The 4d/2d correspondence in twistor space and holomorphic Wilson line

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Based on 2208.06334 with Eduardo Casali

Brief Review of the 4d/2d correspondence (For SDYM)

$$
\begin{tikzcd}
 & 6d\ S^2\times S^3\times\mathbb{R}_{>0}\subset \mathbb{PT}:=\mathbb{CP}^3\setminus\{\lambda_\alpha=0\}\cong\mathbb{R}^4\times\mathbb{CP}^1 \arrow[dl, "S^2"'] \arrow[dr, "S^3"] & \\
\mathbb{R}^4 & & S^2\times\mathbb{R}_{>0}
\end{tikzcd}
$$

$Z^I=(\lambda_\alpha,\mu^{\dot\alpha})$ $\qquad Z^I\sim sZ^I\in\mathbb{C}^4$

$\quad=(\lambda_\alpha,\lambda_\alpha x^{\alpha\dot\alpha})$

$x^{\alpha\dot\alpha}$

$z=\dfrac{\lambda_1}{\lambda_0}$

Form factors involving local operators $\xleftrightarrow[\text{coupling on }\mathbb{CP}^1]{\text{gauge invariant}}$ Correlators of chiral algebra generators with local operators

$\mathrm{tr}\,B_{\alpha\beta}B^{\alpha\beta}(x)\,;\ \mathrm{tr}\,F_{\alpha\beta}F^{\alpha\beta}(x)\,;$

$\mathrm{tr}\,B_{\alpha\beta}B_{\gamma\delta}B^{\alpha\beta}B^{\gamma\delta}(x)\,;$

$\mathrm{tr}\,\partial_{\alpha\dot\alpha}B_{\gamma\delta}\,\partial^{\delta\dot\alpha}B^{\gamma}{}_{\sigma}B^{\sigma\alpha}(x)\cdots$

$\langle\mathrm{tr}B^2|J^{a_1}(z_1)\cdots\tilde J^{a_i}(z_i)\cdots\tilde J^{a_j}(z_j)\cdots J^{a_n}(z_n)\rangle$

$\langle\mathrm{tr}B^4|J^{a_1}(z_1)\cdots\tilde J^{a_i}(z_i)\cdots\tilde J^{a_j}(z_j)\cdots\tilde J^{a_k}(z_k)\cdots\tilde J^{a_l}(z_l)\cdots J^{a_n}(z_n)\rangle$

$\langle\mathrm{tr}\,\partial B\partial BB|J^{a_1}(z_1)\cdots\tilde J^{a_i}[1,0](z_i)\cdots\tilde J^{a_j}[1,0](z_j)\cdots\tilde J^{a_k}(z_k)\cdots\rangle$

[Costello-Paquette-Sharma '22
Costello-Paquette '22]

On the level of the action ...

Start with holomorphic BF theory on $\mathbb{PT}$:

$$\int_{\mathbb{PT}} D^3Z \left(g \wedge \bar\partial a + g \wedge a \wedge a \right) = \int_{\mathbb{PT}} D^3 Z \, g \wedge F(a)$$

$a \in H^{0,1}(\mathbb{PT}, \mathrm{End}(E))$

$g \in H^{0,1}(\mathbb{PT}, \mathcal{O}(-4) \otimes \mathrm{End}(E))$

$\swarrow S^2$ $\qquad\qquad$ $S^3 \searrow$

$$\int_{\mathbb{R}^4} d^4x \, \mathrm{tr}\left(B_{\alpha\beta} F^{\alpha\beta}(A) \right)$$

$\mathbb{R}^4$ component of $g(\lambda, \lambda x)$:

$$B_{\alpha\beta}(x) = \int_{\mathbb{CP}^1} \langle \lambda d\lambda \rangle \lambda_\alpha \lambda_\beta \, g(\lambda, \lambda x)$$

$$\xleftrightarrow[\text{gauge invariant coupling}]{\int_{\mathbb{CP}^1} a J(z), \ \int_{\mathbb{CP}^1} g \tilde{J}(z)}$$

$$\int_{\mathbb{R}_{>0} \times \mathbb{CP}^1} dz \wedge \left(\mathrm{tr} \, B \wedge F(A) + \eta D_A \phi \right) + KK \text{ modes}$$

chiral algebra generator

zero modes $J^a(z)$, $\tilde{J}^a(z)$

$J^a[k,l](z)$

$\tilde{J}^a[k,l](z)$

[Costello-Paquette '22, Ward '77, Chalmers-Siegel '96]

