Gauge Theory, Sigma Models and Generalised Geometry

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Kapustin-Witten theory (2007) is a twisted $N=4$ gauge theory in 4d

- reduces to 2d sigma model with a hyper-Kähler target;
- explains geometric Langlands by electric-magnetic duality.

<u>Today</u>: reduction from a 4-manifold which is not a global product.

Based on Wu (2018) + to appear

Gauge theory in 4d

- gauge group G, a compact simple Lie group.
- $\pi_1(G)$ and $Z(G)$ are finite Abelian groups
- spacetime $X^4 = \mathbb{R} \times Y^3$

canonical quantisation, Hilbert space $\mathcal{H}$

electricity vs. magnetism

18 September 2022 20:04

Discrete magnetic fluxes

Each non-contractible closed surface $S \subset Y$ carries a discrete magnetic flux in $\pi_1(G)$

eg. 't Hooft, $S = T^2$, $G = PU(n)$, $\pi_1(G) = \mathbb{Z}_n$.

If Y^3 has many such surfaces S, the discrete magnetic flux $m \in H^2(Y, \pi_1(G))$.

$$\mathcal{H} = \bigoplus_{m \in H^2(Y, \pi_1(G))} \mathcal{H}_m .$$

Discrete electric fluxes

For each non-contractible loop $L \subset Y^3$, the theory has a $Z(G)$-symmetry: modify the holonomy along L by $z \in Z(G)$ without changing $F_{\mu\nu}$.

If Y has many such loops, the symmetry group is $H^1(Y, Z(G))$.

21 September 2022 06:16

At the quantum level,

$$\mathcal{H} = \bigoplus_{e \in H^1(Y, Z(G))^\vee} \mathcal{H}_e .$$

where for any Abelian group A, the Pontryagin dual $A^\vee = \mathrm{Hom}(A, U(1))$.

$e \in H^1(Y, Z(G))^\vee$ is a discrete electric flux.

Every simple G has a Langlands dual (1960's) or "magnetic group" (GNO, 1978).

e.g.	G	$SU(n)$	$SO(2n)$	$SO(2n+1)$	E_8
	${}^LG$	$PU(n)$	$Spin(2n)$	$Sp(2n)$	E_8

<u>Fact</u> If Y is an orientable 3-manifold.

$$H^2(Y, \pi_1(G)) \cong H^1(Y, Z({}^LG))^\vee$$

$$H^1(Y, Z(G))^\vee \cong H^2(Y, \pi_1({}^LG)).$$

21 September 2022 06:16

Dimensional reduction $4d \Rightarrow 2d$

$$X^4 = \underset{(\text{large})}{\Sigma^2} \times \underset{(\text{small})}{C^2}$$

At low energies, fields satisfy energy minimising equations along C.

In this case, we have Hitchin's equations

$$F_A = \frac{1}{2}[\phi, \phi], \quad D_A{*}\phi = 0, \quad D_A\phi = 0.$$

A : gauge field on C

ϕ : adjoint valued scalar boson on C

+ fermions

moduli space $\mathcal{M}_H(C,G) = \{(A,\phi) : \text{Hitchin eq'ns}\} / \text{gauge}$

It is hyper-Kähler, three complex structures
I, J, K, three Kähler forms $\omega_I, \omega_J, \omega_K$.

The low energy 2d theory is a
sigma model with worldsheet Σ
and target $\mathcal{M}_H(C,G)$.

Recall $X^4 = \Sigma \times C$.

Kähler target space M $\Rightarrow$ $N=(2,2)$ SUSY.

More generally (Gates-Hull-Roček, 1984),

if M has a metric g, complex structures $J_\pm$,

B-field B, affine connections $\nabla^\pm$ satisfying

$g(J_\pm \cdot, J_\pm \cdot) = g(\cdot,\cdot)$, $\nabla^\pm g = 0$, $\nabla^\pm J_\pm = 0$,

$\mathrm{torsion}(\nabla^\pm) = \pm H$, $H = dB$. then

the sigma model also has $N=(2,2)$ SUSY.

