Palatini variation in supergravity

Jan Vysoký

Workshop on Noncommutative and Generalized Geometry in String Theory, Gauge Theory and Related Physical Models
Corfu, 18-25 September 2022

Joint work with B. Jurčo and F. Moučka

Supergavity

For our purposes, the supergravity (close string effective) action is

$$S[g, B, \phi] = \int_M e^{-2\phi}\{\mathcal{R}(g) - \frac{1}{2}(\mathrm{d}B, \mathrm{d}B)_g + 4(\mathrm{d}\phi, \mathrm{d}\phi)_g\} \cdot \omega_g$$

- (M, g) is an orientable Riemannian manifold;
- $B \in \Omega^2(M)$ is a Kalb-Ramond field a.k.a. B-field;
- $\phi \in C^\infty(M)$ is a dilaton field.

Geometry behind this action?

1. Pairs (g, B) correspond to a generalized metric on $\mathbb{T}M := TM \oplus T^*M$. Generalized geometry is the candidate.
2. Where to put the dilaton? Enlarge $\mathbb{T}M$ or encode it in some other geometrical data?
3. Many people found answers for various version of this action: Coimbra, Strickland-Constable, Waldram, Garcia-Fernandez, Ševera, Valach, Jurčo, Vysoký...

Generalized geometry for supergravity

Ingredient I: Courant algebroids

A **Courant algebroid** is a 4-tuple $(E, \rho, \langle \cdot, \cdot \rangle_E, [\cdot, \cdot]_E)$, where

1. E is a vector bundle over M, $\rho : E \to TM$ is a vector bundle map called **the anchor**;
2. $\langle \cdot, \cdot \rangle_E$ is a fiber-wise metric on E;
3. $[\cdot, \cdot]_E$ is an $\mathbb{R}$-bilinear algebra bracket on $\Gamma(E)$.

Those structures satisfy a bunch of axioms:

1. $[\psi, f\psi']_E = f[\psi, \psi']_E + (\rho(\psi)f)\psi'$;
2. $\rho(\psi)\langle \psi', \psi'' \rangle_E = \langle [\psi, \psi']_E, \psi'' \rangle_E + \langle \psi', [\psi, \psi'']_E \rangle_E$;
3. $[\psi, [\psi', \psi'']_E]_E = [[\psi, \psi']_E, \psi'']_E + [\psi', [\psi, \psi'']_E]_E$;
4. $\langle [\psi, \psi], \psi' \rangle_E = \frac{1}{2}\rho(\psi')\langle \psi, \psi \rangle_E$.

Axioms resemble a quadratic Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}}, [\cdot, \cdot]_{\mathfrak{g}})$ promoted to an "algebroid", except for the peculiar axiom (4).

Example (Dorfman bracket)

Consider $E = \mathbb{T}M$, $\rho(X,\xi) := X$, the canonical pairing of TM and T^*M and the **Dorfman bracket** $[(X,\xi),(Y,\eta)]_E = ([X,Y], \mathcal{L}_X\eta - i_Y \mathrm{d}\xi)$.

Ingredient II: Generalized metrics

Let $(E, \langle\cdot,\cdot\rangle_E)$ be a quadratic vector bundle. A **generalized metric** is a maximal positive definite subbundle $V_+ \subseteq E$ w.r.t. $\langle\cdot,\cdot\rangle_E$.

1. E decomposes as $E = V_+ \oplus V_-$, where $V_- := V_+^\perp$; V_- is a maximal negative definite subbundle w.r.t. $\langle\cdot,\cdot\rangle_E$.
2. $V_\pm$ are ± 1 eigenbundles for a unique orthogonal involution $\tau : E \to E$, $\tau^2 = 1$, such that

$$\mathbf{G}(\psi,\psi') := \langle \psi, \tau(\psi')\rangle_E$$

   defines a positive definite fiber-wise metric on E.
3. A generalized metric exists on every E. It corresponds to the reduction of a structure group from $\mathrm{O}(p,q)$ to $\mathrm{O}(p)\times\mathrm{O}(q)$.

