X}}$, there is no explicit symmetry breaking. Adhering to the condition of ordering the values of the order parameter, bosonic masses have to be ordered too.[17]

Smaller masses correspond to larger extends, therefore the vector m_{boson} is usually chosen with increasingly ordered components.

[15] Correspondingly, for $\mathrm{SU}_N\mathbb{C}$, $\ell_{abc} \otimes \ell_{ade} = \delta_{be}\delta_{cd} - N^{-1}\delta_{bc}\delta_{de}$

[16] The one free index is suppressed here to avoid confusion with Einstein summation indexing.

[17] This term is precisely the Gaussian term $S_0|_{\text{boson}}$ found in the GEM, see section §7.1. The Gaussian Expansion Method (GEM) for details.

Fermionic singularity and mass shifting

As seen by the differential equation (7.2.2) of the Langevin process A, the drift is singular due to the fermion matrix $\mathcal{M}$ being manifestly non–invertible as antihermitian and (multiple times in each fermion sub–block) traceless. The eigenvalue spectrum for $\mathcal{M}$ is centered at the complex 0. To circumvent the singularity, the spectrum of $\mathcal{M}$ is separated away from 0, along one direction with the use of an fermionic mass deformation [14, 15, 18]

$$\Delta S_{\text{fermion}} = \frac{1}{2} N m_{\text{fermion}} \operatorname{tr}_{T\times\Sigma}(\overline{\psi}_\alpha \gamma_{\alpha\beta}(\Gamma)\psi_\beta), \tag{7.2.9}$$

where γ is generally a 10–dimensional polynomial matrix function of Γ matrices, depending only on the fermion indices. The most typical examples are linear choices in one specific direction, for example $\gamma = \Gamma_{\dim \mathcal{X}}$ or $\gamma = \Gamma_{\dim \mathcal{X}-3}\Gamma_{\dim \mathcal{X}-2}\Gamma_{\dim \mathcal{X}-1}$ combining 3 directions in one cubic term. Absorbing the effect of the fermion mass into the fermion matrix $\overline{\mathcal{M}}$ as in (6.3.12),

$$\overline{\mathcal{M}}_{\alpha a a' \beta b b'}(m) = \overline{\mathcal{M}}_{\alpha a a' \beta b b'} + m_{\text{fermion}} \gamma_{\alpha\beta} \delta_{ab'} \delta_{a'b},$$

it is understood that γ has to preserve the anti–symmetry of $\mathcal{M}$ as well.

In figure 8.1.2 and figure 8.2.2 in chapter 8. The Euclidean IKKT matrix model, examples of how the spectrum of the fermion matrix $\mathcal{M}$ is shifted towards a specific direction in the study of toy models.

The purely bosonic model

Deformations like (7.2.9) have a theory–landscape effect. The limit $m_{\text{fermion}} \to 0$ corresponds to the original theory, while the limit $m_{\text{fermion}} \to \infty$ decouples the fermions from the theory resulting in a purely bosonic IKKT model, which is expected not to show any spontaneous symmetry breaking.

Primarily, the purely bosonic Euclidean IKKT model

$$S_{\text{boson}} = -\frac{1}{4} N \operatorname{tr}_{T\times\Sigma}([A_\mu|A_\nu][A_\mu|A_\nu])$$

has no complex action (problem). Numerical simulations have shown that both the bosonic, and the full but phase–quenched model show no spontaneous symmetry breaking indicating that (at least) for the Euclidean IKKT model, the imaginary part of the fermion determinant/Pfaffian is responsible for the spontaneous symmetry breaking.

Degenerate configurations

It has been shown in [19] that the expected broken symmetry of the IKKT model does not go below SO_2. This is evident for $\dim \mathcal{X} = 10$ by the result $\operatorname{pf} \mathcal{M} = 0$ for configurations with $A_3 = A_4 = \ldots = A_{\dim \mathcal{X}} = 0$ [16]. More generally, $\operatorname{pf} \mathcal{M}$ gradually takes a simpler form as the dimensionality of the configurations is decreased. Adopting the notation in [19], $\forall D \in \mathbb{N}_{\dim \mathcal{X}}$, the set of all D–dimensional (degenerate) configurations can be expressed as

$$\Omega_D = \{A_\mu | \exists n \in \mathbb{R}^{\dim \mathcal{X} - D} \otimes \mathbb{R}^{\dim \mathcal{X}} \text{ such that } \forall i \in \mathbb{N}_{\dim \mathcal{X} - D},\, n_i|_\mu A_\mu = 0\}.$$

These configurations are by definition spacetime directional, in a specific subspace of lower dimension D.

In [19] it is shown that:

- for $A \in \Omega_9$, $\operatorname{pf} \mathcal{M} \in \mathbb{R}$,
- for $A \in \Omega_6$, $\operatorname{pf} \mathcal{M} \geq 0$,
- for $A \in \Omega_2$, $\operatorname{pf} \mathcal{M} = 0$.

Similar arguments hold for $\dim \mathcal{X} = 6$:

- for $A \in \Omega_5$, $\det \mathcal{M} \in \mathbb{R}$,
- for $A \in \Omega_2$, $\det \mathcal{M} = 0$.

The method used in [19] was to replace the fermionic part of the effective action by

$$\Gamma = \Re\Gamma + \imath\Im\Gamma = \log \operatorname{pf}_{(\mathcal{U}\times T\times\Sigma)^2} \mathcal{M}$$

with $\lim_{\nu\to\infty}(\Re\Gamma+\imath\nu\Im\Gamma)$, in effect extremizing the imaginary part of the action and forcing configurations for which $\exp \imath\nu\Im\Gamma$ becomes stationary. Specifically for $2 \leq D \leq \dim \mathcal{X} - 2$

$$\prod_{i=2}^{\dim \mathcal{X}-D} \frac{\partial}{\partial A_{\mu_i}} \Im\Gamma = 0,$$

which implies that the lower the dimensionality of the configuration A, the more stationary the complex phase of the effective action is [14, 19]. For $D = 2$, $\operatorname{pf}\mathcal{M} = 0$ and $\lim_{\nu\to\infty}(\Re\Gamma + \imath\nu\Im\Gamma)$ is ill–defined, indicating that Ω_2 is a lower bound for degenerate configurations. With this argument, the symmetry breaking mechanism $\mathrm{SO}_{\dim \mathcal{X}} \to \mathrm{SO}_D$ is not expected to go below SO_2. In fact, SO_2 is also prohibited.

Holomorphic Observables

As has been mentioned earlier, the complexified moment of inertial Λ is non–holomorphic, which leaves a question about the applicability and correctness of the CLM, in accordance with the strong criterion developed in (7.2.7) [38]. Since the criterion is only sufficient, there is room for the applicability of the CLM provided an equivalent (and holomorphic) observable is found. For the Euclidean model [14, 15, 72], the diagonal elements where used as observables

$$\lambda_\mu = \Lambda_{\mu\mu} \text{ (no summation implied)}$$

instead of the true eigenvalues of Λ. however there exists an equivalent and holomorphic extraction of the (ordered) eigenvalues of Λ with the help of the characteristic polynomial of Λ and its Vandermonde matrix.

Let the characteristic polynomial of Λ reads $\forall\lambda \in \mathbb{C}$,

$$\chi_\Lambda(z) = \det(\lambda\mathbb{1} - \Lambda) = \prod_{n=1}^{\dim \mathcal{X}} (\lambda - \lambda_n) = \lambda^{\dim \mathcal{X}} + \sum_{n=1}^{\dim \mathcal{X}} a_n \lambda^{\dim \mathcal{X}-n},$$

meaning

$$a_0 = 1,\ a_1 = -\operatorname{tr}\Lambda = \Lambda_{\mu\mu} \text{ (summation implied as usual) and } a_{\dim \mathcal{X}} = (-1)^{\dim \mathcal{X}} \det\Lambda. \tag{7.2.10}$$

The Vandermonde matrix approximation

The Vandermonde matrix $\forall z \in \mathbb{C}^{\dim \mathcal{X}}$,

$$V(z) = \begin{pmatrix} z_1 & \cdots & z_1^{\dim \mathcal{X}} \\ \vdots & \ddots & \vdots \\ z_{\dim \mathcal{X}} & \cdots & z_{\dim \mathcal{X}}^{\dim \mathcal{X}} \end{pmatrix}.$$

The remaining coefficients after (7.2.10) are then the solution of the trimmed Vandermonde system

$$\begin{pmatrix} z_2 & \cdots & z_2^{\dim \mathcal{X}-1} \\ \vdots & \ddots & \vdots \\ z_{\dim \mathcal{X}-1} & \cdots & z_{\dim \mathcal{X}-1}^{\dim \mathcal{X}-1} \end{pmatrix} \begin{pmatrix} a_2 \\ \vdots \\ a_{\dim \mathcal{X}-1} \end{pmatrix} = \begin{pmatrix} \chi^*_\Lambda(a_2) \\ \vdots \\ \chi^*_\Lambda(a_{\dim \mathcal{X}-1}) \end{pmatrix},$$

with

$$z_n = z_{\max} + (z_{\min} - z_{\max})(\dim \mathcal{X} - 3)^{-1}(n-1) \text{ with } z_{\min} = (1/2)^{1/(n-2)} \text{ and } z_{\max} = (3/2)^{1/(n-2)},$$

and $\forall\lambda \in \mathbb{C}$,

$$\begin{aligned}\chi^*_\Lambda(\lambda) = \chi_\Lambda(\lambda) - (\lambda^{\dim \mathcal{X}} + a_1\lambda^{\dim \mathcal{X}-1} + a_{\dim \mathcal{X}}) &= \chi_\Lambda(\lambda) - (\lambda^{\dim \mathcal{X}} - \operatorname{tr}\Lambda\lambda^{\dim \mathcal{X}-1} + (-1)^{\dim \mathcal{X}}\det\Lambda) \\ &= \sum_{n=2}^{\dim \mathcal{X}-1} a_n \lambda^{\dim \mathcal{X}-n}.\end{aligned}$$

Exact solution

In [73], recursively exact solutions of the characteristic polynomial coefficients are extracted as summations and products of trace expressions $b_n = \operatorname{tr} \Lambda^n$,[18] $\forall n \in \mathbb{N}$, special cases of which are found in (7.2.10).

As shown in [73], $\forall m \in \mathbb{Z}_{\dim \mathcal{X}+1}$,

$$\sum\nolimits_{n=0}^{m} \alpha_n b_{m-n} = 0, \tag{7.2.11}$$

leading to a recursive over the matrix size $\dim \mathcal{X} \in \mathbb{N}$ definition of characteristic polynomials,

$$\begin{aligned}
&\lambda \ - b_1, && \dim \mathcal{X} = 1\\
&\lambda^2 - b_1\lambda \ + 2^{-1}(b_1^2 - b_2), && \dim \mathcal{X} = 2\\
&\lambda^3 - b_1\lambda^2 + 2^{-1}(b_1^2 - b_2)\lambda \ - 6^{-1}(b_1^3 - 3b_1b_2 + 2b_3), && \dim \mathcal{X} = 3\\
&\dots
\end{aligned}$$

It is interesting as observed by [73], characteristic polynomial coefficients grow statically as the matrix size $\dim \mathcal{X}$ grows, forming a sequence over $\dim \mathcal{X}$. Expanding equations (7.2.11),

$$\begin{aligned}
-1a_1 &= b_1,\\
-2a_2 &= b_1a_1 + b_2,\\
-3a_3 &= b_1a_2 + b_2a_1 + b_3,\\
&\dots
\end{aligned}$$

Grouping the last set of equations into a power series,

$$\sum\nolimits_{n\in\mathbb{N}} na_nx^n = -\sum\nolimits_{m\in\mathbb{N}} b_{m+1}x^m \sum\nolimits_{n\in\mathbb{N}} a_nx^n. \tag{7.2.12}$$

In accordance with [73], let the generating functions for the sequences a and b of the characteristic polynomial and traces of powers of the matrix Λ be

$$f(x) = \sum\nolimits_{n\in\mathbb{N}} a_nx^n \text{ and } g(x) = \sum\nolimits_{m\in\mathbb{N}} b_{m+1}x^m, \tag{7.2.13}$$

such that

$$a_n = (n!)^{-1}\frac{\partial^n}{\partial x^n}f(0) \text{ and } b_{n+1} = (n!)^{-1}\frac{\partial^n}{\partial x^n}g(0),\ \forall n \in \mathbb{N}.$$

Replacing (7.2.13), (7.2.12) becomes the Cauchy problem

$$\frac{\partial}{\partial x}f(x) = -g(x)f(x),\ f(0) = 1,$$

with solution

$$f(x) = \exp\Big(-\int_0^x g(t)dt\Big),$$

which gives

$$f(x) = \exp\Big(-\sum\nolimits_{n\in\mathbb{N}_*} n^{-1}b_nx^n\Big) = \prod\nolimits_{n\in\mathbb{N}} \exp(-n^{-1}b_nx^n) = \prod\nolimits_{n\in\mathbb{N}_*} \sum\nolimits_{m_n\in\mathbb{N}} (m_n!)^{-1}(-n^{-1}b_nx^n)^{m_n}.$$

Grouping terms of equal power in x,

$$f(x) = \sum\nolimits_{n\in\mathbb{N}} x^n \sum\nolimits_{m\in\mathcal{S}_n\subseteq\mathbb{Z}_{n+1}^n} \prod\nolimits_{k=1}^{n} (m_k!)^{-1}(-k^{-1}b_k)^{m_k},$$

[18] For $n = 0$, $\operatorname{tr}\Lambda^0 = \operatorname{tr}\mathbb{1} = \dim \mathcal{X}$.

where $\forall n \in \mathbb{N}$, $\mathcal{S}_n \subseteq \mathbb{Z}_{n+1}^n$ is the set of all natural number vector solutions $m = (m_k)_{k=1}^n$ satisfying

$$\sum_{k=1}^n k m_k = n.$$

The coefficients are then

$$a_n = \sum\nolimits_{m \in \mathcal{S}_n \subseteq \mathbb{Z}_{n+1}^n} \prod\nolimits_{k=1}^n (m_k!)^{-1} (-k^{-1} b_k)^{m_k}, \, \forall n \in \mathbb{Z}_{\dim \mathcal{X}+1},$$

a formula that can be used for finite cutoff $n = \dim \mathcal{X}$ to calculate the coefficients of the characteristic polynomial χ_Λ of the moment of inertia matrix Λ.

A example included in [73], applies the formula for calculating determinants. In particular,

$$\det \Lambda = (-1)^{\dim \mathcal{X}} a_{\dim \mathcal{X}} = (-1)^{\dim \mathcal{X}} \sum\nolimits_{m \in \mathcal{S}_{\dim \mathcal{X}} \subseteq \mathbb{Z}_{\dim \mathcal{X}+1}^{\dim \mathcal{X}}} \prod\nolimits_{k=1}^{\dim \mathcal{X}} (m_k!)^{-1} (-k^{-1} b_k)^{m_k}. \tag{7.2.14}$$

Expanding (7.2.14) for $\dim \mathcal{X} = 10$ of type IIB superstring theory,

$$\begin{aligned}
\det \Lambda = &-\frac{1}{10} b_{10} + \frac{1}{9} b_9 b_1 + \frac{1}{16} b_8 b_2 - \frac{1}{16} b_8 b_1^2 + \frac{1}{21} b_7 b_3 - \frac{1}{14} b_7 b_2 b_1 + \frac{1}{42} b_7 b_1^3 + \frac{1}{24} b_6 b_4 - \frac{1}{18} b_6 b_3 b_1 - \frac{1}{48} b_6 b_2^2 + \frac{1}{24} b_6 b_2 b_1^2 \\
&- \frac{1}{144} b_6 b_1^4 + \frac{1}{50} b_5^2 - \frac{1}{20} b_5 b_4 b_1 - \frac{1}{30} b_5 b_3 b_2 + \frac{1}{30} b_5 b_3 b_1^2 + \frac{1}{40} b_5 b_2^2 b_1 - \frac{1}{60} b_5 b_2 b_1^3 + \frac{1}{600} b_5 b_1^5 - \frac{1}{64} b_4^2 b_2 + \frac{1}{64} b_4^2 b_1^2 - \frac{1}{72} b_4 b_3^2 \\
&+ \frac{1}{24} b_4 b_3 b_2 b_1 - \frac{1}{72} b_4 b_3 b_1^3 + \frac{1}{192} b_4 b_2^3 - \frac{1}{64} b_4 b_2^2 b_1^2 + \frac{1}{192} b_4 b_2 b_1^4 - \frac{1}{2880} b_4 b_1^6 + \frac{1}{162} b_3^3 b_1 + \frac{1}{144} b_3^2 b_2^2 - \frac{1}{72} b_3^2 b_2 b_1^2 + \frac{1}{432} b_3^2 b_1^4 \\
&- \frac{1}{36} b_3 b_2^3 b_1 + \frac{1}{144} b_3 b_2^2 b_1^3 - \frac{1}{720} b_3 b_2 b_1^5 + \frac{1}{15120} b_3 b_1^7 - \frac{1}{240} b_2^5 + \frac{1}{96} b_2^4 b_1^2 - \frac{1}{288} b_2^3 b_1^4 + \frac{1}{5760} b_2^2 b_1^6 - \frac{1}{80640} b_2 b_1^8 + \frac{1}{3628800} b_1^{10}.
\end{aligned}$$

Vieta formula

Another approach is to use Vieta formula, which relates the roots of a polynomial to its coefficients. In particular for the characteristic polynomial χ_Λ and the eigenvalues $(\lambda_n)_{n=1}^{\dim \mathcal{X}}$ of the moment of inertia Λ (roots of χ_Λ),

$$\sum\nolimits_{1 \leq m_1 < \ldots < m_n \leq \dim \mathcal{X}} \prod\nolimits_{k=1}^n \lambda_{m_k} = (-1)^n a_n, \, \forall n \in \mathbb{Z}_{\dim \mathcal{X}+1},$$

which provides another recursive analytic recipe for evaluating the ordered eigenvalues of Λ.

