HELMHOLTZ ZENTRUM
DRESDEN ROSSENDORF

Noncommutativity and curved momentum space

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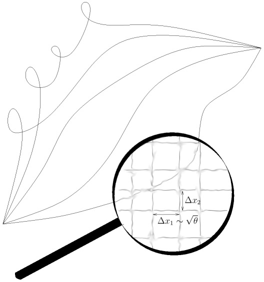
Outline

Noncommutativity

$$[\hat{x}_a, \hat{x}_b] = 2\, i\, \Theta_{ab} \sim \ell_P^2.$$

The idea was conceived in the ’40s as a minimum distance could lead to the cure of ultraviolet divergencies (Snyder, 1946).



LIV vs. DSR

Lorentz invariance violation

- There is a loss of the relativity principle.
- There is a privileged observer $\rightarrow$ physical laws depending on the observer.

Doubly Special Relativity (DSR)

- There is a relativity principle.
- Two fundamental constants: speed of light c and Planck's length ℓ_P.

Snyder

anti-Snyder phase-space:

$$[\hat{J}_{ij},\hat{J}_{kl}] = i\left(\delta_{ik}\hat{J}_{jl} - \delta_{il}\hat{J}_{jk} - \delta_{jk}\hat{J}_{il} + \delta_{lj}\hat{J}_{ik}\right),$$

$$[\hat{J}_{ij},\hat{p}_k] = i\left(\delta_{ik}\hat{p}_j - \delta_{jk}\hat{p}_i\right),$$

$$[\hat{J}_{ij},\hat{x}_k] = i\left(\delta_{ik}\hat{x}_j - \delta_{jk}\hat{x}_i\right),$$

$$[\hat{x}_i,\hat{p}_j] = i\left(\delta_{ij} - \beta^2\hat{p}_i\hat{p}_j\right),$$

$$[\hat{x}_i,\hat{x}_j] = -i\beta^2\hat{J}_{ij}, \qquad [\hat{p}_i,\hat{p}_j] = 0,$$

Lorentz generators ($\hat{J}_{ij} = \hat{x}_i\hat{p}_j - \hat{x}_j\hat{p}_i$).

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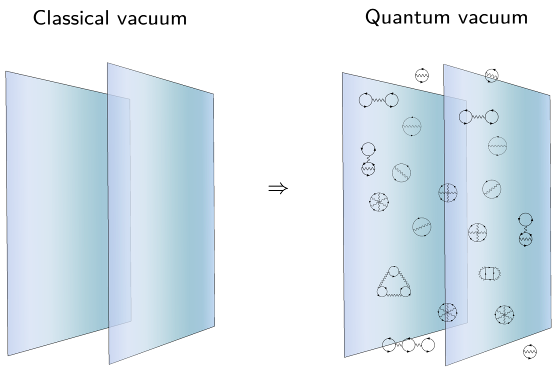
Lorentz generators ($\hat{J}_{ij} = \hat{x}_i\hat{p}_j - \hat{x}_j\hat{p}_i$).

Possible observable: vacuum energy



Experiments

Mathematics. — *On the attraction between two perfectly conducting plates.* By H. B. G. CASIMIR.

(Communicated at the meeting of May 29, 1948.)

In a recent paper by POLDER and CASIMIR 1) it is shown that the interaction between a perfectly conducting plate and an atom or molecule with

meaning but the *difference* between these sums in the two situations, $\frac{1}{2}(\Sigma\hbar\omega)_I - \frac{1}{2}(\Sigma\hbar\omega)_{II}$, will be shown to have a well defined value and this value will be interpreted as the interaction between the plate and the *xy* face.

The possible vibrations of a cavity defined by

$$0 \leqslant x \leqslant L, \quad 0 \leqslant y \leqslant L, \quad 0 \leqslant z \leqslant a$$

have wave numbers

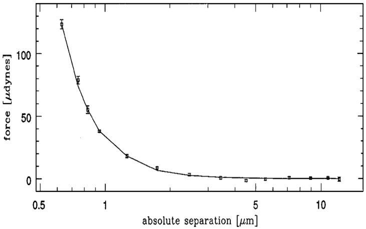
$$k_x = \frac{\pi}{L} n_x, \quad k_y = \frac{\pi}{L} n_y, \quad k_z = \frac{\pi}{a} n_z,$$

$$\frac{F}{A} = \frac{1}{2}\sum_n \omega_n = \frac{\pi^2}{240}\frac{\hbar c}{a^4}$$

$$\sim \frac{0.016}{a^4}\frac{\text{dyn}\,\mu\text{m}^4}{\text{cm}^2}$$

$\Rightarrow$

Lamoreaux (1997)



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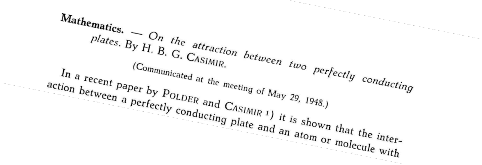

