Metric Compatibility and Levi-Civita Connections on Quantum Groups

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What is this talk about?

Goal: Give a mathematical model for quantum gravity general enough to cover some of our favorite noncommutative spaces.

Problem: While noncommutative differential geometry is rather well-understood there is no straightforward approach to quantum Riemannian geometry.

Ingredients:

- Space of observables: A noncommutative algebra (associative, unital)
- Differential calculus: $(\Omega^\bullet, \wedge, \mathrm{d})$ DGA on $\Omega^0 = A$
- Connection on A: $\nabla\colon \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ satisfying

$$\nabla(\omega a) = \nabla(\omega)a + \omega \otimes_A \mathrm{d}a$$

- Torsion, curvature: $\mathrm{Tor}^\nabla = \wedge \circ \nabla + \mathrm{d}$, $\mathrm{R}^\nabla = \nabla^2$
- ...

But:

- What is a metric $\mathbf{g} \in \Omega^1 \otimes_A \Omega^1$?
- How to make sense of metric-compatibility $\nabla(\mathbf{g}) = 0$?

Solutions: Drinfel'd Twist/Triangular Structure

Quantum Levi-Civita connections have been studied in the twisted/triangular setting [Wess et al. '05], [Aschieri-Castellani '09], [TW '20], [Aschieri '20],...

- A has a triangular Hopf algebra symmetry $(H, \mathcal{R})$
- There is a canonical calculus $(\Omega^\bullet_R, \wedge_\mathcal{R}, \mathrm{d})$ on A dual to the braided derivations

$$X(ab) = X(a)b + (\mathcal{R}^i \triangleright a)(\mathcal{R}_i \triangleright X)(b)$$

  where $X \in \mathrm{End}(A)$, $a, b \in A$, $\mathcal{R}^{-1} = \mathcal{R}^i \otimes \mathcal{R}_i \in H \otimes H$

- A metric $\mathbf{g} \in \Omega^1_\mathcal{R} \otimes_A \Omega^1_\mathcal{R}$ is defined as $\sigma^\mathcal{R}$-symmetric and non-degenerate
- Connections are extended as braided derivations
  $\nabla(\omega \otimes_A \eta) = \nabla(\omega) \otimes_A \eta + (\mathcal{R}^i \triangleright \omega) \otimes_A (\mathcal{R}_i \triangleright \nabla)(\eta)$

Theorem (Levi-Civita for A with $(H, \mathcal{R})$-symmetry)

For every metric $\mathbf{g}$ on A $\exists!$ torsion-free connection ∇ satisfying $\nabla(\mathbf{g}) = 0$.

Solutions: Bimodule Connections and Weak Levi-Civita

[DuboisViolette-Michor '96], [DuboisViolette-Masson '96], [Majid '99], [Beggs-Majid '11,'14],...

Bimodule connection: $\nabla\colon \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ right connection with A-bimodule map $\sigma\colon \Omega^1 \otimes_A \Omega^1 \to \Omega^1 \otimes_A \Omega^1$ such that

$$\nabla(a\omega) = a\nabla(\omega) + \sigma(\mathrm{d}a \otimes_A \omega)$$

corresponding to *central metrics* $\mathbf{g} \in \Omega^1 \otimes_A \Omega^1$.

If there is a 'framing' of the differential calculus $\Omega^1 \cong (A \otimes V)^{\mathrm{co}H}$, V an H-comodule (together with a dual 'coframing') there is a weak metric-compatibility condition

$$(\mathrm{d} \otimes \mathrm{id} - \mathrm{id} \otimes \nabla)\mathbf{g}$$

Theorem (Majid)

For $\mathrm{SL}_q(2)$ there is a unique weak Levi-Civita connection.

Other Solutions

- NC Riemannian geometry for central metrics on fuzzy spaces
  [Madore '93, '96, '97]

- Connections on central modules ${}_{Z(A)}\mathcal{M}_{Z(A)}$ and 'tame' differential calculi
  [DuboisViolette-Michor '96], [DuboisViolette-Masson '96],
  [Bhowmick-Goswami-Landi '19,'20]

- Pseudo-Riemannian calculi and LC connections for NC spaces with preferred frame of derivations
  [Arnlind '17]

- Covariant connections on the matrix quantum groups $\mathrm{SL}_q(N), \mathrm{O}_q(N), \mathrm{Sp}_q(N)$
  [Heckenberger-Schmüdgen '99], [Bhowmick-Mukhopadhyay '19]

- ...

