# Permutation invariant matrix systems and partition algebras

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Corfu workshop on non-commutative and generalized geometry in string theory, gauge theory and related models

# Based on

S. Ramgoolam, "**Permutation invariant Gaussian matrix Models**," **2018** arXiv:1809.07559 [hep-th], Nuclear Physics B, 2019
G. Barnes, A. Padellaro, S. Ramgoolam, " **Permutation invariant Gaussian 2-Matrix models,"** **2021-04** arXiv:2104.03707
G. Barnes, A. Padellaro, S. Ramgoolam, "**Hidden symmetries and large N factorisation for permutation invariant matrix observables**," **2021-12** arXiv:2112.00498 [hep-th]
G. Barnes, A. Padellaro, S. Ramgoolam, " **Permutation symmetry in large N Matrix Quantum Mechanics and Partition Algebras,"** arXiv:2207.02166 [hep-th]

# Introduction

Physical systems with matrix variables $X$ with $U(N)$ symmetry are ubiquitous in gauge-string duality : IKKT model, BFSS matrix model, $N = 4$ SYM with $U(N)$ gauge symmetry for $\text{AdS}_5$/CFT4.

One of the lessons : Representation theory methods are very useful in understanding the map between large composite CFT operators in the CFT4 and giant gravitons, LLM geometries etc. in AdS5 $\times$ S5

Corley, Jevicki, Ramgoolam, "Exact correlators of giant gravitons from dual N=4 SYM" 2001

Lin, Lunin and Maldacena, "Bubbling AdS space and 1/2 BPS geometries," 2004

Berenstein, "Toy model of AdS/CFT" 2004

Examples $\mathrm{Det}(X)$ : More generally composite operators $\mathcal{O}_R(X)$ labelled by Young diagrams.

# Introduction

Underlying mathematical questions :

- Matrix $X$ of size $N$.
- Classify polynomial functions of $X^i_j$, of degree $k$, which are invariant under $X \to UXU^\dagger$.
- Invariant functions are multi-traces : for $k = 2$ we have $\mathrm{tr}(X^2), \mathrm{tr}(X)\mathrm{tr}(X)$. The Young-diagram-labelled operators are $\mathcal{O}_R(X)$ which are linear combinations of multi-traces.

**Remark:** The multi-trace basis and the Young diagram basis can be understood in a way that admits generalisation when we consider physical problems where the invariance

$$X \to UXU^\dagger \qquad U \text{ is a general matrix in } \qquad U(N)$$

is replaced by

$$X \to U_\sigma X U_\sigma^\dagger \qquad U_\sigma \text{ matrix in the natural representation of } \qquad \sigma \in S_N$$
$$X^i_j \to X^{\sigma(i)}_{\sigma(j)}$$

# Trace basis of $U(N)$ invariants as equivalence classes of permutation tensors

The trace basis can be understood as follows. To get $U(N)$ invariants we must contract upper with lower indices - in any order. Pick a permutation $\sigma \in S_k$

$$\begin{aligned} \mathcal{O}_\tau(X) &= \sum_{i_i, i_2, \cdots, i_k} X^{i_1}_{i_{\tau(1)}} X^{i_2}_{i_{\tau(2)}} \cdots X^{i_k}_{i_{\tau(k)}} \\ &= \sum_{i_i, \cdots, i_k} \sum_{j_i, \cdots, j_k} X^{i_1}_{j_1} \cdots X^{i_k}_{j_k} \delta^{j_1}_{i_{\tau(1)}} \cdots \delta^{j_k}_{i_{\tau(k)}} \end{aligned}$$

$\tau \in S_k$ defines an operator on

$$\tau : V_N^{\otimes k} \rightarrow V_N^{\otimes k}$$

$$\begin{aligned} \mathcal{O}_\tau(X) &= \sum_{\vec{i}, \vec{j}} X^{i_1}_{j_1} \cdots X^{i_k}_{j_k} (\tau)^{j_1, \cdots, j_k}_{i_1 \cdots i_k} \\ &= tr_{V_N^{\otimes k}} (X^{\otimes k} \tau) \end{aligned}$$