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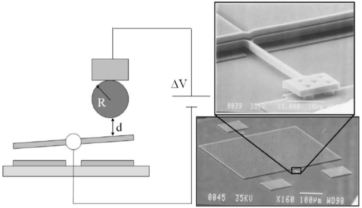

Capasso et al (2007)

$\Rightarrow$

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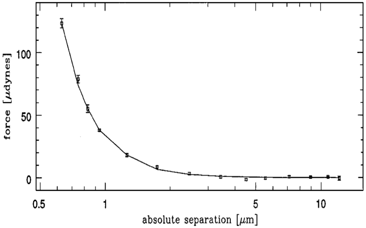
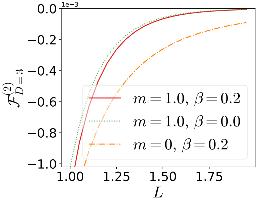


"Soft confinement" through a potential (SF and Mignemi, 2020)

$$H_V = \hat{p}^2 + VH(\hat{x}_\perp - L) + VH(-\hat{x}_\perp - L).$$

For example, choose a realization of the algebra,

$$\hat{p}_i = \frac{p_i}{\sqrt{1+\beta^2 p^2}}, \qquad \hat{x}_i = i\sqrt{1+\beta^2 p^2}\,\frac{\partial}{\partial p_i}.$$



$\Downarrow$ scalar field, time $\times$ anti-Snyder$_{D-1}$, sum over frequencies

$$\mathcal{F} = \frac{\Omega_{D-2}}{2(2\pi)^{D-1}} \int_0^{1/\beta} \frac{dp\, p^{D-2}}{(1-\beta^2 p^2)^{D/2-1/2}} \left(\sum_{n=1}^{\infty} f_n(p) - \int_0^{\infty} dn\, f_n(p) \right),$$

$$f_n(p) = \frac{k_n}{\beta L} \frac{\tanh(\beta k_n)}{\cosh^2(\beta k_n)\sqrt{p^2 + \beta^{-2}\tanh^2(\beta k_n)(1-\beta^2 p^2) + m^2}}, \quad k_n := \frac{n\pi}{2L}. \tag{1}$$

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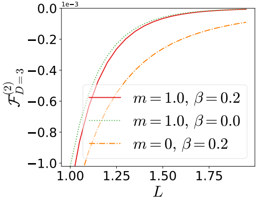

- Momentum space

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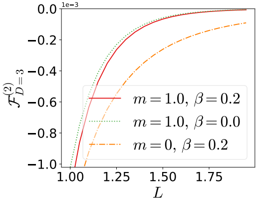

- Momentum space
- Divergences

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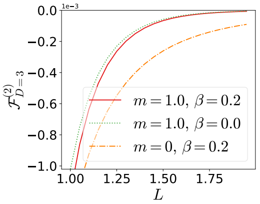

- Momentum space
- Divergences
- Invariant under realizations

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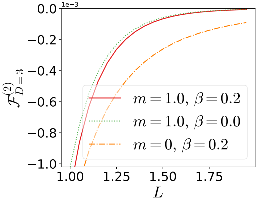

- Momentum space
- Divergences
- Invariant under realizations
- Spectral geometry

For example, the vacuum energy can be expanded as

$$\mathcal{E} = \sqrt{1+\beta^2 m^2}\left[\frac{2L}{(2\pi)^D\beta}\mathrm{Vol}(\mathbb{H}_D) + \frac{1}{4(2\pi)^{D-1}\beta}\mathrm{Vol}(\mathbb{H}_{D,w=0}) + \cdots\right]. \tag{2}$$

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Can one hear the shape of momentum space?

Experiments

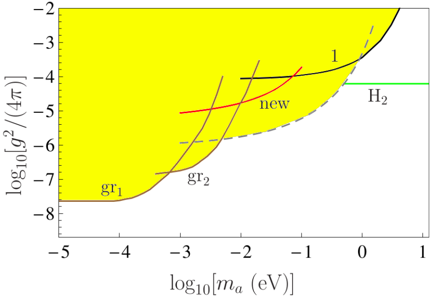
- Noncommutativity, see previous slide, $\mathcal{F} = -\frac{\pi^2}{7680}\frac{1}{L^4} - \frac{\pi^4}{48384}\frac{\beta^2}{L^6} + \mathcal{O}(\beta^4)$.
- First graviton corrections (Donoghue et al), $V(r) = -\frac{Gm_1 m_2}{r}\left(1 - \frac{G(m_1+m_2)}{rc^2} - \frac{127}{30\pi^2}\frac{G\hbar}{c^3 r^2}\right)$
- Dark scalar ϕ (Brax et al., 2021), $\mathcal{O}_1 = \Lambda^{-3}\bar{N}N\phi^2$, $\mathcal{O}_2 = \Lambda^{-2}\bar{N}\gamma^\mu N\phi^* i\partial_\mu\phi, \ldots$): UV $r^{-\alpha}$, IR $e^{-2mr}r^{-\beta/2}$
- Axion a (Klimchitskaya et al., 2021) , $-ig_a\bar{\psi}\gamma^5\psi a(x)$, $V(r) = -\frac{g_{an}^4}{32\pi^3 m^2}\frac{m_a}{r^2}K_1(2m_a r)$