Example (**Generalized tangent bundle**)

Let $E = \mathbb{T}M$ with the canonical fiber-wise metric $\langle \cdot, \cdot \rangle_E$. Every generalized metric $V_+ \subseteq E$ is of the form

$$\Gamma(V_+) = \{(X, (g + B)(X)) \mid X \in \Gamma(TM)\}$$

for a unique pair (g, B), for a Riemannian metric g and $B \in \Omega^2(M)$. The induced fiber-wise metric $\mathbf{G}$ has the block form

$$\mathbf{G} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}.$$

Ingredient III: Courant algebroid connections

Let $(E, \rho, \langle \cdot, \cdot \rangle_E, [\cdot, \cdot]_E)$ be Courant algebroid. A **Courant algebroid connection** is an $\mathbb{R}$-bilinear map $\nabla : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ satisfying

1. $\nabla(f\psi, \psi') = f\nabla(\psi, \psi')$, $\nabla(\psi, f\psi') = f\nabla(\psi, \psi') + (\rho(\psi)f)\psi'$;
2. $\rho(\psi)\langle \psi', \psi'' \rangle_E = \langle \nabla(\psi, \psi'), \psi'' \rangle_E + \langle \psi', \nabla(\psi, \psi'') \rangle_E$.

We write $\nabla_\psi \psi' := \nabla(\psi, \psi')$.

Example (CA connections always do exist)

There always exists a vector bundle connection compatible with $\langle \cdot, \cdot \rangle_E$:

$$\nabla' : \Gamma(TM) \times \Gamma(E) \to \Gamma(E).$$

Then $\nabla(\psi, \psi') := \nabla'(\rho(\psi), \psi')$ is a CA connection

- CA connections have both inputs from $\Gamma(E)$. There should be a torsion operator. Naive one fails. Instead, one defines

$$T_\nabla(\psi, \psi', \psi'') := \langle \nabla_\psi \psi' - \nabla_{\psi'} \psi - [\psi, \psi']_E, \psi'' \rangle_E + \langle \nabla_{\psi''} \psi, \psi' \rangle_E.$$

  T_∇ is completely skew-symmetric and $C^\infty(M)$-linear in every input, hence called a **torsion 3-form** (Gualtieri 2007).

- Each connection ∇ induces a **divergence operator** given by

$$\mathrm{div}_\nabla(\psi) := \mathrm{Tr}(\nabla(\cdot, \psi)).$$

  $\mathrm{div}_\nabla : \Gamma(E) \to C^\infty(M)$ satisfies $\mathrm{div}_\nabla(f\psi) = f\,\mathrm{div}_\nabla(\psi) + \rho(\psi)f$.

- The question of a curvature tensor is a bit more complicated. The naive curvature tensor

$$R^0_\nabla(\phi', \phi, \psi, \psi') := \langle [\nabla_\psi, \nabla_{\psi'}]\phi - \nabla_{[\psi,\psi']_E}\phi, \phi' \rangle_E$$

fails to be $C^\infty(M)$-multilinear and with reasonable symmetries.

- Instead one considers the following peculiar definition (originally by Hohm and Zwiebach in DFT):

$$R_\nabla(\phi', \phi, \psi, \psi') := \frac{1}{2}\{R^0_\nabla(\phi', \phi, \psi, \psi') + R^0_\nabla(\psi', \psi, \phi, \phi') + \mathrm{Tr}(\langle \nabla(-, \psi), \psi' \rangle_E \cdot \langle \nabla(g_E(-), \phi), \phi' \rangle_E)\},$$

where $g_E : \Gamma(E) \to \Gamma(E^*)$ is induced by $\langle \cdot, \cdot \rangle_E$. It is $C^\infty(M)$-linear in all inputs and enjoys the symmetries:

$$\begin{aligned} R_\nabla(\phi', \phi, \psi, \psi') &= -R_\nabla(\phi, \phi', \psi, \psi'), \\ R_\nabla(\phi', \phi, \psi, \psi') &= -R_\nabla(\phi', \phi, \psi', \psi), \\ R_\nabla(\phi', \phi, \psi, \psi') &= R_\nabla(\psi, \psi', \phi', \phi), \end{aligned}$$

plus an algebraic Bianchi. Deeper meaning of R_∇ is a mystery.