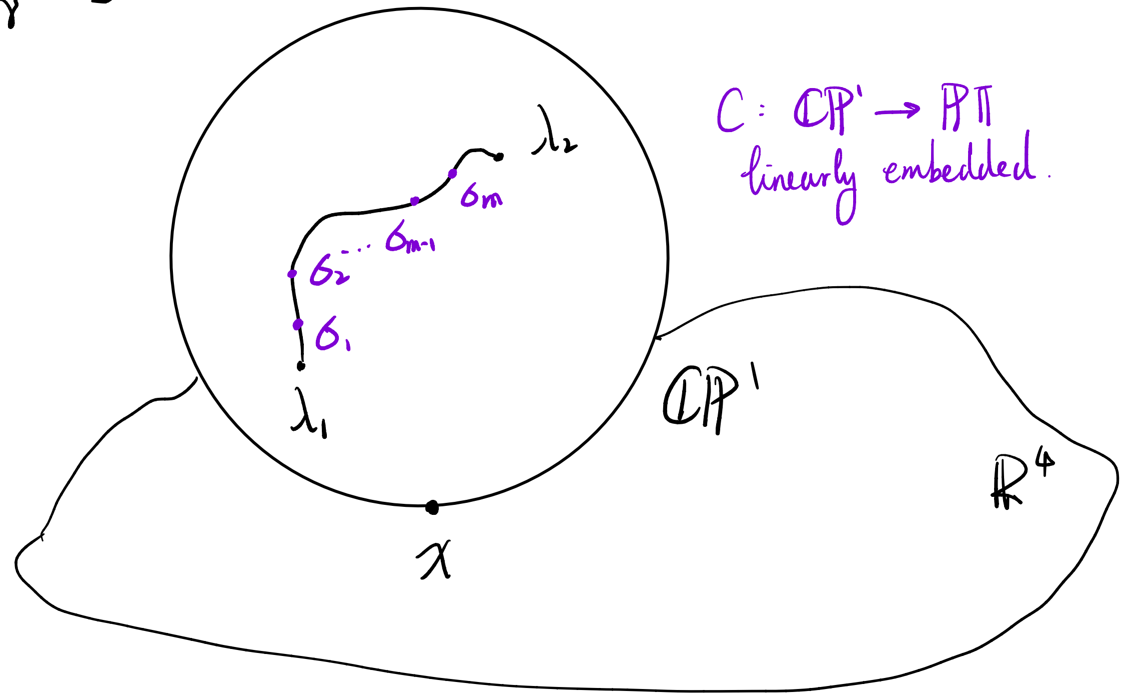
Lifting with holomorphic Wilson line

Start with local operator $\mathrm{tr} B^2(x)$ on $\mathbb{R}^4$,

Lifting to $\mathbb{PT} \cong \mathbb{R}^4 \times \mathbb{CP}^1$, graphically:

Usual Wilson line

$$W[y_1, y_2] = P \exp\left[-\int_\gamma A\right]$$



Lifting with holomorphic Wilson line

Usual Wilson line

$$W[y_1, y_2] = P\exp\left[-\int_\gamma A\right]$$

Here we only have $(0,1)$-part of the connection, to integrate over $\mathbb{CP}^1$, pair with $(1,0)$-form:

$$\omega_{\lambda_1,\lambda_2}(\lambda) = \frac{1}{2\pi i}\frac{(\lambda_1-\lambda_2)\,d\lambda}{(\lambda_1-\lambda)(\lambda-\lambda_2)} ;$$

$$\Longrightarrow \quad W[\lambda_1,\lambda_2] = P\exp\left[-\int_{\mathbb{CP}^1}\omega_{\lambda_1,\lambda_2}\wedge a\right]$$

$$= 1 + \sum_{m=1}^{\infty}(-1)^m\int_{(\mathbb{CP}^1)^m}\bigwedge_{i=1}^{m}\omega_i\wedge a_i(\sigma_i)$$

$$= 1 + \sum_{m=1}^{\infty}(-1)^m\int_{(\mathbb{CP}^1)^m}\frac{\lambda_2-\lambda_1}{(\lambda_2-\sigma_m)(\sigma_m-\sigma_{m-1})\cdots(\sigma_1-\lambda_1)}\bigwedge_{i=1}^{m}a_i(\sigma_i)\,d\sigma_i$$

[Bullimore - Skinner '11]

Reading the conformal block

$$\operatorname{tr} B_{\alpha\beta} B^{\alpha\beta}(x) \longrightarrow \int_{(\mathbb{CP}^1)^2} \langle \lambda_1 d\lambda_1 \rangle \langle \lambda_2 d\lambda_2 \rangle \langle \lambda_1 \lambda_2 \rangle^2$$

$$\operatorname{tr}\left(g(\lambda_1, \lambda_1 x)\, W[\lambda_1, \lambda_2]\, g(\lambda_2, \lambda_2 x)\, W[\lambda_2, \lambda_1] \right)$$

integrate over λ_1, λ_2 since $\operatorname{tr} B^2(x)$ doesn't depend on them.