$\vdots$

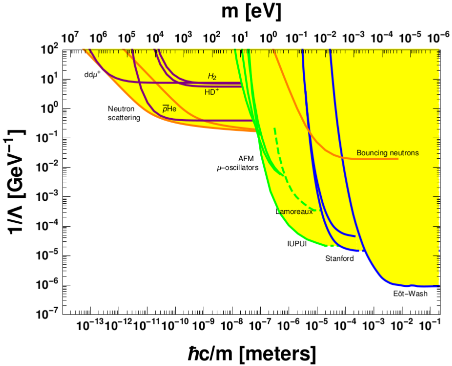
Experiments

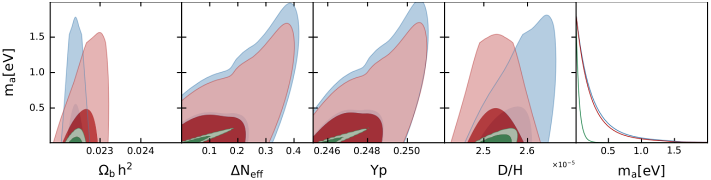
Klimchitskaya et al. (2021),

$$V(r) = -\frac{g_{an}^4}{32\pi^3 m^2}\frac{m_a}{r^2}K_1(2m_a r)$$



Fichet et al. (2021), $\mathcal{O}_1 = \Lambda^{-3}\bar{N}N\phi^2$





D'Eramo et al. (2022), BBN, Kim-Shifman-Vainshtein-Zakharov axion, $\mathcal{L} \sim \mathcal{J}_\pi \partial_\mu a$

Outline

Kinematics in DSR

- Free particles described by a (modified) dispersion relation

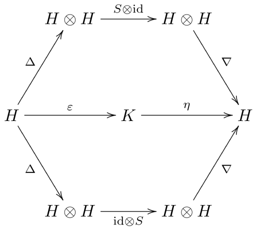
$$C(k) = k_0^2 - \mathbf{k}^2 + \frac{k_0^3}{\Lambda} + \ldots = m^2.$$

- A (modified) conservation of total momentum holds when interaction takes place

$$\text{Total momentum} = (p \oplus q)_\mu = p_\mu + q_\mu + \frac{p_\mu q_0}{\Lambda} + \cdots .$$

- Dispersion relation and conservation law compatible with relativity principle → deformed Lorentz transformations.

Kinematics in DSR



- Most known, κ-Poincaré, customarily studied in the context of Hopf algebras [**Majid1994**]. (κ-Minkowski, $[x_0, x_i] = \mathrm{i}x_i$)

  Product $\nabla = [\cdot, \cdot]$, unit η, coproduct Δ, counit ε, antipode S.

- Particular example: symmetric basis [**Lukierski:1992dt**]

  Deformed dispersion relation

$$C_{\mathrm{A}}^{(S)}(P) = \left(2\Lambda \sinh\left(\frac{P_0}{2\Lambda}\right)\right)^2 - P_i P_i .$$

  Deformed conservation law

$$(P \oplus Q)_0 = P_0 + Q_0 , \qquad (P \oplus Q)_i = P_i e^{Q_0/2\Lambda} + Q_i e^{-P_0/2\Lambda} .$$

For the usual Poincaré symmetry, one introduces fermions by

$$J_{AB} = M_{AB} \otimes I + I \otimes \Sigma_{AB}. \tag{3}$$

In the κ-Poincaré setup, one can take into account the nontrivial coproduct (symmetric basis)

$$\Delta L_i = L_i \otimes e^{\frac{P_0}{2\kappa}} + e^{-\frac{P_0}{2\kappa}} \otimes L_i + \frac{1}{2\kappa}\epsilon_{ijk}\left(P_j \otimes M_k e^{\frac{P_0}{2\kappa}} + e^{-\frac{P_0}{2\kappa}} M_k \otimes P_j\right), \tag{4}$$

so that replacing the generators on the right side of the tensor products by a finite dimensional representation one gets [**Nowicki:1992if**]

$$\mathcal{L}_i = L_i + e^{\frac{-P_0}{2\kappa}} l_i - \frac{1}{2\kappa}\epsilon_{ijk} m_j P_k. \tag{5}$$

The Dirac operator is

$$\mathcal{D}_{\text{Hopf}} := \gamma^0\left(\Lambda \sinh\left(\frac{P_0}{\Lambda}\right) - \frac{\mathbf{P}^2}{2\Lambda}\right) + e^{-P_0/2\Lambda} P_i \gamma^i.$$

Geometry in momentum space

Our perspective [**AmelinoCamelia:2011bm**, **Carmona:2019fwf**], see also [**Kowalski-Glikman:2002oyi**], [**Majid:1999td**]:

- Dispersion relation $\rightarrow$ Squared distance from the origin to k.
- To have (modified) Poincaré invariance: translations $\oplus$ (or T_μ), Lorentz generators $J_{\mu\nu}$, all acting on momentum space $\rightarrow$ 10 isometries of the metric

  $\rightarrow$ maximally symmetric momentum space (Minkowski, dS, AdS).
- Connection determined by the composition law

$$\partial_{p_\mu} \partial_{q_\nu} (p \oplus q)_\sigma \Big|_{q=p=0} = -\Gamma^{\mu\nu}_{\sigma}(0). \tag{6}$$

Geometry in momentum space

- Consider a momentum metric

$$g_{00}(p) = 1\,, \quad g_{0i}(p) = g_{i0}(p) = \frac{p_i}{2\Lambda}\,, \quad g_{ij}(p) = -\delta_{ij} e^{-p_0/\Lambda} + \frac{p_i p_j}{4\Lambda^2}\,.$$

- Compute the composition law, defined from the translation symmetry

$$g_{\mu\nu}(p \oplus q) = \frac{\partial (p \oplus q)_\mu}{\partial q_\rho} g_{\rho\sigma}(q) \frac{\partial (p \oplus q)_\nu}{\partial q_\sigma}\,,$$

$$(p \oplus q)_0 = p_0 + q_0\,, \qquad (p \oplus q)_i = p_i e^{q_0/2\Lambda} + q_i e^{-p_0/2\Lambda}\,. \tag{7}$$

- This metric is related to the kinematics of κ-Poincaré in the symmetric basis.

Deformed relativistic wave equations

- Klein-Gordon equation derived from the Casimir (squared distance) (SF and J. Relancio, 2022).

$$(C_{\mathrm{D}} - m^2)\phi = \left[\Lambda^2 \operatorname{arccosh}^2\left(\cosh\left(\frac{p_0}{\Lambda}\right) - \frac{\mathbf{p}^2}{2\Lambda^2}\right) - m^2\right]\phi(p) \; = \; 0$$

- Klein–Gordon equation derived from the Casimir (Hopf algebra) [**Lukierski:1992dt**]

$$(C_{\mathrm{A}} - m^2)\phi = \left[\left(2\Lambda \sinh\left(\frac{p_0}{2\Lambda}\right)\right)^2 - \mathbf{p}^2 - m^2\right]\phi(p) \; = \; 0$$

- $C_{\mathrm{D}} = f(C_{\mathrm{A}})$

Remarks

- Different kinematics can be obtained from the same metric: same dispersion relation but different composition laws
  Example:

$$g_{\mu\nu}(p) = \eta_{\mu\nu} + p_\mu p_\nu/\Lambda^2.$$

Snyder kinematics [**Battisti:2010sr**]

$$(p \oplus q)_\mu^{\mathrm{Snyder}} = p_\mu \left(\sqrt{1 + \frac{q^2}{\Lambda^2}} + \frac{p_\mu \eta^{\mu\nu} q_\nu}{\Lambda^2 \left(1 + \sqrt{1 + p^2/\Lambda^2} \right)} \right) + q_\mu .$$

κ-Poincaré in classical basis [**Borowiec2010**]

$$(p \oplus q)_\mu^{\kappa-\text{Poincaré}} = p_\mu \left(\sqrt{1 + \frac{q^2}{\Lambda^2}} + \frac{q_0}{\Lambda} \right) + q_\mu + n_\mu \left[\frac{\sqrt{1 + p^2/\Lambda^2} - p_0/\Lambda}{1 - \mathbf{p}^2/\Lambda^2} \left(q_0 + \frac{q_\alpha \eta^{\alpha\beta} p_\beta}{\Lambda} \right) - q_0 \right],$$

where $n_\mu := (1, 0, 0, 0)$.

Remarks

- Action in momentum space

$$S_{\mathrm{KG}} := \int \mathrm{d}^4 p \, \sqrt{-g} \, \phi^*(p) \left(C_{\mathrm{D}}(p) - m^2 \right) \phi(p).$$

- Covariance.
- Invariance under deformed Lorentz transformations of the metric assuming the field transforms as a scalar

$$\phi'(p') = \phi(p),$$

since

$$C_{\mathrm{D}}(p) = C_{\mathrm{D}}(p').$$

Dirac equation

- We want to find the equivalent of the Dirac equation in this framework:

$$\text{“}\,(\gamma^{\mu}\mathcal{P}_{\mu} - m)\,\psi(p) \;=\; 0.\text{”}$$

Dirac equation

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- What is $\mathcal{P}_{\mu}$?

$$\mathcal{P}_{\mu} = f_{\mu}(p) \;:=\; g_{\mu\nu}(p) f^{\nu}(p) \;=\; \frac{1}{2} g_{\mu\nu}(p) \frac{\partial C_{\mathrm{D}}(p)}{\partial p_{\nu}}.$$

