A study of Leptonic CP Violation

Workshop on the Standard Model and Beyond

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Based on arXiv:2205.02796, Y.H.Ahn, R. Ramos, SK, M. Tanimoto

Outline

- Introduction
- Neutrino Mixing Matrix
- Predicting Leptonic CP violation
- Numerical Results & Future Prospect
- A model based on modular A_4 symmetry
- Conclusion

Introduction

- Brief summary of neutrino oscillation parameters:

  - Discovery of neutrino oscillations →
  massive neutrinos & lepton flavor mixing

  - Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{\mathrm{CP}}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{\mathrm{CP}}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\alpha_1} & 0 & 0 \\ 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$|\nu_\alpha> = \Sigma\, U_{\alpha i}\, |\nu_i>$ (flavor α)

-Oscillation probability in vacuum :

$$P_{\alpha\beta}(E,L) = \sum_{k,j} U^*_{\alpha k} U_{\beta k} U_{\alpha j} U^*_{\beta j} e^{i\frac{\Delta m^2_{kj}}{2E}L}$$

Introduction

- Brief summary of neutrino oscillation parameters:

  - From ν oscillation exps. we can determine

Δm^2_{21}

θ_{12}

KamLAND+ SOL

$|\Delta m^2_{31}|$

ATM + REACT+ LBL

θ_{23}

ATM + LBL

θ_{13}

REACT

δ : LBL / ATM

Mass Ordering : LBL+REACT / ATM

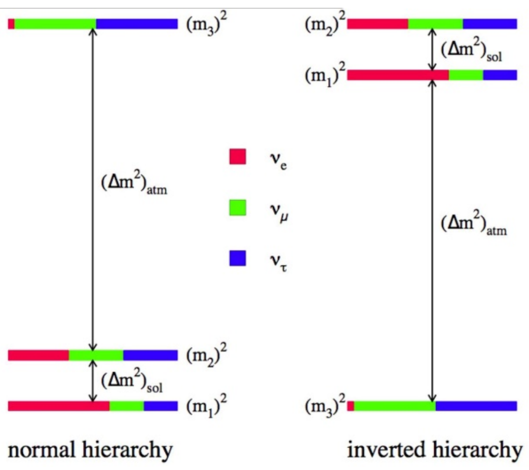
Unknown

Global fit to neutrino data

M.C.Gonzalez-Garcia, M. Maltoni, T. Schwetz, arXiv:2111.03086

Parameter	Best fit$\pm 1\sigma$ (NO)	3σ range (NO)	Best fit$\pm 1\sigma$ (IO)	3σ range (IO)
$\sin^2\theta_{12}$	0.304 ± 0.012	$[0.269, 0.343]$	$0.304^{+0.013}_{-0.012}$	$[0.269, 0.343]$
$\sin^2\theta_{13}$ $[10^{-2}]$	2.246 ± 0.062	$[2.060, 2.435]$	$2.241^{+0.074}_{-0.062}$	$[2.055, 2.457]$
$\sin^2\theta_{23}$	$0.450^{+0.019}_{-0.016}$	$[0.408, 0.603]$	$0.570^{+0.016}_{-0.022}$	$[0.410, 0.613]$
δ_{CP} [deg]	230^{+36}_{-25}	$[144, 350]$	278^{+22}_{-30}	$[194, 345]$
Δm^2_{21} $[10^{-5}\ \mathrm{eV}^2]$	$7.42^{+0.21}_{-0.20}$	$[6.82, 8.04]$	$7.42^{+0.21}_{-0.20}$	$[6.82, 8.04]$
Δm^2_{3k} $[10^{-3}\ \mathrm{eV}^2]$	2.510 ± 0.027	$[2.430, 2.593]$	$-2.490^{+0.26}_{-0.28}$	$[-2.574, -2.410]$

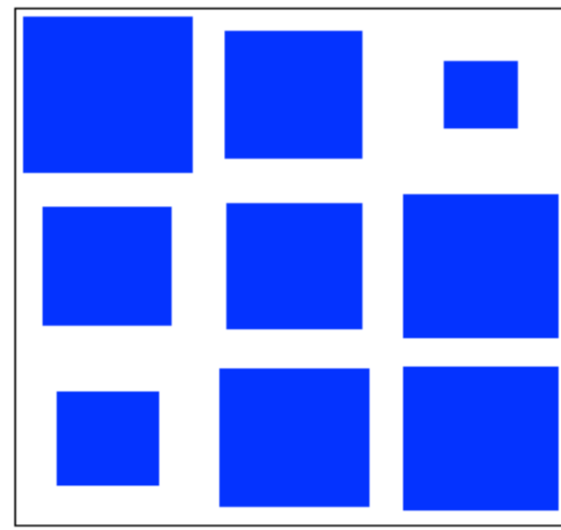
TABLE I. Oscillation parameters for three neutrino flavors as reported normal ordering (NO) ($\Delta m^2_{3k} = \Delta m^2_{31}$) and inverted ordering (IO) (Δm^2_{3k} = tabulated χ^2 data from Super-Kamiokande.



Neutrino Mixing Matrix

- From fit to neutrino data in 3-neutrino paradigm

$$|U_{PMNS}| = \begin{pmatrix} 0.800-0.844 & 0.515-0.581 & 0.139-0.155 \\ 0.229-0.516 & 0.438-0.699 & 0.614-0.790 \\ 0.249-0.528 & 0.462-0.715 & 0.595-0.776 \end{pmatrix}$$



Looks different from quark mixing matrix !!

$$|V_{CKM}| = \begin{pmatrix} 0.97434 & 0.22506 & 0.00357 \\ 0.22492 & 0.97351 & 0.0414 \\ 0.00875 & 00403 & 0.99915 \end{pmatrix}$$

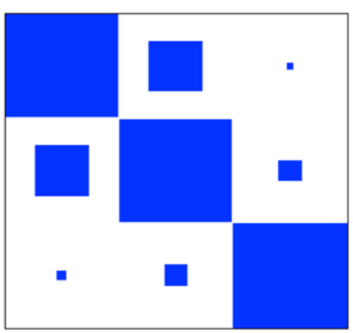


Image from Mark Messier

Neutrino Mixing Matrix

- Before measuring θ_{13} , tri-bimaximal mixing hypothesis :

$$- U^{TBM} = \begin{pmatrix} \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Harrison & Perkins & Scott (2002)

$$\theta_{13} \approx 0; \quad \theta_{23} \approx 45^0; \quad \theta_{12} = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 35.3^0$$