Substitute in the holomorphic Wilson line and expand to order $n-2$.

$$\operatorname{tr} B^2(x) \longrightarrow \int_{(\mathbb{CP}^1)^n} \prod_{p=1}^{n} \langle \lambda_p d\lambda_p \rangle \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

$$\operatorname{tr}\left(\cdots a(\lambda_{i-1})\, g(\lambda_i)\, a(\lambda_{i+1}) \cdots a(\lambda_{j-1})\, g(\lambda_j) \cdots \right)$$

From the Wilson line and weight factor in linearised Penrose transform

Compared with conformal block in 2d:

$$\langle \operatorname{tr} B^2 | J^{a_1}(z_1) \cdots \tilde{J}^{a_i}(z_i) \cdots \tilde{J}^{a_j}(z_j) \cdots J^{a_n}(z_n) \rangle = \operatorname{tr}\left(T^{a_1} \cdots T^{a_n} \right) \frac{z_{ij}^4}{z_{12}\, z_{23} \cdots z_{n1}}$$

More examples...

$$\operatorname{tr} B_{\alpha\beta} B_{\gamma\delta} B^{\alpha\beta} B_{\gamma\delta}(x) \longrightarrow \int_{(\mathbb{CP}^1)^4} \prod_{i=1}^{4} \langle \lambda_i d\lambda_i \rangle \langle 13\rangle^2 \langle 24\rangle^2$$

$$\operatorname{tr}\Big(g(\lambda_1)\, W[\lambda_1,\lambda_2]\, g(\lambda_2)\, W[\lambda_2,\lambda_3]\, g(\lambda_3)\, W[\lambda_3,\lambda_4]\, g(\lambda_4)\, W[\lambda_4,\lambda_1] \Big)$$

$$\operatorname{tr} B_{\alpha}{}^{\beta} B_{\beta}{}^{\gamma} B_{\gamma}{}^{\delta} B_{\delta}{}^{\alpha}(x) \longrightarrow \langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle$$

$\Longrightarrow$ Schematically:

Any combination of $\langle\,,\rangle$ with weight 2 in each label.

$$\langle \operatorname{tr} B^4 \,|\, J^{a_1} \cdots \tilde{J}^{a_i} \cdots \tilde{J}^{a_j} \cdots \tilde{J}^{a_k} \cdots \tilde{J}^{a_l} \cdots \rangle$$

$$= \operatorname{tr}(T^{a_1} \cdots T^{a_n}) \, \frac{\langle ij\rangle\langle jk\rangle\langle kl\rangle\langle li\rangle}{\langle 12\rangle\langle 23\rangle \cdots \langle n-1\, n\rangle\langle n1\rangle} \, N(i,j,k,l)$$

Example involving derivatives

For local operators with derivatives, linearised Penrose transform:

$$\frac{\partial}{\partial x^{\alpha\dot\alpha}} B_{\beta\gamma}(x) = \int_{\mathbb{CP}^1} \langle \lambda d\lambda \rangle \lambda_\alpha \lambda_\beta \lambda_\gamma \frac{\partial}{\partial \mu^{\dot\alpha}} g(\lambda,\mu)\Big|_{\mu^{\dot\alpha}=\lambda_\alpha x^{\alpha\dot\alpha}}$$

Instead of coupling with $\tilde{J}^a(z)$ on $\mathbb{CP}^1$, it couples to $\tilde{J}^a[m,n](z)$

$$\int_{\mathbb{CP}^1} \tilde{J}^a(z)\, g(z) \longrightarrow \int_{\mathbb{CP}^1} \underbrace{\tilde{J}^a[m,n](z)}_{=\tilde{J}^a(z)(\mu^{\dot 0})^m(\mu^{\dot 1})^n} \underbrace{\partial^m_{\mu^{\dot 0}} \partial^n_{\mu^{\dot 1}}}_{\text{components of } \mu^{\dot\alpha}} g(z)$$