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- Our guess is "local" Lorentz representation

$$\underline{\gamma}^\mu \;:=\; \gamma^a e^\mu{}_a(p).$$

The gamma matrices satisfy

$$\{\underline{\gamma}^\mu, \underline{\gamma}^\nu\} \;=\; 2 g^{\mu\nu}(p) \mathbb{1}.$$

Choices

- There exist different tetrads that lead to the same metric; which one should we use? The composition law identifies one and only one tetrad:

$$g_{\mu\nu}(p \oplus q) = \frac{\partial (p \oplus q)_\mu}{\partial q_\rho} g_{\rho\sigma}(q) \frac{\partial (p \oplus q)_\nu}{\partial q_\sigma},$$

so for $q \to 0$

$$g_{\mu\nu}(p) = \left.\frac{\partial (p \oplus q)_\mu}{\partial q_\rho}\right|_{q\to 0} \eta_{\rho\sigma} \left.\frac{\partial (p \oplus q)_\nu}{\partial q_\sigma}\right|_{q\to 0},$$

i.e.

$$e_\mu{}^a(p) := \delta^a_\nu \left.\frac{\partial (p \oplus q)_\mu}{\partial q_\nu}\right|_{q\to 0}.$$

Dirac equation

- For the symmetric basis we find

$$\mathcal{D}_{\mathrm{D}}^{(S)} := \frac{\sqrt{\frac{C_{\mathrm{D}}^{(S)}(p)}{\Lambda^2}}}{2\Lambda \sinh\left(\sqrt{\frac{C_{\mathrm{D}}^{(S)}(p)}{\Lambda^2}}\right)} \left[2\Lambda e^{-\frac{p_0}{2\Lambda}} \gamma^i p_i + \gamma^0 \left(2\Lambda^2 \sinh\left(\frac{p_0}{\Lambda}\right) - \mathbf{p}^2\right)\right].$$

- If we use instead $C_{\mathrm{A}}^{(S)}(p)$

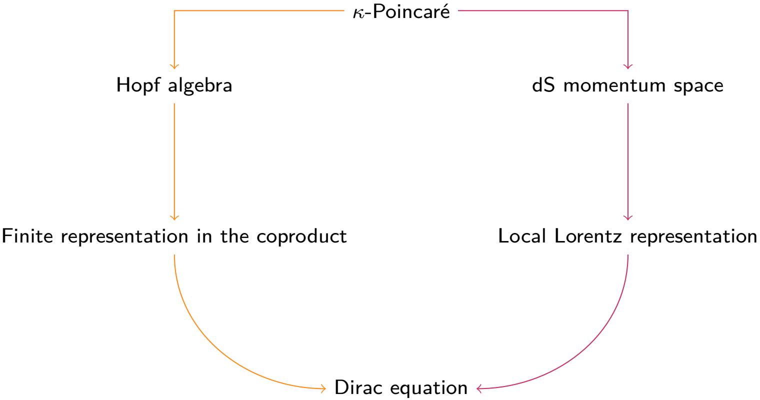
$$\mathcal{D}_{\mathrm{A}}^{(S)} := \gamma^0 \left(\Lambda \sinh\left(\frac{p_0}{\Lambda}\right) - \frac{\mathbf{p}^2}{2\Lambda}\right) + e^{-p_0/2\Lambda} p_i \gamma^i,$$

which is the same result as that obtained within the Hopf-algebraic approach [**Nowicki:1992if**].

- Our construction leads to

$$\left(\mathcal{D}_{\mathrm{D}}^{(S)}\right)^2 = C_{\mathrm{D}}^{(S)}.$$

Dirac equation



Remarks

- Klein–Gordon equation is obtained straightforwardly

$$\left(\underline{\gamma}^{\nu} f_{\nu}(p) - m\right)\left(\underline{\gamma}^{\nu} f_{\nu}(p) + m\right) = C_{\mathrm{D}}(p) - m^2 .$$

- The Dirac equation can be obtained from the action

$$S_{\mathrm{Dirac}} := \int \mathrm{d}^4 p \, \sqrt{-g}\, \bar{\psi}(-p)\left(\underline{\gamma}^{\mu} f_{\mu}(p) - m\right)\psi(p).$$

- Spinors in Snyder

Remarks

- Invariant under deformed Lorentz transformations.
- Covariant.
- Introduce discrete symmetries

$$\mathcal{P}_0 := \mathrm{i}\gamma^0 , \qquad \tilde{\psi}_{\mathcal{P}} := \mathrm{i}\gamma^0\tilde{\psi}(p_0, -\mathbf{p}) ,$$
$$\mathcal{T}_0 := \mathrm{i}\gamma^1\gamma^3\mathcal{K} , \qquad \tilde{\psi}_{\mathcal{T}} := \mathrm{i}\gamma^1\gamma^3\tilde{\psi}^*(p_0, -\mathbf{p}) ,$$
$$\mathcal{C}_0 := \mathrm{i}\gamma^2\mathcal{K} , \qquad \tilde{\psi}_{\mathcal{C}} := \mathrm{i}\gamma^2\tilde{\psi}^*(-p) .$$