Generalised geometry

Gualtieri (2003)

Replace $T\mathcal{M}$ by $T\mathcal{M} \oplus T^*\mathcal{M}$.

Courant bracket $[X+\alpha, Y+\eta] = [X,Y] + \cdots$

Then take an isotropic subspace of $T\mathcal{M} \oplus T^*\mathcal{M}$

<u>Fact</u> The GHR data is equivalent to a generalised Kähler structure on $\mathcal{M}$ (pair of commuting generalised complex structures with positivity condition)

$$\mathcal{J}_\pm = \frac{1}{2}\begin{pmatrix} J_+ \pm J_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \pm \omega_- & -(J_+^t \pm J_-^t) \end{pmatrix} \text{ on } T\mathcal{M} \oplus T^*\mathcal{M}.$$

where $\omega_\pm = g J_\pm$.

Back to 4d $N=4$ gauge theory $\tau = \frac{\theta}{2\pi} + \frac{4\pi\sqrt{-1}}{e^2}$

After twisting, two SUSY's δ_ℓ, δ_r survive on X^4.

Choose $\delta_t = \delta_\ell + t\,\delta_r$, $t \in \mathbb{C} \sqcup \{\infty\} = \mathbb{CP}^1$.

The 2d theory has target $\mathcal{M} = \mathcal{M}_H(C, G)$ with

$$\mathcal{J}_t = \frac{1}{1+\bar{t}t}\begin{pmatrix} -\sqrt{-1}(t-\bar{t})J & -\tau_2^{-1}((1-\bar{t}t)\omega_I^{-1} - (t+\bar{t})\omega_K^{-1}) \\ -\tau_2((1-\bar{t}t)\omega_I - (t+\bar{t})\omega_K & \sqrt{-1}(t-\bar{t})J^t \end{pmatrix}$$

- $t = \pm\sqrt{-1}$, B-model with J
- $t \in \mathbb{R} \sqcup \{\infty\}$. A-model, $\omega = a\omega_I + b\omega_K$.
- Other t : B-field transform of an A-model.

Choose $\Sigma = T^1 \times S^1$ (torus), $X = T^1 \times Y$, $Y = S^1 \times C$

dis mag flux $H^2(Y, \pi_1 G) = H^2(C, \pi_1 G) \oplus H^1(C, \pi_1 G)$

$$m = m_0 + m_1$$

$$\pi_0(\mathcal{M}_H) \qquad \pi_1(\mathcal{M}_H)$$

dis elec flux $H^1(Y, Z(G))^\vee = H^1(C, Z(G))^\vee \oplus H^0(C, Z(G))^\vee$

$$e = e_1 + e_0$$

$H^1(C, Z(G))$ acts on $\mathcal{M}_H$ as Sym

$\Downarrow$ B-field on $\mathcal{M}_H$

Now, X^4 contains some non-orientable surface C'
is not a global product $\Sigma \times C$ or $\Sigma \times C'$.
is still orientable.

Consider orientation double cover $\overset{\circlearrowleft \iota}{C} \xrightarrow[\pi]{} C'$

$\tilde{\Sigma}$ orientable surface with orientation reversing ι
which has fixed points

$\Sigma = \tilde{\Sigma}/\iota = \mathring{\Sigma} \cup \partial\Sigma$, $\partial\Sigma$ from fixed pts.