- generates specific neutrino mass matrix

$$UM_\nu^D U^T = \begin{pmatrix} m_1 & m_2 & m_2 \\ \cdot & \frac{1}{2}(m_1+m_2+m_3) & \frac{1}{2}(m_1+m_2-m_3) \\ \cdot & \cdot & \frac{1}{2}(m_1+m_2+m_3) \end{pmatrix}$$

$$= \frac{m_1+m_3}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{m_2-m_1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{m_1-m_3}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

– Integer (inter–family related) matrix elements hint at discrete symmetry

(E. Ma, G. Rajasekaran, 2001)

Neutrino Mixing Matrix

- Those mixings are theoretically well motivated, but challenged by the current experimental data.
- Modifying Tri-Bimaximal mixing matrix

  - Minimal possible forms deviated from Tri-Bimaximal :

$$V=\begin{cases} U^{TBM} \cdot U_{ij}(\theta,\phi) \\ {U_{ij}}^{+}(\theta,\phi) \cdot U^{TBM} \end{cases}$$

- θ possibly gives rise to non-zero θ_{13} and possible deviation from maximal for θ_{23}.
- We call those forms modified TBM parameterization.

(Ge, Dicus,Repko, Phys. Lett. B 702(2011), SK, CSKim, PRD90(2014), SK, Tanimoto, PRD91(2015), Shimizu,Tanimoto,Yamamoto, MPLA30(2015)Girardi, Petcov, Titov, NPB894(2015), NPB902(2016), Delgadillo etal, PRD97(2018)…)

Neutrino Mixing Matrix

- Among possible 6 forms, the following 4 are eligible for our aim :

$$V = \begin{cases} U^{TBM} \cdot U_{23}(\theta, \phi) & (case\ A) \\ U^{TBM} \cdot U_{13}(\theta, \phi) & (case\ B) \\ {U_{12}}^{+}(\theta, \phi) \cdot U^{TBM} & (case\ C) \\ {U_{13}}^{+}(\theta, \phi) \cdot U^{TBM} & (case\ D) \end{cases}$$

$$U_{12}(\theta,\phi) = \begin{pmatrix} \cos\theta & -\sin\theta e^{i\phi} & 0 \\ \sin\theta e^{-i\phi} & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},\ U_{13}(\theta,\phi) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta e^{i\phi} \\ 0 & 1 & 0 \\ \sin\theta e^{-i\phi} & 0 & \cos\theta \end{pmatrix},\ U_{23}(\theta,\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta e^{i\phi} \\ 0 & \sin\theta e^{-i\phi} & \cos\theta \end{pmatrix}$$

- $$U^{TBM} \cdot U_{12} = \begin{pmatrix} \blacksquare & \blacksquare & 0 \\ \blacksquare & \blacksquare & -\frac{1}{\sqrt{2}} \\ \blacksquare & \blacksquare & \frac{1}{\sqrt{2}} \end{pmatrix}$$ unchanged $$U^{TBM} \cdot U_{13} = \begin{pmatrix} \blacksquare & -\frac{1}{\sqrt{3}} & \blacksquare \\ \blacksquare & \frac{1}{\sqrt{3}} & \blacksquare \\ \blacksquare & \frac{1}{\sqrt{3}} & \blacksquare \end{pmatrix}$$

- $$U^{TBM} \cdot U_{23} = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & \blacksquare & \blacksquare \\ \frac{1}{\sqrt{6}} & \blacksquare & \blacksquare \\ \frac{1}{\sqrt{6}} & \blacksquare & \blacksquare \end{pmatrix}$$ etc...

Unchanged rows and columns of TBM may reflect the remnants of flavor symmetry

CP violation in U_{PMNS}

- Can we predict possible value of LCPV?

  -- From the viewpoint of calculability, it is conceivable that a LCP phase can be predicted in terms of some observables.

  -- What observables can be responsible for the prediction ?

  ν masses, mixing angles, or all of them ?

- We propose a simple scheme to calculate the possible value of LCP phase in terms of two ν mixing angles only, in the standard parameterization of neutrino mixing matrix.

Calculability of LCPV

- Any forms of neutrino mixing matrix should be equivalent to the PMNS matrix presented in the (PDG) standard para :
- $U^{ST} = U_{23}(\theta_{23})U_{13}(\theta_{13}, \delta_D)U_{12}(\theta_{12})P_{\phi}$

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_D} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_D} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_D} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_D} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_D} & c_{23}c_{13} \end{pmatrix} P$$

$= P_{\alpha} \cdot \underbrace{V(\theta, \phi)}_{\text{(modified TBM)}} \cdot P_{\beta}$: neutrino mixing matrix proposed before

$\Rightarrow$ $\boxed{V_{ij}e^{i(\alpha_i+\beta_j)} = (U^{ST})_{ij}}$

Predicting LCPV

- For $V = \begin{cases} U^{TBM} \cdot U_{23}(\theta, \phi) & (case\ A) \\ U^{TBM} \cdot U_{13}(\theta, \phi) & (case\ B) \end{cases}$

- Using $|V_{13}| = |U_{13}{}^{ST}|$ and $\left|{}^{V_{11}}/_{V_{12}}\right| = \left|{}^{U_{11}{}^{ST}}/_{U_{12}{}^{ST}}\right|$, we get

$$s_{12}{}^2 = \begin{cases} 1 - \frac{2}{3(1-s_{13}{}^2)} & (case\ A) \\ \frac{1}{3(1-s_{13}{}^2)} & (case\ B) \end{cases}$$

- From the explicit form of V for case A, we see that

$$\frac{V_{23}-V_{33}}{V_{22}-V_{32}} = \frac{V_{13}}{V_{12}} \quad \text{and} \quad V_{21} = -V_{31}$$

Predicting LCPV

- Using $V_{ij}e^{i(\alpha_i+\beta_j)} = (U^{ST})_{ij}$, we can get

$$\frac{U_{13}{}^{ST}}{U_{12}{}^{ST}} = \frac{U_{23}{}^{ST}U_{31}{}^{ST} + U_{33}{}^{ST}U_{21}{}^{ST}}{U_{22}{}^{ST}U_{31}{}^{ST} + U_{32}{}^{ST}U_{21}{}^{ST}}$$

- Plugging the explicit form of $U_{ij}{}^{ST}(\theta_{ij}, \delta_D)$,

$$\boxed{\cos\delta_D = \frac{1}{\tan 2\theta_{23}} \cdot \frac{1-5s_{13}{}^2}{s_{13}\sqrt{2-6s_{13}{}^2}}}$$