- There is an unambiguous definition of a *symmetric* **Ricci tensor**

$$\mathrm{Ric}_{\nabla}(\psi, \psi') := \mathrm{Tr}(R_{\nabla}(g_E(-), \psi, -, \psi')).$$

  This definition needs only ∇ and the underlying CA.
- Using an arbitrary fiber-wise metric $\mathbf{G}$, one can take the trace to obtain the **scalar curvature of ∇ with respect to $\mathbf{G}$**:

$$\mathcal{R}^{\mathbf{G}}_{\nabla} := \mathrm{Tr}_{\mathbf{G}}(\mathrm{Ric}_{\nabla}) \equiv \mathrm{Ric}_{\nabla}(\psi_{\mu}, \mathbf{G}^{-1}(\psi^{\mu})).$$

- One can impose some conditions on CA connections ∇:
  1. ∇ is **torsion-free**, if $T_{\nabla} = 0$.
  2. ∇ is **compatible with a generalized metric** $V_+ \subseteq E$, if

$$\nabla_{\psi}(\Gamma(V_+)) \subseteq \Gamma(V_+) \text{ for all } \psi \in \Gamma(E).$$

     Equivalently $\nabla_{\psi} \circ \tau = \tau \circ \nabla_{\psi}$, or

$$\rho(\psi)\mathbf{G}(\psi', \psi'') = \mathbf{G}(\nabla_{\psi}\psi', \psi'') + \mathbf{G}(\psi', \nabla_{\psi}\psi'').$$

  3. ∇ is a **Levi-Civita connection with respect to V_+**, if it is torsion-free and compatible with V_+. Write $\nabla \in \mathrm{LC}(E, V_+)$.

Suppose $\nabla \in \mathrm{LC}(E, V_+)$. Any other CA connection ∇' is related to ∇ as

$$\langle \nabla'_\psi \psi', \psi'' \rangle_E = \langle \nabla_\psi \psi, \psi'' \rangle_E + \mathcal{K}(\psi, \psi', \psi''),$$

where $\mathcal{K} \in \Omega^1(E) \otimes \Omega^2(E)$ is unique. It is easy to see that

1. ∇' is torsion-free, iff $\mathcal{K}_a = 0$.
2. ∇' is compatible with V_+, iff $\mathcal{K}(\psi, \psi_+, \psi_-) = 0$.
3. $\mathrm{div}_{\nabla'} = \mathrm{div}_\nabla$, iff $\mathcal{K}'(\psi) := \mathrm{Tr}[\mathcal{K}(-, g_E(-), \psi)] = 0$ for all $\psi \in \Gamma(E)$.

Proposition (Abundance of LC connections)

- $\mathrm{LC}(E, V_+) \neq \emptyset$ *and it is infinite (except low dimensions).*
- *Let* $\mathrm{div} : \Gamma(E) \to C^\infty(M)$ *be a given divergence operator, that is*

$$\mathrm{div}(f\psi) = f\,\mathrm{div}(\psi) + \rho(\psi) f.$$

Let $\mathrm{LC}(E, V_+, \mathrm{div})$ *denote the set of LC connections such that*

$$\mathrm{div}_\nabla = \mathrm{div}.$$

Then $\mathrm{LC}(E, V_+, \mathrm{div}) \neq \emptyset$ *and it is infinite (except low dimensions).*

Supergravity using generalized geometry

Theorem (Jurčo, Vysoký 2016)

Let $E = \mathbb{T}M$.

1. *Let* $V_+ \subseteq E$ *be a generalized metric corresponding to a pair* (g, B).
2. *Define a divergence operator* $\mathrm{div} : \Gamma(E) \to C^\infty(M)$ *by the formula*

$$\mathrm{div}(X, \xi) := \mathrm{div}_\omega(X),$$

where $\mathrm{div}_\omega(X) := (\mathcal{L}_X \omega) \cdot \omega^{-1}$ *and* $\omega = e^{-2\phi} \omega_g$.

3. *Let* $\nabla \in \mathrm{LC}(E, V_+, \mathrm{div})$ *be arbitrary.*

Then (g, B, ϕ) *satisfies the equations of motion of* S, *iff*

1. $\mathcal{R}^{\mathbf{G}}_\nabla = 0$;
2. ∇ *is Ricci compatible with* V_+, *that is* $\mathrm{Ric}_\nabla(V_+, V_-) = 0$.