Assuming $a_0 = 1$, a recursive formula is possible via the 2–loop algorigthm

$$a_0 \to 1 : \forall n \in \mathbb{Z}_{\dim \mathcal{X}+1} \{ a_n \to 0 : \forall m \in \mathbb{Z}_n \{ a_m \to a_m + \lambda_n a_{m-1} \} \}.$$

Eigenvalue expectation values as polynomials of expectations of holomorphic expressions

The characteristic polynomial is holomorphic, as is the polynomial solutions for its coefficients presented above, and finding the eigenvalues of the moment of inertia is reduced to finding the roots of p_Λ after (emphasis here) the expectation values of the (holomorphic) coefficients via

$$\langle \chi_\Lambda \rangle = \langle \det(\lambda \mathbb{1} - \Lambda) \rangle = \lambda^{\dim \mathcal{X}} + \sum\nolimits_{n=1}^{\dim \mathcal{X}} \langle a_n \rangle \lambda^{\dim \mathcal{X} - n}.$$

Note that, while the eigenvalues of a matrix are interchangeable, the coefficients of the characteristic polynomial are not, therefore the late ordering of the eigenvalues can and actually (based on simulations) match with those averaged from straightforward measurement and ordering of the eigenvalues on each step of the (thermalized) Langevin process.

In each step of the process, the (unordered) eigenvalues are measured, then the coefficients of the characteristic polynomial are evaluated via Vieta formula. At the end of the simulations, expectation values of the coefficients are estimated, and then used to extract estimates for the expectation values of the eigenvalues. This process compared to estimating the ordered eigenvalues outright gives similar results, indicating that the applicability of the CLM is not affected by the non–holomorphicity of Λ.

8. The Euclidean IKKT matrix model

As discussed in section §6.3. Matrix models, the (ordered) eigenvalues of the moment of inertia tensor (6.3.13) represent the extends of spacetime. To avoid using non–holomorphic complexified observables, the ordered set of the diagonal elements of the moment of inertia tensor are assumed instead. The estimated observable

$$\rho_\mu = \frac{\langle \lambda_\mu \rangle}{\sum_\nu \langle \lambda_\nu \rangle}$$

is a function of the finite size N, the global symmetry breaking parameter ε and the fermion deformation mass m_{fermion}. The dependence on ε is reduced between the numerator and the denominator, which makes the $\varepsilon \to 0$ extrapolation more reliable and is the reason for using ρ as the set of order parameters [15]. In what follows, the following ordered composite limit is estimated by extrapolation of the simplest possible fitting polynomial:

$N^{-1} \to 0$ Elimination of finite size effects comes first for every deformed and explicitly symmetry–broken model. The model is defined for $N \to \infty$ similarly to the thermodynamic limit in statistical mechanics, therefore is it the first parameter to extrapolate. Linear

$$a_1 N^{-1} + a_0$$

and quadratic

$$a_2 N^{-2} + a_1 N^{-1} + a_0$$

extrapolations were attempted.

$\varepsilon \to 0$ Vanishing of the bosonic orders parameter to look for the spontaneous symmetry breaking of m_{fermion}–deformed model. Quadratic extrapolation

$$a_2 \varepsilon^2 + a_1 \varepsilon + a_0$$

as used for the bosonic order parameter.

$m_{\text{fermion}} \longrightarrow 0$ Elimination of the m_{fermion}–deformation to retrieve the original (Euclidean) IKKT model. Quartic (even) extrapolation

$$a_4 m_{\text{fermion}}^4 + a_2 m_{\text{fermion}}^2 + a_0$$

was used for the deformation fermionic mass.

The following result sections are organized as follows:

- Samples of the drift norm histories are given to show where the CLM fails or not, based on the fall–off speed of the drift norm.
- Samples of the hermiticity norm histories are given to show the effect of gauge cooling in constraining the drift of A in antihermitian direction.
- Samples of the eigenvalue spectrum of the fermion matrix $\mathcal{M}$ is given to show the effect of the fermion deformation in eliminating the singular drift problem of the complex Langevin process A, due to the zero eigenvalues of $\mathcal{M}$.
- Observable ρ plots in order of appearance:
  - The selection of the symmetry–breaking bosonic masses m_{boson} and fermion deformation γ matrix
  - A sample depicting the $N^{-1} \to 0$ extrapolation.
  - A few samples depicting the spontaneous symmetry breaking (or symmetry recovery) by $\varepsilon \to 0$.
  - The final plot where $m_{\text{fermion}} \to 0$ to examine the spontaneous symmetry breaking of the original model.

Next to the simulation results, the corresponding GEM search space results are displayed for comparison.

8.1. Euclidean Gaussian matrix model with $\dim \mathcal{X} = 4$

It has been shown that $\det \mathcal{M} \geq 0$ in the 4–dimensional IKKT matrix model [7], meaning that customary Monte Carlo methods are applicable. In that same study, it was shown that the model presents no spontaneous symmetry breaking, which is a further hint that (at least for the Euclidean IKKT model) the imaginary part of the fermionic contribution $\propto \log\det \mathcal{M}$ on the effective action might be responsible for the spontaneous symmetry breaking.

In the GEM study [74] a Gaussian matrix model was used as an example with $\dim \mathcal{X} = 4$ that does have a complex action and presents spontaneous symmetry breaking, which was later studied via the CLM in [18], namely,[1]

$$S_{\text{boson}} \propto N \operatorname{tr}(A_\mu A_\mu) \text{ and } S_{\text{fermion}} \propto -N\overline{\psi}_\alpha \Gamma_{\mu\alpha\beta} A_\mu \psi_\beta. \tag{8.1.1}$$

This section presents and compares these results as a precursor to the $\dim \mathcal{X} = 6, 10$ exploration that followed.

Gaussian Expansion Method

For $\dim \mathcal{X} = 4$, $\dim \mathcal{U} = 2$, m_{fermion} is recoded from a fermion mass matrix to a boson mass vector via the Γ matrices as

$$S_0|_{\text{fermion}} = m_{\text{fermion}}|_\mu \overline{\psi}_\alpha \Gamma_{\mu\alpha\beta} \psi_\beta,$$

thus the parametric freedom is $4 + 4$ and the ansatz search space is practically very small. In [74]the SO_3 and $\mathrm{SO}_2 \times \mathbb{Z}_2$ ansätze are explored.

$\mathbf{SO_3}$ ansatz (3 parameters)

$m_{\text{boson}}|_1 = m_{\text{boson}}|_2 = m_{\text{boson}}|_3$. (2 parameters including $m_{\text{boson}}|_4$)

$m_{\text{fermion}}|_1 = m_{\text{fermion}}|_2 = m_{\text{fermion}}|_3 = 0$. (1 parameters including $m_{\text{fermion}}|_4$)

This vacuum is realized for arbitrarily large N.

$\mathbf{SO_2 \times \mathbb{Z}_2}$ ansatz (3 parameters)

$m_{\text{boson}}|_1 = m_{\text{boson}}|_2$, $m_{\text{boson}}|_3 = m_{\text{boson}}|_4$. (2 parameters)

$m_{\text{fermion}}|_1 = m_{\text{fermion}}|_2 = 0$, $m_{\text{fermion}}|_3 = m_{\text{fermion}}|_4$. (1 parameters)

This vacuum is realized for arbitrarily many fermion flavors N_{fermion} with fixed (and finite) ratio N_{fermion}/N for $N \to \infty$, indicating that SO_2 is a surviving symmetry after the spontaneous symmetry breaking of SO_4.

In particular, for $N_{\text{fermion}}/N = 1$, the observable values

$$\lambda_1 = \lambda_2 = 2.1,\ \lambda_3 = 1.0 \text{ and } \lambda_4 = 0.8,$$

approximately correspond to

$$\rho_1 = \rho_2 = 0.35,\ \rho_3 = 0.17 \text{ and } \rho_4 = 0.13, \tag{8.1.2}$$

indicating that the surviving SO_2 symmetry is also the extended one, with the broken dimensions being shrunken in comparison.

[1] A variant of the model was used in [74] originally found in [75], containing N_{fermion} fermion families instead of just one. In said study, the large–N limit was taken with N_{fermion}/N fixed.

Complex Langevin Method

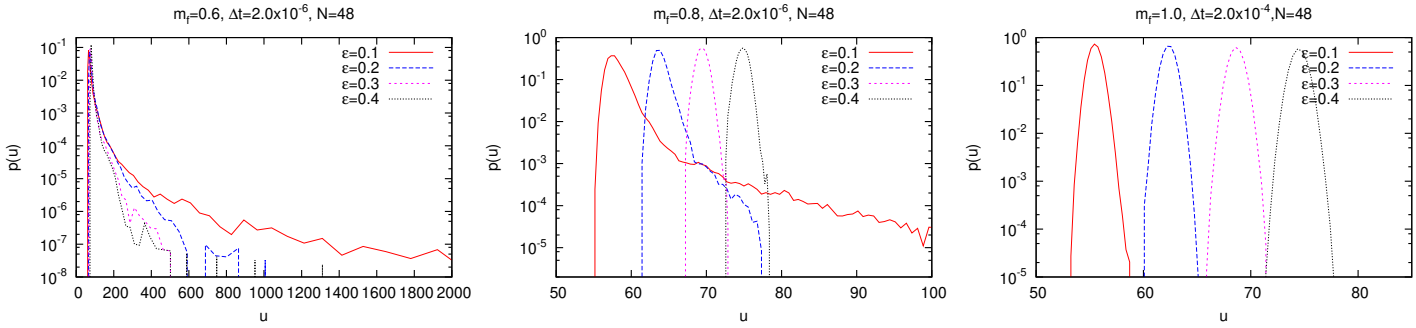


Figure 8.1.1.: Three sets of runs with varying ε and increasing m_{fermion} that show how reducing both leads to problematic drift fall–off for the Gaussian matrix model (8.1.1) [18].

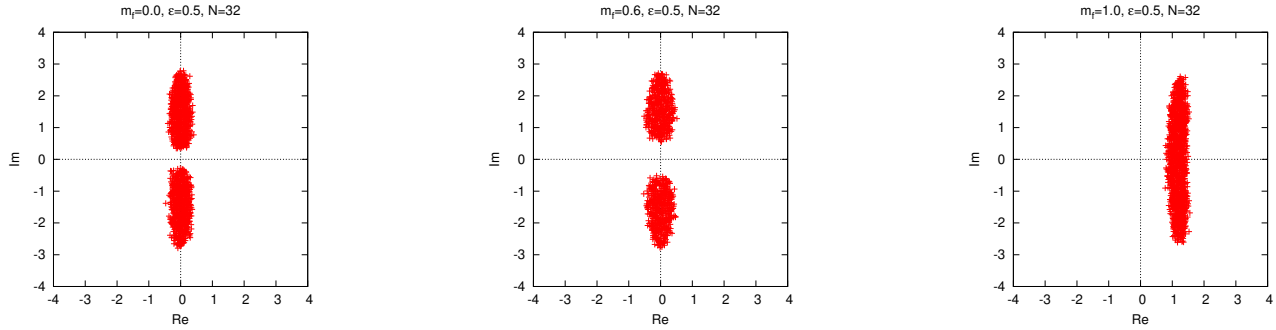


Figure 8.1.2.: A case for specific ε that shows the effect of the fermion mass deformation in eliminating zero eigenvalues in $\mathcal{M}$ of the Gaussian matrix model (8.1.1) , as shown from left to right in increasing m_{fermion} [18].

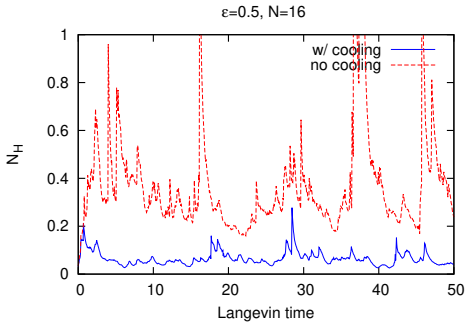


Figure 8.1.3.: The hermiticity norm shown for a run with and without gauge cooling for a run of the Gaussian matrix model (8.1.1) [18].

In [18], $m_{\text{boson}} = (2^0, 2^{+1}, 2^{+2}, 2^{+3})$ and $\gamma = \Gamma_4$ were chosen.

In figure 8.1.1 some (discarded) cases are shown where the drift norm fall–off is not sub–exponential.

In figure 8.1.2 the fermion matrix eigenvalue spectrum is shown shifted away from the origin with tuning the deformation mass m_{fermion}, which results in a non–singular fermion matrix $\mathcal{M}$.

In figure 8.1.3 shows an example of a run using gauge cooling and one without. While both runs do not seem to escape in the imaginary direction, the constraint of A towards hermiticity is evident.

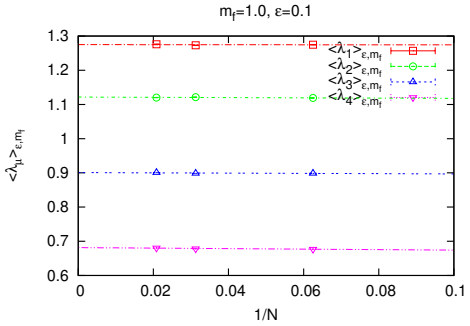


Figure 8.1.4.: The $N^{-1} \to 0$ limit of runs of the Gaussian matrix model (8.1.1) with linear extrapolation $\alpha_1 N^{-1} + \alpha_0$ from finite size [18].

In figure 8.1.4 an example is shown where results seems consistent across various finite matrix sizes N. However a linear extrapolation is taken for the limit $N^{-1} \to 0$ regardless.

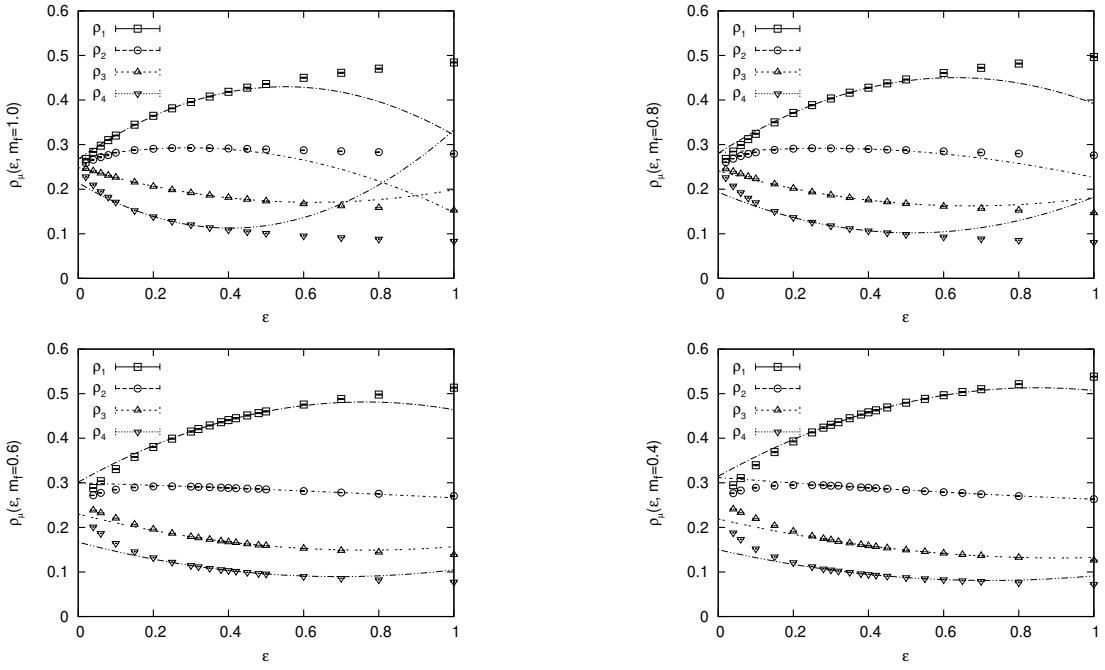


Figure 8.1.5.: The $\varepsilon \to 0$ limit with quadratic extrapolation $\beta_2 \varepsilon^2 + \beta_1 \varepsilon + \beta_0$ showing the symmetry breaking mechanism ΔS_{boson} in the Gaussian matrix model (8.1.1). As the fermion deformation $\Delta S_{\text{fermion}}$ starts to vanish, the extends of spacetime observables start to separate [18].

In figure 8.1.5, a breakdown of the rotational symmetry breaking of $\mathcal{X}$ is shown, using the $N^{-1} \to 0$ extrapolated values from each parameter run. A quadratic (fit) extrapolation is used to extract the $\varepsilon \to 0$ limits. An interesting effect is that, while the fitted polynomials extrapolate to separate values, the corresponding $\varepsilon > 0$ simulations seem to converge to one value. This seems to indicate that if the original model were to be simulated, the SO_4 spacetime symmetry would be preserved.

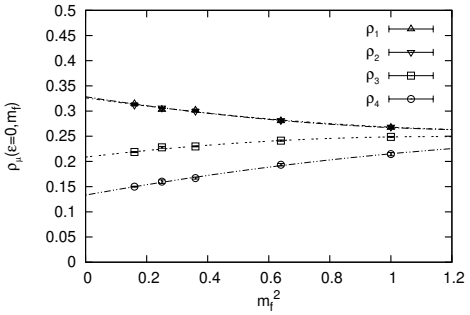


Figure 8.1.6.: The $m_{\text{fermion}} \to 0$ limit with quadratic extrapolation $\gamma_2 m_{\text{fermion}}^4 + \gamma_1 m_{\text{fermion}}^2 + \gamma_0$ on m_{fermion}^2, showing the surviving symmetry in the original model in the Gaussian matrix model (8.1.1) [18].

Finally, figure 8.1.6 the $\varepsilon \to 0$ extrapolated results are collected with respect to m_{fermion}, and by extrapolation, an estimate for the original undeformed model is obtained. Towards the bosonic limit, SO_4 seems to be restored, while towards the original model, SO_2 appears to be the maximal surviving symmetry, with the rest of spacetime extends shrunk to a smaller scale.

In particular, approximately,

$$\rho_1 = \rho_2 = 0.33,\ \rho_3 = 0.21 \text{ and } \rho_4 = 0.13, \tag{8.1.3}$$

which results relatively agree with the estimates (8.1.2) by the Gaussian Expansion Method on the SO_2 part of the $\mathrm{SO}_2 \times \mathbb{Z}_2$ ansatz.