Approach of this talk

Goals:

- Provide a metric compatibility condition $\nabla(\mathbf{g}) = 0$ for $A = H$ a quantum group (Hopf algebra). In particular
  - H is NOT (quasi-)triangular
  - $\mathbf{g}$ is NOT central
- Prove existence and uniqueness of ∇^{LC} for a certain class of metrics. In particular for $\mathrm{SL}_q(2)$.

Main tools:

- 'Sum of connections' (or 'braided derivation') formula for the tensor product extension of a connection
- Direct sum decomposition

$$\Omega^1_H \otimes_H \Omega^1_H \cong (\Omega^1_H \vee \Omega^1_H) \oplus (\Omega^1_H \wedge \Omega^1_H)$$

  in case the canonical quantum double braiding $\sigma^{\mathcal{W}} \colon \Omega^1_H \otimes_H \Omega^1_H \to \Omega^1_H \otimes_H \Omega^1_H$ is diagonalisable (e.g. for $\mathrm{SL}_q(N), \mathrm{O}_q(N), \mathrm{Sp}_q(N), \ldots$).
  Inspired by ideas of [Bhowmick-Mukhopadhyay '20]

Plan of the talk

1. A 'braided'-tensor product for connections
   - closed monoidal category $(\mathcal{M}^H, \otimes, \mathrm{HOM}^{\mathrm{Ad}})$
   - rational morphisms $\mathrm{HOM}^{\mathrm{Ad}}$
   - bicovariant bimodules ${}^H_H\mathcal{M}^H_H$
2. Decomposition of $\Omega^1_H \otimes_H \Omega^1_H$ for diagonalisable $\sigma^{\mathcal{W}}$
3. Sum of connections
4. Quantum Riemannian geometry on quantum groups

Comodules and rational morphisms

(H, Δ, ϵ, S) Hopf algebra (with invertible antipode) over a field $\Bbbk$.
$(\mathcal{M}^H, \otimes)$ monoidal category of right H-comodules. M, $\Delta_M \colon M \to M \otimes H$

The external Hom-functor Hom^H is given by right H-colinear maps

$$\phi \colon M \to N \text{ s.t. } \Delta_N \circ \phi = (\phi \otimes \mathrm{id}) \circ \Delta_M.$$

The internal Hom-functor $\mathrm{HOM}_{\Bbbk}^{\mathrm{Ad}}$ is given by $\Bbbk$-linear maps which admit a right adjoint coaction, the so-called right **rational morphisms**:

$$\Delta^{\mathrm{Ad}} \colon \mathrm{Hom}_{\Bbbk}(M, N) \to \mathrm{Hom}_{\Bbbk}(M, N \otimes H), \qquad \Delta^{\mathrm{Ad}}(\phi)(m) = \phi(m_0)_0 \otimes \phi(m_0)_1 S(m_1)$$

Then $\mathrm{HOM}_{\Bbbk}^{\mathrm{Ad}}(M, N) := (\Delta^{\mathrm{Ad}})^{-1}(\mathrm{Hom}_{\Bbbk}(M, N) \otimes H)$.

Proposition (Ulbrich '90)

Let M, N be objects in $\mathcal{M}^H$. Then

1. $\mathrm{HOM}_{\Bbbk}^{\mathrm{Ad}}(M, N)$ *is an object in* $\mathcal{M}^H$
2. $\mathrm{HOM}_{\Bbbk}^{\mathrm{Ad}}(M, N)^{\mathrm{co}H} = \mathrm{Hom}^H(M, N)$
3. *The evaluation and concatenation for* $\mathrm{HOM}_{\Bbbk}^{\mathrm{Ad}}$ *are morphisms in* $\mathcal{M}^H$

Covariant modules

Remark

$(\mathcal{M}^H, \otimes, \mathrm{HOM}_{\Bbbk}^{\mathrm{Ad}})$ *is a closed monoidal category, i.e. the currying*

$$\mathrm{Hom}^H(M \otimes \cdot, \cdot) \Rightarrow \mathrm{Hom}^H(M, \mathrm{HOM}_{\Bbbk}^{\mathrm{Ad}}(\cdot, \cdot))$$

is a natural isomorphism.