- Invariant under $\mathcal{P}$ and $\mathcal{T}$,

$$e^{\mu}{}_a(p_0, -\mathbf{p}) f_\mu(p_0, -\mathbf{p}) = -e^{\mu}{}_a(p) f_\mu(p) , \quad a = 1, 2, 3 . \tag{8}$$

- Invariant under $\mathcal{C}$ when $\Lambda \to -\Lambda$,

$$e^{\mu}{}_a(-p) f_\mu(-p) = -e^{\mu}{}_a(p) f_\mu(p) , \quad a = 0, 1, 2, 3 . \tag{9}$$

Outline

Conclusions

- DSR as curved momentum space.
- Vacuum energies in Snyder spaces, spectral properties.
- DSR Fermions as local realizations of the Lorentz group in momentum space.
- ...

Appendix

Snyder representation [**Snyder:1946qz**, **Lu:2011fh**], is defined by the operators

$$\hat{p}_i = p_i, \quad \hat{x}_i = i\left(\delta_{ij} - \beta^2 p_i p_j\right)\frac{\partial}{\partial p_j}. \tag{10}$$

acting on a Hilbert space of functions $\psi(p)$ with measure $d\mu = \frac{d^D p}{(1-\beta^2 p^2)^{(D+1)/2}}$ [**Lu:2011fh**].
A different realization was introduced in [**Mignemi:2011gr**]:

$$\hat{p}_i = \frac{p_i}{\sqrt{1+\beta^2 p^2}}, \qquad \hat{x}_i = i\sqrt{1+\beta^2 p^2}\,\frac{\partial}{\partial p_i}. \tag{11}$$

The measure on the Hilbert space is in this case $d\mu = \frac{d^D p}{\sqrt{1+\beta^2 p^2}}$.
transformation of variables $p = \frac{q}{\sqrt{1+\beta^2 q^2}}$

$$
\begin{aligned}
\mathcal{E} &= \frac{\Omega_{D-2}}{2\beta} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{dq}{(2\pi)^{D-1}}\, q^{D-2} \sqrt{1+\beta^2 m^2 - \frac{1}{(1+\beta^2 q^2)\cosh^2(\beta k_n)}} \\
&= \frac{\Omega_{D-2}}{2\beta} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{dq}{(2\pi)^{D-1}}\, q^{D-2} \sqrt{1+\beta^2 m^2} \left[1 - \frac{1}{2u} - \frac{1}{8u^2} + \cdots\right],
\end{aligned}
\tag{12}
$$

where $u = (1+\beta^2 m^2)(1+\beta^2 q^2)\cosh^2(\beta k_n)$. After the expansion, there exists only a finite number of divergent terms in expression (18) for a fixed dimension D. Using an adequate regularization one can make use of Abel-Plana formula to approximate the series with an integral plus a constant contribution, that in conjunction with the change of variables (??) gives

$$
\frac{\Omega_{D-2}}{2\beta} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{dq}{(2\pi)^{D-1}}\, q^{D-2} = \frac{2L}{(2\pi)^D \beta} \mathrm{Vol}(\mathbb{H}_D) + \frac{1}{4(2\pi)^{D-1}\beta} \mathrm{Vol}(\mathbb{H}_{D,w=0}). \tag{13}
$$

Appendix

$$\mathcal{F} = \frac{\Omega_{D-2}}{2(2\pi)^{D-1}} \int_0^{1/\beta} \frac{dp\, p^{D-2}}{(1-\beta^2 p^2)^{D/2-1/2}} \left(\sum_{n=1}^{\infty} f_n(p) - \int_0^{\infty} dn\, f_n(p) \right), \tag{14}$$

where we have introduced the functions

$$f_n(p) = \frac{k_n}{\beta L} \frac{\tanh(\beta k_n)}{\cosh^2(\beta k_n) \sqrt{p^2 + \beta^{-2} \tanh^2(\beta k_n)(1-\beta^2 p^2) + m^2}}. \tag{15}$$

Snyder spaces

Noncommutative coordinates $\bar{x}_\mu$ and p_μ satisfy the following commutation relations[1]

$$\begin{aligned} [\bar{x}_\mu, \bar{x}_\nu] &= i\beta M_{\mu\nu}\psi(\beta p^2), \\ [p_\mu, p_\nu] &= 0, \\ [p_\mu, \bar{x}_\nu] &= -i\phi_{\mu\nu}(\beta p^2), \end{aligned} \tag{16}$$

while the Lorentz generators $M_{\mu\nu}$ have the same commutation relations as in the usual case, i.e.