- Leptonic Jarlskog invariant :

$$J_{CP}{}^2 = (\mathrm{Im}[U_{11}{}^{ST}U_{12}{}^{ST*}\,U_{21}{}^{ST}U_{11}{}^{ST*}\,])^2$$
$$= \frac{1}{12^2}\left(8s_{13}{}^2(1-3s_{13}{}^2) - \cos 2\theta_{23}\, s_{13}{}^2\right)$$

Predicting LCPV

- For $V = \begin{cases} U_{23}(\theta,\phi)U^{BM} & (case\ C) \\ U_{13}(\theta,\phi)U^{BM} & (case\ D) \end{cases}$
- Using $|V_{13}| = |U_{13}{}^{ST}|$ and $\left|{}^{V_{23}}/_{V_{33}}\right| = \left|{}^{U23^{ST}}/_{U33^{ST}}\right|$, we get

$$s_{23}{}^2 = \begin{cases} 1 - \dfrac{1}{2(1-s_{13}{}^2)} & (case\ C) \\ \dfrac{1}{2(1-s_{13}{}^2)} & (case\ D) \end{cases}$$

Predicting LCPV

- Taking the same procedure shown for case A,

Cases		$\cos\delta_D$	$\mathrm{J}^2_{\mathrm{CP}}$
B	$\frac{V_{21}-V_{31}}{V_{23}-V_{33}} = \frac{V_{11}}{V_{13}}$	$-\eta_{23}\frac{1-2s^2_{13}}{s_{13}\sqrt{2-3s^2_{13}}}$	$\frac{1}{6^2}[s^2_{13}(2-3s^2_{13})-\kappa_{23}]$
C	$\frac{V_{12}-V_{11}}{V_{21}-V_{22}} = \frac{V_{13}}{V_{23}}$	$\eta_{12}\frac{1-3s^2_{13}}{s_{13}\sqrt{1-2s^2_{13}}}$	$\frac{1}{8^2}[4s^2_{13}(1-2s^2_{13})-\kappa_{12}]$
D	$\frac{V_{11}-V_{12}}{V_{32}-V_{31}} = \frac{V_{13}}{V_{33}}$	$-\eta_{12}\frac{1-3s^2_{13}}{s_{13}\sqrt{1-2s^2_{13}}}$	$\frac{1}{8^2}[4s^2_{13}(1-2s^2_{13})-\kappa_{12}]$

$$\eta_{ij} = \frac{1}{2\tan 2\theta_{ij}} \text{ and } \kappa_{ij} = \cos^2 2\theta_{ij}\cdot c^4_{13}$$

Numerical Results

- Plugging 1 σ data for s_{13}^2 into the previous expressions,

  we predict

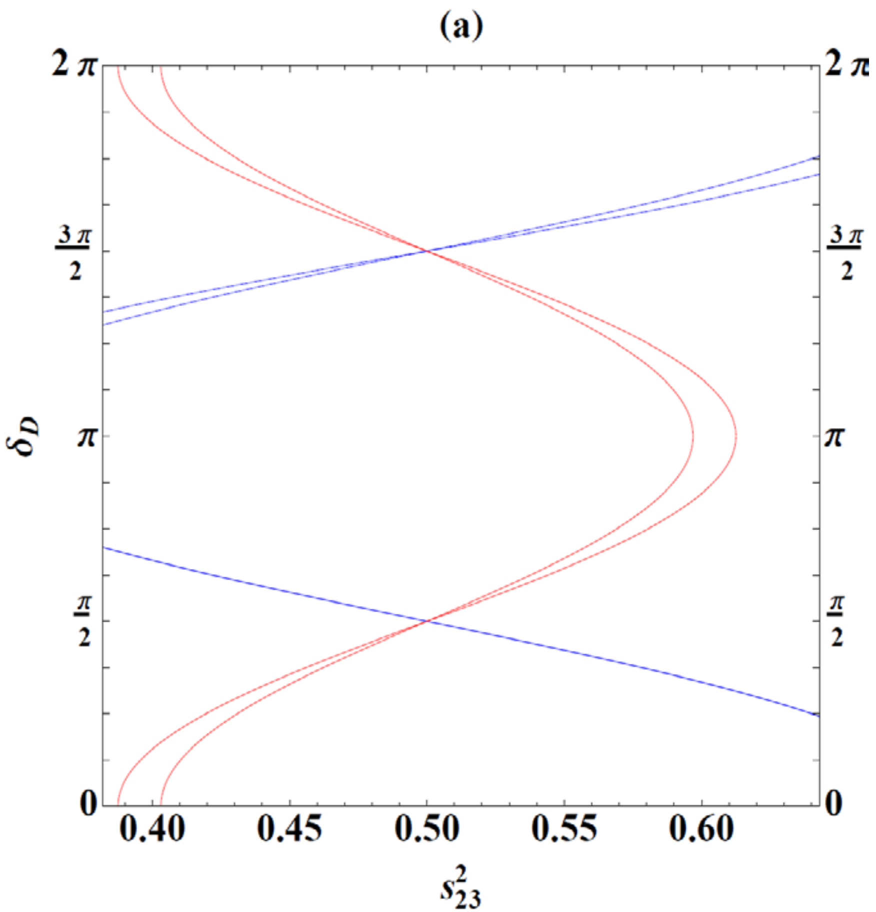
$$s_{12}^2 = \begin{cases} 0.318 - 0.319(\mathrm{CaseA}), \\ 0.340 - 0.341(\mathrm{CaseB}), \end{cases}$$

$$s_{23}^2 = \begin{cases} 0.488 - 0.489(\mathrm{CaseC}), \\ 0.510 - 0.511(\mathrm{CaseD}). \end{cases}$$

- It turns out that case A is in consistent with global fit results at ~1 σ, whereas the others are so at 3 σ.