Let M be a right H-covariant right H-module, i.e. $M \in \mathcal{M}_H \cap \mathcal{M}^H$ such that $\Delta_M(ma) = \Delta_M(m)\Delta(a)$, $m \in M$, $a \in H$.

Proposition (Caenepeel-Guedenon '07)

For $M, N \in \mathcal{M}_H^H$ *we have*

$$\mathrm{HOM}_H^{\mathrm{Ad}}(M, N) = \mathrm{Hom}_H(M, N)$$

if M *is finitely generated as a right* H*-module.*

In the following we write $\mathrm{HOM}_{\Bbbk} := {}^{\mathrm{Ad}}\mathrm{HOM}_{\Bbbk}^{\mathrm{Ad}}$ and $\mathrm{HOM}_H := {}^{\mathrm{Ad}}\mathrm{HOM}_H^{\mathrm{Ad}}$.

Bicovariant bimodules

A bicovariant H-bimodule is an H-bimodule and H-bicomodule M such that

$$\Delta_M(amb) = \Delta(a)\Delta_M(m)\Delta(b) \quad \text{and} \quad {}_M\Delta(amb) = \Delta(a){}_M\Delta(m)\Delta(b).$$

Category ${}^H_H\mathcal{M}^H_H$ with morphisms H-bilinear and H-bicolinear.

For $M, N \in {}^H_H\mathcal{M}^H_H$ the tensor product $M \otimes_H N$ becomes object in ${}^H_H\mathcal{M}^H_H$ via $a \cdot (m \otimes_H n) \cdot b = (am) \otimes_H (nb)$ and the diagonal coactions.

Proposition

$({}^H_H\mathcal{M}^H_H, \otimes_H, \sigma^{\mathcal{W}})$ is a braided monoidal category with Woronowicz braiding

$$\sigma^{\mathcal{W}}_{M,N} \colon M \otimes_H N \to N \otimes_H M, \qquad m \otimes n \mapsto m_{-2} n_0 S(m_{-1} n_1) \otimes_H m_0 n_2.$$

The restriction to finitely generated bicovariant bimodules is closed braided monoidal with internal Hom*-functor* $\mathrm{HOM}_H = \mathrm{Hom}_H$.

Note that the H-module actions on $\phi \in \mathrm{HOM}_H(M, N)$ are $(a \cdot \phi \cdot b)(m) = a\phi(bm)$.

We write $\sigma^{\mathcal{W}}(m \otimes_H n) = {}_\alpha n \otimes_H {}^\alpha m$.

Braided tensor product of internal homomorphisms

For $M, M', N, N' \in {}^H_H\mathcal{M}^H_H$ we define

$$\otimes_{\sigma\mathcal{W}} : \mathrm{Hom}_H(M, M') \otimes_H \mathrm{Hom}_H(N, N') \to \mathrm{Hom}_H(M \otimes_H N, M' \otimes_H N') \qquad (1)$$

via

$$(\phi \otimes_{\sigma\mathcal{W}} \psi)(m \otimes_H n) = \phi({}_\alpha m) \otimes_H ({}^\alpha\psi)(n) = \phi(\psi_{-2} m_0 S(\psi_{-1} m_1)) \otimes_H \psi_0(m_2 n).$$

Proposition (Majid)

The operation (1) is a morphism in ${}^H_H\mathcal{M}^H_H$ and associative, i.e.

$$(\phi \otimes_{\sigma\mathcal{W}} \psi) \otimes_{\sigma\mathcal{W}} \chi = \phi \otimes_{\sigma\mathcal{W}} (\psi \otimes_{\sigma\mathcal{W}} \chi).$$

This works in any closed braided monoidal category.

Problem: Connections are NOT right H-linear. Can we generalize $\otimes_{\sigma\mathcal{W}}$ to connections?!