The bosonic nature of $X$ means that for any $\gamma \in S_k$, we have

$$\mathcal{O}_\sigma(X) = \mathcal{O}_{\gamma\sigma\gamma^{-1}}$$

$\sigma$ form a basis for the group algebra $\mathbb{C}(S_k)$. For elements of $A \in \mathbb{C}(S_k)$ :

$$A = \sum_{\sigma} a_{\sigma}\ \sigma$$

we define

$$\mathcal{O}_A(X) = \sum_{\sigma} a_{\sigma} \mathcal{O}_{\sigma}(X)$$

Example at $n = 3$ :

$$X^{i_1}_{i_1} X^{i_2}_{i_2} X^{i_3}_{i_3} = (\mathrm{tr}(\mathrm{X}))^3$$
$$X^{i_1}_{i_2} X^{i_2}_{i_1} X^{i_3}_{i_3} = (\mathrm{tr}\mathrm{X}^2)(\mathrm{tr}\mathrm{X})$$
$$X^{i_1}_{i_2} X^{i_2}_{i_3} X^{i_3}_{i_1} = (\mathrm{tr}\mathrm{X})^3$$

3 observables ; 3 conjugacy classes in $S_3$ :

$$(1)(2)(3)$$
$$(1,2)(3); (2,3)(1); (1,3)(2)$$
$$(1,2,3), (1,3,2)$$

# Large $N$ factorisation

Two-point functions of invariants comes from free field limit of the CFT4. It is also a property of 2-point functions in complex matrix model with Gaussian action. Nice formula follows easily from the index-free tensor operators description of invariant functions :

$$\mathcal{Z} = \int [dX] e^{-\frac{1}{2} tr(XX^\dagger)}$$

$$\langle \mathcal{O}_{\sigma_1}(X) \mathcal{O}_{\sigma_2}(X^\dagger) \rangle = \sum_{\gamma, \sigma_3 \in S_k} \delta(\sigma_1 \gamma \sigma_2 \gamma^{-1} \sigma_3) N^{C_{\sigma_3}}$$

An important property of the trace basis ( for AdS/CFT) is large $N$ factorisation follows

$$\langle \widehat{\mathcal{O}}_{\sigma_1}(X) \widehat{\mathcal{O}}_{\sigma_2}(X^\dagger) \rangle = \delta_{[\sigma_1],[\sigma_2]} + \text{ order } 1/N$$

## Trace-like basis of $S_N$ invariants as equivalence classes of partition algebra tensors

$$\begin{aligned} \mathcal{O}_d(X) &= \sum_{i_i,\cdots,i_k} \sum_{j_i,\cdots,j_k} X^{i_1}_{j_1} \cdots X^{i_k}_{j_k} (d)^{j_1,\cdots,j_k}_{i_1,\cdots,i_k} \\ &= tr_{V_N^{\otimes k}}(X^{\otimes k} d) \end{aligned}$$

$d$ is one of a set of basis elements of the partition algebra $P_k(N)$ :

Manifest symmetry of matrix models /Hamiltonians :

$U(N) \to S_N$

Hidden symmetry organising observables :

$$\bigoplus_{k=0}^{\infty} \mathbb{C}(S_k) \to \bigoplus_{k=0}^{\infty} P_k(N)$$

Equivalence by $S_k$ permutations due to bosonic nature of $X$ as before :

$$\mathcal{O}_d(X) = \mathcal{O}_{\gamma d \gamma^{-1}} \quad \text{for all} \quad \gamma \in S_k$$

## Young Diagram basis of $U(N)$ invariants and $S_k$ characters

There is Young diagram basis for 1-matrix $U(N)$ gauge invariants which are linear combinations of the permutation basis operators

$$\mathcal{O}_R(X) = \frac{1}{n!} \sum_{\sigma \in S_k} \chi^R(\sigma) \mathcal{O}_\sigma(X)$$

where the weights $\chi^R(\sigma)$ are characters of $S_n$ element $\sigma$ in irrep of $_n$ associated with Young diagram $R$.