8.2. Euclidean IKKT matrix model with $\dim \mathcal{X} = 6$

Gaussian Expansion Method

For $\dim \mathcal{X} = 6$, $\dim \mathcal{U} = 4$, m_{fermion} is recoded from a fermion mass matrix to a boson mass vector plus a self–dual 3–form via the Γ matrices as

$$S_0|_{\text{fermion}} = m_{\text{fermion}}|_\mu \overline{\psi} \Gamma_\mu \psi + m_{\text{fermion}}|_{\mu\nu\xi} \overline{\psi} \Gamma_\mu \Gamma_\nu^\dagger \Gamma_\xi \psi, \text{ with } m_{\text{fermion}}|_{\mu\nu\xi} = \imath m_{\text{fermion}}|_{[\mu\nu\xi]}, \tag{8.2.1}$$

where the self–duality

$$m_{\text{fermion}}|_{\mu\nu\xi} = \frac{1}{6} \imath \varepsilon_{\mu\nu\xi\mu'\nu'\xi'} m_{\text{fermion}}|_{\mu'\nu'\xi'},$$

manifests from the Weyl condition of ψ [74]. In this case the 6–dimensional IKKT bosonic term is used.

At this point it is important to elaborate the parameter counting of (8.2.1), as it is the most general form that can appear in IKKT model variants. In the general case, the rank–1 m_{fermion} always counts $\dim \mathcal{X}$ parameters. The 3–form m_{fermion} on the other hand as a rank–3 tensor originally has $(\dim \mathcal{X})^3$ parameters. As a 3–form however,

$$m_{\text{fermion}}|_{[\mu\nu\xi]} = m_{\text{fermion}}|_{\mu\nu\xi},$$

which means that the independent components come down to

$$\binom{\dim \mathcal{X}}{3} = \frac{(\dim \mathcal{X})!}{3!(\dim \mathcal{X} - 3)!} = \frac{1}{6} \dim \mathcal{X}(\dim \mathcal{X} - 1)(\dim \mathcal{X} - 2),$$

which is further trimmed–down by the aforementioned self–duality in the case of $\dim \mathcal{X} = 6$ by 2, resulting in $6 + 6 + 10$ total parametric freedom of S_0.[2]

[2] For $\dim \mathcal{X} = 6$,

$$\frac{1}{2} \frac{1}{6} \dim \mathcal{X}(\dim \mathcal{X} - 1)(\dim \mathcal{X} - 2) = 10.$$

In [22], ansätze $\mathrm{SO}_6 \to \mathrm{SO}_D$ with $D = 5$, $D = 4$ and $D = 3$ were explored, with the most prominent examples given in table 8.2.1. The corresponding free energy analysis shows a preference for SO_3 surviving, implying the spontaneous $\mathrm{SO}_6 \to \mathrm{SO}_3$ symmetry breaking.

	f	$\langle\rho_1\rangle$	$\langle\rho_2\rangle$	$\langle\rho_3\rangle$	$\langle\rho_4\rangle$	$\langle\rho_5\rangle$	$\langle\rho_6\rangle$
SO_5	−1.70472	0.18993	0.18993	0.18993	0.18993	0.18993	0.05037
SO_4	−1.79599	0.22999	0.22999	0.22999	0.22999	0.04190	0.03816
$\mathrm{SO}_4 \times \mathbb{Z}_2$	−1.78072	0.22365	0.22365	0.22365	0.22365	0.05952	0.04589
$\mathrm{SO}_3 \times \mathbb{Z}_3$	−1.81743	0.30457	0.30457	0.30457	0.02990	0.02990	0.02650
$\mathrm{SO}_3 \times \mathrm{SO}_2$	−1.84330	0.30497	0.30497	0.30497	0.02673	0.02673	0.03162

Table 8.2.1.: List of broken symmetries explored by GEM on $\dim \mathcal{X} = 6$ IKKT model in [22]. Identical values are grouped together to outline the surviving symmetries after the spontaneous symmetry breaking.

Following is a breakdown of the parametric freedom of fundamental ansätze.

SO_5 ansatz (3 parameters)

$m_{\text{boson}}|_1 = m_{\text{boson}}|_2 = m_{\text{boson}}|_3 = m_{\text{boson}}|_4 = m_{\text{boson}}|_5$. (2 parameters including $m_{\text{boson}}|_6$)
$m_{\text{fermion}}|_1 = m_{\text{fermion}}|_2 = m_{\text{fermion}}|_3 = m_{\text{fermion}}|_4 = m_{\text{fermion}}|_5 = 0$. (1 parameters including $m_{\text{fermion}}|_6$)
$\forall \mu, \nu, \xi \in \mathbb{N}_{\dim \mathcal{X}}$, $m_{\text{fermion}}|_{\mu\nu\xi} = 0$.

SO_4 ansatz (5 parameters)

$m_{\text{boson}}|_1 = m_{\text{boson}}|_2 = m_{\text{boson}}|_3 = m_{\text{boson}}|_4$. (3 parameters including $m_{\text{boson}}|_5$, $m_{\text{boson}}|_6$)
$m_{\text{fermion}}|_1 = m_{\text{fermion}}|_2 = m_{\text{fermion}}|_3 = m_{\text{fermion}}|_4 = 0$. (2 parameters including $m_{\text{fermion}}|_5$, $m_{\text{fermion}}|_6$)
$\forall \mu, \nu, \xi \in \mathbb{N}_{\dim \mathcal{X}}$, $m_{\text{fermion}}|_{\mu\nu\xi} = 0$.

SO_3 ansatz (8 parameters)

$m_{\text{boson}}|_1 = m_{\text{boson}}|_2 = m_{\text{boson}}|_3$. (4 parameters including $m_{\text{boson}}|_4$, $m_{\text{boson}}|_5$, $m_{\text{boson}}|_6$)
$m_{\text{fermion}}|_1 = m_{\text{fermion}}|_2 = m_{\text{fermion}}|_3 = 0$. (3 parameters including $m_{\text{fermion}}|_4$, $m_{\text{fermion}}|_5$, $m_{\text{fermion}}|_6$)
$\forall \mu, \nu, \xi \in \mathbb{N}_{\dim \mathcal{X}}$, $m_{\text{fermion}}|_{\mu\nu\xi} = 0$ except for:

- $m_{\text{fermion}}|_{123} = -\imath m_{\text{fermion}}|_{456}$. (1 parameters)

SO_2 ansatz (13 parameters)

$m_{\text{boson}}|_1 = m_{\text{boson}}|_2$. (5 parameters including $m_{\text{boson}}|_3$, $m_{\text{boson}}|_4$, $m_{\text{boson}}|_5$, $m_{\text{boson}}|_6$)
$m_{\text{fermion}}|_1 = m_{\text{fermion}}|_2 = 0$. (4 parameters including $m_{\text{fermion}}|_3$, $m_{\text{fermion}}|_4$, $m_{\text{fermion}}|_5$, $m_{\text{fermion}}|_6$)
$\forall \mu, \nu, \xi \in \mathbb{N}_{\dim \mathcal{X}}$, $m_{\text{fermion}}|_{\mu\nu\xi} = 0$ except for:

- $m_{\text{fermion}}|_{123} = -\imath m_{\text{fermion}}|_{456}$, (1 parameters)
- $m_{\text{fermion}}|_{124} = -\imath m_{\text{fermion}}|_{356}$, (1 parameters)
- $m_{\text{fermion}}|_{125} = -\imath m_{\text{fermion}}|_{346}$, (1 parameters)
- $m_{\text{fermion}}|_{126} = -\imath m_{\text{fermion}}|_{345}$. (1 parameters)

Complex Langevin Method

As part of this study, the next step was taken, by studying the 6–dimensional Euclidean IKKT model, which, based on the GEM result presented here [74], the spontaneous symmetry breaking is expected to be $SO_6 \longrightarrow SO_3$. In [14], $m_{\text{boson}} = (2^{-1}, 2^{-1}, 2^{0}, 2^{+1}, 2^{+2}, 2^{+3})$ and $\gamma = \Gamma_6$ were chosen. The rationale behind having a manifest minimal rotational symmetry is because it is unlikely to observe symmetry breaking beyond SO_2, based on the argument about degenerate configurations presented in section §7.2. Complex Langevin Method (CLM). Technically, the lowest expected surviving symmetry is in fact SO_3 based on that argument, however in order to observe it using Monte Carlo and the CLM, and assuming it is one of the possibilities (and in fact it is), SO_2 only is preserved in the symmetry breaking term ΔS_{boson}.

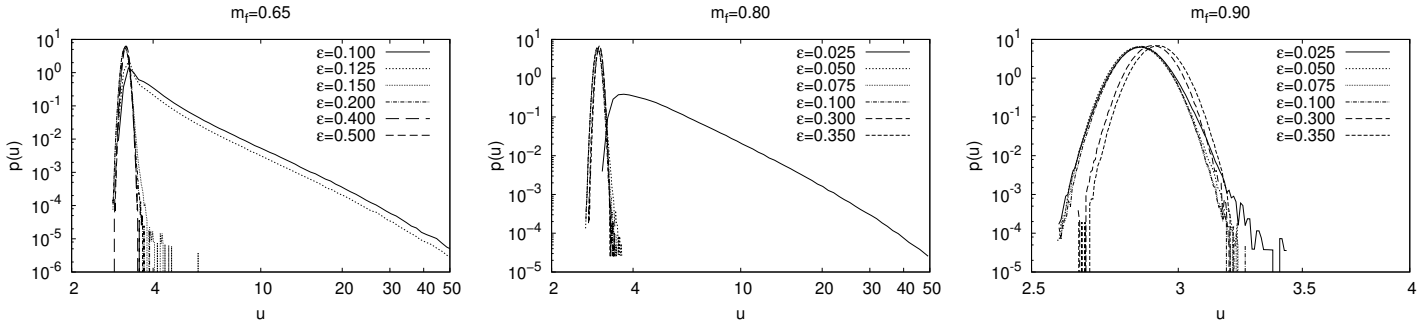


Figure 8.2.1.: Three sets of runs with varying ε and increasing m_{fermion} that show how reducing both leads to problematic drift fall–off for the 6–dimensional IKKT matrix model [14].

In figure 8.2.1 the effect of both vanishing ε and m_{fermion} on the drift of the Langevin (stochastic) process A is shown. While the effect of $m_{\text{fermion}} \to 0$ relates to the singularity of the drift, it appears low values of ε impede convergence of the process A as well.

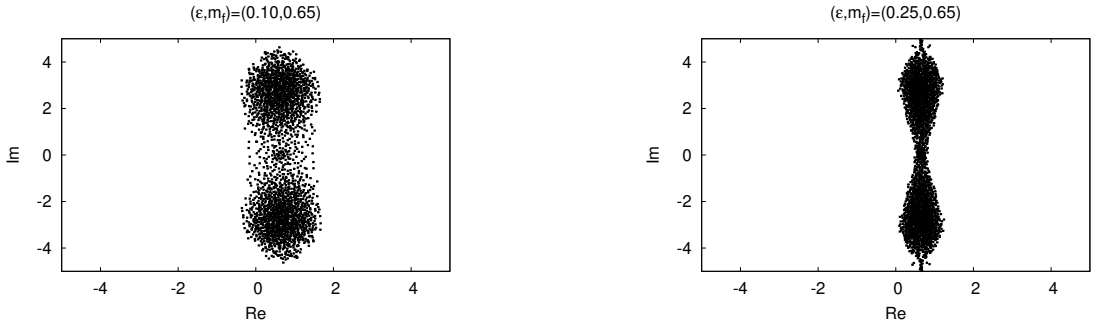


Figure 8.2.2.: A case for specific m_{fermion} in the deformed 6–dimensional IKKT matrix model, that shows that vanishing symmetry order parameter ε has a detrimental effect on the eigenvalue distribution of the fermion matrix as it is affected by the Langevin process A [14].

In figure 8.2.2, the effect of vanishing ε on the eigenvalue spectrum of the (deformed) fermion matrix is apparent as well, relating to the effect of $\varepsilon \to 0$ has on the drift norm.

In figure 8.2.3 an example is shown where results for finite matrix sizes N are once more linearly extrapolated to the limit $N^{-1} \to 0$. This time however, since SO_2 is expected to minimally survive, the respective observables $\langle\lambda_1\rangle$ and $\langle\lambda_2\rangle$ may be grouped together for more statistics on a unified observable $(\langle\lambda_1\rangle + \langle\lambda_2\rangle)/2$ or

$$\rho_{1+2} = \frac{\rho_1 + \rho_2}{2}.$$

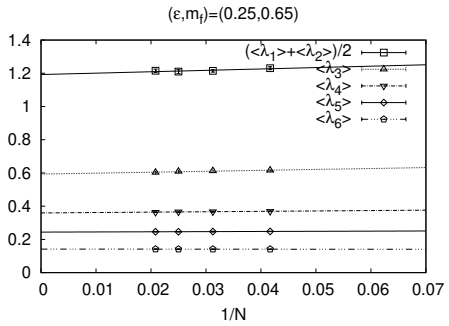


Figure 8.2.3.: The $N^{-1} \to 0$ limit or runs of the 6–dimensional IKKT matrix model using a linear extrapolation $\alpha_1 N^{-1} + \alpha_0$ from finite size [14].

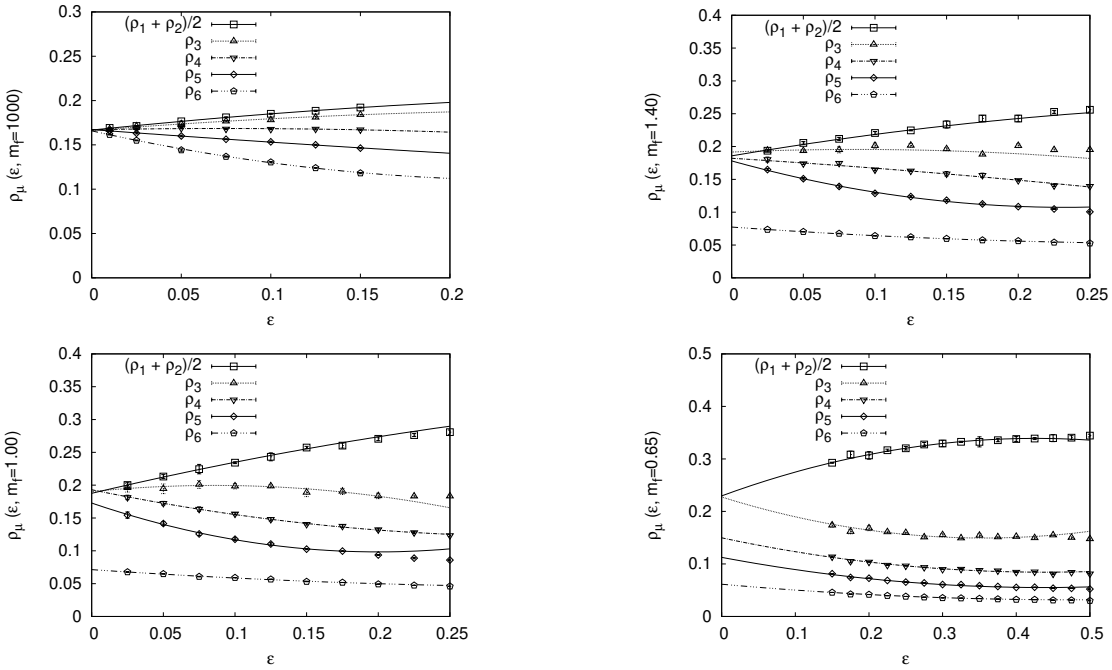


Figure 8.2.4.: The $\varepsilon \to 0$ limit with quadratic extrapolation $\beta_2 \varepsilon^2 + \beta_1 \varepsilon + \beta_0$ showing the symmetry breaking in the 6–dimensional IKKT matrix model, as the effect of ΔS_{boson} is reduced. As the fermion deformation $\Delta S_{\text{fermion}}$ starts to vanish, the extends of spacetime observables start to separate [14].

In figure 8.2.4, a breakdown of the rotational symmetry of $\mathcal{X}$ is shown, using the $N^{-1} \to 0$ extrapolated values from each parameter run. A quadratic (fit) extrapolation is used to extract the $\varepsilon \to 0$ limits. The near–bosonic model $m_{\text{fermion}} = 1000$ restores SO_6 at $\varepsilon \to 0$. As m_{fermion} becomes smaller, the restored symmetry becomes lesser; for 1.40 it is SO_5 that survives, while for 0.60 it is SO_3 that is restored, with all other extends of spacetime taking lower values than the three dominant extends.

Finally, in figure 8.2.5 the $\varepsilon \to 0$ extrapolated results are collected with respect to m_{fermion}. It is clear that ρ_{1+2} and ρ_3 converge to the value predicted by the GEM analysis of the 6–dimensional model, while ρ_4, ρ_5 and ρ_6 converge to the lower one [22]. Specifically, said values are

$$\rho_1(m_{\text{fermion}} \to 0) = \rho_2(m_{\text{fermion}} \to 0) = \rho_3(m_{\text{fermion}} \to 0) = 0.30,$$
$$\rho_4(m_{\text{fermion}} \to 0) = \rho_5(m_{\text{fermion}} \to 0) = \rho_6(m_{\text{fermion}} \to 0) = 0.035.$$

Grouping the observables are such, and using the same extrapolation scheme as in the original observables, the projected values obtained by the complex Langevin simulation are a little higher, [14],

$$\rho_1(m_{\text{fermion}} \to 0) = \rho_2(m_{\text{fermion}} \to 0) = \rho_3(m_{\text{fermion}} \to 0) = 0.33,$$
$$\rho_4(m_{\text{fermion}} \to 0) = \rho_5(m_{\text{fermion}} \to 0) = \rho_6(m_{\text{fermion}} \to 0) = 0.046.$$

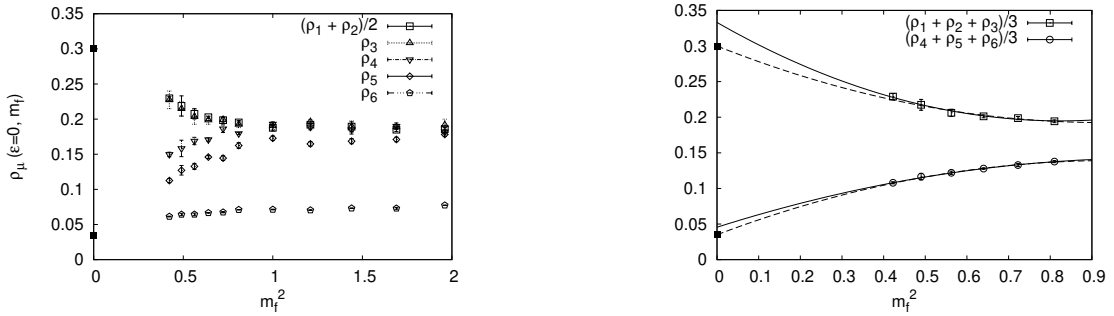


Figure 8.2.5.: The $m_{\text{fermion}} \to 0$ limit with quadratic extrapolation $\gamma_2 m_{\text{fermion}}^4 + \gamma_1 m_{\text{fermion}}^2 + \gamma_0$ on m_{fermion}^2, showing the surviving symmetry in the original 6–dimensional IKKT matrix model. The points at $m_{\text{fermion}} = 0$ represent the GEM results [14].