Lifting of Woronowicz braiding

Consider the following lifting $\sigma\colon M\otimes N\to N\otimes M$ of Woronowicz braiding

$$\begin{array}{ccccc} m\otimes n & & M\otimes N \xrightarrow{\ \sigma\ } N\otimes M & & m_{-2}n_0S(m_{-1}n_1)\otimes m_0n_2 \\ & & \downarrow\pi_H \qquad\qquad \pi_H\downarrow & & \\ m\otimes_H n & & M\otimes_H N \xrightarrow{\ \sigma^{\mathcal{W}}\ } N\otimes_H M & & m_{-2}n_0S(m_{-1}n_1)\otimes_H m_0n_2 \end{array}$$

Lemma

1. *σ is a morphism in ${}^H\mathcal{M}_H^H$*
2. $\sigma(ma\otimes n)=\sigma(m\otimes an)$
3. $\sigma_{M\otimes N,O}=(\sigma_{M,O}\otimes\mathrm{id}_N)\circ(\mathrm{id}_M\otimes\sigma_{N,O})$

Definition

For M,M',N,N' in ${}^H_H\mathcal{M}_H^H$ we define

$$\otimes_\sigma\colon \mathrm{HOM}_{\Bbbk}(M,M')\otimes\mathrm{HOM}_{\Bbbk}(N,N')\to\mathrm{HOM}_{\Bbbk}(M\otimes N,M'\otimes N')$$

by $(\phi\otimes_\sigma\psi)(m\otimes n)=\phi({}_\alpha m)\otimes({}^\alpha\psi)(n)=\phi(\psi_{-2}m_0S(\psi_{-1}m_1))\otimes\psi_0(m_2n)$.

Braided tensor product for rational morphisms

Theorem (Aschieri-TW)

The operation

$$\otimes_\sigma\colon \mathrm{HOM}_\Bbbk(M,M')\otimes\mathrm{HOM}_\Bbbk(N,N')\to\mathrm{HOM}_\Bbbk(M\otimes N,M'\otimes N')$$

is associative and a morphism in ${}^H_H\mathcal{M}^H_H$.

Since $(\phi\otimes_\sigma\psi)(ma\otimes n)=(\phi\otimes_\sigma\psi)(m\otimes an)$ the map $\phi\otimes_\sigma\psi\colon M\otimes N\to M'\otimes N'$ descends to a map $\widetilde{\phi\otimes_\sigma\psi}\colon M\otimes_H N\to M'\otimes N'$.
Define $\phi\hat{\otimes}\psi=\pi_H\circ(\widetilde{\phi\otimes_\sigma\psi})$.

Corollary

The operation

$$\hat{\otimes}\colon \mathrm{HOM}_\Bbbk(M,M')\otimes\mathrm{HOM}_\Bbbk(N,N')\to\mathrm{HOM}_\Bbbk(M\otimes_H N,M'\otimes_H N'),$$

is associative and a morphism in ${}^H_H\mathcal{M}^H_H$.

If ϕ and ψ are right H-linear we obtain $\phi\hat{\otimes}\psi=\phi\otimes_{\sigma\mathcal{W}}\psi$.

Connections on bicovariant bimodules

Fix a bicovariant FODC (Ω^1, d) on H and an object M in ${}^H_H\mathcal{M}^H_H$.

Definition

A right connection on M is a $\mathbb{k}$-linear map $\triangledown \colon M \to M \otimes_H \Omega^1$ such that $\triangledown(ma) = \triangledown(m)a + m \otimes_H \mathrm{d}a$.

Its extension $\triangledown^\bullet \colon M \otimes_H \Omega^\bullet \to M \otimes_H \Omega^{\bullet+1}$ to higher orders is defined by

$$\triangledown^\bullet = \wedge_{23} \circ (\triangledown \otimes_H \mathrm{id}_{\Omega^\bullet}) + \mathrm{id}_M \otimes_H \mathrm{d} \tag{2}$$

and satisfies

$$\triangledown^\bullet(m \otimes_H \omega a) = \triangledown^\bullet(m \otimes_H \omega)a + (-1)^{|\omega|} m \otimes_H \omega \wedge \mathrm{d}a.$$

Remark

The individual terms on the right of (2) are not well-defined on $\otimes_H$. However, we have an equality

$$\triangledown^\bullet = \wedge_{23} \circ (\triangledown \hat{\otimes} \mathrm{id}_{\Omega^\bullet}) + \mathrm{id}_M \hat{\otimes} \mathrm{d}$$

where all individual terms are well-defined on $M \otimes_H \Omega^\bullet$.