Example involving derivatives

Lifting becomes:

$$\mathrm{tr}\, \partial_{\alpha\dot\alpha} B_{\gamma\delta}\, \partial^{\delta\dot\alpha} B^{\gamma}{}_{\sigma}\, B^{\sigma\alpha}(x) \longrightarrow \int_{(\mathbb{CP}^1)^3} \prod_{i=1}^{3} \langle \lambda_i d\lambda_i \rangle \langle 12\rangle^2 \langle 13\rangle\langle 23\rangle$$

$$\mathrm{tr}\left(\left.\frac{\partial g(\lambda_1,\mu_1)}{\partial \mu_1^{\dot\alpha}}\right|_L W[\lambda_1,\lambda_2] \left.\frac{\partial g(\lambda_2,\mu_2)}{\partial \mu_2^{\dot\alpha}}\right|_L W[\lambda_2,\lambda_3]\, g(\lambda_3)\, W[\lambda_3,\lambda_1] \right)$$

Conformal block:

$$\langle \mathrm{tr}\, \partial_{\alpha\dot 0} B_{\gamma\delta}\, \partial^{\delta\dot 0} B^{\gamma}{}_{\sigma}\, B^{\sigma\alpha} \,|\, J^{a_1} \cdots \tilde J^{a_i}[1,0] \cdots \tilde J^{a_j}[1,0] \cdots \tilde J^{a_k} \cdots \rangle$$

$$= \mathrm{tr}(T^{a_1} \cdots T^{a_n})\, \frac{\langle ij\rangle\langle jk\rangle\langle ki\rangle}{\langle 12\rangle\langle 23\rangle \cdots \langle n-1\, n\rangle\langle n1\rangle} \langle ij\rangle^2 \langle ik\rangle\langle jk\rangle$$

From the 2d point of view

Gauge invariant coupling on $\mathbb{CP}^1$:

$$\int_{\mathbb{CP}^1} \mathrm{tr}\big(J^a(z)\, a(z)\big) \quad ; \quad \int_{\mathbb{CP}^1} \mathrm{tr}\big(\tilde{J}^a(z)\, g(z)\big)$$

$J^a(z)$ section of K

$a(z) = a(z, z\lambda)$ section of $\bar{K}$

$\tilde{J}^a(z)$ section of K twisted by $\mathcal{O}(+4)$

$g(z) = g(z, z\lambda)$ section of $\bar{K} \otimes \mathcal{O}(-4)$

$$\Longrightarrow J^a(z_1) J^b(z_2) \sim \frac{f^{ab}{}_c}{z_{12}} J^c(z_2) \; ; \quad J^a(z_1) \tilde{J}^b(z_2) \sim \frac{f^{ab}{}_c}{z_{12}} \tilde{J}^c(z_2) \; ;$$

$$\tilde{J}\tilde{J} \sim 0$$

Let:
- $J^a(z) = \hat{j}^a(z)$ Kac-Moody @ level 0
- $\tilde{J}^a(z) = M(z)\, \hat{j}^a(z)$ with $M(z)$ value in $\mathcal{O}(+4) \cong T^2$.

Identification between bundles over $\mathbb{CP}^1$

Element of section of K over $\mathbb{CP}^1$ or $(1,0)$ form on $\mathbb{CP}^1$:

$f(z)dz$ or in homogeneous coordinate $\lambda = \begin{pmatrix} 1 \\ z \end{pmatrix}$:

$$f(\lambda)\langle \lambda d\lambda \rangle = f(z) \det\left(\begin{pmatrix} 1 \\ z \end{pmatrix} \begin{pmatrix} 0 \\ dz \end{pmatrix} \right)$$

$\Rightarrow f(\lambda)$ is a section of $\mathcal{O}(-2)$ over $\mathbb{CP}^1$.

$\Rightarrow K \cong \mathcal{O}(-2)$, $T \cong \mathcal{O}(+2)$

Representation for chiral algebra with $\mathrm{tr}\, B^2$

The requirement for the operator we construct:

① $M(z)$ takes value in T^2;

② $\tilde{J}\tilde{J} \sim 0$ suggests $M(z_1) M(z_2) \sim z_{12} + \mathcal{O}(z_{12}^2)$;

③ Helicity selection rule requires exactly 2 $\tilde{J}$ insertions to obtain non-zero correlators.

Representation for chiral algebra with $\mathrm{tr}\, B^2$

$\beta\gamma$-system with $\gamma \in T^{3/2}$, $\beta \in K^{5/2}$.

Then $M(z) = \gamma \partial \gamma$ is in T^2; ✓

by Riemann-Roch, γ has $H^0(\mathbb{CP}^1, \mathcal{O}(3)) = 4$ zero modes. ✓

$$\gamma\partial\gamma(z_1)\, \gamma\partial\gamma(z_2) \sim z_{12}^4 \, ;$$ ✓

$$\langle J^{a_1}(z_1) \cdots \tilde{J}^{a_i}(z_i) \cdots \tilde{J}^{a_j}(z_j) \cdots J^{a_n}(z_n) \rangle = \langle \prod_{i=1}^{n} j^{a_i}(z_i) \rangle \langle \gamma\partial\gamma(z_1)\, \gamma\partial\gamma(z_2) \rangle$$

$$= \mathrm{tr}(T^{a_1} \cdots T^{a_n}) \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n-1\, n \rangle \langle n 1 \rangle}$$

For higher order local operators, twist γ by $T^n \cong \mathcal{O}(2n)$ to have $2n+4$ zero modes.