# Young Diagram basis of $U(N)$ invariants and Schur-Weyl duality

$$\mathcal{O}_R(X) = \sum_{\vec{i};\vec{j}} (P_R)^{j_1 \cdots j_k}_{i_1, \cdots, i_k} X^{i_1}_{j_1} \cdots X^{i_k}_{j_k}$$

$P_R$ is a projector in $\mathbb{C}(S_k)$ :

$$P_R \sim \sum_{\sigma \in S_k} \chi^R(\sigma)\sigma$$

Give a basis for the centre of $\mathbb{C}(S_k)$.

Number of Young diagrams $R$ with $k$ boxes equals the number of cycle structures for permutations in $S_k$.

In the half-BPS sector the Young diagram basis is exactly orthogonal in the CFT4 inner product :

$$\langle \mathcal{O}_R(X) \mathcal{O}_S(X^\dagger) \rangle = \delta_{RS} f_R$$

Can understand the formula and the orthogonality using Schur-Weyl duality.

## Understanding $\mathcal{O}_R(X)$ using Schur-Weyl duality

The upper indices of $X^{\otimes k}$ transform as $V_N^{\otimes k}$ and the lower indices as $\bar{V}_N^{\otimes k}$

$$V_N^{\otimes k} = \bigoplus_{\substack{R \vdash n \\ l(R) \leq N}} V_R^{U(N)} \otimes V_R^{S_k}$$

There is a basis for $V_N^{\otimes k}$

$$|R, M_R\rangle \otimes |R, m_R\rangle \equiv |R, M_R, m_R\rangle$$

Multiplicities of irreps of the group of interest are controlled by the representation theory of the hidden symmetry.

Clebsch-Gordan coefficients

$$\langle i_1, \cdots, i_k | R, M_R, m_R\rangle = C^{i_1, \cdots, i_k}_{R, M_R, m_R}$$

Construction of $\mathcal{O}_R(X)$ using SW duality :

$$\mathcal{O}_R(X) \quad = \sum_{\vec{i}, \vec{j}, R, M_R, m_R} C^{j_1, \cdots, j_k}_{M_R, m_R} C^{R, M_R, m_R}_{i_1, \cdots, i_k} X^{i_1}_{j_1} \cdots X^{i_k}_{j_k}$$

## Outline

Holographic and many-body-quantum-physics features of permutation invariance in matrix quantum mechanics.

Part I : Standard matrix harmonic oscillator and its permutation invariant sector. Properties of partition algebra $P_k(N)$ : the geometrical basis and the representation theory basis. Large $N$ factorisation. Orthogonal basis using rep theory.
Part IIA : The general (11-parameter) permutation invariant harmonic oscillator.
Part IIB: Dynamically in the spectrum, separating the permutation invariant states using Casimir Hamiltonians. Large $N$ simplifications using $P_k(N)$.
Part IIC : Realization of quantum many body scars in a permutation invariant setting.

## Simplest Matrix SHO

Hamiltonian has $U(N)$ symmetry and therefore $S_N$ symmetry. Look at all states ( not just the $U(N)$ invariant states ) and within the full Hilbert space, describe the $S_N$ invariant sector. Lagrangian :

$$L_0 = \frac{1}{2}\left( \sum_{i,j=1}^{N} \partial_t X_{ij} \partial_t X_{ij} - X_{ij} X_{ij} \right)$$