8.3. Euclidean IKKT matrix model with $\dim \mathcal{X} = 10$

Gaussian Expansion Method

For $\dim \mathcal{X} = 10$, $\dim \mathcal{U} = 16$, m_{fermion} is recoded from a fermion mass matrix to a (non self–dual) 3–form only (in contrast to $\dim \mathcal{X} = 6$) via the Γ matrices as

$$S_0|_{\text{fermion}} = m_{\text{fermion}}|_{\mu\nu\xi}\overline{\psi}\Gamma_\mu\Gamma_\nu^\dagger\Gamma_\xi\psi, \text{ with } m_{\text{fermion}}|_{\mu\nu\xi} = \imath m_{\text{fermion}}|_{[\mu\nu\xi]}, \tag{8.3.1}$$

In this case, only the 3–form symmetry $m_{\text{fermion}}|_{[\mu\nu\xi]} = m_{\text{fermion}}|_{\mu\nu\xi}$ affects parameter counting, yielding 120 free parameters for m_{fermion}. For m_{boson} it is 10 as expected.

In [23], ansätze $\mathrm{SO}_{10} \to \mathrm{SO}_D$ with $2 \leq D \leq 7$ were explored, where the parameter freedom of each ansatz was restricted to 5. This is possible $\forall D$ by imposing extra symmetries in the form of coordinate permutations in the remaining directions corresponding to the broken symmetry, with the only requirement that the resulting group is a subgroup of the original SO_{10} symmetry group. The corresponding free energy analysis shows a preference for broken symmetries with SO_3 surviving, implying the spontaneous $\mathrm{SO}_{10} \to \mathrm{SO}_3$ breaking.

Complex Langevin

The next and final step in studying the Euclidean IKKT model is the physical 10–dimensional model in which, based on the GEM result, the spontaneous symmetry breaking is expected to be $\mathrm{SO}_{10} \longrightarrow \mathrm{SO}_3$.

For the 10–dimensional model, $\gamma = \imath\Gamma_8\Gamma_9^\dagger\Gamma_{10}$ was chosen. This type of deformation manifestly breaks SO_{10} down to SO_7. As the fermion deformation term contains the last of the gamma matrices, and the observables λ are ordered before taking the expectation value $\langle\lambda\rangle$, the part involved in the spontaneous symmetry breaking is SO_7.

In [15],

$$m_{\text{boson}} = (2^{-1}, 2^{-1}, 2^{-1}, 2^0, 2^{+1}, 2^{+2}, 2^{+3}, 2^{+3}, 2^{+3}, 2^{+3}) \text{ for } m_{\text{fermion}} = 3 \text{ and}$$
$$m_{\text{boson}} = (2^{-1}, 2^{-1}, 2^0, 2^{+1}, 2^{+2}, 2^{+3}, 2^{+3}, 2^{+3}, 2^{+3}, 2^{+3}) \text{ for } m_{\text{fermion}} < 3$$

were chosen respectively. As shown by the 4–dimensional and 6–dimensional cases respectively, greater deformation fermion mass m_{fermion} leads to higher surviving symmetry at the extrapolated $\varepsilon \to 0$ limit, therefore for $m_{\text{fermion}} \geq 3$, the breaking terms were grouped accordingly to maximize statistics and focus on the dimensionality of the expected symmetry breaking. For lower m_{fermion} values, the minimal SO_2 is expected as discussed in section §7.2. Complex Langevin Method (CLM) [19].

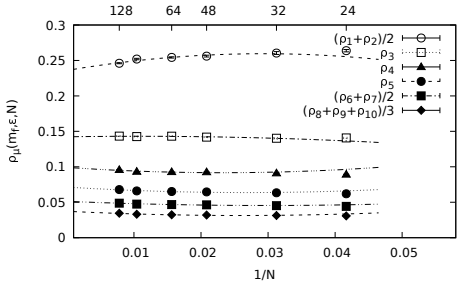


Figure 8.3.1.: The $N^{-1} \to 0$ limit of runs of the 10–dimensional IKKT matrix model with linear extrapolation $\alpha_1 N^{-1} + \alpha_0$ from finite size [15].

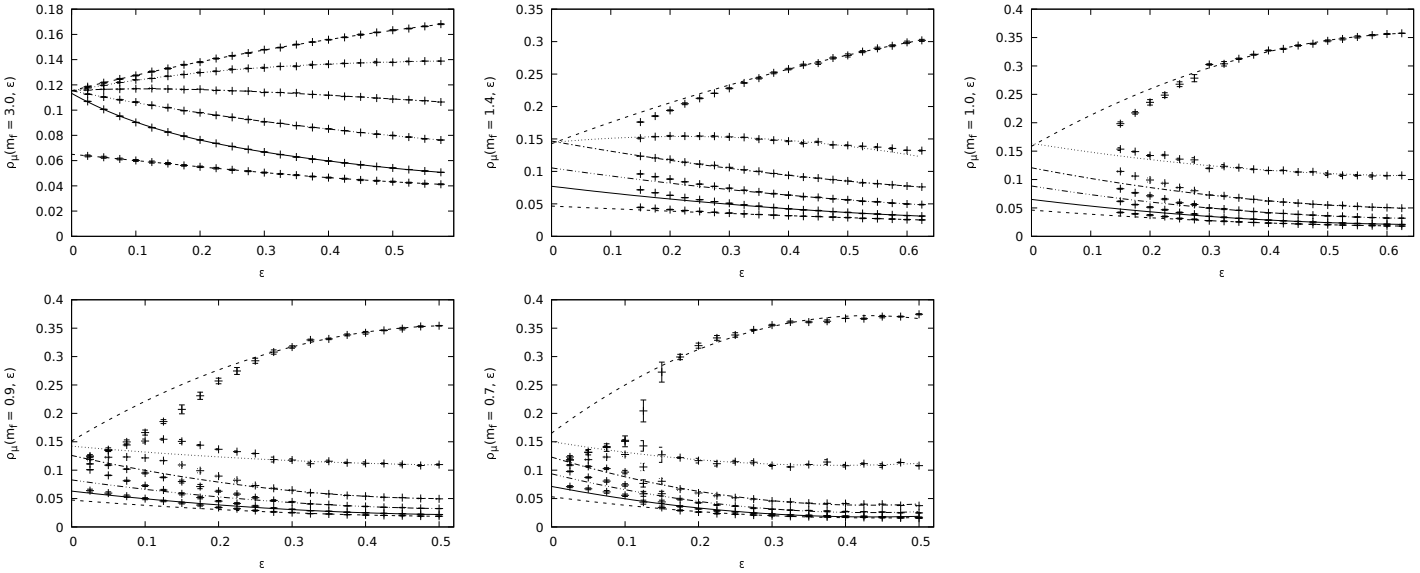


Figure 8.3.2.: The $\varepsilon \to 0$ limit with quadratic extrapolation $\beta_2 \varepsilon^2 + \beta_1 \varepsilon + \beta_0$ showing the symmetry breaking mechanism in the 10–dimensional IKKT matrix model as the effect of ΔS_{boson} is reduced. As the fermion deformation $\Delta S_{\text{fermion}}$ starts to vanish, the extends of spacetime observables start to separate [15].

The corresponding grouping of observables is

$$\frac{\rho_1+\rho_2+\rho_3}{3}, \rho_4, \rho_5, \rho_6, \rho_7, \frac{\rho_8+\rho_9+\rho_{10}}{3} \text{ for } m_{\text{fermion}} = 3, \text{ and}$$
$$\frac{\rho_1+\rho_2}{2}, \rho_3, \rho_4, \rho_5, \frac{\rho_6+\rho_7}{2}, \frac{\rho_8+\rho_9+\rho_{10}}{3} \text{ for } m_{\text{fermion}} < 3.$$

The same techniques used in the lower–dimensional models were used here, namely, monitoring the drift norm and discarding runs when the drift fall-off is not subexponential, gauge cooling and the ordered limit $N \to \infty$, $\varepsilon \to \infty$ and $m_{\text{fermion}} \to 0$.

In figure 8.3.1, an example with $m_{\text{fermion}} < 3$ is shown, where results for finite matrix sizes N are quadratically extrapolated to the limit $N^{-1} \to 0$, i.e. like $\alpha_2 N^{-2} + \alpha_1 N^{-1} + \alpha_0$.

In figure 8.3.2, a breakdown of the rotational symmetry of $\mathcal{X}$ is shown, using the $N^{-1} \to 0$ extrapolated values from each parameter run. A quadratic (fit) extrapolation is used to extract the $\varepsilon \to 0$ limits. At $m_{\text{fermion}} = 3$, the manifestly broken SO_7 remains intact. As m_{fermion} is reduced, the simulations themselves start to collapse back to a fully restored SO_{10} symmetry. Recall that the manifest $\text{SO}_{10} \to \text{SO}_7$ breaking is due to the fermion mass deformation, so this separation vanishes with m_{fermion} close to 0. Accounting only for the simulations before the aforementioned "collapse", the quadratic fits and corresponding extrapolations lead to a gradual separation of the observed extents in spacetime, with $m_{\text{fermion}} = 0.7$ exhibiting a restoration of a dominant SO_3 over all other smaller extents. This is in agreement to the GEM $\text{SO}_{10} \to \text{SO}_3$ result.

9. The Lorentzian IKKT matrix model

When studying the Euclidean IKKT model,[1] it has been shown with both the analytical GEM and the numerical CLM methods that the rotational symmetry SO_{10} breaks down to SO_3 instead of SO_4, and with the dominant spacetime extents being finite instead of infinite. This prompted the study of the Lorentzian IKKT model.

On the other hand, the Lorentzian IKKT model presents a few challenges in itself. By construction, all Monte Carlo studies of quantum field theories rely on the Wick–rotated Euclidean counterparts for a well–defined–as–a–probability Boltzmann factor $\exp(-S)$ in contrast to $\exp \imath S$. It has been shown in (but not restricted to) the Euclidean IKKT model, there are cases where even then, the action is complex and the corresponding Boltzmann factor is ill–defined as a probability. As the CLM offers an alternative stochastic process for sampling effectively the configuration space of such a theory, models with their original metric signature may become approachable again, as the Boltzmann factor they defined has a strong complex phase by construction.

However, the Boltzmann factor complex phase is not the only concern. Returning to the Lorentzian IKKT model, its action is unbounded from below, leading to divergences in the partition function [24, 25, 26, 27].

In this chapter, the primary technical elements of the Lorentzian IKKT CLM study are presented, and relevant simulation results included.

A note on index notation: greek indices starting from $\mu, \nu, \ldots$ refer to all spacetime components, which are raised/lowered by the Lorentzian metric η. Latin indices starting from $i, j, k \ldots$ refer to space components only which are contracted by the positive definite Kronecker δ and therefore positioning will be fixed in upper. Time index is labeled with 0. Matrix operator abbreviations like that of the trace will be used for the internal degrees of freedom only, which are indexed with latin starting from $a, b, c, d, \ldots$.

9.1. Background

Gauge fixing in the Lorentzian IKKT matrix model

The band–diagonal gauge fixing

It has been shown in section §7.2. Complex Langevin Method (CLM), that the bosonic matrices of the Euclidean IKKT model have an internal SU_N symmetry that can be exploited to modify the configuration search path during Monte Carlo simulations. For instance, in the Euclidean IKKT, the symmetry was used to search for a configuration that has minimal deviation from hermiticity, — a process labeled as gauge cooling [41, 42, 44] — to prevent the search path from spending too many simulation steps in the antihermitian direction..

An alternative gauge fixing stems from the idea of approximating simultaneous diagonalization of all bosonic matrices [7, 8]. As explained in section §6.3. Matrix models, the fundamental interpretation of the matrix model through Connes' operator approach to geometry [4], is dynamically generating spacetime via the $N \times N$ bosonic matrices representing N spacetime points. If all bosonic matrices were diagonalizable, this would result in a classical geometry with their eigenvalues serving as the spacetime points. In the case of the IKKT matrix model however (and in general), they are not, leading to a fuzzy geometry where not all $\dim \mathcal{X}$ coordinates of a spacetime point can be known simultaneously.

Much like in gauge cooling, where in the context of the CLM, hermiticity is approximated via the SU_N internal matrix symmetry, the same can be done towards approximating simultaneous diagonalization, by bringing all $\dim \mathcal{X}$ bosonic matrices

[1] See chapter 7. Methodology and references therein for more.

to as thin a band–diagonal structure as possible. [6] devised an $\mathrm{SO}_{\dim \mathcal{X}-1,1}$ (Lorentz) and SU_N symmetric norm

$$\Delta^2 = N^{-1} \sum_{n=1}^{N} (A_\mu A_\mu)_{nn} - \max_{U \in \mathrm{SU}_N} \left(N^{-1} \sum_{n=1}^{N} (U A_\mu U^\dagger)_{nn} (U A_\mu U^\dagger)_{nn} \right)$$

measuring proximity to simultaneous diagonalization of all bosonic matrices, also used in [7, 8].

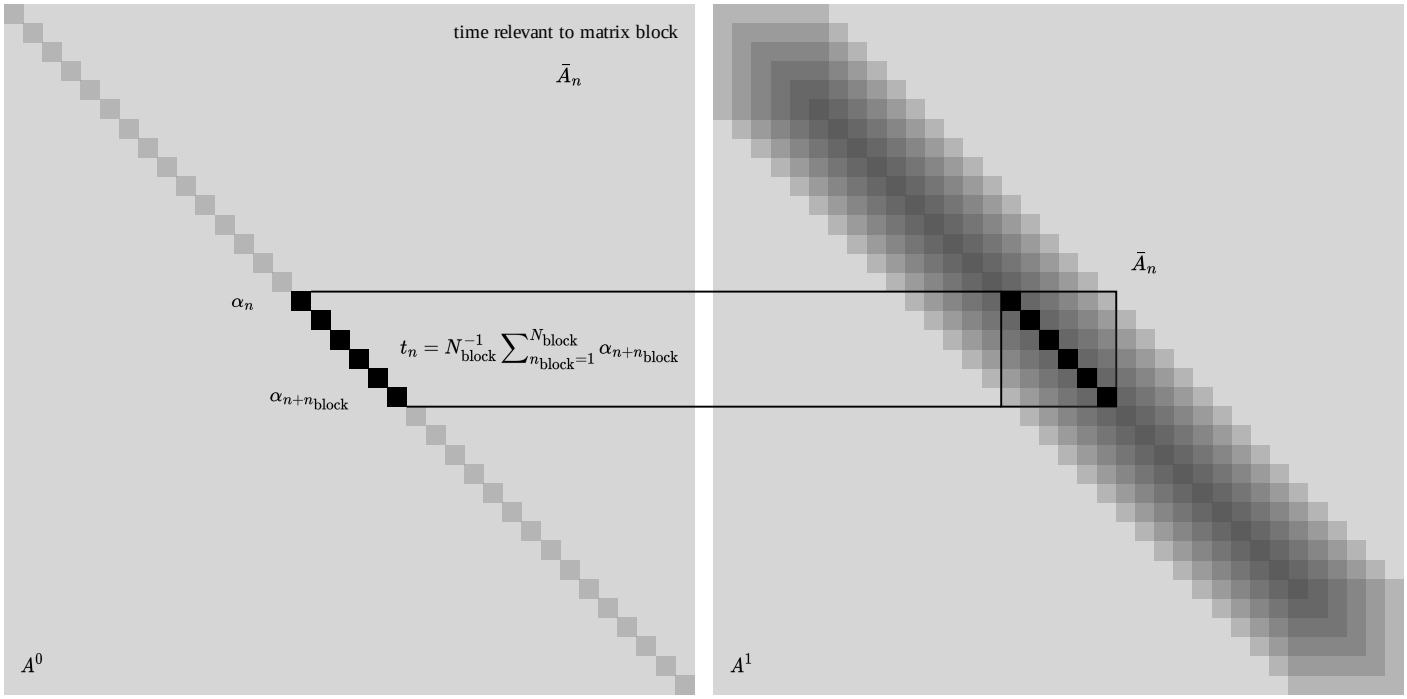


Figure 9.1.1.: Two approaches of matching time with space, one matching all spatial blocks that contain the indices of a particular time and one matching all times with indices of a particular spatial block.

The time matrix A_0

As demonstrated in section §7.2. Complex Langevin Method (CLM), the SU_N — becoming SL_N after complexification — gauge symmetry of the N–size Euclidean IKKT matrix model is exploited to select a gauge that minimized the hermiticity deviation of the bosonic matrices all over the CLM configuration path. The Lorentzian IKKT model presents another possibility. The fundamental aspect of the non–commutative geometry of the IKKT is that $\forall \mu \in \mathbb{Z}_{\dim \mathcal{X}}$, A^μ are pair–wise not simultaneously diagonalizable, which also translates to the fact that the coordinates diagonalizing one of them always leaves the others non–diagonal. Since time is special in the Lorentzian IKKT model in that it has a different sign in the signature, it becomes apparent that the gauge can be fixed by diagonalizing the time bosonic matrix,

$$A^0 = \operatorname{diag} \alpha = \begin{pmatrix} \alpha_1 & & \cdots & \\ & \alpha_2 & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ & & \cdots & \alpha_N \end{pmatrix}$$

Reordering is allowed within a gauge symmetry, therefore the eigenvalues of A^0 are chosen in ascending order, resembling time advance.

In this gauge, the spatial bosonic matrices are generally expected to have a band–diagonal structure of block size $N_{\mathrm{block}} \in \mathbb{N}_N$ [76]. In practice what is observed in simulations is that the structure is approximately band–diagonal, with off–band-diagonal

elements being below a practical threshold, which in turn defines N_{block}. This allows the interpretation of a possible time and space matching, two mainstream possibilities of which are shown in figure 9.1.1, the latter of which is applied in [24, 25, 26, 27] as well as in our current work.