Numerical Results

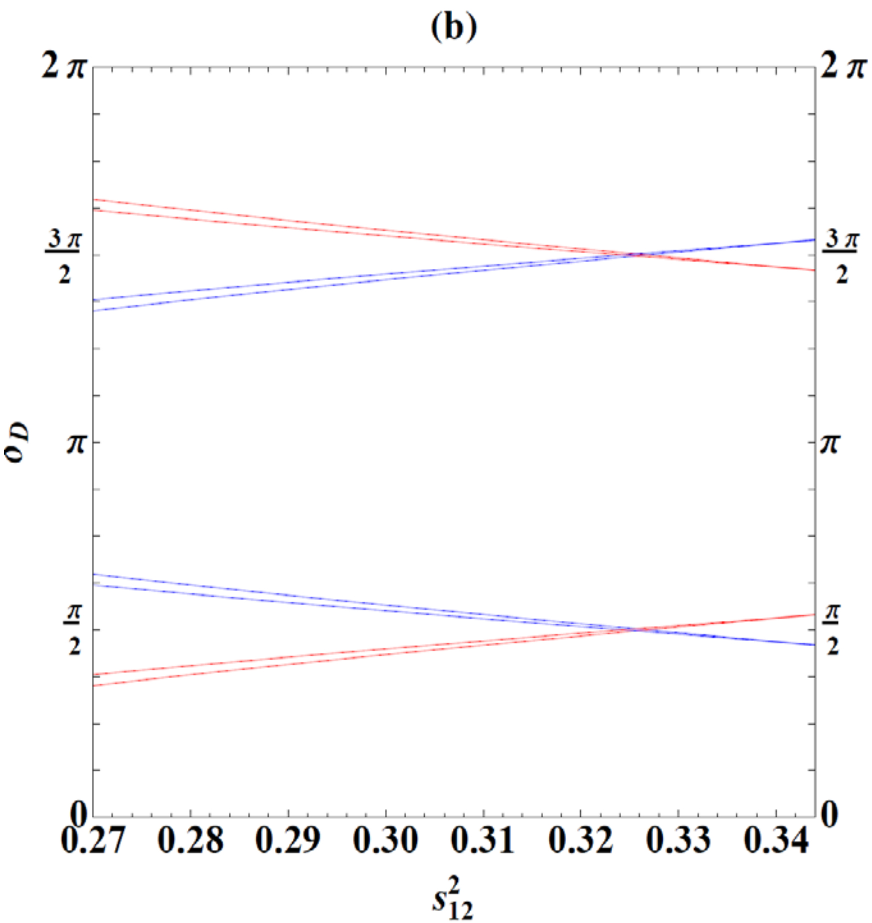
- Extending the data up to 3 σ



Blue : Case A
Red : Case B

Numerical Results

- Extending the data up to 3 σ



Blue : Case C
Red : Case D

Statistical analysis

- Probability density of $\cos\delta_D$

$$P^{(\mathrm{A,B})}_{\cos\delta_{\mathrm{CP}}}(z) = \int dx\,dy\,\delta(f_{\mathrm{A,B}}(x,y) - z)P_{s^2_{13}}(x)P_{s^2_{23}}(y),$$

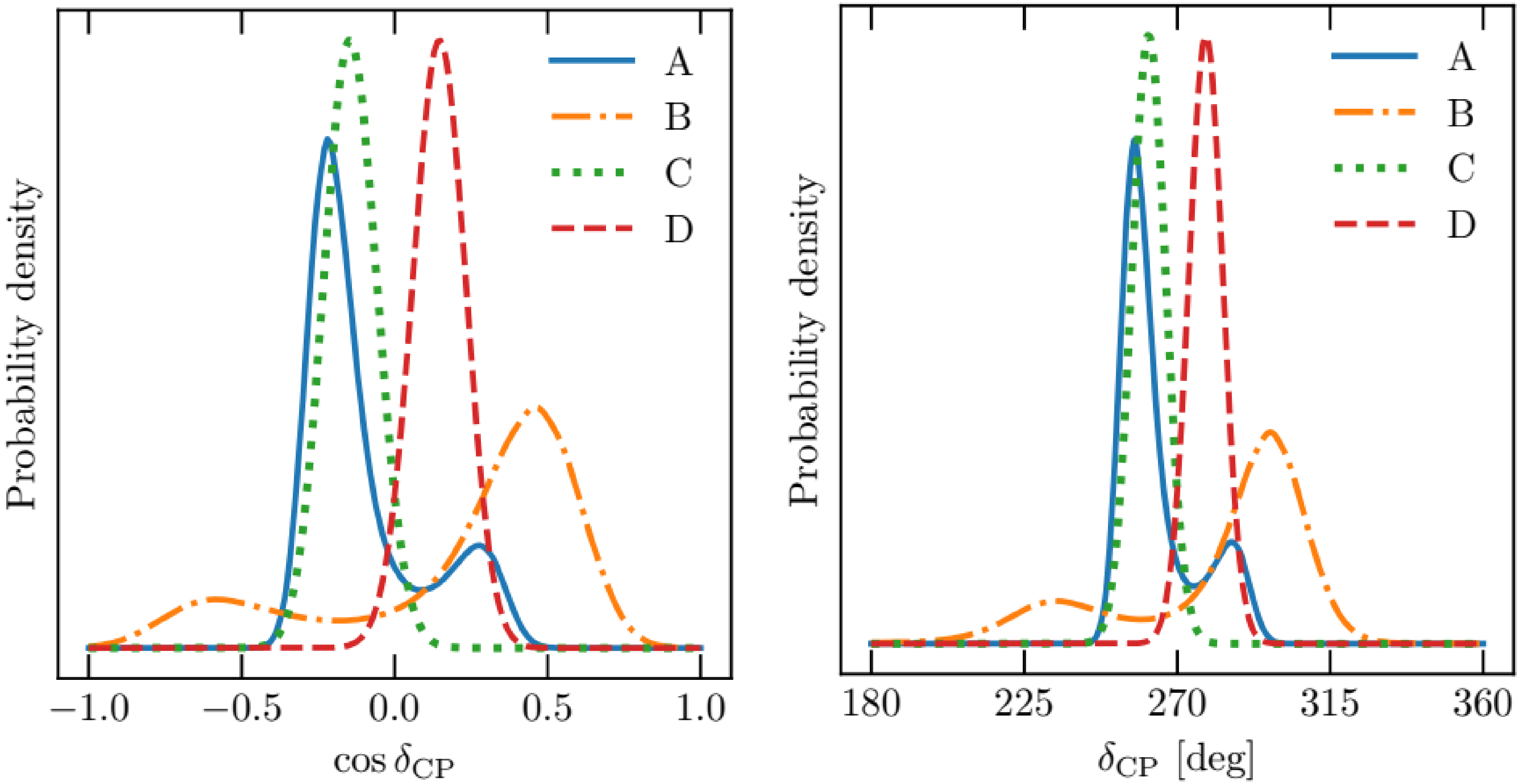
$$P^{(\mathrm{C,D})}_{\cos\delta_{\mathrm{CP}}}(z) = \int dx\,dw\,\delta(f_{\mathrm{C,D}}(x,w) - z)P_{s^2_{13}}(x)P_{s^2_{12}}(w),$$

w, x, y represent values of s^2_{12}, s^2_{13}, s^2_{23}, respectively.