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}), \\ [M_{\mu\nu}, p_\lambda] &= i(\eta_{\mu\lambda}p_\nu - \eta_{\lambda\nu}p_\mu), \\ [M_{\mu\nu}, \bar{x}_\lambda] &= i(\eta_{\mu\lambda}\bar{x}_\nu - \eta_{\lambda\nu}\bar{x}_\mu). \end{aligned} \tag{17}$$

[1] Mignemi et al.: 2017

Appendix

Expansion of Casimir:

$$\mathcal{E} = \frac{\Omega_{D-2}}{2\beta} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{dq}{(2\pi)^{D-1}} \, q^{D-2} \sqrt{1+\beta^2 m^2} \left[1 - \frac{1}{2u} - \frac{1}{8u^2} + \cdots \right], \tag{18}$$

$u = (1+\beta^2 m^2)(1+\beta^2 q^2) \cosh^2(\beta k_n)$

Appendix

$$T_\rho := {T'}_\rho + c_\rho^{\mu\nu} J_{\mu\nu} \implies$$

$$[T^0, T^i] = \frac{c_1}{\Lambda} T^i + \frac{c_2}{\Lambda^2} J^{0i}, \qquad [T^i, T^j] = \frac{c_2}{\Lambda^2} J^{ij} . \tag{19}$$

$$(\phi, \psi)_{\mathrm{KG}} = \int \frac{\mathrm{d}^3 p}{\omega_p} \left[\phi_+^*(p)\psi_+(p) - \phi_-^*(p)\psi_-(p)\right] , \tag{20}$$

$$(\phi, \psi)_{C_{\mathrm{D}}} := 2 \int \mathrm{d}^4 p \sqrt{-g}\, \delta\left(C_{\mathrm{D}} - m^2\right) \Theta(f^\nu t_\nu)\, \phi^*(p)\psi(p) , \tag{21}$$

$$\psi'(p') = \mathcal{S}(\Lambda(p))\psi(p)\,. \tag{22}$$

$$\left(\mathcal{S}\underline{\gamma}^{\mu}\mathcal{S}^{-1}f_{\mu}(p) - m\right)\mathcal{S}\psi(p) = \left(\mathcal{S}\gamma^{a}\mathcal{S}^{-1}e'^{\rho}{}_{b}(p')f'_{\rho}(p') - m\right)\psi'(p') = 0\,, \tag{23}$$

where we have used appropriate transformations for the vielbein and the functions f^{μ}, cf. the definition (??) (all the quantities in the new system of coordinates are denoted with a prime).
Then, since the proposed diffeomorphism is an isometry of the metric, the vielbein satisfies

$$e'^{\rho}{}_{a}(p') = e^{\rho}{}_{b}(p')\Lambda^{b}{}_{a}(p')\,, \tag{24}$$

where $\Lambda^{b}{}_{a}(p')$ is a (local) Lorentz transformation that may depend on the point p'. Note that this transformation is not the deformed Lorentz transformation obtained from the isometries of the metric. Considering (24) and the symmetry of f_{μ} (cf. the definition (??)), we obtain

$$\left(\mathcal{S}\gamma^{a}\mathcal{S}^{-1}\Lambda^{b}{}_{a}(p')e^{\rho}{}_{b}(p')f_{\rho}(p') - m\right)\psi'(p') = 0\,. \tag{25}$$

Of course, local Lorentz transformations near the identity can be expanded in terms of antisymmetric parameters $\epsilon_{ab} := \eta_{ac}\epsilon^{c}{}_{b}$ as customarily,

$$\Lambda^{b}{}_{a}(p') = \delta^{b}_{a} + \epsilon^{b}{}_{a}(p') + \cdots\,. \tag{26}$$

Thus, we can also expand the matrix $\mathcal{S}$ in Eq. (23) for transformations around the identity; this allows us to determine the infinitesimal form of $\mathcal{S}$ as a function of the Lorentz coefficients $\epsilon_{ab}(p')$:

$$\mathcal{S} = \mathbb{I} - \frac{\mathrm{i}}{4}\sigma^{ab}\epsilon_{ab}(p') + \cdots\,,\sigma^{ab} := \frac{\mathrm{i}}{2}\left[\gamma^{a},\gamma^{b}\right]\,. \tag{27}$$

More generally, for an arbitrary metric in momentum space, the expression (??) is covariant once we postulate

NC QFT

Consider the simpler situation (Moyal plane)

$$[\hat{x}_a, \hat{x}_b] = 2i\Theta_{ab}.$$

Functions are promoted to operators by means of Weyl's transform:

$$\hat{f} := \Omega(f)(\hat{x}) := \int d\tilde{p}\, \tilde{f}(p)\, e^{ip\hat{x}}. \tag{31}$$

The product of these operators

$$\begin{aligned} \hat{f}\hat{g} &= \int d\tilde{p} d\tilde{q}\, \tilde{f}(p)\, e^{ip\hat{x}} \tilde{g}(q)\, e^{iq\hat{x}} \\ &= \int d\tilde{p} \left(\int d\tilde{q}\, \tilde{f}(p-q) \tilde{g}(q)\, e^{ip\Theta q} \right) e^{ip\hat{x}}, \end{aligned}$$

is such that its Weyl's antitransformation is nothing but the Moyal product:

$$\Omega^{-1}(\hat{f}\hat{g}) = \int d\tilde{p} d\tilde{q}\, \tilde{f}(p-q) \tilde{g}(q)\, e^{ip\Theta q}\, e^{ipx} =: (f \star g)(x).$$