Hamiltonian:

$$H_0 = \sum_{i,j=1}^{N} (a^\dagger)^i_j a^i_j,$$

## Simplest Matrix SHO

The degree $k$ subspace is given by

$$\mathcal{H}^{(k)} \cong \mathrm{Span}_{\mathbb{C}} \{(a^\dagger)^{i_1}_{j_1} \dots (a^\dagger)^{i_k}_{j_k} |0\rangle\},$$

$$T(e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_k}) = \sum_{j_1, j_2, \dots, j_k = 1}^{N} T^{j_1 \dots j_k}_{i_1 \dots i_k} e_{j_1} \otimes e_{j_2} \otimes \dots \otimes e_{j_k}.$$

a general state $|T\rangle \in \mathcal{H}^{(k)}$ can be written as a trace

$$\boxed{|T\rangle = \mathrm{Tr}_{V_N^{\otimes k}}(T(a^\dagger)^{\otimes k})|0\rangle = \sum_{\substack{i_1, \dots, i_k \\ j_1, \dots, j_k}} T^{j_1 \dots j_k}_{i_1 \dots i_k} (a^\dagger)^{i_1}_{j_1} \dots (a^\dagger)^{i_k}_{j_k} |0\rangle,}$$

Exchange action in tensor space $V_N^{\otimes k}$ :

$$\mathcal{L}_\tau(e_{i_1} \otimes \cdots \otimes e_{i_k}) = e_{i_{\tau^{-1}(1)}} \otimes \cdots \otimes e_{i_{\tau^{-1}(k)}}$$

Exchange symmetry of state-tensors $T$

$$\boxed{\mathcal{L}_\tau T \mathcal{L}_{\tau^{-1}} = T, \quad \forall \tau \in S_k,}$$

Dimension of space of oscillator states at degree $k$ :

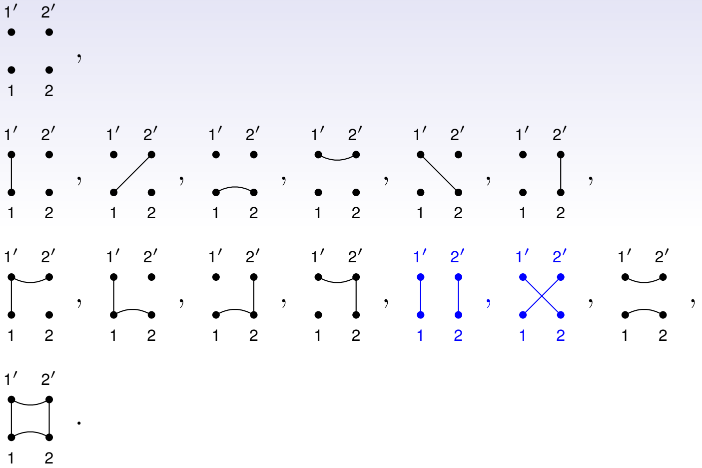
$$\mathrm{Dim}\mathcal{H}^{(k)} = \binom{N^2 + k - 1}{k} = \frac{N^2(N^2+1)\dots(N^2+k-1)}{k!}.$$

Now impose the invariance under the $S_N$ action on the indices (which label a basis of $V_N$)

$$\mathrm{End}_{S_N \times S_k}(V_N^{\otimes k}) =$$
$$\mathrm{Span}_{\mathbb{C}}\{T \in \mathrm{End}(V_N^{\otimes k}) \,:\, \mathcal{L}(\sigma)T\mathcal{L}(\sigma^{-1}) = \mathcal{L}_\tau T \mathcal{L}_{\tau^{-1}} = T,\ \forall \sigma \in S_N, \tau \in S_k\}.$$

The condition :

$$\begin{aligned} \mathcal{L}(\sigma)T\mathcal{L}(\sigma^{-1}) &= T \\ \implies \mathcal{L}(\sigma)T &= T\mathcal{L}(\sigma) \\ \implies T &= d \in P_k(N) \end{aligned}$$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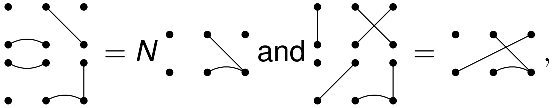
## Multiplication of partition algebra diagrams