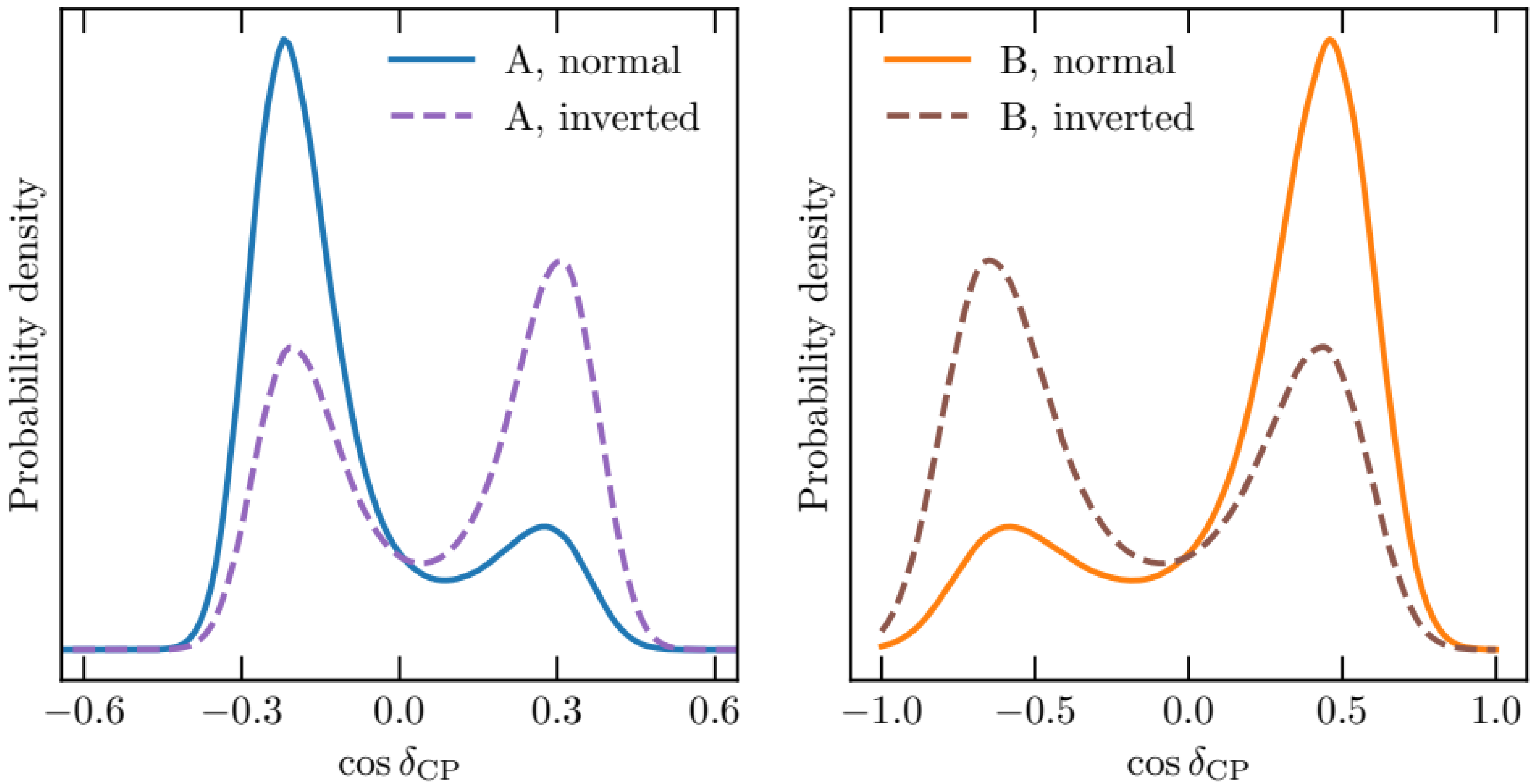
The functions f_j, with $j \in \{\mathrm{A,B,C,D}\}$, represent $\cos\delta_{\mathrm{CP}}$ for each case

$P(\alpha) = N\exp(-\chi^2(\alpha)/2)$, where $N = (\int d\alpha \exp(-\chi^2(\alpha)/2))^{-1}$

We use χ^2 table provided by NuFIT web.



A
B
C
D
Probability density
−1.0
−0.5
0.0
0.5
1.0
cos δCP
180
225
270
315
360
δCP [deg]


A, normal
A, inverted
Probability density
−0.6
−0.3
0.0
0.3
0.6
$\cos\delta_{\rm CP}$
B, normal
B, inverted
Probability density
−1.0
−0.5
0.0
0.5
1.0
$\cos\delta_{\rm CP}$

Prospects at future experiments

- We consider DUNE, T2HK, ESSnuSB
- For numerical analysis, we use GLoBES.
- To compare observations with predictions, GLoBES χ^2_G function

$$\chi^2_{\mathrm{G}}(\theta,\phi) = \sum_i \left[N_i^{\mathrm{th}}(\theta,\phi) - N_i^{\mathrm{obs}} + N_i^{\mathrm{obs}} \ln\left(\frac{N_i^{\mathrm{obs}}}{N_i^{\mathrm{th}}(\theta,\phi)} \right) \right]$$

- We compute # of electron events in the i-th E bin,

$$N_i = T\, n_n\, \epsilon \int_0^{E_{\max}} dE \int_{E_{A_i}^{\min}}^{E_{A_i}^{\max}} dE_A\, \phi(E)\, \sigma_{\nu_e}(E)\, R(E, E_A)\, P_{\mu e}(E)$$

T : total running time, ϵ : detector efficiency
$\phi(\mathrm{E})$: neutrino flux spectrum
$R(\mathrm{E}, \mathrm{E_A})$: Gaussian energy resolution function of the detector