Under the reasonable assumption $\mathrm{d}\phi|_{\partial M} = 0$, S itself can be written as

$$S[g, B, \phi] = \int_M e^{-2\phi} \mathcal{R}^{\mathbf{G}}_\nabla \cdot \omega_g.$$

The claim of the theorem does not depend on the particular choice of ∇. We impose three non-trivial conditions on ∇:

1. It must be torsion-free;
2. It must be compatible with V_+;
3. Its divergence must be defined by the above equation.

- We required (1) and (2) "apriori" as a starting point.
- We have calculated the quantities $\mathcal{R}^{\mathbf{G}}_{\nabla}$ and $\mathrm{Ric}_{\nabla}(V_+, V_-)$ for the most general $\nabla \in \mathrm{LC}(\mathbb{T}M, V_+)$.
- We have chosen a particular ∇ to obtain the EOM of S. Later it turned out that this can be written as the above divergence condition.

Question: Are those requirements really necessary?

Palatini variation

One can mimic the famous trick (supposedly by Einstein in 1920). We will start with the following data:

1. An *arbitrary* Courant algebroid $(E, \rho, \langle \cdot, \cdot \rangle_E, [\cdot, \cdot]_E)$; This will determine our "generalized geometry".
2. A **generalized metric** $V_+ \subseteq E$ inducing a fiber-wise metric $\mathbf{G}$ on E.
3. An **arbitrary Courant algebroid connection** ∇ on E;
4. An **arbitrary volume form** ω on M;

One can use those *unrelated* data as fields for the following action:

$$S[V_+, \nabla, \omega] := \int_M \mathcal{R}_\nabla^{\mathbf{G}} \cdot \omega$$

Recall that $\mathcal{R}_\nabla^{\mathbf{G}} \equiv \mathbf{G}^{\mu\nu}[\mathrm{Ric}_\nabla]_{\mu\nu}$. Let us call this action a **Palatini action**.

How do the extremal fields of this functional look like?

Step 1: the variation of the volume form

Consider the variation
$$\omega'_\epsilon := e^{\epsilon\lambda}\omega,$$
for an arbitrary $\lambda \in C^\infty(M)$ satisfying $\lambda|_{\partial M} = 0$. Then
$$S[V_+, \nabla, \omega'_\epsilon] = S[V_+, \nabla, \omega] + \epsilon \int_M \mathcal{R}^{\mathbf{G}}_\nabla \cdot \lambda\omega + O(\epsilon^2)$$

Whence ω is an extremal field for S, iff $\mathcal{R}^{\mathbf{G}}_\nabla = 0$.

- This explains (but that is obvious in this case), why the SUGRA equation of motion for the dilaton ϕ is equivalent to $\mathcal{R}^{\mathbf{G}}_\nabla = 0$.

Step 2: the variation of the generalized metric

- Let $V_+ \subseteq E$ be a given generalized metric.
- Any other generalized metric V'_+ can be written as

$$\Gamma(V'_+) = \Gamma(\mathrm{gr}(\varphi_+)) \equiv \{(\psi_+, \varphi_+(\psi_+)) \mid \psi_+ \in \Gamma(V_+)\},$$

  for a unique vector bundle map $\varphi_+ : V_+ \to V_-$.
- V'_- is given using $\varphi_- : V_- \to V_+$ determined uniquely by φ_+.

We thus perform the variation V_+ as follows. Let $\varphi_+ : V_+ \to V_-$ be an arbitrary vector bundle map with $\varphi_+|_{\partial M} = 0$. Set

$$V'_+(\epsilon) := \mathrm{gr}(\epsilon\varphi_+),$$

where $\epsilon > 0$ is small enough for $V'_+(\epsilon)$ to define a generalized metric. Such ϵ always exists if M or $\mathrm{supp}(\varphi_+)$ are compact.