Let $d_\pi$ and $d_{\pi'}$ be two diagrams in $P_k(N)$. The composition $d_{\pi''} = d_\pi d_{\pi'}$ is constructed by placing $d_\pi$ above $d_{\pi'}$ and identifying the bottom vertices of $d_\pi$ with the top vertices of $d_{\pi'}$. The diagram is simplified by following the edges connecting the bottom vertices of $d_{\pi'}$ to the top vertices of $d_\pi$. Any connected components within the middle rows are removed and we multiply by $N^c$, where $c$ is the number of these complete blocks removed. For example,




where the factor of $N$ in the first equation comes from removing the middle component at vertex 1 and 2.

Bosonic symmetry $\rightarrow$ sub-algebra of $P_k(N)$.

The exchange symmetry $S_k$ leads to

$$d \sim \gamma d \gamma^{-1}$$

This leads to sub-algebra $SP_k(N)$ of $P_k(N)$.

$$[d] = \frac{1}{k!} \sum_{\tau \in S_k} \gamma d \gamma^{-1}$$

# Schur-Weyl basis and Representation basis

$$V_N^{\otimes k} = \bigoplus_{\Lambda_1 \in \mathcal{Y}_S(k)} V_{\Lambda_1}^{S_N} \otimes V_{\Lambda_1}^{P_k(N)}$$

$$P_k(N) \cong \mathrm{End}_{S_N}(V_N^{\otimes k}) \cong \bigoplus_{\Lambda_1 \in \mathcal{Y}_S(k)} V_{\Lambda_1}^{P_k(N)} \otimes V_{\Lambda_1}^{P_k(N)}$$

$$\boxed{Q_{\alpha\beta}^{\Lambda_1} = \sum_{i=1}^{B(2k)} \mathrm{Dim}(V_{\Lambda_1}^{S_N}) D_{\beta\alpha}^{\Lambda_1}((b_i^*)^T) b_i .}$$

This is an associative algebra generalisation of :

$$Q_{ij}^R = \frac{d_R}{|G|} \sum_{g \in G} D_{ji}^R(g) g^{-1}$$

The equation (1) is derived in the representation theory literature :

Ram, Dissertation, Chapter 1, Representation theory (available online)

Halverson and Ram, “Partition algebras” 2004

# Representation basis : projecting to $S_k$ invariants

$$Q^{\Lambda_1}_{\Lambda_2,\mu\nu} = \sum_{\alpha,\beta,p} Q^{\Lambda_1}_{\alpha\beta} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\alpha\to\Lambda_2,p;\mu} B^{P_k(N)\to\mathbb{C}[S_k]}_{\Lambda_1,\beta\to\Lambda_2,p;\nu} ,$$

These ways of thinking about $S_N$ invariant polynomials of degree $k$ lead to two useful ways of counting the dimension of the $S_N$ invariant subspace of the oscillator Hilbert space :

$$\mathrm{Dim}(N,k) = \frac{1}{D!k!} \sum_{\sigma\in S_D} \sum_{\tau\in S_k} \prod_{i=1}^{k} (\sum_{l|i} l C_l(\sigma))^{2C_i(\tau)}$$

$$= \frac{1}{D!k!} \sum_{p\vdash D} \sum_{q\vdash k} \frac{D!}{\prod_{j=1}^{D} j^{p_j} p_j!} \prod_{i=1}^{k} \frac{k!}{i^{q_i} q_i!} \left( \sum_{l|i} l p_l \right)^{2q_i}$$

In terms of rep theory multiplicities related to $Q$ :

$$\mathrm{Dim}(N,k) = \sum_{\Lambda_1\vdash N;\Lambda_2\vdash k} Mult(V_N^{\otimes k}, V^{S_N}_{\Lambda_1} \otimes V^{S_k}_{\Lambda_2})^2$$