$X^4 = \tilde{\Sigma} \times_2 C$ smooth, without boundary
orientable

X is not a global product, but has a projection map $\pi_X : X \to \Sigma$

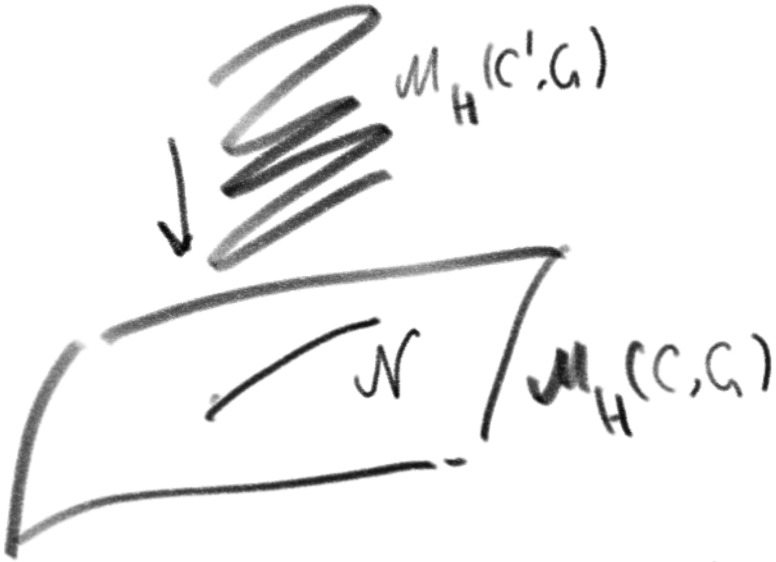
For $\sigma \in \Sigma$, $\pi_X^{-1}(\sigma) = \begin{cases} C, & \text{if } \sigma \in \mathring{\Sigma} \\ C', & \text{if } \sigma \in \partial\Sigma. \end{cases}$

For large $\tilde{\Sigma}, \Sigma$ and small C, C', the low energy is a sigma model

$$\begin{array}{ccc} \Sigma & \to & \mathcal{M}_H(C, G) \\ \cup & & \\ \partial\Sigma & \to & \mathcal{M}_H(C', G) \end{array}$$

Same bulk theory on $\mathring{\Sigma}$, but $\partial\Sigma$ lives on branes!

Geometry of $\mathcal{M}_H(C,G)$ and $\mathcal{M}_H(C',G)$



$\mathcal{N}$ is complex wrt J

but Lagrangian wrt ω_I, ω_K.

$\Downarrow$

brane of type (A,B,A).

Moreover $\mathcal{M}_H(C'.G) \to \mathcal{N}$ is a $Z(G)_{[2]}$-sheeted covering.

For Abelian group A, $A_{[2]} = \{a \in A : 2a = 0\}$.

Recall $t = \pm\sqrt{-1}$ B-model wrt J

$t \in \mathbb{R} \amalg \{\infty\}$, A-model wrt $\omega = a\omega_I + b\omega_K$.

The boundary condition is consistent with SUSY.

Alternative perspective by generalised geometry:

The generalised tangent space of $\mathcal{N}$ is

$\mathrm{Ker}(\tau_* - 1) \oplus \mathrm{Ker}(\tau^* + 1) \subset (T \oplus T^*)\mathcal{M}_H$.

It is preserved by $\mathcal{J}_t$. ~~for~~ all $t \in \mathbb{CP}^1$.

$\mathcal{N}$ is a generalised complex submanifold in $\mathcal{M}_H$

Suppose $\tilde{\Sigma} = T^1 \times S^1$, $\Sigma =$ 

Then $X = T^1 \times Y$, $Y = S^1 \times_{\mathbb{Z}_2} C$.

dis mag flux $\in H^2(Y, \pi_1(G))$ has two parts

$$\begin{array}{ccc} \dfrac{H^1(C, \pi_1 G)}{\pi^* H^1(C', \pi_1 G)} & \text{and} & \left(\dfrac{\pi_1 G}{2\pi_1 G}\right)^{\oplus 2} \\ \| & & \| \\ \text{relative } \pi_1(\mathcal{M}_H, \mathcal{N}) & & \pi_0(\mathcal{M}_H(C', G))^{\oplus 2} \end{array}$$

N

dis elec flux $\in H^1(Y, Z(G))^\vee$ has two parts

$\pi^* H^1(C', Z(G))^\vee$ and $\left(Z(G)^{\oplus 2}_{[2]}\right)^\vee$

from symm on $\mathcal{M}_H(C,G)$ preserving $\mathcal{N}$

deck transformation of $\mathcal{M}_H(C',G) \to \mathcal{N}$

Absence of m_0, e_0

Only $\mathcal{M}_H(C,G)_{m_0=0}$ supports branes

If $e_0 \neq 0$, the 2d theory is anomalous.

20

21 September 2022 06:17

Thank you!