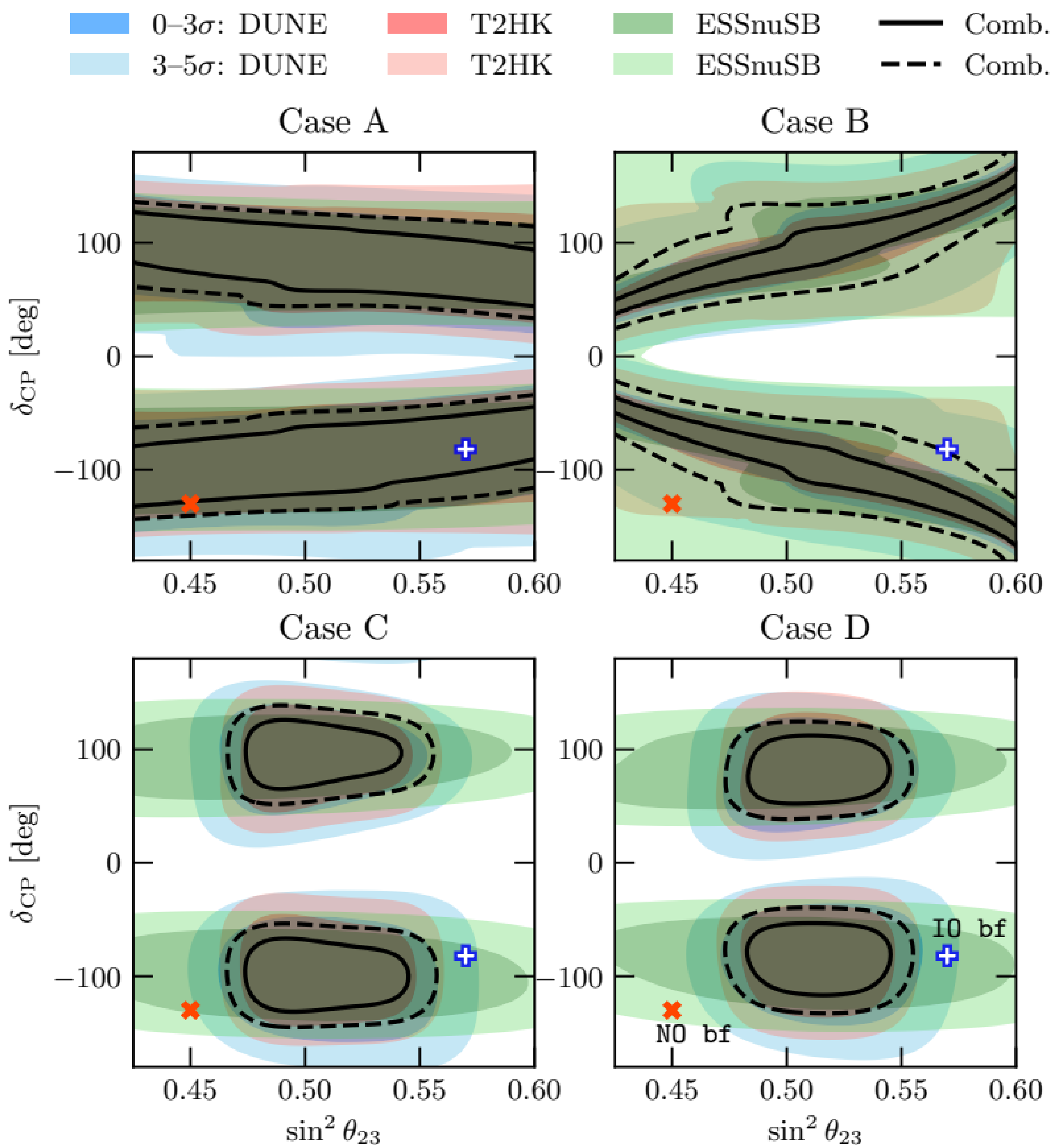
Prospects at future experiments

- We also consider Gaussian prior contributions to total χ^2

$$\chi^2_{\rm pr}(\theta,\phi) = \left(\frac{s^2_{12}(\theta,\phi) - s^2_{12,\rm obs}}{\sigma_{12}}\right)^2 + \left(\frac{s^2_{13}(\theta,\phi) - s^2_{13,\rm obs}}{\sigma_{13}}\right)^2 + \chi^2_{23,\rm NuFIT}(s^2_{23}(\theta,\phi))$$

$$\chi^2(\theta,\phi) = \chi^2_{\rm G}(\theta,\phi) + \chi^2_{\rm pr}(\theta,\phi). \qquad \Delta\chi^2 \equiv \chi^2_{min} - \chi^2_0$$

- For DUNE, we adopt simulation given in arXiv:2002.03005
  (for a total run time of 7 years, 0.5 GeV < E<10 GeV)
- For T2HK, we adopt simulation given in arXiv:1611.06118
  (for a total run time of 10 years for the first detector
  0.1 GeV < E < 7 GeV)
- For ESSnuSB, we use the results in http://essnusb.eu/DocDB
  (for a total of 10 years, 0.1 GeV < E < 2.5 GeV)



0–3σ: DUNE
3–5σ: DUNE
T2HK
T2HK
ESSnuSB
ESSnuSB
Comb.
Comb.
Case A
Case B
Case C
Case D
δCP [deg]
sin² θ23
100
0
−100
0.45
0.50
0.55
0.60
IO bf
NO bf

• A Model with A4 Modular Symmetry

- Modular transformation :

- linear fractional transformations of the upper half complex plane H

$$z \mapsto \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbb{Z}, ad-bc=1$$

- We can identify the modular transformation described by (a, b, c, d) with a 2x2 matrix with integer entries, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

- Modular group is defined as

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z), \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