It is straightforward that the inverse to the fiber-wise metric $\mathbf{G}'^{-1}_{\epsilon}$ used to define the scalar curvature has the block form

$$\mathbf{G}'^{-1}_{\epsilon} = \mathbf{G}^{-1} + \epsilon \begin{pmatrix} 0 & 2\mathbf{g}_{+}^{-1}\varphi_{+}^{T} \\ 2\varphi_{+}\mathbf{g}_{+}^{-1} & 0 \end{pmatrix} + O(\epsilon^2),$$

where $\mathbf{g}_{+}$ is the (positive definite) restriction of $\langle \cdot, \cdot \rangle_E$ to V_{+}.
It is then easy to calculate the variation

$$\begin{aligned} S[V'_{+}(\epsilon), \nabla, \omega] = S[V_{+}, \nabla, \omega] \\ + 4\epsilon \int_M \mathrm{Tr}_{V_{+}}[(\mathrm{Ric}_{\nabla})_{+-}(-, \varphi_{+}\mathbf{g}_{+}^{-1}(-))] \cdot \omega + O(\epsilon^2). \end{aligned}$$

This proves that V_{+} is an extremal of S, iff there holds the condition

$$Ric_{\nabla}(V_{+}, V_{-}) = 0.$$

This explains why the SUGRA equations for (g, B) correspond to the Ricci-compatibility condition.

Step 3: the (Palatini) variation of the connection

- Let ∇ be a given CA connection. One defines a variation
$$\langle \nabla'_\epsilon(\psi,\psi'),\psi''\rangle_E := \langle \nabla(\psi,\psi'),\psi''\rangle_E + \epsilon\cdot\mathcal{L}(\psi,\psi',\psi''),$$
where $\mathcal{L}\in\Omega^1(E)\otimes\Omega^2(E)$ is arbitrary tensor satisfying $\mathcal{L}|_{\partial M}=0$.
- One can define a divergence operator $\mathrm{div}:\Gamma(E)\to C^\infty(M)$ by
$$\mathrm{div}(\psi) := \mathrm{div}_\omega(\rho(\psi)) \equiv (\mathcal{L}_{\rho(\psi)}\omega)\cdot\omega^{-1}.$$
Equivalently, it then satisfies the integral equation
$$\int_M \mathrm{div}(\psi)\omega = \int_{\partial M} i_{\rho(\psi)}\omega.$$
- Fix $\nabla^0\in \mathrm{LC}(E,V_+,\mathrm{div})$. Such a connection (except for lowest dimensions) always exists.

- One can then write ∇ using ∇^0 and a tensor $\mathcal{K}$:

$$\langle \nabla(\psi, \psi'), \psi'' \rangle_E = \langle \nabla^0(\psi, \psi'), \psi'' \rangle_E + \mathcal{K}(\psi, \psi', \psi'').$$

  In other words, we change $\mathcal{K}$ to $\mathcal{K}'_\epsilon = \mathcal{K} + \epsilon \mathcal{L}$.
- Using the fact that $\mathrm{div}_{\nabla^0} = \mathrm{div}$ and $\mathcal{L}|_{\partial M} = 0$, we get

$$S[V_+, \nabla'_\epsilon, \omega] = S[V_+, \nabla, \omega] + \epsilon \int_M \mathcal{L}^{\mu\nu\kappa} \mathcal{C}_{\mu\nu\kappa}[\mathcal{K}, V_+] \cdot \omega + O(\epsilon^2),$$

  where $\mathcal{C}[\mathcal{K}, V_+] \in \Omega^1(E) \otimes \Omega^2(E)$ is a tensor containing only $\mathcal{K}$ and the generalized metric V_+. Amazingly, we got rid of derivatives of $\mathcal{L}$ pretty easily.
- In other words, ∇ is an extremal of S, iff $\mathcal{C}[\mathcal{K}, V_+] = 0$.

Question: how to interpret $\mathcal{C}[\mathcal{K}, V_+] = 0$?