Torsion, curvature and canonical connections

Definition

The curvature of ∇ is the right H-linear map defined by

$$\mathrm{R}^{\nabla} = \nabla^{\bullet} \circ \nabla \colon M \to M \otimes_H \Omega^2.$$

If $M = \Omega^1$ the torsion of ∇ is the right H-linear map defined by

$$\mathrm{Tor}^{\nabla} = \wedge \circ \nabla + \mathrm{d} \colon \Omega^1 \to \Omega^2.$$

Example

On every bicovariant bimodule there is a canonical flat connection $\mathrm{d}^M \colon M \to M \otimes_H \Omega^1$ given by

$$\mathrm{d}^M(m) = m_0 S(m_1) \otimes_H \mathrm{d}m_2.$$

For a basis $\{\omega^i\}$ of left coinvariant 1-forms the structure constants $C^k_{ij} \in \mathbb{k}$ determined by the Cartan-Maurer formula $\mathrm{d}\omega^k = C^k_{ij}\omega^i \wedge \omega^j$ give a canonical torsion-free connection

$$\nabla^C(\omega^k) = -C^k_{ji}\omega^i \otimes_H \omega^j.$$

Sum of connections via canonical connections

Lemma

Let M be a finitely generated bicovariant bimodule. Any right connection $\nabla\colon M \to M \otimes_H \Omega^1$ is a rational morphism.

Proof.

All right H-linear maps $\phi\colon M \to M \otimes_H \Omega^1$ are rational and also d^M is rational since it is right H-colinear. Thus, $\nabla = \mathrm{d}^M + \phi$ is rational. □

Remark

The canonical connections satisfy

$$\mathrm{d}^{M\otimes_H N} = \sigma_{23}^{\mathcal{W}} \circ (\mathrm{d}^M \otimes_H \mathrm{id}_N) + \mathrm{id}_M \otimes_H \mathrm{d}^N$$

where the individual terms are not well-defined.

Using the canonical connections there is a natural notion of "sum of connection":

$$\nabla^{M\otimes_H N} := \sigma_{23}^{\mathcal{W}} \circ ((\nabla^M - \mathrm{d}^M) \otimes_{\sigma^{\mathcal{W}}} \mathrm{id}_N) + \mathrm{id}_M \otimes_{\sigma^{\mathcal{W}}} (\nabla^N - \mathrm{d}^N) + \mathrm{d}^{M\otimes_H N}$$

Sum of connections independent from can. connections

Using the braided tensor product $\hat{\otimes}$ of rational morphism we define

$$\nabla^M \oplus \nabla^N := \sigma_{23}^{\mathcal{W}} \circ (\nabla^M \hat{\otimes} \mathrm{id}_N) + \mathrm{id}_M \hat{\otimes} \nabla^N$$

with all individual terms well-defined.

Theorem (Aschieri-TW)

$\nabla^M \oplus \nabla^N \colon M \otimes_H N \to M \otimes_H N \otimes_H \Omega^1$ is a right connection on $M \otimes_H N$ and both notions of sum of connection coincide: $\nabla^M \oplus \nabla^N = \nabla^{M \otimes_H N}$.

The curvature of $\nabla^{M \otimes_H N}$ is

$$\begin{aligned}
\mathrm{R}^{\nabla^M \oplus \nabla^N} &= \sigma_{23}^{\mathcal{W}} \circ (\mathrm{R}^{\nabla^M} \otimes_{\sigma^{\mathcal{W}}} \mathrm{id}_N) + \mathrm{id}_M \otimes_{\sigma^{\mathcal{W}}} \mathrm{R}^{\nabla^N} \\
&+ \wedge_{34} \circ \sigma_{23}^{\mathcal{W}} \circ ((\nabla^M - \mathrm{d}^M) \otimes_{\sigma^{\mathcal{W}}} (\nabla^N - \mathrm{d}^N)) \\
&+ \wedge_{34} \circ (\mathrm{id}_M \otimes_{\sigma^{\mathcal{W}}} ((\nabla^N - \mathrm{d}^N) \otimes_{\sigma^{\mathcal{W}}} \mathrm{id}_{\Omega^1})) \circ \sigma_{23}^{\mathcal{W}} \circ ((\nabla^M - \mathrm{d}^M) \otimes_{\sigma^{\mathcal{W}}} \mathrm{id}_N).
\end{aligned}$$

