Celestial Amplitudes as AdS-Witten Diagrams and the Leading Soft Algebra

Based on [arXiv:2204.10249] with E. Casali and A. Strominger

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Outline

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  - Translations in cCFT and low-point functions.
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- YM and Massive Scalar Theory
- All-$+$ gluon amplitudes as Witten Diagrams
- Soft- and collinear- structure
- Deformations of the Leading Soft Algebra

Translations in cCFT

- cCFTs posess the full symmetry group of flat space amplitudes, including the Poincaré group.
- Celestial amplitudes therefore obey new Ward identities of the form

$$\sum_j \epsilon_j P_j^\mu \mathcal{A}(1^{\epsilon_1}_{\Delta_1,a_1} \cdots n^{\epsilon_n}_{\Delta_n,a_n}) = 0 \tag{1}$$

  where P_j^μ generates a translation acting on the j-th particle.
- For massless particles $P^\mu = q^\mu(z,\bar{z})e^{\partial_\Delta}$.

Translations in cCFT

- Ward identities force two-, three-, and four-point amplitudes to be vanishing or nonsingular.
- Three point functions can only be non-vanishing in (2,2) signature and when $z_{12} = z_{23}$ or $\bar{z}_{12} = \bar{z}_{23}$.
- Four-point functions have support only where the cross-ratio is real

$$\mathcal{A}_4 \propto \delta(z - \bar{z}) \tag{2}$$

- This does not match the expectations from standard conformal field theories.

What to do?

- Some of these divergences may arrise from having non-compact spectrum
- We can also try to find celestial CFTs that don't have translation symmetry but preserve as much structure as possible.

Celestial Holography and AdS/CFT

- We would also like to understand connections with other forms of holography, in particular AdS/CFT.
- In AdS/CFT, boundary correlation functions can be built up perturbatively by Witten diagrams, which integrate products of propogators over the AdS bulk.
- Klein space and regions of Minkowski space are foliated by AdS slices, so we'd like to understand if we can build up celestial amplitudes from Witten diagrams integrated over these AdS slices.

YM + Massive Scalar

- We work with the action

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{4}\mathrm{Tr}\ F_{\mu\nu} F^{\mu\nu} - \frac{\mu}{4}\phi \mathrm{Tr}\ F^+_{\mu\nu} F^{+\mu\nu} \qquad (3)$$

- Here $F^+ = \frac{1}{2}(F + *F)$ is the self-dual component of the field strength.
- ϕ couples to positive-helicity gluons.
- Giving ϕ a position-dependent background breaks translation invariance, rendering low-point functions non-singular.

Amplitudes in Non-Trivial Background

- For $\overline{\mathrm{MHV}}$ and all-$+$ helicity configurations, $\phi + \mathrm{gluon}$ amplitudes are simple

$$\begin{aligned} A(\phi 1^+ 2^+ \cdots n^+) &= \frac{\mu m^4}{\langle 12 \rangle \cdots \langle n1 \rangle} \delta^{(4)}(p_\phi + p_1 + \cdots p_n) \\ A(\phi 1^+ 2^+ 3^- \cdots n^-) &= \frac{\mu [12]^4}{[12][23] \cdots [n1]} \delta^{(4)}(p_\phi + p_1 + \cdots p_n) \end{aligned} \tag{4}$$

Amplitudes in Non-Trivial Background

- We want to consider scattering in a massive scalar background $\phi(x)$ that
  1. Solves the massive wave equation: $(\partial^2 - m^2)\phi = 0$
  2. Is damped for large $x^2 > 0$
  3. Is a function of $x^2 = -\tau^2$ only
- Because x^2 is invariant under Lorentz transformations but not translations, scattering in this background will be Lorentz invariant but not translation invariant.
- We choose the background wavefunction

$$\phi_B(x) = \phi^+_{\Delta=0,m}(x) + \phi^-_{\Delta=0,m}(x) = \frac{8\pi^2}{im\tau} K_1(-im\tau) \tag{5}$$

Amplitudes in Non-Trivial Background

- We calculate the scattering at linear order in μ by considering amplitudes with a single interaction with the background:

$$A_n^{\phi}(1^+ \cdots n^+) = \int \widetilde{d^3 p_\phi} \phi_B(p_\phi) A_{n+1}(\phi 1^+ \cdots n^+) \tag{6}$$

where $\phi_B(p_\phi)$ is the momentum-space wavefunction of the background field.