Assuming a fixed block size N_{block}, let $\forall n \in \mathbb{N}_{n-n_{\text{block}}+1}$, $\bar{A}_n$ be the nth block of a band–diagonal bosonic matrix A,[2] i.e. such that $\forall a_{\text{block}}, b_{\text{block}} \in \mathbb{N}_{n_{\text{block}}}$,

$$\bar{A}_n|_{a_{\text{block}},b_{\text{block}}} = A_{n+a_{\text{block}},n+b_{\text{block}}}.$$

The time matching such a block is defined as the average time indexed by the block,

$$\bar{\alpha}_n = N_{\text{block}}^{-1} \sum_{n_{\text{block}}=1}^{N_{\text{block}}} \alpha_{n+n_{\text{block}}},$$

as

$$t_n = \sum_{i=1}^{n} |\Delta\bar{\alpha}_i|,\ \Delta\bar{\alpha}_i = \bar{\alpha}_{i+1} - \bar{\alpha}_i \tag{9.1.1}$$

The (s, k) IKKT space

Wick rotation

When studying quantum field theories via Monte Carlo methods that rely on the Boltzmann factor of the partition function, Wick rotation to a Euclidean version may be necessary, because in the original theory, said partition function is formally

$$Z_{\text{lorentzian}} = \int \exp(\imath S) \text{ becoming } Z_{\text{euclidean}} = \int \exp(-S),$$

after a Wick rotation to imaginary time.

In [77] the relationship of the two versions of the IKKT model are first explored simplified down to the pure bosonic model to reduce computational complexity, to allow an exhaustive exploration of the theory space. The bosonic action

$$S_{\text{boson}} = \frac{1}{4} N \operatorname{tr}_{T\times\Sigma} F^{\mu\nu} F_{\mu\nu} = N\left(\frac{1}{4}\operatorname{tr}(F_{ij}F_{ij}) - \frac{1}{2}\operatorname{tr}(F_{0i}F_{0i})\right) \text{ with } F_{\mu\nu} = \imath[A_\mu|A_\nu]$$

is real, therefore the corresponding Boltzmann factor is a pure complex phase. To circumvent the problem, a double–parametric extension of the Wick rotation has been introduced in [48],

$$S_{\text{boson}} = -\imath N\beta \exp\left(\imath s\frac{\pi}{2}\right)\left(\frac{1}{4}\operatorname{tr}(F_{ij}F_{ij}) - \frac{1}{2}\exp(-\imath k\pi)\operatorname{tr}(F_{0i}F_{0i})\right), \tag{9.1.2}$$

such that the partition function is written by the corresponding Euclidean convention

$$Z = \int dA \exp(-S_{\text{boson}})$$

such that it formally has the form of a well–defined Boltzmann factor (ignoring the fact that S_{boson} may be complex. β contains the global coupling constant of the model; usually $\beta = g^{-2}N^{-1}$. Parameter s corresponds to a Wick rotation in the worldsheet while k corresponds to a Wick rotation in the target space.[3]

$(s, k) = (0, 0)$ corresponds to the Lorentzian IKKT matrix model while $(s, k) = (1, 1)$ corresponds to the Euclidean one. This is evident by the form of the Wick rotated $\tilde{A}_\mu$, $\forall\mu$,

$$A_0 = \exp\left(\imath(s-4k)\frac{\pi}{8}\right)\tilde{A}_0 \text{ and } \forall i,\ A_i = \exp\left(\imath s\frac{\pi}{8}\right)\tilde{A}_i.$$

In general s and k define a parametric theory space with the Lorentzian and Euclidean being special cases. In figure 9.1.2, the full phase diagram for s and k is shown, along with domains that lead to ill–defined models.

[2] Suppressing the bosonic index μ here for simplicity.

[3] While the model is zero–volume, it carries the spacetime signature of the corresponding type IIB superstring theory (or the corresponding super Yang–Mills theory). It is the spacetime signature on the IKKT matrix model that is affected by k.

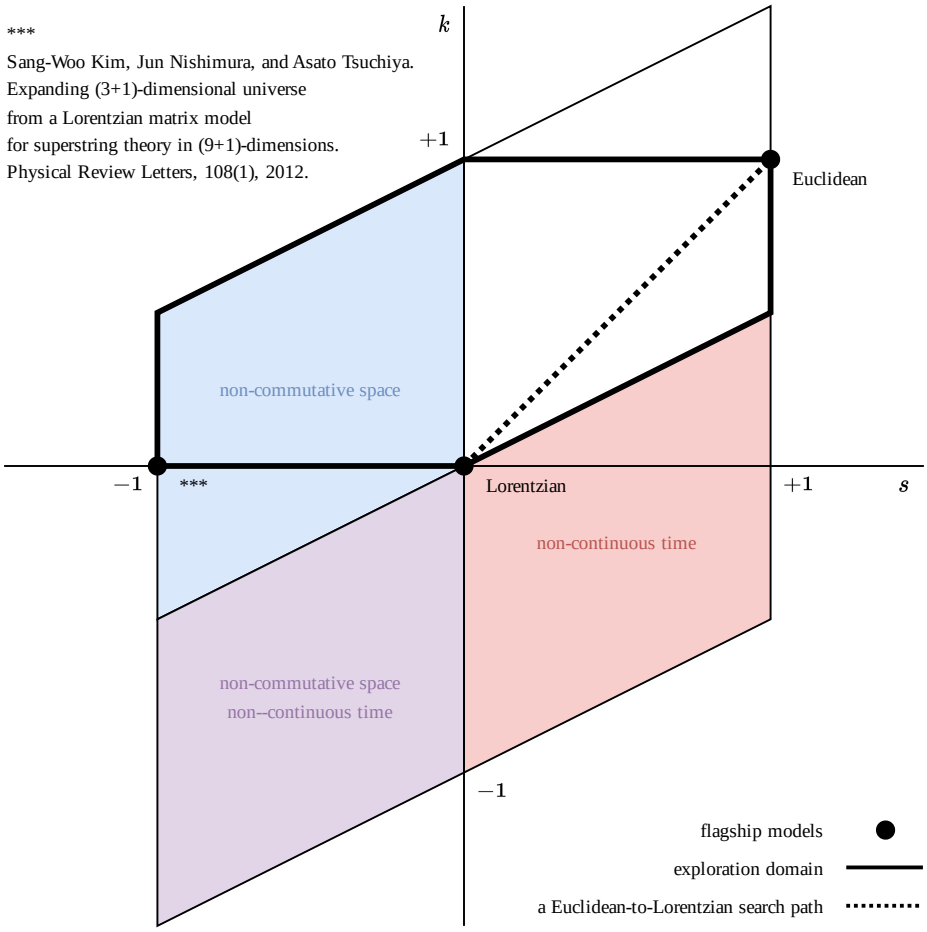


Figure 9.1.2.: The phase diagram of the parameter s and k space prescribing the Wick rotation of the IKKT model [48]. The Wick rotation is periodic beyond the region shown.

The decisive factor is the sign of $\Re S$. Let the Lorentzian IKKT action be

$$S_{\text{boson}} = S_{\text{time}} + S_{\text{space}} \propto \frac{1}{2}\beta \exp\left(\imath(s+1-2k)\frac{\pi}{2}\right) \operatorname{tr}(F_{0i}F_{0i}) + \frac{1}{4}N\beta \exp\left(\imath(s-1)\frac{\pi}{2}\right) \operatorname{tr}(F_{ij}F_{ij}).$$

S_{space} promotes space non–commutativity, while S_{time} promotes a more band–diagonal form for A.

According to (4.4.5), $\Re S_{\text{time}} \geq 0$ becomes

$$\cos\left(|s-1|\frac{\pi}{2}\right) \geq 0 \text{ means } |s-1|\frac{\pi}{2} \leq \frac{\pi}{2} \text{ or } 0 \leq s \leq 2,$$

and $\Re S_{\text{space}} \geq 0$ becomes

$$\cos\left(|s+1-2k|\frac{\pi}{2}\right) \geq 0 \text{ means } |s+1-2k|\frac{\pi}{2} \leq \frac{\pi}{2} \text{ or } \frac{1}{2}s \leq k \leq \frac{1}{2}s+1.$$

The real part signs are given in the table below based on the distinct phase of each term in the action:

s condition	k condition	$\Re S_{\text{time}}$	$\Re S_{\text{space}}$
$0 \leq s \leq +1$	$s/2 \leq k \leq s/2+1$	$+$	$+$
$0 \leq s \leq +1$	$s/2-1 \leq k \leq s/2$	$+$	$-$
$-1 \leq s \leq 0$	$s/2 \leq k \leq s/2+1$	$-$	$+$
$-1 \leq s \leq 0$	$s/2-1 \leq k \leq s/2$	$-$	$-$

In the single–parametric theory subspace defined by $s = k = u$, the Wick rotation takes the form

$$A_0 = \exp\left(-\imath u\frac{3\pi}{8}\right)\tilde{A}_0 \text{ and } \forall i,\ A_i = \exp\left(\frac{1}{8}\imath\pi u\right)\tilde{A}_i, \tag{9.1.3}$$

with $\tilde{F}_{\mu\nu} = \imath[\tilde{A}_\mu|\tilde{A}_\nu]$. In this subspace, $u = 0$ corresponds to the Lorentzian version and $u = 1$ to the Euclidean one. Let

$$\tilde{\lambda}_\mu = N^{-1}\operatorname{tr}\tilde{A}_\mu^2$$

By (9.1.3),

$$\langle\lambda_0\rangle = \exp\left(-\imath u\frac{3\pi}{8}\right)\langle\tilde{\lambda}_0\rangle \text{ and } \langle\lambda_i\rangle = \exp\left(\imath u\frac{\pi}{8}\right)\langle\tilde{\lambda}_i\rangle. \tag{9.1.4}$$

The general concept behind this dual parametrization is to allow getting results for the Lorentzian model (at $s = 0$ and $k = 0$), by simulating variants in the dual parameter space approaching the Lorentzian model along a path starting from a well defined variant, like the Euclidean model, as was done in [77].

The real time constraint

In order to obtain real time in the Lorentzian IKKT matrix model,[4] the

$$\alpha_N = \sqrt{\kappa} \in \mathbb{C}$$

constraint is imposed via an even polynomial action deformation

$$\Delta S_\alpha = \frac{1}{4}\gamma_\alpha(\alpha_N - \sqrt{\kappa})^4,$$

which, however, breaks (9.1.3) and consequently (9.1.4) as well. This constraint stems from the infrared cutoff

$$\lambda_0 = \kappa \text{ and } \forall i,\ \lambda_i = 1,$$

introduced to moderate the unboundedness of the Lorentzian IKKT bosonic action [78].

The logarithmic time

The complex Langevin method is applied to this model as is founded in chapter 4. Stochastic Quantization and applied the same way as in the Euclidean version, as in chapter 7. Methodology.

The ascending α sequence is encoded by a logarithmic time–increment sequence τ as

$$\alpha_b = \sum\nolimits_{c<b} \exp\tau_c,$$

[4] In the sense that the Euclidean IKKT matrix model has imaginary time.

with Jacobian

$$\frac{\partial \alpha_b}{\partial \tau_a} = \begin{cases} \exp \tau_a = \alpha_a & a < b \in \mathbb{N} \\ 0 & a \geq b \in \mathbb{N} \end{cases},$$

that defines the τ–derivative

$$\frac{\partial}{\partial \tau_a} = \sum\nolimits_b \frac{\partial \alpha_b}{\partial \tau_a} \frac{\partial}{\partial \alpha_b} = \alpha_a \sum\nolimits_{b>a} \frac{\partial}{\partial \alpha_b}.$$

Of interest is the τ–derivative of an α–difference,

$$\frac{\partial}{\partial \tau_a}(\alpha_b - \alpha_c) = \begin{cases} +\alpha_a & c \leq a < b \\ 0 & \text{otherwise: } a < b, c \text{ or } b, c \leq a \\ -\alpha_a & b \leq a < c \end{cases},$$

or

$$\frac{\partial}{\partial \tau_a}(\alpha_b - \alpha_c) = \operatorname{sign}(\alpha_b - \alpha_c) \begin{cases} \alpha_a & c \leq a < b \text{ or } b \leq a < c \\ 0 & a < b, c \text{ or } b, c \leq a \end{cases}.$$

Since $\operatorname{sign}(\alpha_b - \alpha_c)^2 = 1$, the τ–derivative of the absolute α–difference is

$$\frac{\partial}{\partial \tau_a}|\alpha_b - \alpha_c| = \operatorname{sign}(\alpha_b - \alpha_c) \frac{\partial}{\partial \tau_a}(\alpha_b - \alpha_c) = \begin{cases} \alpha_a & c \leq a < b \text{ or } b \leq a < c \\ 0 & a < b, c \text{ or } b, c \leq a \end{cases}.$$

These formulas are essential in calculating the drift term of the complex Langevin equations of the bosonic Lorentzian IKKT model.

The bosonic action in the time gauge[5]

The various quantities of S_{boson} are calculated here on the assumption of a diagonal $A^0 = \operatorname{diag} \alpha$ matrix;[6] in index notation it is

$$A^0_{ab} = \alpha_a \delta_{ab} = \delta_{ab} \alpha_b.$$

Let the bosonic squared terms be,

$$C^{ij} = A^i A^j \text{ or } C^{ij}_{ab} = \sum\nolimits_c A^i_{ac} A^j_{cb} \text{ and } D = \sum\nolimits_i C^{ii} \text{ or } D_{ab} = \sum\nolimits_i C^{ii}_{ab}.$$

The bosonic matrix commutators with time components are,

$$[A^0|A^i]_{ab} = C^{0i}_{ab} - C^{0i}_{ba} = \sum\nolimits_c A^0_{ac} A^i_{cb} - \sum\nolimits_c A^i_{ac} A^0_{cb} = \sum\nolimits_c \alpha_a \delta_{ac} A^i_{cb} - \sum\nolimits_c A^i_{ac} \delta_{cb} \alpha_b = \alpha_a A^i_{ab} - A^i_{ab} \alpha_b = (\alpha_a - \alpha_b) A^i_{ab}.$$

The bosonic Jacobi terms with time components are,

$$\begin{aligned} [A^0|[A^0|A^i]]_{ab} &= \sum\nolimits_c A^0_{ac} [A^0|A^i]_{cb} - \sum\nolimits_c [A^0|A^i]_{ac} A^0_{cb} \\ &= \sum\nolimits_c A^0_{ac} (\alpha_c - \alpha_b) A^i_{cb} - \sum\nolimits_c (\alpha_a - \alpha_c) A^i_{ac} A^0_{cb} = \sum\nolimits_c \alpha_a \delta_{ac} (\alpha_c - \alpha_b) A^i_{cb} - \sum\nolimits_c (\alpha_a - \alpha_c) A^i_{ac} \delta_{ab} \alpha_b \\ &= \alpha_a (\alpha_a - \alpha_b) A^i_{ab} - (\alpha_a - \alpha_b) A^i_{ab} \alpha_b = (\alpha_a - \alpha_b)^2 A^i_{ab}. \end{aligned}$$

The bosonic moment of inertia with time components is,

$$\begin{aligned} \sum\nolimits_i \operatorname{tr} F^{0i} F^{0i} &= \sum\nolimits_i \sum\nolimits_a \sum\nolimits_b F^{0i}_{ab} F^{0i}_{ba} = -\sum\nolimits_i \sum\nolimits_a \sum\nolimits_b [A^0|A^i]_{ab} [A^0|A^i]_{ba} \\ &= -\sum\nolimits_i \sum\nolimits_a \sum\nolimits_b (\alpha_a - \alpha_b) A^i_{ab} (\alpha_b - \alpha_a) A^i_{ba} = \sum\nolimits_i \sum\nolimits_a \sum\nolimits_b (\alpha_a - \alpha_b)^2 A^i_{ab} A^i_{ba}. \end{aligned} \quad (9.1.5)$$

[5] Note that, in this subsection only, for the range of calculations in the index–less notation, derivation with the transpose is implied instead of being explicitely stated, i.e. $\forall A$ square matrix,

$$\left.\frac{\partial}{\partial A}\right|_{ab} = \frac{\partial}{\partial A_{ba}},$$

as this is the only derivation used in the calculations, and cluttered notation is avoided this way.

[6] For index notational clarity, the spacetime indices will be rendered raised in this subsection.

The bosonic Jacobi terms with space components are,

$$\sum\nolimits_j [A^j|[A^j|A^i]] = \sum\nolimits_j (A^j[A^j|A^i] - [A^j|A^i]A^j) = \sum\nolimits_j (A^j(A^jA^i - A^iA^j) - (A^jA^i - A^iA^j)A^j)$$
$$= \sum\nolimits_j (A^jA^jA^i - A^jA^iA^j - A^jA^iA^j + A^iA^jA^j) = DA^i - \sum\nolimits_j C^{ij}A^j + A^iD - \sum\nolimits_j A^jC^{ji}.$$

The bosonic moment of inertia with space components is,

$$\sum\nolimits_i \sum\nolimits_j \operatorname{tr} F^{ij}F^{ij} = \operatorname{tr} \sum\nolimits_i \sum\nolimits_j [A^i|A^j][A^i|A^j] = \operatorname{tr} \sum\nolimits_i \sum\nolimits_j (A^iA^j - A^jA^i)(A^iA^j - A^jA^i)$$
$$= \operatorname{tr} \sum\nolimits_i \sum\nolimits_j (A^iA^jA^iA^j - A^iA^jA^jA^i - A^jA^iA^iA^j + A^jA^iA^jA^i)$$
$$= \operatorname{tr} \Big(\sum\nolimits_i \sum\nolimits_j C^{ij}C^{ij} - \sum\nolimits_i A^iDA^i - \sum\nolimits_j A^jDA^j + \sum\nolimits_i \sum\nolimits_j C^{ji}C^{ji} \Big)$$
$$= 2 \operatorname{tr} \sum\nolimits_i \Big(\sum\nolimits_j C^{ij}C^{ij} - A^iDA^i \Big) = 2 \Big(\sum\nolimits_i \sum\nolimits_j \operatorname{tr} C^{ij}C^{ij} - \operatorname{tr} DD \Big), \quad (9.1.6)$$

because matrices in a product can cycle when traced.