Due to sign ambiguity from Det.,

$$\Gamma(N) \sim SL(2,\mathbf{Z})/\{\pm I\} = PSL(2,\mathbf{Z})$$

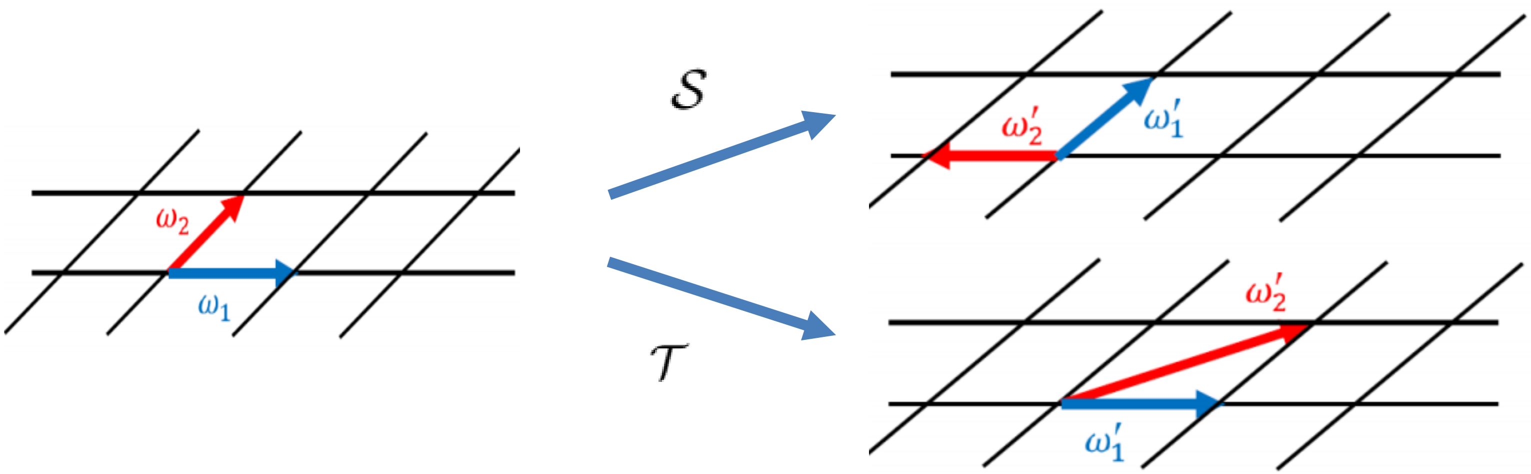
• A Model with A4 Modular Symmetry

- Modular transformation :

  - Generators of modular group :

$$\mathcal{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \mathcal{S}^2 = (\mathcal{S}\mathcal{T})^3 = \mathbb{1}$$

  - Any modular transformation can be obtained by a combination of the powers of $\mathcal{T}$, $\mathcal{S}$



• A Model with A4 Modular Symmetry

- Modular transformation :

  - Transformation of two lattice vectors $\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$

  Equivalent to the modular transformation

  $$\tau \to \frac{\mathrm{a}\tau+\mathrm{b}}{\mathrm{c}\tau+\mathrm{d}}\,, \qquad \tau = \frac{\omega_2}{\omega_1} \rightarrow \text{modulus}$$

  - Imposing $\mathcal{T}^N = \mathbb{I}$, the modular groups for $N \in \{2,3,4,5\}$ are isomorphic to the groups , S_3, A_4, S_4, A_5.

- Modular Form :

  - For a given integer k, , we say a function f is weakly modular of weight $2k$ if $f(z)$ is meromorphic on H and satisfies

  $$f(z) = (cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right) \quad \mathit{for\ all} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

• A Model with A4 Modular Symmetry

- Modular invariant Lagrangian;

$$\mathcal{L}_1 = f(\tau)\phi_1\phi_2\cdots\phi_n$$

$$f(\tau) \to (c\tau + d)^k f(\tau)$$ ⇒ modular form with weight k

$$\phi_i \to (c\tau + d)^{-k_i}\phi_i$$

- When $k = \sum_{\mathrm{i}} k_{\mathrm{i}}$, the Lagrangian is modular invariant.

- Modular forms for $\Gamma(3) \sim A_4$ (Feruglio, arXiv:1706.08749) :

3 independent forms transforming in 3-D rep. of A_4 by using Dedekind η-function defined in the upper complex plane:

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n) \quad , \qquad q \equiv e^{i2\pi\tau}$$

$$Y_1(\tau) = \frac{i}{2\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)} + \frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)} - \frac{27\eta'(3\tau)}{\eta(3\tau)}\right],$$

$$Y_2(\tau) = \frac{-i}{\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)} + \omega^2\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \omega\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}\right], \qquad \omega = (-1+i\sqrt{3})/2.$$

$$Y_3(\tau) = \frac{-i}{\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)} + \omega\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \omega^2\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}\right],$$

- q-expansion :

$$\begin{aligned} Y_1(\tau) &= 1 + 12q + 36q^2 + 12q^3 + \ldots \\ Y_2(\tau) &= -6q^{1/3}(1 + 7q + 8q^2 + \ldots) \\ Y_3(\tau) &= -18q^{2/3}(1 + 2q + 5q^2 + \ldots) \end{aligned}$$

- They are triplet under $A_4 \sim \Gamma_3$ where generators S and T are presented by

$$\rho(S) = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \qquad \rho(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

- There are modular invariant combinations of $Y_i(\tau)$
  - for $k=2$, there are 6 independent $Y_i Y_j$ which are arranged into irreducible rep. of A_4 decomposed as $3+1+1'+1''$

2k

rep.

$$
\begin{aligned}
Y_3^{(4)} &= (Y_1^2 - Y_2Y_3, Y_3^2 - Y_1Y_2, Y_2^2 - Y_1Y_3) \\
Y_1^{(4)} &= Y_1^2 + 2Y_2Y_3 \\
Y_{1'}^{(4)} &= Y_3^2 + 2Y_1Y_2 \\
Y_{1''}^{(4)} &= Y_2^2 + 2Y_1Y_3 = 0
\end{aligned}
$$

- Under CP transformation, $\tau \to -\tau^*$,

  so, complex τ with $Re(\tau) \neq 0$ implies CPV

- CPV can occur through modulus stabilization
  (B. S. Acharya et al., [hep-th/9506143], T. Dent, [hep-ph/0105285], S. Khalil, O. Lebedev and S. Morris, [hep-th/0110063], J. Giedt, [hep-ph/0204017], Tatsuo Kobayashi et al, arXiv:1910.11553.)

- Generalized CP symmetry embedded in modular symmetry from the viewpoint of superstring theory
  (A. Baur, H. P. Nilles, A. Trautner and P. K. S. Vaudrevange, arXiv:1901.03251; arXiv:1908.00805.)