## Large N factorisation of PIMOs in matrix quantum mechanics

For $d_1, d_2 \in P_{kN}$

$$\langle 0|\mathcal{O}_{d_1}(a)\mathcal{O}_{d_2}(a^\dagger)|0\rangle = \sum_{\gamma \in S_{k_1}} \mathrm{Tr}_{V_N^{\otimes k_2}}(\gamma^{-1} d_1^T \gamma d_2)$$

$$= \begin{cases} 1 + O(1/\sqrt{N}) & \text{if } [d_1] = [d_2], \\ 0 + O(1/\sqrt{N}) & \text{otherwise.} \end{cases}$$

## Exact orthogonality of representation basis

Define $S_D$ invariant states using the representation basis :

$$|Q^{\Lambda_1}_{\Lambda_2,\mu\nu}\rangle = \mathrm{Tr}_{V_N^{\otimes k}}(Q^{\Lambda_1}_{\Lambda_2,\mu\nu}(a^\dagger)^{\otimes k})$$

They are exactly orthogonal :

$$\langle Q^{\Lambda_1}_{\Lambda_2,\mu\nu} \mid Q^{\Lambda_1'}_{\Lambda_2',\mu'\nu'}\rangle = k!\delta^{\Lambda_1\Lambda_1'}\delta_{\Lambda_2\Lambda_2'}\delta_{\nu\nu'}\mathrm{Dim}\, V^{S_N}_{\Lambda_1}\mathrm{Dim}\, V^{S_k}_{\Lambda_2}\delta_{\mu\mu'}$$

## Part II-A : General permutation invariant matrix harmonic oscillator

A matrix quantum mechanics with more general potential is described by the Lagrangian

$$L = \frac{1}{2} \sum_{i,j=1}^{N} \partial_t X_{ij} \partial_t X_{ij} - \frac{1}{2} V(X).$$

The general quadratic $S_N$ (permutation) invariant potential defines the general permutation invariant matrix oscillator

$$V(X_{ij}) = V(X_{\sigma(i)\sigma(j)}).$$

There are 11 different quadratic invariants.

# Part II-A : Permutation invariants and graphs - quadratic invariants

There is also a graph description of the invariants (closely related to the partition algebra diagrams)

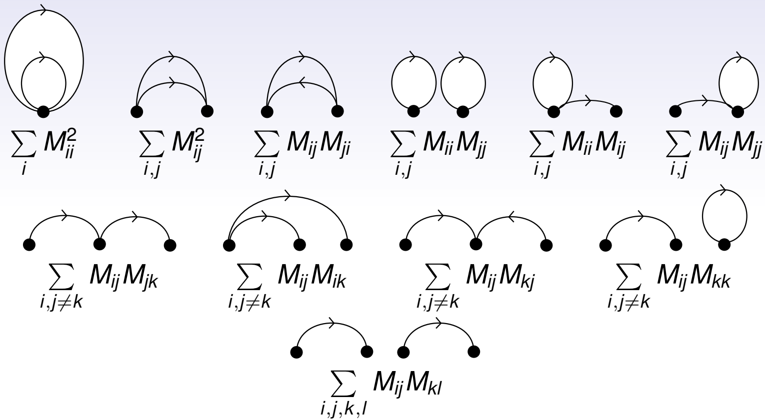



**Figure:** $S_D$ invariant functions and corresponding graphs illustrated for the 11 quadratic invariants

## Part II-A : Quadratic invariants using representation theory and solving the matrix harmonic oscillator

Decomposing $X_{ij} \sim V_N \otimes V_N$ in terms of irreps :

$$V_N \otimes V_N \cong 2\,V^{S_N}_{[N]} \oplus 3\,V^{S_N}_{[N-1,1]} \oplus V^{S_N}_{[N-2,2]} \oplus V^{S_N}_{[N-2,1,1]},$$

$$X^{\Lambda,\alpha}_a = \sum_{i,j} C^{\Lambda,\alpha}_{a,ij} X_{ij}.$$

In the representation basis the potential has a simple form,

$$V(X) = \sum_{\Lambda,\alpha,\beta,a} X_a^{\Lambda,\alpha} g_{\alpha\beta}^{\Lambda} X_a^{\Lambda,\beta},$$