Position-Space Computations of Celestial Amplitudes

- We transform to the celestial sphere by taking the Mellin transform:

$$\mathcal{A}_n^{\phi}(1^{\epsilon_1}_{\Delta_1,+}\cdots n^{\epsilon_n}_{\Delta_n,+}) = \int \prod_{j=1}^{n} d\omega_j \omega_j^{\Delta_j-1} A_n^{\phi}(1^+\cdots n^+) \quad (7)$$
$$p_j = \epsilon_j \omega_j q(z_j,\bar{z}_j)$$

- Expanding $\delta^{(4)}(P) = \frac{1}{(2\pi)^4}\int d^4X e^{iP\cdot X}$ and performing the Mellin transform gives

$$\mathcal{A}_n^{\phi}(1^{\epsilon_1}_{\Delta_1,+}\cdots n^{\epsilon_n}_{\Delta_n,+}) = \frac{\mu m^4}{8(2\pi)^4 z_{12}\cdots z_{n1}}\int d^4X \phi_B(X) \times \prod_{j=1}^{n}\phi^{\epsilon_j}_{\Delta_j-1}(X|z_j,\bar{z}_j) \quad (8)$$

Klein Space

- Klein space has (2,2) signature. With coordinates $x^\mu = (x^0, x^1, x^2, x^3)$, the metric is

$$ds^2 = -dx_0^2 + dx_1^2 - dx_2^2 + dx_3^2 \tag{9}$$

- We can write Klein space in coordinates $(\tau, \hat{x})$ such that

$$ds^2 = \begin{cases} -d\tau^2 + \tau^2 d^2 H & x^2 < 0 \\ d\tau^2 - \tau^2 d^2 H & x^2 > 0 \end{cases} \tag{10}$$

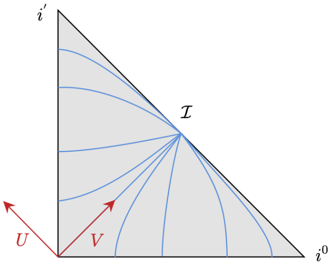
  where $d^2 H$ is the metric on the $\mathrm{AdS}_3/\mathbb{Z}$ constant-x^2 slice of Klein space.

- A generic null momenta takes the form

$$p^\mu(z, \bar{z}) = \pm\omega q(z, \bar{z}) = \pm\omega(1 + z\bar{z}, z + \bar{z}, z - \bar{z}, 1 - z\bar{z}) \tag{11}$$

Foliations of Klein Space

- Klein space can be foliated by slices of constant $x^2 = -\tau^2$.



- Both the $\tau^2 > 0$ and $\tau^2 < 0$ are geometrically $\mathrm{AdS}_3/\mathbb{Z}$.
- We want to integrate over Klein space by integrating over τ^2 and the constant τ^2 slice of Klein space.

Integrating over Klein Space

- For analytic functions f with certain pole configurations, we can deform the $\hat{x}$ integration contour so that integrating over Klein space becomes
$$\int d^4 x f(x) = \int_{i\infty}^{\infty} d\tau \tau^3 \int \widetilde{d^3 \hat{x}} f(\tau \hat{x}) \tag{12}$$
where the τ contour runs down the positive imaginary axis and then along the positive real axis, and $\hat{x}^2 = -1$.
- A suitable choice of $i\epsilon$ procedure guarantees that we can perform this contour rotation.

Celestial Amplitudes as Witten Diagrams

- We now evaluate the n-point all-$+$ amplitude with background as an integral over Klein space:

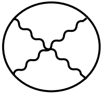
$$
\begin{aligned}
\mathcal{A}_n^{\phi}(1^{\epsilon_1}_{\Delta_1,+}\cdots n^{\epsilon_n}_{\Delta_n,+}) &= \frac{\mu m^4 \prod_{j=1}^n (-i\epsilon_j)^{\Delta_j-1}\Gamma(\Delta_j-1)}{8(2\pi)^4 z_{12}\cdots z_{n1}} \\
&\times \int d^4X \phi_B(X) \prod_{j=1}^n \frac{1}{(-q(z_j,\bar{z}_j)\cdot X)^{\Delta_j-1}} \\
&= \frac{\mu m^4 \prod_{j=1}^n (-i\epsilon_j)^{\Delta_j-1}\Gamma(\Delta_j-1)}{8(2\pi)^4 z_{12}\cdots z_{n1}} \\
&\times \int_{i\infty}^{\infty} d\tau \tau^3 \widetilde{d^3\hat{x}} \phi_B(\tau\hat{x}) \prod_{j=1}^n \frac{1}{(-q(z_j,\bar{z}_j)\cdot\tau\hat{x})^{\Delta_j-1}}
\end{aligned}
\tag{13}
$$