Some linear algebra

In the following we assume that $\sigma^{\mathcal{W}}\colon \Omega^1 \otimes_H \Omega^1 \to \Omega^1 \otimes_H \Omega^1$ is diagonalizable, i.e. $\exists\ \lambda \in \Lambda$ eigenvalues of $\sigma^{\mathcal{W}}$ with eigenspaces V_λ. Then

$$\Omega^1 \otimes_H \Omega^1 = \underbrace{\Omega^1 \vee \Omega^1}_{:=V_1} \oplus \underbrace{\Omega^2}_{=\bigoplus_{\lambda\neq 1} V_\lambda},$$

where we identify $\bigoplus_{\lambda\neq 1} V_\lambda$ with degree 2 of the exterior algebra.

We have a projectors $P_\vee = \Pi_{\lambda\neq 1}\frac{\sigma^{\mathcal{W}}-\lambda\mathrm{id}}{1-\lambda}$ and $\wedge = 1 - P_\vee$ with corresponding projections $\pi_\vee\colon \Omega^1 \otimes_H \Omega^1 \to \Omega^1 \vee \Omega^1$ and $\pi_\wedge\colon \Omega^1 \otimes_H \Omega^1 \to \Omega^2$

The dual bicovariant H-bimodule $\mathfrak{X} := \mathrm{Hom}_H(\Omega^1, H)$ of "vector fields" admits a dual decomposition

$$\mathfrak{X}^1 \otimes_H \mathfrak{X}^1 = \underbrace{\mathfrak{X}^1 \vee \mathfrak{X}^1}_{=V_1^*} \oplus \underbrace{\mathfrak{X}^2}_{=\bigoplus_{\lambda\neq 1} V_\lambda^*}.$$

Example

$\sigma^{\mathcal{W}}$ is diagonalizable for all matrix quantum groups of the A, B, C, D series.

Decomposition of connections and LC

Given a right connection ∇ on Ω^1 we construct the sum of connection $\nabla \oplus \nabla \colon \Omega^1 \otimes_H \Omega^1 \to (\Omega^1 \otimes_H \Omega^1) \otimes_H \Omega^1$. Then we obtain

1. a connection $\nabla_\vee := \pi_\vee^{12} \circ (\nabla \oplus \nabla)|_{\Omega^1 \vee \Omega^1} \colon \Omega^1 \vee \Omega^1 \to (\Omega^1 \vee \Omega^1) \otimes_H \Omega^1$ on $\Omega^1 \vee \Omega^1$
2. a connection $\nabla_\wedge := \pi_\wedge^{12} \circ (\nabla \oplus \nabla)|_{\Omega^2} \colon \Omega^2 \to \Omega^2 \otimes_H \Omega^1$ on Ω^2
3. two right H-linear maps $\nabla_{12} := \pi_\vee^{12} \circ (\nabla \oplus \nabla)|_{\Omega^2} \colon \Omega^2 \to (\Omega^1 \vee \Omega^1) \otimes_H \Omega^1$ and $\nabla_{21} := \pi_\wedge^{12} \circ (\nabla \oplus \nabla)|_{\Omega^1 \vee \Omega^1} \colon \Omega^1 \vee \Omega^1 \to \Omega^2 \otimes_H \Omega^1$

$$\nabla = \begin{pmatrix} \nabla_\vee & \nabla_{12} \\ \nabla_{21} & \nabla_\wedge \end{pmatrix}$$

The dual left connection $\nabla \colon \mathfrak{X} \to \Omega^1 \otimes_H \mathfrak{X}$ with respect to ∇ is defined by

$$\langle \nabla X, \omega \rangle = \mathrm{d}\langle X, \omega \rangle - \langle X, \nabla \omega \rangle.$$

Lemma

It follows that $(\nabla_\vee, \nabla_\vee)$ and $(\nabla_\wedge, \nabla_\wedge)$ are dual connections. Furthermore $(\nabla_{12}, -\nabla_{21})$ and $(\nabla_{21}, -\nabla_{12})$ are transposed maps.