Let the function

$$\Delta(\alpha) = \prod\nolimits_{a \le N} \prod\nolimits_{b<a} (\alpha_a - \alpha_b)$$

accumulate all forward time differences. Then the effective action of the model reads

$$S_{\text{effective}} = S_{\text{boson}} - \sum\nolimits_{a \le N} \sum\nolimits_{b \ne a} \log|\alpha_a - \alpha_b| - \sum\nolimits_{a<N} \tau_a = S_{\text{boson}} - 2\log\Delta(\alpha) - \sum\nolimits_{a<N} \tau_a.$$

Time drift

The time derivative of time component (9.1.5) is

$$\frac{\partial}{\partial \tau_a} \operatorname{tr}[A^0|A^i]^2 = -\frac{\partial}{\partial \tau_a} \sum\nolimits_b \sum\nolimits_c (\alpha_b - \alpha_c)^2 A^i_{bc}A^i_{cb} = -\sum\nolimits_b \sum\nolimits_c A^i_{bc}A^i_{cb} \frac{\partial}{\partial \tau_a}(\alpha_b - \alpha_c)^2$$
$$= -2 \sum\nolimits_b \sum\nolimits_c (\alpha_b - \alpha_c) A^i_{bc}A^i_{cb} \frac{\partial}{\partial \tau_a}(\alpha_b - \alpha_c)$$
$$= -2\alpha_a \sum\nolimits_{b>a} \sum\nolimits_{c \le a} (\alpha_b - \alpha_c) A^i_{bc}A^i_{cb} + 2\alpha_a \sum\nolimits_{b \le a} \sum\nolimits_{c<a} (\alpha_b - \alpha_c) A^i_{bc}A^i_{cb}$$
$$= -2\alpha_a \sum\nolimits_{b>a} \sum\nolimits_{c \le a} (\alpha_b - \alpha_c) A^i_{bc}A^i_{cb} + 2\alpha_a \sum\nolimits_{c \le a} \sum\nolimits_{b<a} (\alpha_c - \alpha_b) A^i_{cb}A^i_{bc}$$
$$= -4\alpha_a \sum\nolimits_{b>a} \sum\nolimits_{c \le a} (\alpha_b - \alpha_c) A^i_{bc}A^i_{cb},$$

therefore

$$\frac{\partial}{\partial \tau_a} S_{\text{boson}} = 2\imath N\beta \exp\left(\imath s \frac{\pi}{2}\right) \exp(-\imath k\pi)\alpha_a \sum\nolimits_i \sum\nolimits_{b>a} \sum\nolimits_{c \le a} (\alpha_b - \alpha_c) A^i_{bc}A^i_{cb}.$$

The effective action volume term is

$$\log\Delta(\alpha) = \frac{1}{2} \sum\nolimits_a \sum\nolimits_{b \ne a} \log|\alpha_a - \alpha_b|.$$

Since $c \le a < b$ or $b \le a < c$ implies $b \ne c$, while $c < b$ implies $a_c \le a_b$,

$$\frac{\partial}{\partial \tau_a} \log\Delta(\alpha) = \frac{1}{2} \sum\nolimits_b \sum\nolimits_{c \ne b} \frac{\partial}{\partial \tau_a} \log|\alpha_b - \alpha_c| = \sum\nolimits_b \sum\nolimits_{c \ne b} \frac{1}{|\alpha_b - \alpha_c|} \frac{\partial}{\partial \tau_a} |\alpha_b - \alpha_c|$$
$$= \alpha_a \sum\nolimits_{b>a} \sum\nolimits_{c \le a} \frac{1}{|\alpha_b - \alpha_c|} + \alpha_a \sum\nolimits_{b \le a} \sum\nolimits_{c<a} \frac{1}{|\alpha_b - \alpha_c|}$$
$$= \alpha_a \sum\nolimits_{b>a} \sum\nolimits_{c \le a} \frac{1}{|\alpha_b - \alpha_c|} + \alpha_a \sum\nolimits_{c \le a} \sum\nolimits_{b<a} \frac{1}{|\alpha_c - \alpha_b|}$$
$$= 2\alpha_a \sum\nolimits_{b>a} \sum\nolimits_{c \le a} \frac{1}{\alpha_b - \alpha_c}.$$

Finally,

$$\frac{\partial}{\partial\tau_a}\sum\nolimits_{b<N}\tau_b = 1-\delta_{aN}.$$

The effective action time derivative is,

$$\frac{\partial}{\partial\tau_a}S_{\text{effective}} = N\beta\exp\left(\imath\frac{\pi}{2}(2+1-2k)\right)\alpha_a\left(\sum\nolimits_{b>a}\sum\nolimits_{c\leq a}\left((\alpha_b-\alpha_c)\sum\nolimits_i A^i_{bc}A^i_{cb}-\frac{2}{\alpha_b-\alpha_c}\right)-1+\delta_{aN}\right),$$

Space drift9.1.3

The space derivative of time component (9.1.5) is

$$-\frac{\partial}{\partial A^i_{ba}}\sum\nolimits_j \operatorname{tr}[A^0|A^j]^2 = 2\sum\nolimits_j\sum\nolimits_c\sum\nolimits_d(\alpha_c-\alpha_d)^2\frac{\partial A^j_{cd}}{\partial A^i_{ba}}A^j_{dc} = 2\sum\nolimits_j\sum\nolimits_c\sum\nolimits_d(\alpha_c-\alpha_d)^2\delta_{ij}\delta_{bc}\delta_{ad}A^j_{dc} = 2(\alpha_a-\alpha_b)^2A^i_{ab}$$

or

$$-\frac{\partial}{\partial A^i}\sum\nolimits_j \operatorname{tr}[A^0|A^j]^2 = 2[A^0|[A^0|A^i]].$$

The space derivative of space component (9.1.6) is,[7]

$$\frac{\partial C^{jk}_{cd}}{\partial A^i_{ba}} = \sum\nolimits_e\left(\frac{\partial A^j_{ce}}{\partial A^i_{ba}}A^k_{ed}+A^j_{ce}\frac{\partial A^k_{ed}}{\partial A^i_{ba}}\right) = \sum\nolimits_e(\delta_{ij}\delta_{bc}\delta_{ae}A^k_{ed}+A^j_{ce}\delta_{ik}\delta_{be}\delta_{ad}) = \delta_{ij}\delta_{bc}A^k_{ad}+A^j_{cb}\delta_{ik}\delta_{ad},$$

and

$$\begin{aligned}\sum\nolimits_j\sum\nolimits_k\sum\nolimits_c\sum\nolimits_d\frac{\partial C^{jk}_{cd}C^{jk}_{dc}}{\partial A^i_{ba}} &= \sum\nolimits_j\sum\nolimits_k\sum\nolimits_c\sum\nolimits_d\left(\frac{\partial C^{jk}_{cd}}{\partial A^i_{ba}}C^{jk}_{dc}+C^{jk}_{cd}\frac{\partial C^{jk}_{dc}}{\partial A^i_{ba}}\right)\\ &= \sum\nolimits_j\sum\nolimits_k\sum\nolimits_c\sum\nolimits_d((\delta_{ij}\delta_{bc}A^k_{ad}+A^j_{cb}\delta_{ik}\delta_{ad})C^{jk}_{dc}+C^{jk}_{cd}(\delta_{ij}\delta_{bd}A^k_{ac}+A^j_{db}\delta_{ik}\delta_{ac}))\\ &= \sum\nolimits_j\sum\nolimits_k\sum\nolimits_c\sum\nolimits_d(\delta_{ij}\delta_{bc}A^k_{ad}C^{jk}_{dc}+A^j_{cb}\delta_{ik}\delta_{ad}C^{jk}_{dc}+\delta_{ij}\delta_{bd}A^k_{ac}C^{jk}_{cd}+A^j_{db}\delta_{ik}\delta_{ac}C^{jk}_{cd})\\ &= \sum\nolimits_k\sum\nolimits_d A^k_{ad}C^{ik}_{db}+\sum\nolimits_j\sum\nolimits_c C^{ji}_{ac}A^j_{cb}+\sum\nolimits_k\sum\nolimits_c A^k_{ac}C^{ik}_{cb}+\sum\nolimits_j\sum\nolimits_d C^{ji}_{ad}A^j_{db}\\ &= 2\sum\nolimits_j\sum\nolimits_c(A^j_{ac}C^{ij}_{cb}+C^{ji}_{ac}A^j_{cb}) = 2\sum\nolimits_j\sum\nolimits_c(A^j_{ac}C^{ij}_{cb}+C^{ji}_{ac}A^j_{cb}),\end{aligned}$$

or

$$\frac{\partial}{\partial A^i}\operatorname{tr}C^{jk}C^{jk} = 2\sum\nolimits_j(C^{ij}A^j+A^jC^{ji}),$$

[7] In general, for a matrix product ABC for example,

$$\frac{\partial}{\partial B_{ji}}\operatorname{tr}(ABC) = \frac{\partial}{\partial B_{ji}}\sum\nolimits_k\sum\nolimits_l\sum\nolimits_m A_{kl}B_{lm}C_{mk} = \sum\nolimits_k\sum\nolimits_l\sum\nolimits_m A_{kl}\frac{\partial B_{lm}}{\partial B_{ji}}C_{mk} = \sum\nolimits_k\sum\nolimits_l\sum\nolimits_m\delta_{im}C_{mk}A_{kl}\delta_{lj} = \sum\nolimits_k C_{ik}A_{kj},$$

or

$$\frac{\partial}{\partial B}\operatorname{tr}(ABC) = CA,$$

so derivating scalar functions of matrices reduces to scalar derivative operations with attention to product ordering when applicable. More (trivial) examples include

$$\frac{\partial}{\partial B_{ji}}\operatorname{tr}B = \frac{\partial}{\partial B_{ji}}\sum\nolimits_k B_{kk} = \sum\nolimits_k\frac{\partial}{\partial B_{ji}}B_{kk} = \sum\nolimits_k\delta_{ik}\delta_{kj} = \delta_{ij} \text{ or } \frac{\partial}{\partial B}\operatorname{tr}B = \mathbb{1},$$

or

$$\begin{aligned}\frac{\partial}{\partial B_{ji}}\operatorname{tr}B^N = \frac{\partial}{\partial B_{ji}}\sum\nolimits_{\mathbf{k}\in\mathbb{N}^N}B_{k_Nk_1}B_{k_1k_2}\dots B_{k_{N-1}k_N} &= N\sum\nolimits_{\mathbf{k}\in\mathbb{N}^N}\frac{\partial B_{k_Nk_1}}{\partial B_{ji}}B_{k_1k_2}\dots B_{k_{N-1}k_N}\\ &= N\sum\nolimits_{\mathbf{k}\in\mathbb{N}^N}\delta_{ik_1}B_{k_1k_2}\dots B_{k_{N-1}k_N}\delta_{k_Nj} = N\sum\nolimits_{\mathbf{k}\in\mathbb{N}^{N-2}}B_{ik_2}\dots B_{k_{N-1}j} \text{ or } \frac{\partial}{\partial B}\operatorname{tr}B^N = NB^{N-1}.\end{aligned}$$

and

$$\frac{\partial D_{cd}}{\partial A^i_{ba}} = \sum_j \frac{\partial C^{jj}_{cd}}{\partial A^i_{ba}} = \sum_j (\delta_{ij}\delta_{bc}A^j_{ad} + A^j_{cb}\delta_{ij}\delta_{ad}) = \delta_{bc}A^i_{ad} + A^i_{cb}\delta_{ad},$$

and

$$\begin{aligned}\sum\nolimits_c \sum\nolimits_d \frac{\partial D_{cd}D_{dc}}{\partial A^i_{ba}} &= \sum\nolimits_c \sum\nolimits_d \left(\frac{\partial D_{cd}}{\partial A^i_{ba}} D_{dc} + D_{cd}\frac{\partial D_{dc}}{\partial A^i_{ba}}\right) \\ &= \sum\nolimits_c \sum\nolimits_d ((\delta_{bc}A^i_{ad} + A^i_{cb}\delta_{ad})D_{dc} + D_{cd}(\delta_{bd}A^i_{ac} + A^i_{db}\delta_{ac})) = \sum\nolimits_d A^i_{ad}D_{db} + \sum\nolimits_c A^i_{cb}D_{ac} + \sum\nolimits_c D_{cb}A^i_{ac} + \sum\nolimits_d D_{ad}A^i_{db} \\ &= 2\sum\nolimits_c (A^i_{ac}D_{cb} + D_{ac}A^i_{cb}),\end{aligned}$$

or

$$\frac{\partial}{\partial A^i} \operatorname{tr} DD = 2(DA^i + A^i D),$$

thus

$$\begin{aligned}\frac{\partial}{\partial A^i}\sum\nolimits_j \sum\nolimits_k \operatorname{tr} F^{jk}F^{jk} &= \frac{\partial}{\partial A^i}\operatorname{tr}\sum\nolimits_j \sum\nolimits_k [A^j, A^k]^2 = 2\frac{\partial}{\partial A^i}\Big(\sum\nolimits_j \sum\nolimits_k \operatorname{tr} C^{jk}C^{jk} - \operatorname{tr} DD\Big) \\ &= 2\Big(2\sum\nolimits_j (C^{ij}A^j + A^j C^{ji}) - 2(DA^i + A^i D)\Big) = -4\Big(DA^i - \sum\nolimits_j C^{ij}A^j + A^i D - \sum\nolimits_j A^j C^{ji}\Big) \\ &= -4\sum\nolimits_j [A^j, [A^j, A^i]].\end{aligned}$$

Finally,

$$\begin{aligned}\frac{\partial S_{\text{effective}}}{\partial A^i}[A] &= -\imath N\beta \exp\left(\imath s\frac{\pi}{2}\right)\left(\frac{1}{2}\exp(-\imath k\pi)\frac{\partial}{\partial A^i}\sum\nolimits_j \operatorname{tr}[A^0, A^j]^2 - \frac{1}{4}\frac{\partial}{\partial A^i}\sum\nolimits_j \sum\nolimits_k \operatorname{tr}[A^j, A^k]^2\right) \\ &= -\imath N\beta \exp\left(\imath s\frac{\pi}{2}\right)\left(\frac{1}{2}\exp(-\imath k\pi)\frac{\partial}{\partial A^i}\sum\nolimits_j \operatorname{tr}[A^0, A^j]^2 - \frac{1}{4}\frac{\partial}{\partial A^i}\sum\nolimits_j \sum\nolimits_k \operatorname{tr}[A^j, A^k]^2\right) \\ &= \imath N\beta \exp\left(\imath s\frac{\pi}{2}\right)(\exp(-\imath k\pi)[A^0, [A^0, A^i]] - \sum\nolimits_j [A^j, [A^j, A^i]]\Big).\end{aligned}$$

Time constraint drift

The constraint term is,

$$V = \frac{1}{4}(\alpha_N - \sqrt{\kappa})^4\gamma_\alpha.$$

The time derivative of constraint term is,

$$\frac{\partial V}{\partial \tau_a} = \sum\nolimits_b \frac{\partial V}{\partial \alpha_b}\frac{\partial \alpha_b}{\partial \tau_a} = \alpha_a \sum\nolimits_{b>a} \frac{\partial V}{\partial \alpha_b} = \alpha_a \sum\nolimits_{b>a} \delta_{bN}(\alpha_N - \sqrt{\kappa})^3\gamma_\alpha = \alpha_a(\alpha_N - \sqrt{\kappa})^3\gamma_\alpha.$$

9.2. The Euclidean–Lorentzian IKKT matrix model correspondence

The pure bosonic case

From (9.1.3), the difference between Euclidean and Lorentzian observables linear in the bosonic matrices is a phase of $-(3/8)\pi$. For time α in particular with the constraint turned off ($\gamma_\alpha = 0$),

$$\langle \alpha_{\text{lorentzian}} \rangle = \langle \alpha_{\text{euclidean}} \rangle \exp\left(-\imath\frac{3\pi}{8}\right).$$

Let $\forall i \in \mathbb{N}_{N-N_{\text{block}}-1}$,

$$\Delta\alpha_i = \alpha_{i+1} - \alpha_i$$

be the physical time step.

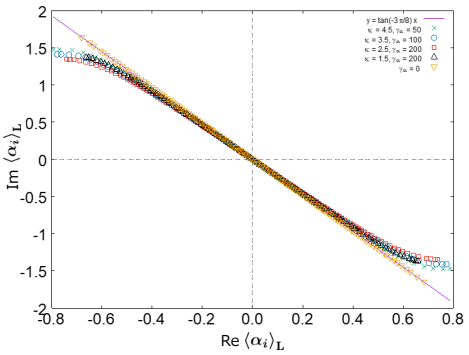


Figure 9.2.1.: Distribution of complex time α, for various constrain settings (offset κ and scale γ_α) of the Lorentzian IKKT model with a time cutoff $\alpha_n = \sqrt{\kappa}$. The line corresponds to Euclidean time [77].

For

$$\langle\Delta\alpha_i\rangle \propto \exp\left(-\imath\frac{3\pi}{8}\right),$$

the emergent time is Euclidean, while for $\langle\Delta\alpha_i\rangle \in \mathbb{R}$, the emergent time is Lorentzian, i.e. corresponding to the proper metric signature. In figure 9.2.1, α is plotted for various constraint values of γ_α and κ. In all settings the emergent time is Euclidean near times with low $|\alpha|$, but becomes manifestly Lorentzian for large $|\alpha|$.

An observable of space extent can be defined from the specific space extent moment of inertia as

$$N^{-1}\sum\nolimits_{i=1}^{\dim\mathcal{X}-1}\langle \mathrm{tr}_{\mathrm{block}}(\bar{A}_i)^2\rangle,$$

where

$$\mathrm{tr}_{\mathrm{block}} = \sum\nolimits_{n_{\mathrm{block}}=1}^{N_{\mathrm{block}}}.$$

To obtain a time evolution of the space extent, we are limited by the block size allowed by the non–commutativity of spacetime [77], so time (9.1.1) is used and averaging on space block (band–diagonal) matrices is the maximum "resolution" that can be achieved when estimating time dependence,

$$R^2(t) = NN_{\mathrm{block}}^{-1}\sum\nolimits_{i=1}^{\dim\mathcal{X}-1}\langle\bar{\lambda}_i\rangle \text{ with } \Lambda^2 = |R^2(0)| \text{ and } \theta(t) = \frac{1}{2}\arg R^2(t). \tag{9.2.1}$$

In [77] it was shown that spacetime of the Lorentzian bosonic IKKT matrix model exhibits a small expansion in times away from $t = 0$. The main result is that without constraints the Euclidean and Lorentzian IKKT models are equivalent, while introducing said constraints on time ($\alpha_N = \sqrt{\kappa}$), Lorentzian time emerges at the edges of time.