$\mathcal{N}=4$ Supersymmetric story

Exactly the same procedure, take $\mathrm{tr}\,\Phi^4(x)$ as example:

$$\mathrm{tr}\,\Phi^4(x) \longrightarrow \int_{(\mathbb{CP}^1)^4} \langle\lambda_i d\lambda_i\rangle\langle\lambda_j d\lambda_j\rangle\langle\lambda_k d\lambda_k\rangle\langle\lambda_l d\lambda_l\rangle\, \varepsilon^{ACDE}\varepsilon^{BFGH}$$

$$\mathrm{tr}\Big(\phi_{AB}(\lambda_1)W[\lambda_1,\lambda_2]\phi_{CD}(\lambda_2)W[\lambda_2,\lambda_3]\phi_{EF}(\lambda_3)W[\lambda_3,\lambda_4]\underset{\sim}{\phi_{GH}}(\lambda_4)W[\lambda_4,\lambda_1]\Big)$$

R-symmetry index

Expanding Wilson lines:

$$\langle \mathrm{tr}\,\Phi^4(x)\,|\,\varepsilon_{ACDE}\varepsilon_{BFGH} J^{a_1}(z_1)\cdots \underset{\sim}{J_\phi^{a_i,AB}(z_i)}\cdots J_\phi^{a_j,CD}(z_j)\cdots J_\phi^{a_k,EF}(z_k)\cdots J_\phi^{a_l,GH}(z_l)\cdots\rangle$$

$$\int_{\mathbb{CP}^1} J_\phi^{a,AB}(z)\,\phi_{AB}(z,zx)$$

$$= \mathrm{tr}(T^{a_1}\cdots T^{a_n})\,\frac{\langle ij\rangle\langle jk\rangle\langle kl\rangle\langle li\rangle}{\langle 12\rangle\langle 23\rangle\cdots\langle n\text{-}1\, n\rangle\langle n1\rangle}$$

[Boels-Mason-Skinner '06, Koster-Mitev-Staudacher-Wilhelm '16]

Representation of the chiral algebra

Let η^A be section of $T^{1/2}$, $A=1,2,3,4$. Then $H^0(\mathbb{CP}^1, \mathcal{O}(1)) = 2$ zero modes for each η^A.

$$\int \mathrm{tr}\, J^a a \;;\; \int \mathrm{tr}\, J_\psi^{a,A} \psi_A \;;\; \int \mathrm{tr}\, J_\phi^{a,AB} \phi_{AB} \;;\; \int \mathrm{tr}\, J_{\bar\psi,A}^{a} \bar\psi^A \;;\; \int \mathrm{tr}\, \tilde{J}^a g$$

$$J_\phi^{a,AB} = \eta^A \eta^B j^a$$

Twistor string at MHV.

$$\langle \varepsilon_{ACDE}\varepsilon_{BFGH} J^{a_1}(z_1) \cdots J_\phi^{a_i,AB}(z_i) \cdots J_\phi^{a_j,CD}(z_j) \cdots J_\phi^{a_k,EF}(z_k) \cdots J_\phi^{a_l,GH}(z_l) \cdots \rangle$$

$$= \prod_{i=1}^{n} \langle j^{a_i}(z_i)\rangle \langle \varepsilon_{ACDE}\varepsilon_{BFGH}\, \eta^A(z_i)\eta^B(z_i)\eta^C(z_j)\eta^D(z_j)\eta^E(z_k)\eta^F(z_k)\eta^G(z_l)\eta^H(z_l)\rangle$$

$$= \mathrm{tr}(T^{a_1} \cdots T^{a_n}) \frac{\langle ij\rangle\langle jk\rangle\langle kl\rangle\langle li\rangle}{\langle 12\rangle\langle 23\rangle \cdots \langle n-1\, n\rangle\langle n1\rangle}$$

[Witten '03, Berkovits-Witten '04]

Further direction

1. One-loop all-plus amplitude (on a line in twistor space) from lifting?
2. Role of the recently discovered scalar ormation of SDYM. in this framework
3. Generalizing to a "Wilson line" lifting for gravity.
4. Generalizing to higher degree embeddings.