Definition (Levi-Civita connection)

An element $\mathbf{g} \in \mathfrak{X} \vee \mathfrak{X}$ is said to be a (pseudo-Riemannian) **metric** if $\mathbf{g}^\# \colon \Omega^1 \to \mathfrak{X}$, $\mathbf{g}^\#(\omega) = \langle \mathbf{g}, \omega \rangle$ is an isomorphism in $\mathcal{M}_H$.

A right connection ∇ on Ω^1 is said to be **Levi-Civita** if $\nabla_\vee(\mathbf{g}) = 0$ and $\mathrm{Tor}^\nabla = 0$.

Existence and uniqueness theorem

Given a metric $\mathbf{g}$ we define a right H-linear map

$$\Phi_{\mathbf{g}} \colon \mathrm{Hom}_H(\Omega^1, \Omega^1 \vee \Omega^1) \to \mathrm{Hom}_H(\Omega^1 \vee \Omega^1, \Omega^1)$$

by $\Phi_{\mathbf{g}}(\phi) = \langle \mathbf{g}, \cdot \vee \cdot \rangle \circ (\phi \oplus \phi)_\vee$, where

$$\phi \oplus \phi = \sigma_{23}^{\mathcal{W}} \circ (\phi \otimes_{\sigma\mathcal{W}} \mathrm{id}_{\Omega^1}) + \mathrm{id}_{\Omega^1} \otimes_{\sigma\mathcal{W}} \phi$$

mimics the sum of connections.

Theorem (Aschieri-TW)

Let $\mathbf{g} \in \mathfrak{X} \vee \mathfrak{X}$ be a metric. If $\Phi_{\mathbf{g}}$ is invertible then

$$\nabla^{\mathrm{LC}} = \nabla^{C} + \Phi_{\mathbf{g}}^{-1}\left(\left(\mathrm{d} \circ \langle \mathbf{g}, \cdot \vee \cdot \rangle - \langle \mathbf{g}, \nabla^{C}(\cdot \vee \cdot) \rangle \right) \Big|_{\Omega^1 \vee \Omega^1} \right)$$

is the unique Levi-Civita connection for $\mathbf{g}$.

Sketch of the proof

Existence:

1. $\psi := \left(\mathrm{d}\circ\langle\mathbf{g},\cdot\vee\cdot\rangle - \langle\mathbf{g},\nabla^C(\cdot\vee\cdot)\rangle\right)\Big|_{\Omega^1\vee\Omega^1}\colon \Omega^1\vee\Omega^1\to\Omega^1$ is right H-linear
2. $\phi := \Phi_{\mathbf{g}}^{-1}(\psi)\in\mathrm{Hom}_H(\Omega^1,\Omega^1\vee\Omega^1)$ and $\nabla^{\mathrm{LC}} := \nabla^C + \phi$ is a right connection on Ω^1
3. Torsion-free: $\wedge\circ\nabla^{\mathrm{LC}} = \wedge\circ\nabla^C + \wedge\circ\phi = -\mathrm{d} + 0$
4. Metric compatible:

$$\begin{aligned}\langle\mathbf{g},\nabla^{\mathrm{LC}}_\vee(\cdot\otimes_S\cdot)\rangle &= \langle\mathbf{g},\nabla^C_\vee(\cdot\vee\cdot)\rangle + \langle\mathbf{g},(\phi\oplus\phi)_\vee(\cdot\vee\cdot)\rangle\\ &= \langle\mathbf{g},\nabla^C_\vee(\cdot\vee\cdot)\rangle + \psi(\cdot\vee\cdot)\\ &= \mathrm{d}\circ\langle\mathbf{g},\cdot\vee\cdot\rangle\end{aligned}$$