Hamiltonian can be written in terms of decoupled oscillators after diagoanlising the low-dimensional coupling matrices :

$$\boxed{H = \sum_{\Lambda,\alpha,a} \omega_\alpha^\Lambda (A^\dagger)_a^{\Lambda,\alpha} A_a^{\Lambda,\alpha}.}$$

$$Z(\beta) = \mathrm{Tr}_{\mathcal{H}}\ e^{-\beta H}$$

Write $x = e^{-\beta}$

$$Z(\beta) = \frac{1}{(1-x^{\omega_1^{[N]}})(1-x^{\omega_2^{[N]}})} \frac{1}{(1-x^{\omega_1^{[N-1,1]}})^{N-1}(1-x^{\omega_2^{[N-1,1]}})^{N-1}(1-x^{\omega_3^{[N-1,1]}})^{N-1}} \times \frac{1}{(1-x^{\omega^{[N-2,2]}})^{(N-1)(N-2)/2}(1-x^{\omega^{[N-2,1,1]}})^{N(N-3)/2}}.$$

## Part II-B : Dynamically separating the invariant states

In the matrix HOs - there is no separation of the $S_N$ invariant states from the remaining states. $S_N$ action is the adjoint one :

$$X \rightarrow M_\sigma X M_{\sigma^{-1}}$$

Equivalently

$$X_{ij} \rightarrow X_{\sigma(i)\sigma(j)}$$

Can construct Hamiltonians which produce a separation - using Casimirs of this $S_N$ action. These are elements in the centre of the group algebra $\mathbb{C}(S_N)$, e.g. sum of all permutations in a fixed conjugacy class.

$$T_2 = \sum_{i \neq j = 1}^{n} (ij)$$

## Part II-B : Dynamically separating the invariant states

In the $k$-oscillator subspace this acts as

$$\sum_{ij} (ij)^{\otimes 2k}$$

The eigenvalues are known from $S_N$ group theory.
Schur-Weyl duality implies that these operators can be written as sums of elements in $P_{2k}(N)$.
The explicit computation of the $Q^{\Lambda_1}_{\Lambda_2,\mu,\nu}$ basis of the invariant states can be done using eigenvalue equations in the partition algebra $P_k(N)$. (Good programmes for working with partition algebras in SAGE).

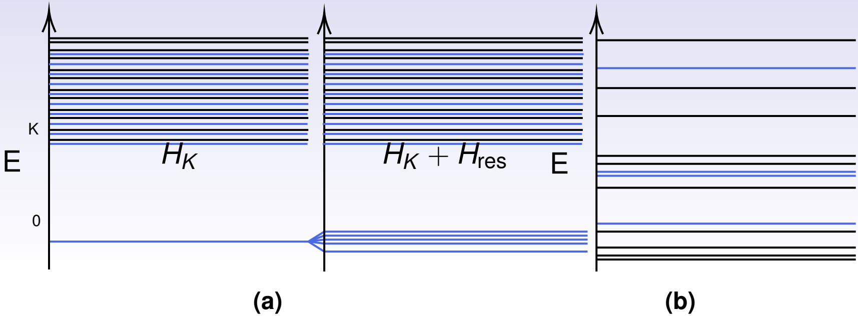



**Figure:** The figure illustrates the type of spectra that can be engineered using the algebraic Hamiltonians discussed in this section. Blue lines correspond to states that are invariant under the adjoint action of $S_N$. Black lines are non-invariant states.