Celestial Amplitudes as Witten Diagrams

- Closing the contour in τ gives

$$\begin{aligned}
\mathcal{A}^{\phi} &= \frac{\mu m^4 \prod_{j=1}^{n}(-i\epsilon_j)^{\Delta_j-1}\Gamma(\Delta_j-1)}{2^n(2\pi)^4 z_{12}\cdots z_{n1}} \\
&\times \int_0^{i\infty} d\tau \tau^{3-\beta}\phi_B(\tau)(e^{-2\pi i\beta}-1) \\
&\times \int_{AdS_3/Z} \widetilde{d^3x}\frac{1}{\prod_{j=1}^{n}(-q_n\hat{x})^{\Delta_j-1}} \\
\beta &= \sum_{j=1}^{n}(\Delta_j-1) \qquad (14)
\end{aligned}$$

$$\mathcal{A}_n = Kin(z_i, \bar{z}_i) \times f(\beta) \times$$



Three-Point Function

- The three-point function takes the form

$$\mathcal{A}^{\phi}(1^{\epsilon_1}_{\Delta_1,+}2^{\epsilon_2}_{\Delta_2,+}3^{\epsilon_3}_{\Delta_3,+}) = \frac{\mu m^4(e^{-2\pi i\beta}-1)(1+(-1)^{\beta})}{16(2\pi)^4 z_{12}z_{23}z_{31}}$$
$$\times \int_0^{i\infty} d\tau \tau^{3-\beta}\phi_B(\tau)$$
$$\times \frac{\pi}{2}\Gamma\left(\frac{\Delta_1+\Delta_2+\Delta_3-5}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_2-\Delta_3-1}{2}\right) \quad (15)$$
$$\times \Gamma\left(\frac{\Delta_1-\Delta_2+\Delta_3-1}{2}\right)\Gamma\left(\frac{-\Delta_1+\Delta_2+\Delta_3-1}{2}\right)$$
$$\times \frac{1}{(z_{12}\bar{z}_{12})^{\frac{\Delta_1+\Delta_2-\Delta_3-1}{2}}(z_{23}\bar{z}_{23})^{\frac{\Delta_2+\Delta_3-\Delta_1-1}{2}}(z_{31}\bar{z}_{31})^{\frac{\Delta_3+\Delta_1-\Delta_2-1}{2}}}$$

Soft and Collinear Structure

- The integral formula for the connected celestial amplitudes at first order in the background imply that they obey the soft theorem

$$\lim_{\epsilon\to 0} \epsilon \mathcal{A}_n(1^{\epsilon_1}_{1+\epsilon,+}\cdots) = \frac{z_{n2}}{2z_{n1}z_{12}} \mathcal{A}_{n-1}(2^{\epsilon_2}_{\Delta_2,+}\cdots) \tag{16}$$

- and the collinear limit

$$\mathcal{A}_n(1^{+}_{\Delta_1,+}2^{+}_{\Delta_2+}\cdots) \sim \frac{1}{2z_{12}} \sum_{m=0}^{\infty} \frac{B(\Delta_1+m-1,\Delta_2-1)}{m!} \times \bar{z}_{12}^{m} \bar{\partial}_2^{\,m} \mathcal{A}_n(2^{+}_{\Delta_1+\Delta_2-1,+}) + \mathcal{O}(z_{12}^0) \tag{17}$$

Background Deformations to the Leading Soft Algebra

- The connected diagrams for $n > 3$ seem to imply that the soft-algebra is undeformed, but examining low-point functions shows that there is now a nontrivial level to the Kac-Moody algebra describing the leading soft theorem.
- Taking the soft-limit of the full amplitude

$$\lim_{\epsilon\to 0} \epsilon^3 \mathcal{A}(1^{+a}_{1+\epsilon,+} 2^{+b}_{1+\epsilon,+} 3^{+c}_{1+\epsilon,+}) = \frac{4\mu f^{abc}}{z_{12}z_{23}z_{31}} \tag{18}$$

implies that the $J_1^a J_1^b$ OPE takes the form

$$J_1^a(z_1)J_1^b(z_2) \sim \frac{4i\mu\delta^{ab}}{z_{12}^2} + \frac{if^{ab}{}_c J_1^c(z_2)}{z_{12}} \tag{19}$$

so that the Kac-Moody algebra has a nontrivial level $k = 4i\mu$.

Background Deformations to the Leading Soft Algebra

- Calculating the n-point soft gluon correlator, either directly or using the leading soft theorem recursively, reveals that

$$\langle J_1^{a_1}(z_1) \cdots J_1^{a_n}(z_n) \rangle = \frac{-4i\mu \mathrm{Tr}\left[T^{a_1} \cdots T^{a_n}\right]}{2^{n-1} z_{12} \cdots z_{n1}} + \cdots . \tag{20}$$

  where $\cdots$ includes other color orderings and multi-trace terms at higher order in k

- These correlation functions can be derived by fixing the singularities of the $J^a J^b$ currents in CFTs with Kac-Moody current algebras with level $k = 4i\mu$.

Future Work

- Adding the background gives the leading soft algebra for position-space celestial amplitudes a central term. We would like to understand possible central terms for the full soft algebra and compute the background-deformed soft algebra for modes at $\Delta = 0, -1, -2, \ldots$
- Correlation functions of soft currents correspond exactly to correlation functions of holomorphic currents in a WZW model of level $k = 4i\mu$. We would like to understand if there is a broader correspondence between the soft sector of cCFTs and WZW models, including, for example, negative helicity gluons and antiholomorphic currents in WZW models, and with other representations that appear in WZW models when $k = 4i\mu \in \mathbb{N}$.