$\Rightarrow \nabla_\vee(\mathbf{g}) = 0$

Uniqueness: Assume ∇ is another LC connection for $\mathbf{g}$. Then $\nabla^{\mathrm{LC}} - \nabla\colon\Omega^1\to\Omega^1\vee\Omega^1\subseteq\Omega^{\otimes_H 2}$ since $\wedge\circ(\nabla^{\mathrm{LC}}-\nabla) = -\mathrm{d}+\mathrm{d} = 0$. Now

$$\Phi_{\mathbf{g}}(\nabla^{\mathrm{LC}}-\nabla) = \langle\mathbf{g},\nabla^{\mathrm{LC}}(\cdot\vee\cdot)\rangle - \langle\mathbf{g},\nabla(\cdot\vee\cdot)\rangle = \mathrm{d}\circ\langle\mathbf{g},\cdot\vee\cdot\rangle - \mathrm{d}\circ\langle\mathbf{g},\cdot\vee\cdot\rangle = 0$$

implies $\nabla^{\mathrm{LC}} = \nabla$ by the injectivity of $\Phi_{\mathbf{g}}$.

Another existence and uniqueness theorem

We call $\mathbf{g}$ σ-central if $\sigma^{\mathcal{W}}(\mathbf{g} \otimes_H \omega) = \omega \otimes_H \mathbf{g}$ and $\sigma^{\mathcal{W}}(\omega \otimes_H \mathbf{g}) = \mathbf{g} \otimes_H \omega \ \forall \omega \in \Omega^1$.

Lemma

1. *If $\mathbf{g}$ is σ-central then $\mathbf{g}$ is a central element.*
2. *If $\mathbf{g}$ is central and bi-coinvariant then $\mathbf{g}$ is σ-central.*

Theorem (Aschieri-TW)

If $\mathbf{g}$ is σ-central then $\Phi_{\mathbf{g}}$ is invertible if and only if

$$\pi^{23}_{\vee} \colon (\Omega^1 \vee \Omega^1) \otimes_H \Omega^1 \to \Omega^1 \otimes_H (\Omega^1 \vee \Omega^1)$$

is invertible.

Proof.

$$\begin{array}{ccc} \mathrm{Hom}_H(\Omega^1, \Omega^1 \vee \Omega^1) \cong (\Omega^1 \vee \Omega^1) \otimes_H \mathfrak{X} & \xrightarrow{\mathrm{id}_{\Omega^1 \vee \Omega^1} \otimes_{\sigma^{\mathcal{W}}} \mathbf{g}^{\#-1}} & (\Omega^1 \vee \Omega^1) \otimes_H \Omega^1 \\ \Big\downarrow \tfrac{1}{2}\Phi_{\mathbf{g}} & & \Big\downarrow \pi^{23}_{\vee} \\ \mathrm{Hom}_H(\Omega^1 \vee \Omega^1, \Omega^1) \cong \Omega^1 \otimes_H (\Omega^1 \vee \Omega^1)^* & \xleftarrow{\mathrm{id}_{\Omega^1} \otimes_{\sigma^{\mathcal{W}}} \mathbf{g}^{\#2}} & \Omega^1 \otimes_H (\Omega^1 \vee \Omega^1) \end{array}$$

□

Extension and examples

Theorem (Aschieri-TW)

Let $\mathbf{g}$ *be a metric such that* $\Phi_{\mathbf{g}}$ *is invertible. For any metric* $\mathbf{g}'$ *which is conformally equivalent to* $\mathbf{g}$ *(i.e.* $\mathbf{g}' = f\mathbf{g}$ *for an* $f \in H$ *invertible) there is the unique Levi-Civita connection*

$$\nabla'^{\mathrm{LC}} = \nabla^{\mathrm{LC}} + \Phi_{\mathbf{g}}^{-1}\left(f^{-1}\mathrm{d}f\ \langle \mathbf{g}, \cdot \otimes_H \cdot \rangle \Big|_{\Omega^1 \vee \Omega^1} \right).$$

Example

For the 4-dim bicovariant calculus on $\mathrm{SL}_q(2)$ it was shown that $\pi_{\vee}^{23}$ is invertible.
$\Rightarrow$ for any σ-central metric on $\mathrm{SL}_q(2)$ there is a unique LC connection.

Consider for example the bi-coinvariant central (so in particular σ-central) metric

$$\mathbf{g} = e_c \otimes_H e_b + q^2 e_b \otimes_H e_c + \frac{q^2}{(2)_q}(e_z \otimes_H e_z + \theta \otimes_H \theta).$$

Then there is a unique LC connection for $\mathbf{g}$ on $\mathrm{SL}_q(2)$.

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Thank you for your attention!