The extents of space

In [78] a detailed exploration of the extents of space in our simulations was done, using a modification of the spatial moment of inertia tensor Λ defined $\forall i, j$ as

$$\bar{\Lambda}_{ij}(t) = \mathrm{tr}_{\mathrm{block}}(\Re\bar{A}_i(t)\Re\bar{A}_j(t))$$

where $\forall A \in \mathbb{M}_N\mathbb{C}$,

$$\Re A = \frac{1}{2}(A + A^\dagger) \text{ and } \Im A = \frac{1}{2}(A - A^\dagger)$$

are the hermitian and antihermitian parts of A respectively, and once again, tracing is done on the block submatrices $\bar{A}$ of A, to allow exploring the dependence of space expansion with time at the fuzzy resolution of the emergent non–commutative spacetime. The reason behind regarding the hermitian part of the complex bosonic matrices only is that, the original moment of inertia tensor (6.3.13) is non–holomorphic. However, due to gauge cooling intermixed in the simulation process, the hermiticity norm of configurations is minimal. Also, as explained in section §7.2. Complex Langevin Method (CLM), an alternative holomorphic method was used that could give identical results within statistical error.

The moment of inertia Λ is a matrix of its own with components that are block–traces of the spatial bosonic matrices. As a matrix of its own, its space–only trace labeled as

$$R^2(t) = \langle \mathrm{tr}_{\mathcal{X}/X^0}\bar{\Lambda}(t)\rangle = \sum\nolimits_{i=1}^{\dim\mathcal{X}-1}\langle \mathrm{tr}_{\mathrm{block}}(\Re\bar{A}_i(t))^2\rangle$$

in (9.2.1), where

$$\mathrm{tr}_{\mathcal{X}/X^0} = \sum\nolimits_{i=1}^{\dim\mathcal{X}-1}.$$

In [78], we used a "detailed variant"

$$Q(t) = \sum\nolimits_{i=1}^{\dim\mathcal{X}-1}(\Re\bar{A}_i(t))^2,$$

to measure the radial spread of space. Interestingly

$$\mathrm{tr}_{\mathcal{X}/X^0}\,\bar{\Lambda}(t) = \mathrm{tr}_{\mathrm{block}}\,Q(t),$$

but studying the distribution of the eigenvalues as event radii may give more information about how the space expands. In figure 9.2.2 some examples are shown with different radial behavior. Some spacetimes show tendency to be singular at the large N limit, while others show a more spread distribution of radial extends (eigenvalues of Q). The hermiticity norm h in figure 9.2.2 is a normalized variant of the one used in our simulations, namely

$$h(A) = -\|A_i\|^{-2}\,\mathrm{tr}(\Re\bar{A}_i)^2 = \frac{\|A_i^\dagger - A_i\|^2}{4\|A_i\|^2},\ \|A\| = \mathrm{tr}(A^\dagger A),\ \forall A \in \mathbb{M}_N\mathbb{C},$$

such that $h = 0$ corresponds to hermitian A and $h = 1$ corresponds to antihermitian A.

As is explained in [24, 48], terms $[A_0|A_i]$ favor close to diagonal structure, while terms $[A_i|A_j]$ favor non–commutativity of the bosonic matrices, arguments that apply on their block versions as well. Maximal commutativity according to [24], leads to a so–called Pauli–like structure of $\bar{A}_i$ configurations, which can be evident by the radial extent tensor Q; if only two eigenvalues of Q separate from the rest, a Pauli–like structure is evident.

The parametric path chosen in [78] is for fixed $k = 0$ starting at $s = -1$ (figure 9.2.3). No spontaneous symmetry breaking is observed.

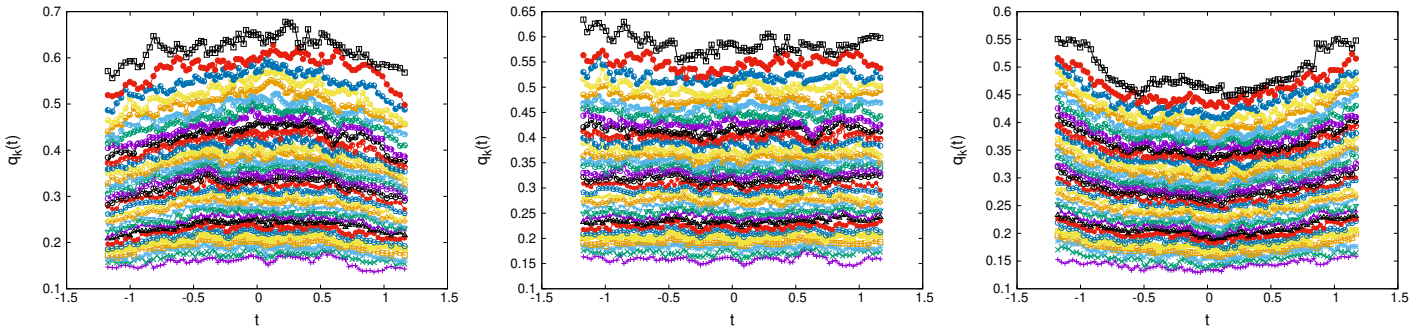


Figure 9.2.3.: Plots of the eigenvalues of Q respectively with respect to (block–averaged) time along the $k = 0$ path of the Wick–rotated Lorentzian IKKT matrix model [78], with $N = 128$, $\beta = 2.5$ and $\kappa = .8$ and from left to right, $s = -.8$, $s = -.6$ and $s = 0$ respectively.

The eigenvalue distribution of the moment of inertia Λ (figure 9.2.4) shows a uniform change through time t, hinting at no spontaneous symmetry breaking either, which hints that it might be an effect of supersymmetry in the Lorentzian IKKT model as well as it show to be in the Euclidean case [15, 79].

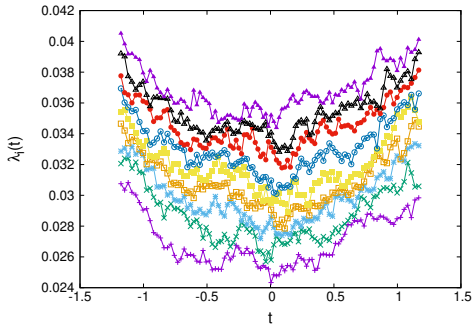
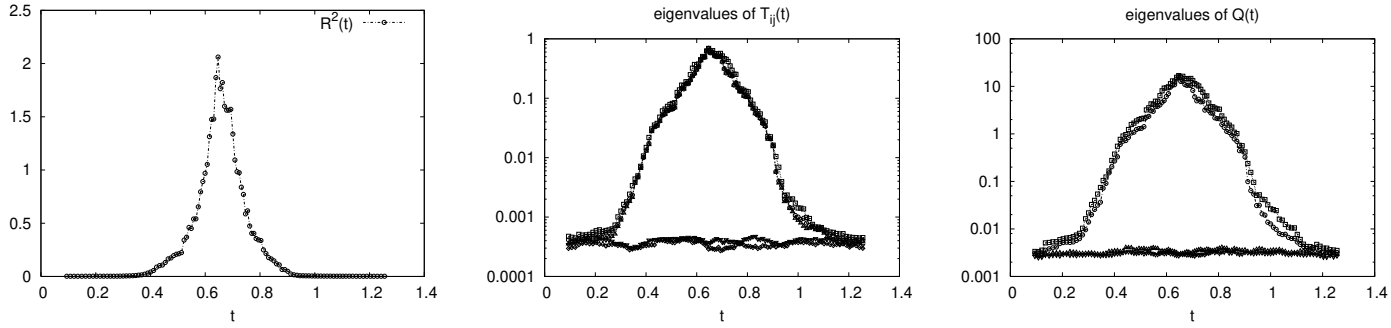
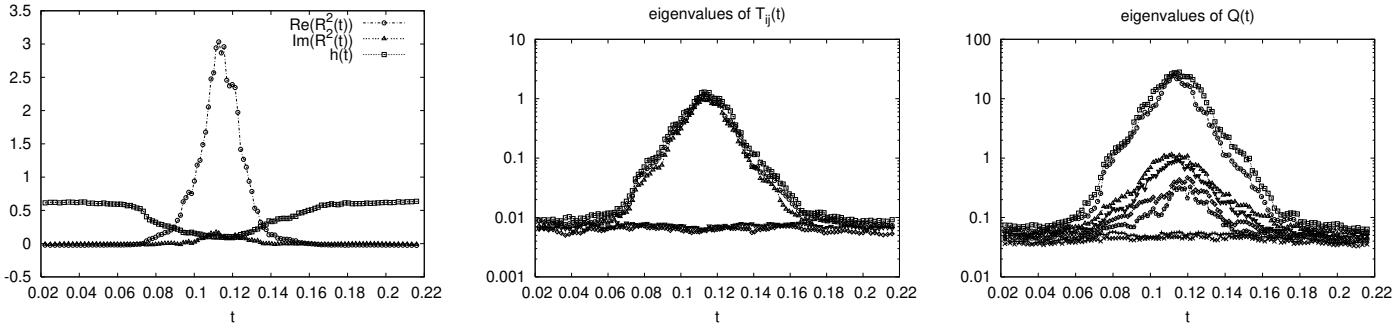


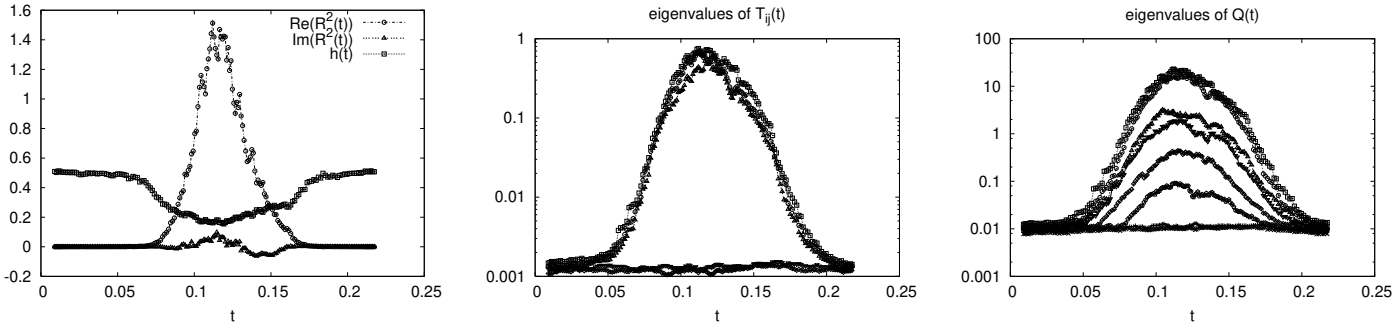
Figure 9.2.4.: Plot of the eigenvalues of Λ respectively with respect to (block–averaged) time for the Lorentzian IKKT matrix model (at $s = 0$ and $k = 0$) [78], with $N = 128$, $\beta = 2.5$ and $\kappa = .8$.



(a) Model with $s = -1.0000$ and $k = 0.0000$, $N = 128$ with $N_{\text{block}} = 16$, $\kappa = 0.1300$ and $\beta = 2$.
Only the largest 4 eigenvalues of Q are plotted.
This spacetime is expected to be singular in the $N \to \infty$ limit, as some of the eigenvalues of the radial expansion tensor Q are orders of magnitute distinct from the rest. Pauli–like structure is evident here.



(b) Model with $s = 0.0076$ and $k = 0.5038$, $N = 128$ with $N_{\text{block}} = 16$, $\kappa = 0.0037$ and $\beta = 32$.
Only the largest 8 eigenvalues of Q are plotted.
This spacetime is expected to be less singular than its $s = -1$ and $k = 0$ counterpart in the $N \to \infty$ limit, as the eigenvalues of the radial expansion tensor Q have more gradual differences. Departure from the Pauli-like structure is evident.



(c) Model with $s = 0.0118$ and $k = 0.5059$, $N = 192$ with $N_{\text{block}} = 24$, $\kappa = 0.0044$ and $\beta = 64$.
Only the largest 8 eigenvalues of Q are plotted.
This spacetime is a further improvement in the $N \to \infty$ limit, as the eigenvalues of the radial expansion tensor Q are even more spread out.The departure from the Pauli-like structure is more evident here.

Figure 9.2.2.: Plots (logarithmic) of the radial R, the eigenvalues of the moment of inertia Λ and the radial extend tensor Q for one thermalized configuration A (no expectation value) respectively with respect to (block–averaged) time [48], for various settings of the $\dim \mathcal{X} = 6$ bosonic Lorentzian IKKT matrix model (9.1.2).

Lorentz–invariant mass term

It is apparent from the previous results that the introduction of constraints induced a dynamical signature change in the emergent spacetime, near time $t = 0$, from Euclidean to Lorentzian. In [49], a new Lorentz-invariant term scaled by γ is introduced in the pure bosonic action (9.1.2) in further exploring the dynamic emergence of a Lorentzian spacetime.,

$$S_{\text{mass}} = -\frac{1}{2}\imath N\gamma \exp\left(\imath s\frac{\pi}{4}\right)\left(\exp\left(-\imath k\pi\right)\operatorname{tr}(A_0A_0) - \operatorname{tr}(A_iA_i)\right),$$

where the s and k dependence stems from the derived wick rotation (9.1.3), where the tilde notation is omitted for simplicity, since we always refer to the Wick–rotated fields in the context of s and k model parametrization. A variant of the γ term was first introduced in [80], further explored in [53, 55, 56, 76, 81, 82, 83, 84, 85, 86, 87, 88, 89].

In [49] we studied the Lorentzian bosonic IKKT model ($s = 0$, $k = 0$ and $\beta = 1$) with a final action

$$S_{\text{boson}} = \frac{1}{4}\imath N\left(\frac{1}{2}\operatorname{tr}(F_{0i}F_{0i}) - \frac{1}{4}\operatorname{tr}(F_{ij}F_{ij})\right) - \frac{1}{2}\imath N\gamma(\operatorname{tr}(A_0A_0) - \operatorname{tr}(A_iA_i)). \tag{9.2.2}$$

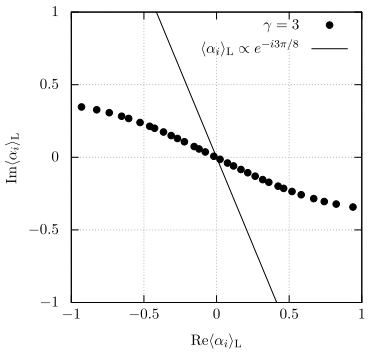


Figure 9.2.5.: Complex phase diagram of the expectation values $\langle\alpha_i\rangle$ $\forall i$, $N = 32$ and $\gamma = 3$ for the Lorentzian IKKT matrix model with Lorentz–invariant mass term (9.2.3) [49]. The solid line represents the Euclidean signature.

As shown in figure 9.2.5, the resulting spacetime deviates from Euclidean signature towards a closer–to–Lorentzian signature, with near–Lorentzian signature at late times.

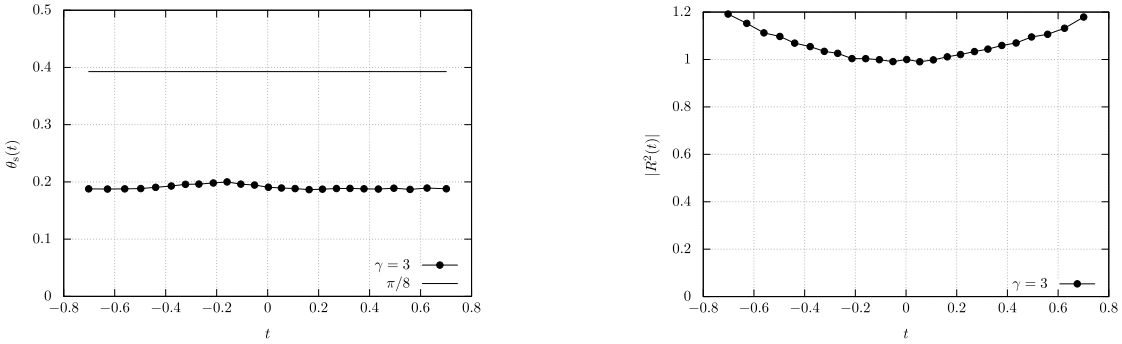


Figure 9.2.6.: Plots of expectation values of $\theta = \langle\arg R(t)\rangle$ and $|R^2(t)|$ respectively for the Lorentzian IKKT matrix model with Lorentz–invariant mass term (9.2.3) and $\gamma = 3$ [49]. The Euclidean signature (solid line) corresponds to $\theta = \pi/8$.

In figure 9.2.6, the deviation of the phase θ of R^2 from Euclidean spacetime is evident, while a slight expansion of space at late times is observed as well. These results indicate that the presence of the Lorentz–invariant mass term introduced in (9.2.2) with $\gamma > 0$, affects the signature of the emergent spacetime towards a Lorentzian signature, which brings the model closer to one generating a proper spacetime.