Construction of a model with modular $A_4 \times U(1)_X$

	$e^c,$	$\mu^c,$	τ^c	N^c	$L_e,\ L_\mu,\ L_\tau$	H_d	H_u	χ
$SU(2)_L \times U(1)_Y$		$(1,+1)$		$(1,0)$	$(2,-1/2)$	$(2,-1/2)$	$(2,+1/2)$	$(1,0)$
A_4	$\mathbf{1},$	$\mathbf{1}'',$	$\mathbf{1}'$	$\mathbf{3}$	$\mathbf{1},\ \mathbf{1}',\ \mathbf{1}''$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$U(1)_X$	$-\frac{1}{2}-f_e,$	$-\frac{1}{2}-f_\mu,$	$-\frac{1}{2}-f_\tau$	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	1
k_I		6		2	0	0	0	0

The superpotential

$$W = \alpha_1\, e^c L_e Y_{\mathbf{1}}^{(6)} \left(\frac{\chi}{\Lambda}\right)^{f_e} H_d + \alpha_2\, \mu^c L_\mu Y_{\mathbf{1}}^{(6)} \left(\frac{\chi}{\Lambda}\right)^{f_\mu} H_d + \alpha_3\, \tau^c L_\tau Y_{\mathbf{1}}^{(6)} \left(\frac{\chi}{\Lambda}\right)^{f_\tau} H_d$$
$$+ \beta_1 (N^c Y)_{\mathbf{1}} L_e H_u + \beta_2 (N^c Y)_{\mathbf{1}''} L_\mu H_u + \beta_3 (N^c Y)_{\mathbf{1}'} L_\tau H_u$$
$$+ \gamma_1 (N^c N^c)_{\mathbf{1}} Y_{\mathbf{1}}^{(4)} \chi + \gamma_2 (N^c N^c)_{\mathbf{3}} Y_{\mathbf{3}}^{(4)} \chi + \gamma_3 (N^c N^c)_{\mathbf{1}''} Y_{\mathbf{1}'}^{(4)} \chi\,,$$

$$Y_{\mathbf{1}}^{(4)} = Y_1^2 + 2Y_2Y_3, \qquad Y_{\mathbf{3}}^{(4)} = (Y_1^2 - Y_2Y_3,\, Y_3^2 - Y_1Y_2,\, Y_2^2 - Y_1Y_3),$$
$$Y_{\mathbf{1}'}^{(4)} = Y_3^2 + 2Y_1Y_2, \qquad Y_{\mathbf{1}}^{(6)} = Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1Y_2Y_3.$$

- Charged lepton masses:

$$\mathcal{M}_\ell = Y_1^3\langle H_d\rangle(1+a^3+b^3-3ab)\,\mathrm{diag}\left(\alpha_1\left(\frac{\langle\chi\rangle}{\Lambda}\right)^{f_e},\alpha_2\left(\frac{\langle\chi\rangle}{\Lambda}\right)^{f_\mu},\alpha_3\left(\frac{\langle\chi\rangle}{\Lambda}\right)^{f_\tau}\right)$$

$$a \equiv Y_2/Y_1 \text{ and } b \equiv Y_3/Y_1$$

- Neutrino masses:

$$m_D = Y_1\langle H_u\rangle\beta_1\begin{pmatrix} 1 & \beta b & \beta' a \\ 1b & \beta a & \beta' \\ 1a & \beta & \beta' b \end{pmatrix},$$

$$\beta \equiv \beta_2/\beta_1 \quad \beta' \equiv \beta_3/\beta_1$$

$$\gamma \equiv \gamma_2/\gamma_1 \quad \gamma' \equiv \gamma_3/\gamma_1$$

$$M_R = Y_1^2\langle\chi\rangle\gamma_1\begin{pmatrix} 1+\frac{4}{3}\gamma+ab\left(2-\frac{4}{3}\gamma\right) & 2\gamma b & -\frac{2}{3}\gamma\left(b^2-a\right)+\gamma'(b^2+2a) \\ (M_R)_{1,2} & \frac{4}{3}\gamma\left(b^2-a\right)+\gamma'(b^2+2a) & 1-\frac{2}{3}\gamma+ab\left(2+\frac{2}{3}\gamma\right) \\ (M_R)_{1,3} & (M_R)_{2,3} & -4\gamma b \end{pmatrix},$$

$$\mathcal{M}_\nu = -m_D^T M_R^{-1} m_D \propto \langle H_u\rangle^2\beta_1^2/\langle\chi\rangle\gamma_1 = 5.4572\times 10^{-12}\ \mathrm{GeV}$$

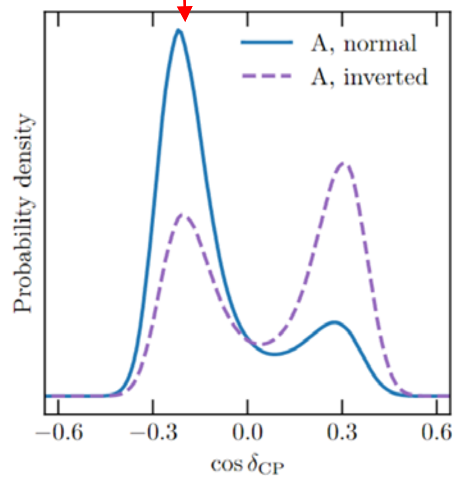
Numerical Results

	τ	β	β'	γ	γ'
A1	$0.27859 + i0.98630$	1.7191	2.2104	0.26016	0.25510
A2	$0.28491 + i0.99292$	1.6562	2.0412	0.30203	0.22781
A3	$0.28747 + i0.99414$	1.6288	1.9900	0.31567	0.21780
TBM	i	2.0	2.0	free	1.0

- BPs for the model predicting a pattern consistent with case A
- Taking $\beta, \beta' \gamma, \gamma'$ to be real, TBM is achieved at $\tau = i$ & $\beta = \beta'=2$
- Deviations from TBM reflect $Re(\tau) \neq 0$, breaking of $\beta = \beta'=2$, and change of γ'
- Complex τ with $Re(\tau) \neq 0$ is the source of CP violation

	s_{12}^2	s_{13}^2	s_{23}^2	$\cos\delta_{\rm CP}$	Δm_{21}^2 [eV^2]	Δm_{31}^2 [eV^2]	$m_{\nu 1}$ [eV]	$\sum m_\nu$ [eV]
A1	0.3183	0.02210	0.4370	-0.2780	7.282	2.499	0.009432	0.07303
A2	0.3184	0.02191	0.4629	-0.1637	7.418	2.502	0.008453	0.07125
A3	0.3180	0.02242	0.4674	-0.1419	7.241	2.519	0.008203	0.07088

- Predictions of oscillation parameters for BPs shown before
- The values of $s_{13}^2, s_{23}^2, \Delta m_{21}^2, \Delta m_{31}^2$ are within their 1σ ranges
- $\sum m_\nu$ is below the cosmological upper bound of 0.12 eV.
- $\delta_{CP} \sim \frac{3}{2}\pi$



Conclusion

- We have shown how the Dirac type CP phase can be estimated in terms of neutrino mixing angles in the standard parameterization of the PMNS mixing matrix.
- Performing numerical analysis based on the current experimental data, we have estimated the values of the CP phase and leptonic Jarlskog invariant for several cases.
- We have shown how future experiments can test the prediction of CPV in this study.
- We have proposed a model with modular A_4 that leads to case A and accommodates experimental results.