- By contracting the first two inputs using $\langle \cdot, \cdot \rangle_E$, we immediately obtain (except for some low dimensions) that

$$\mathcal{K}'(\psi) := \mathrm{Tr}(\mathcal{K}(g_{\mathcal{E}}(-), -, \psi)), \;\; \mathcal{K}'_{\mathbf{G}}(\psi) := \mathrm{Tr}(\mathcal{K}(\mathbf{G}(-), -, \psi)).$$

  vanish identically. This implies $\mathrm{div}_{\nabla} = \mathrm{div}_{\nabla^0}$.
- The vanishing of $(+ + -)$ and $(- + -)$ components of $\mathcal{C}[\mathcal{K}, V_+]$ implies that $\mathcal{K}(\psi, \psi_+, \psi_-) = 0$, that is ∇ is compatible with V_+;
- The vanishing of $(+ + +)$ and $(- - -)$ components of $\mathcal{C}[\mathcal{K}, V_+]$ implies that $\mathcal{K}_a = 0$, that is ∇ is torsion-free.

In fact, we can go the other way round, that is

Answer: $\mathcal{C}[\mathcal{K}, V_+] = 0$, iff $\nabla \in \mathrm{LC}(E, V_+, \mathrm{div})$;
The connection ∇ is an extremal of S, iff $\nabla \in \mathrm{LC}(E, V_+, \mathrm{div})$, where div is the divergence operator determined as above by ω.

Palatini variation: conclusion

Theorem (Palatini variation)

- *Let* $(E, \rho, \langle\cdot,\cdot\rangle_E, [\cdot,\cdot]_E)$ *be any Courant algebroid. Suppose we are giveng the following data:*
  1. *a generalized metric* $V_+ \subseteq E$*;*
  2. *a Courant algebroid connection* ∇ *on* E*;*
  3. *a volume form* ω *on* M*.*
- *Let* div *be the divergence operator* $\mathrm{div}(\psi) := (\mathcal{L}_{\rho(\psi)}\omega)\cdot\omega^{-1}$.

Then (V_+, ∇, ω) *extremalize the Palatini action, iff*

1. $\mathcal{R}^{\mathbf{G}}_{\nabla} = 0$;
2. $\mathrm{Ric}_{\nabla}(V_+, V_-) = 0$;
3. $\nabla \in \mathrm{LC}(E, V_+, \mathrm{div})$.

- First two conditions equations do not depend on the solution of the third one. One can "integrate out" ∇.
- Morally, ∇ is the "gauge field" of $S[V_+, \nabla, \omega]$ with div_{∇} a corresponding "field strength".

Definition

Einstein-Hilbert action is given by

$$S_{EH}[V_+, \omega] := \int_M \mathcal{R}^{\mathbf{G}}_{\nabla} \cdot \omega,$$

where $\nabla \in \mathrm{LC}(E, V_+, \mathrm{div})$ is fixed but arbitrary.
It's EOM are (1) and (2) above.

Example (**Back to supergravity**)

For $E = \mathbb{T}M$ over a connected M and $V_+ \approx (g, B)$, we may use g to write any volume form as

$$\omega = \pm e^{-2\phi} \omega_g$$

for a unique $\phi \in C^\infty(M)$. Note that g plays just an auxiliary role.
An apriori assumption $\nabla \in \mathrm{LC}(\mathbb{T}M, V_+, \mathrm{div})$ is thus obtained by plugging the EOM for ∇ from the Palatini action.

Appliactions, outlooks

- One can choose any Courant algebroid. For $E = TM \oplus \mathfrak{g}_P \oplus T^*M$, where $\mathfrak{g}_P$ is the adjoint bundle of a principal G-bundle $\pi : P \to M$ for G compact, one obtains a **heterotic supergravity**.
- We have explicit formulas for $\mathcal{R}^{\mathbf{G}}_{\nabla}$ and $\mathrm{Ric}_{\nabla}(V_+, V_-) = 0$ for any Lie quasi-bialgebroid. This allows one to find theories equivalent to (g, B, ϕ) supergravity:
  1. For $B^{-1} = \theta$, this gives "symplectic gravity".
  2. If $\Pi \in \mathfrak{X}^2(M)$ is a fixed Poisson tensor, one may find an equivalent effective action in terms of the fields (G, Φ, ϕ), where

$$\frac{1}{g + B} = \frac{1}{G + \Phi} + \Pi$$

     are Seiberg-Witten open-closed relations.
- We aim to a "generalized Palatini action", which would give us a "generalized supergravity" by Tseytlin.

Thank you for your attention!