## Part II-C : Quantum many-body scars

( Quantum many-body scars are of active interest in theoretical condensed matter + Klebanov and collaborators - see refs. in the paper )
Non-trivial collective states in a complex quantum system (not an integrable model) show periodic behaviour. This has been modelled - in symmetry-based mechanicsm - as follows.
A Hamiltonian $H$ such that for all states $|d\rangle \in \mathcal{H}_{\text{inv}}$

$$H|d\rangle \in \mathcal{H}_{\text{inv}},$$

and the time-evolution of $|d\rangle$ using $H$ is periodic.

Construct a total Hamiltonian

$$H_{\mathrm{tot}} = H + H_{\mathrm{s}}$$

The new term will completely break the symmetry of $H$, but is required to satisfy

$$H_{\mathrm{s}}|d\rangle = 0 \ \text{ for all } |d\rangle \in \mathcal{H}_{\mathrm{inv}}.$$

This ensures that time-evolution of $|d\rangle$ using $H_{\mathrm{tot}}$ is equivalent to time-evolution using $H$, which was periodic by construction.

$$H|e_i\rangle = E_i|e_i\rangle$$

The state $|d\rangle$ exhibits revival with periodicity $T$ if the quantum fidelity (return probability)

$$f(t) = |\langle d|e^{-iHt}|d\rangle|^2$$

satisfies $f(mT) = 1$ for $m = 0, 1, \ldots$.

Expanding $|d\rangle$ in the eigenbasis

$$|d\rangle = \sum_i d_i |e_i\rangle$$

and computing $f(t)$ gives

$$f(t) = \sum_{i,j} |d_i|^2 |d_j|^2 e^{-i(E_i - E_j)t}.$$

If all energy differences $\Delta E_{ij} = E_i - E_j$ have a greatest common divisor $E$, that is

$$\Delta E_{ij} = E_i - E_j = E(\varepsilon_i - \varepsilon_j)$$

and $\varepsilon_i - \varepsilon_j$ is an integer for all $i, j$, then $f(mT) = 1$ for $T = 2\pi/E$.

$$H_\sigma = (1 - \mathrm{Ad}(\sigma^{-1}))h_\sigma(1 - \mathrm{Ad}(\sigma)),$$

Partition algebras give a description of the invariant Hilbert space (irrespective of the Hamiltonian) . Can be used to classify invariant Hamiltonians $\mathcal{H}$.

Can realize the symmetry-based mechanism with $S_N$ symmetry in matrix systems.

The matrix oscillators $a_{ij}^\dagger$ can be viewed as creating bosons on sites of a square lattice labelled by $(i, j)$. Combining this interpretation with the above mechanism, we get deformations of Hubbard model which realise quantum many-body scars.

## Conclusions

Summary : Focused on Matrix quantum mechanics here, with motivations from holography. And found some features we have seen in algebraic approaches to matrix systems with continous symmetries, or of interest in TCMP :

- Large N factorisation (in trace-like basis).
- Orthogonal bases labelled by representation theory data ( for general permutation invariant sector of simplest matrix HO)
- Casimir Hamiltonians - algebraic degeneracies.
- Realisations of quantum many-body scars (QMBS).

We can also do permutation invariant matrix models (zero-dimensional). The 13-parameter Gaussian model has been applied to real world ensembles of matrices ( coming from natural language data) to find approximate Gaussianity in language.

Kartsaklis, Ramgoolam, Sadrzadeh, **"Linguistic Matrix Theory,"** Annales de linstitut Henri Poincare D (AIHPD) , vol. 6, 2019

Ramgoolam, Sadrzadeh, Sword, **"Gaussianity and typicality in matrix distributional semantics,"**

arXiv:1912.10839v1 [hep-th] ; AIHPD Vol 9, 2022

Future: Computational/analytic approaches to correlators in Gaussian matrix models + perturbations.

Future: Orthogonal bases for the general 11-parameter matrix Harmonic oscillator.

Future : computational/analytic appraoched to the scar models

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