The supersymmetric case

In [90, 91], we added the contribution of fermions in studying the Lorentzian IKKT model. The $s = k = u$ IKKT model parametric subspace is chosen once more, as in (9.1.3). For the sake of simplicity the tilde is again not shown. The full supersymmetric Lorentzian IKKT matrix model action is,

$$S_{\text{effective}} = \imath N \exp\left(\imath u\frac{\pi}{2}\right)\left(\frac{1}{2}\exp(-\imath u\pi)\,\mathrm{tr}(F_{0i}F_{0i}) - \frac{1}{4}\,\mathrm{tr}(F_{ij}F_{ij})\right) \\ - \frac{1}{2}\imath N\gamma\exp\left(\imath u\frac{\pi}{4}\right)(\exp(-\imath u\pi)\,\mathrm{tr}(A_0A_0) - \mathrm{tr}(A_iA_i)) - \log\mathrm{pf}\,\mathcal{M}, \quad (9.2.3)$$

where the fermion matrix $\mathcal{M}$ is wick–rotated relative to its original Lorentzian definition as

$$A_0 \to \imath A_0 = \exp\left(\imath u\frac{\pi}{2}\right)A_0.$$

When applying the new Lorentz-invariant deformation, the sign of the γ coupling plays an important role in the sign of $\Re S_{\text{effective}}$, which is why $\gamma > 0$ is explored in particular.[8] Once again, the term is split into two phases

$$S_\gamma = S_{\gamma\text{–time}} + S_{\gamma\text{–space}} = \frac{1}{2}N\gamma\exp\left(-\imath(3u+2)\frac{\pi}{4}\right)\mathrm{tr}(A_0A_0) + \frac{1}{2}N\gamma\exp\left(\imath(u+2)\frac{\pi}{4}\right)\mathrm{tr}(A_iA_i).$$

$0 \le u \le 1$ is the range from the Euclidean to the Lorentzian model. The period of u with a $\pi/4$ factor is 8, and since $\cos\varphi \le 0$ means $|\varphi| \le 2$, the range of φ is chosen around 0 for each phase explored.

According to (4.4.5), $\forall\gamma \ge 0$, $\Re S_{\gamma\text{–time}} \ge 0$ if

$$\cos\left(|3u+2|\frac{\pi}{4}\right) \ge 0 \text{ meaning } |3u+2| \le 2 \text{ or } -\frac{4}{3} \le u \le 0 \text{ with a period } \frac{8}{3},$$

and $\Re S_{\gamma\text{–space}} \ge 0$ if

$$\cos\left(|u+2|\frac{\pi}{4}\right) \ge 0 \text{ meaning } |u+2| \le 2 \text{ or } -4 \le u \le 0 \text{ with a period } 8,$$

All inequalities overlap only for $u = 0$, which stands for the Lorentzian IKKT model, and for which the gamma term becomes 0 too. Therefore, on the parameter path $s = k$, $\Re S_\gamma \ge 0$ for $\gamma \le 0$, which secures the equivalence of the models. On the other hand, $\gamma > 0$ poses an interesting domain where the equivalence is broken, as the model is ill–defined along the search path $s = k = u > 0$.

The large–N limit however of the Lorentzian IKKT matrix model is expected to be nonequivalent to the Euclidean IKKT matrix model. Furthermore, the authors in [76] have discovered classical solutions with an expanding (3+1)–dimensional space for $\gamma > 0$, we thus focused our Monte Carlo study around that domain.

The fermionic contribution presents singularities much like its Euclidean counterpart,[9], therefore a correspondent fermion mass deformation

$$\Delta S_{\text{fermion}} = -Nm_{\text{fermion}}\,\mathrm{tr}(\overline{\psi}\Gamma_7\Gamma_8^\dagger\Gamma_9\psi)$$

is necessary to shift the eigenvalue distribution away from the origin.

In [90], we applied the so–called dynamical stabilization of the complex Langevin process by interjecting $\forall A$ bosonic matrix,

$$\Re_\eta A \to (1+\eta)^{-1}(A + \eta A^\dagger).$$

Note that $\Re_0 A = A$ and $\Re_1 A = \Re A$, thus $\Re_\eta$ for $0 \le \eta \le 1$ interpolated between the original matrix and its hermitized version. This method was first introduced in [92]. In the study that followed, $\eta = .01$ was used.

[8] $\gamma = 0$ defaults to the original Wick–rotated Lorentzian IKKT matrix model.
[9] See section §7.2. Complex Langevin Method (CLM) for details.

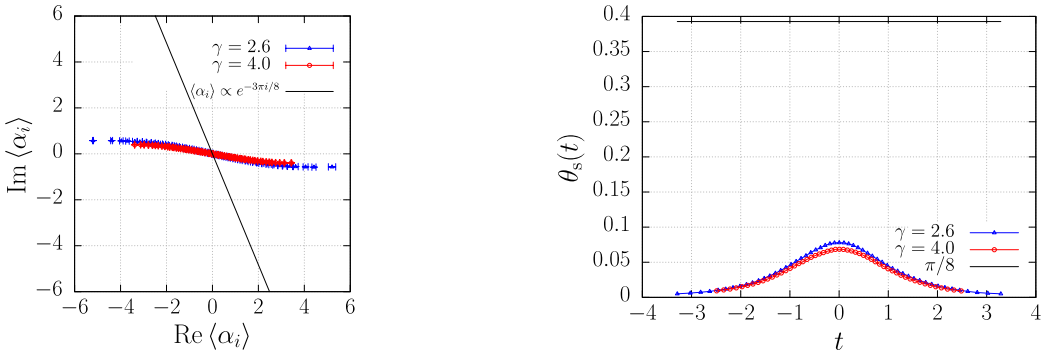
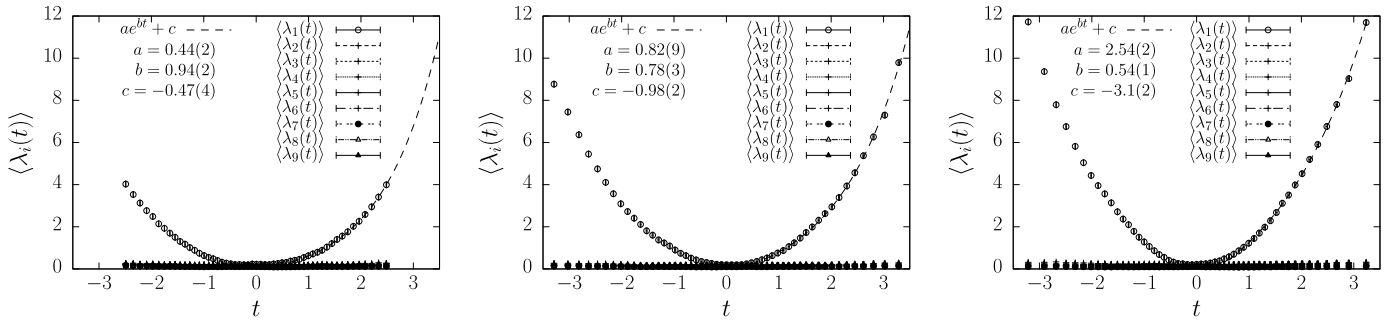


Figure 9.2.7.: Complex phase diagram of the expectation values $\langle \alpha_i \rangle$ $\forall i$, $N = 64$ and γ values shown for the Lorentzian IKKT matrix model with Lorentz–invariant mass term (9.2.3) [90, 91]. The solid line represents the Euclidean signature.

The effect of the γ mass term speculated in [49] to affect the signature of spacetime is shown here to evidently shift the signature from Euclidean to Lorentzian, with $1.8 < \gamma < 2.6$ identified as the region of this phase transition (figure 9.2.7).



(a) Model for $\gamma = 4.0$ and $m_{\text{fermion}} = 10$.

(b) Model for $\gamma = 2.6$ and $m_{\text{fermion}} = 10$.

(c) Model for $\gamma = 2.6$ and $m_{\text{fermion}} = 5$.

Figure 9.2.8.: Plots of the eigenvalues of Λ for the supersymmetric Lorentzian IKKT matrix model with Lorentz–invariant mass term (9.2.3) [90]. The expanding eigenvalue is fitted with $a \exp(bt) + c$.

On account of figure 9.2.8, a spontaneous symmetry breaking of SO_9 is observed, however with one broken dimension being the dominant one. The expansion of this one dimension becomes more pronounced as γ decreases and m_{fermion} decreases. While lower γ generally implies "less" Lorentzian spacetime signature, these runs where done within the overall Lorentzian signature phase, not going into the phase transition range. It is speculated in [90, 91], that this spontaneous symmetry breaking shown for finite m_{fermion} may lead to a higher dimensionality symmetry breaking as m_{fermion} is decreased.

[50] has a good breakdown of the current development in exploring the Lorentzian model. [71] contains an overarching review of the history and successes of the IKKT model from its date of origin to date.

Summary

First, in [18] a 4–dimensional matrix model and in[14, 15], the 6–dimensional and 10–dimensional IKKT matrix models were studied via the CLM.

The CLM applied to the IKKT model presents three challenges:

- the singularities of the drift of the complex Langevin process,
- the excursions of the complex Langevin process in the imaginary direction and
- the correctness of convergence of the complex Langevin process.

Each of these challenges were addressed by:

- the deformation of the original model:
  - finite size N approximation
  - explicit symmetry–breaking term of order parameter ε
  - fermion mass m_{fermion} deformation of the Dirac operator
- gauge cooling
- the drift norm diminishing faster than exponentially [38]

The deformation of the original model is necessary for technical reasons. The finite size N makes simulations computationally possible, so the $N \to \infty$ extrapolation provides insight to the original model. The explicit symmetry–breaking term is necessary to probe the SSB by explicitly breaking it, then gradually turning the order parameter ε down and observing the surviving symmetries in the original model via appropriate observables. Finally the artificial fermion mass is eliminated last, which in fact, acted as a decoupler of fermions, with the $m_{\text{fermion}} \to \infty$ model in fact being identical to the pure bosonic model.

The excursions in the imaginary direction were solved by gauge cooling, an exploit of the underlying symmetries of the action, to maintain the imaginary part of the complexified configurations at minimum.

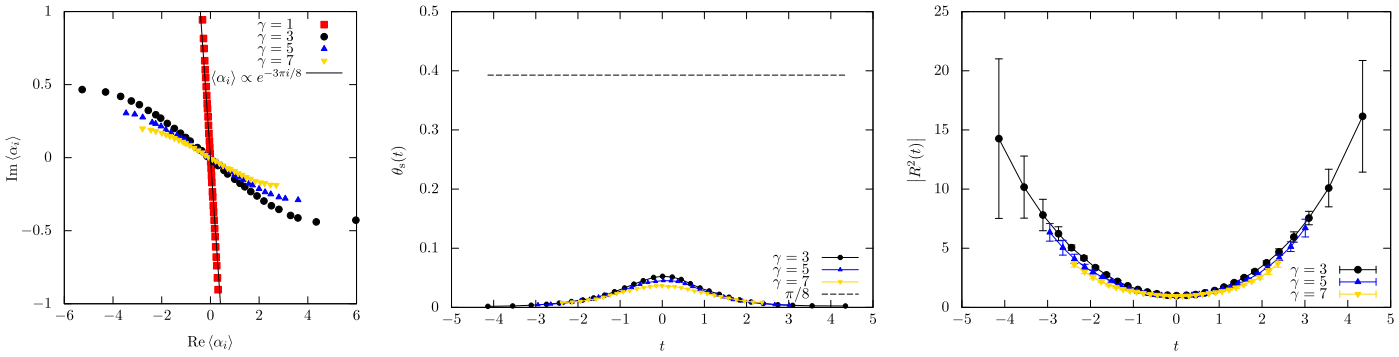
Finally, correctness is established by monitoring the drift norm histogram and making sure its falloff is subexponential, ensuring applicability of the CLM [38].

In [18], the authors successfully applied the CLM to Gaussian matrix model based on the aforementioned strategy and methodology, in which model the SSB of $SO_4 \to SO_2$ is expected to occur due to the phase of the complex fermion determinant. The SSB did not occur with the phase quenched, which implies that the overlap problem in the reweighting–based Monte Carlo methods is severe.

In the 4–dimensional model, the authors in [18] treated the drift singularity by shifting the Dirac operator of the original model to a custom model with no singularities in two ways, which were gradually reduced to approximate the original model in a series of simulations. The results were self–consistent and consistent with the corresponding GEM result [74].

In [14], we applied the CLM on the 6–dimensional IKKT model, with the same recipe. One type of fermion deformation was applied, and all results stem from gradual extrapolations $N \to \infty$, $\varepsilon \to 0$ and $m_{\text{fermion}} \to 0$, leading to an $SO_6 \to SO_3$ with observable expectations that are consistent with the corresponding GEM result [22].

In [15], we applied the same methodology to study the true 10–dimensional IKKT model. The fermion mass deformation was applied in the (expected) compactified directions, and the same process and order of extrapolations were applied to yield an $SO_{10} \to SO_3$ SSB which is consistent with the corresponding GEM result [23] as well.



(a) The expectation values of α_i are plotted $\forall i$ in the complex plane for $\gamma = 1, 3, 5, 7$. The solid line corresponds to the Euclidean model.

(b) $\theta(t) = \arg R(t)$, is plotted against the time t for the non–Euclidean values of $\gamma = 3, 5, 7$. The dashed line $\theta(t) = \pi/8$ corresponds to the Euclidean mode.

(c) The extent of space $|R^2(t)|$ is plotted against the time t again for $\gamma = 3, 5, 7$.

Figure 1.: Observables plotted for the modified Lorentzian IKKT model (9.2.3) for various $\gamma > 0$. Quoted from [50, 71].

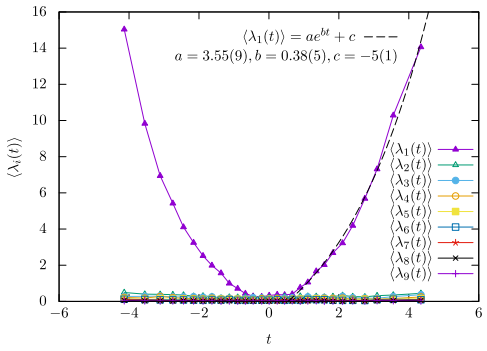


Figure 2.: Expansion of one spatial dimension exponentially with a $\langle\lambda_1(t)\rangle \sim a\exp(b|t|) + c$ fit.

Next, attention was drawn to the Lorentzian IKKT matrix model which despite having an ill–defined (complex) Boltzmann factor, its study is possible with the CLM, provided it satisfies the strong conditions of correctness [38, 42].Equivalence between the original Lorentzian and Euclidean IKKT matrix models is found, and a Lorentz–invariant $\gamma > 0$ mass term is introduced to account for the generation of a Lorentzian spacetime from the dynamics of the model. Different phases are identified across the γ spectrum:

- For finite N and as γ is reduced, we have a Euclidean signature generated spacetime [71] (figure 1a). For approximately $1.8 < \gamma < 2.6$, we have a phase transition from a strictly Euclidean signature spacetime to a mixed signature spacetime that is manifestly Lorentzian at late times, while not clearly Euclidean.
- These mixed with time spacetime signatures are more evident in the phase $\theta = \arg R$ of the radial expansion of spacetime R (figure 1b), where $\theta(t) \xrightarrow{|t|\to\infty} 0$, while $\theta(0) < \pi/8$ meaning there is a finite but small shift towards a Euclidean signature at early times.
- The mixed signature phase presents an overall expansion tendency $|R(t)| \xrightarrow{|t|\to\infty} \infty$ of space at late times (figure 1c).

A detailed exploration of how space expands can be given by the moment of inertia Λ of the bosonic matrices encoding spacetime. Preliminary studies show that 1 out of 9 dimensions expand exponentially as indicated by a sample run in figure 2.

Conclusions

The numerical simulations of the Euclidean IKKT model using the CLM yield results that are consistent with the $\mathrm{SO}_{10} \to \mathrm{SO}_3$ rotational SSB result of the GEM. While this is an interesting dynamical property, its relevance to the real world is unclear. The initial expectation has been $\mathrm{SO}_{10} \to \mathrm{SO}_4$ corresponding to an emergent 4–dimensional spacetime. The discrepancy with $\mathrm{SO}_{10} \to \mathrm{SO}_3$ is not only the difference in the dimension of the surviving symmetry. Both the GEM and the CLM results yield a finite ratio of the extended dimensions versus the "compactified" ones, in contrast with the fact that the observable 4–dimensional spacetime is manifestly infinite (even if time–expanding). Both of these concerns eventually motivate the study of the Lorentzian IKKT model using the CLM, where we expect to see real time emerging and 3–dimensional space expanding.

The Lorentzian IKKT model on the other hand also has a sign problem, it stems however from a different source. As the Boltzmann factor is now $\exp \imath S$, the phase stems from the bosonic part of the action as $\exp \imath S_{\mathrm{boson}}$, and the problem is thus manifest in both the pure bosonic Lorentzian IKKT model and the full model. The fermionic part S_{fermion} is still responsible for the drift singularities, thus similar to the Euclidean version fermion deformations are necessary.

The Lorentzian version however has one more issue: its action is unbounded from below. Without any cutoffs however, the Lorentzian model was found to be equivalent to the Euclidean model, and that the emergent spacetime obtained from the Lorentzian model should be interpreted as Euclidean.

To overcome this situation, we proposed to add a Lorentz–invariant mass term γ. Our preliminary results for the bosonic model are very promising. When the mass parameter γ is large enough, the path integral is dominated by one of the classical solutions, having Lorentzian signature and expanding behavior. As γ is decreased, the extent of the emergent time increases and the emergent space is expanding more at late times. The expansion at late times is consistent with an exponential behavior. The signature of spacetime is Lorentzian at late times, while it seems to change to Euclidean at early times. We speculate that an expanding spacetime with Lorentzian signature emerges at late times in the $\gamma \to +0$ limit after taking the large–N limit.

When space has an expanding behavior, we observe a clear block–diagonal structure, which is important in extracting the time–evolution from the matrix configurations that we obtain from the model. We also observe that space appears to be continuous instead of having the Pauli–matrix structure that was observed previously by using another approximation to avoid the sign problem.

In the bosonic model, we observed that only 1 out of 9 spatial directions expands. This may be understood from the action of the original type IIB matrix model. Since the spatial directions expand exponentially, the $\mathrm{tr}_{T\times\Sigma}[A_I|A_J][A_I|A_J]$ term becomes dominant. The fluctuation of this term can be made small by having only one expanding direction.

As a future prospect, it is important to study the impact of the fermionic matrices on the dynamical generation of spacetime. We expect supersymmetry to play an essential role in realizing the expansion of 3 spatial directions. It is known that $\mathrm{pf}\,\mathcal{M}$ vanishes if we set $A_\mu = 0\ \forall \mu$ apart for two of them [13,15], which strongly suppresses the $(1+1)$–dimensional spacetime and possibly also $(2+1)$–dimensional spacetime considering the exponential expansion of space. It remains to be seen whether we can reduce the fermionic mass deformation m_{fermion} to the extent that enables us to see the effects of supersymmetry needed to make the emergent spacetime $(3+1)$–dimensional.

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