NC QFT

An approach to NC QFT is to introduce the $\star$ noncommutative product (Groenewold-Moyal) to deform the algebra of functions (coordinates are not operators anymore):

$$f(\hat{x})g(\hat{x}) = (f \star g)(x) = e^{i\Theta_{ab}\partial_a^f \partial_b^g}\, f(x)\, g(x)$$

$$= f(x_a + i\Theta_{ab}\partial_b^g)\, g(x)$$

$$= f(x)\, g(x_a - i\Theta_{ab}\partial_b^f).$$

The $\star$ product is NC and associative; of course

$$[x_a, x_b]_\star = 2i\Theta_{ab}.$$

Snyder spaces

One can find a $\star$-product; for two plane waves (Mignemi et al.: 2017):

$$e^{ik\cdot x} \star e^{iq\cdot x} = e^{iD(k,q)\cdot x + iG(k,q)}, \tag{32}$$

where

$$\begin{aligned} D^\mu(k,q) &= k^\mu + q^\mu + \beta\left[k^\mu\left(s_1 q^2 + \left(s_1 + \frac{s_2}{2}\right)k\cdot q\right) + q^\mu s_2\left(k\cdot q + \frac{k^2}{2}\right)\right] + \mathcal{O}(\beta^2), \\ G(k,q) &= -i\beta\left(s_1 + \frac{D+1}{2}s_2\right)k\cdot q + \mathcal{O}(\beta^2). \end{aligned} \tag{33}$$

One can show that

$$\int f(x)\star g(x) = \int f(x)g(x). \tag{34}$$

The $\star$-product is nonassociative (but the Jacobi identities are still satisfied).

Snyder $\lambda\phi^4_\star$

Consider the action (Meljanac, Mingemi et al., 2017; SAF and Mignemi: 2018)

$$S[\varphi] = \int \frac{1}{2}\partial_\mu\varphi\star\partial^\mu\varphi + \frac{m^2}{2}\varphi\star\varphi - \frac{\lambda}{4!}\varphi\star(\varphi\star(\varphi\star\varphi)). \tag{35}$$

so that

$$\Gamma_{1-loop}[\phi] = S[\phi] + \frac{\mu^{-\epsilon}}{2}\mathrm{Tr}\log A, \quad Af(x) = \int dy\,\frac{\delta^2 S}{\delta\varphi(x)\delta\varphi(y)}[\phi]f(y). \tag{36}$$

The interaction term gives a contribution

$$\frac{\delta^2 S_I}{\delta\varphi_x\delta\varphi_y} \approx -\frac{\lambda}{4!}\frac{1}{2}\int\left(\prod_{m=1}^{3}\frac{dq_m}{(2\pi)^D}\right)\tilde{\phi}_1\tilde{\phi}_2 e^{-iq_3 y+i(q_1+q_2+q_3)x}\Bigg[4! + 4\beta(s_1+s_2) \\ \times\left(-2i\sum_k q_k^2 q_k\cdot x + (2+D)\left(-\sum_{i=1}^{3}q_i^2 + q_4^2\right)\right)\Bigg]_{q_4=-(q_1+q_1+q_3)}. \tag{37}$$

Snyder-de Sitter

The commutation relations are (SAF and Mignemi: 2019, 2021)

$$[\hat{x}_i, \hat{x}_j] = i\beta^2 J_{ij}, \qquad [\hat{p}_i, \hat{p}_j] = i\alpha^2 J_{ij},$$
$$[\hat{x}_i, \hat{p}_j] = i[\delta_{ij} + \alpha^2 \hat{x}_i \hat{x}_j + \beta^2 \hat{p}_j \hat{p}_i + \alpha\beta(\hat{x}_j \hat{p}_i + \hat{p}_i \hat{x}_j)]. \tag{38}$$

with the "usual" Lorentz generators

$$J_{ij} = \frac{1}{2}(\hat{x}_i \hat{p}_j - \hat{x}_j \hat{p}_i + \hat{p}_j \hat{x}_i - \hat{p}_i \hat{x}_j). \tag{39}$$

An interacting theory with action

$$S_{SdS} = \frac{1}{2}\int d^D x\ \varphi\,(\hat{p}^2 + m^2)\,\varphi + \frac{\lambda}{4!}\int d^D x\ \varphi(\hat{x})\left[\varphi(\hat{x})\left(\varphi^2(\hat{x})\right)\right], \tag{40}$$

so that the relevant operator at the one-loop level is

$$\delta^2 S_W \approx p^2 + \omega^2 x^2 + \alpha_{\text{eff}}^2\,(x_i x_j p_j p_i)_S + m^2 + V_W, \quad \omega^2 = \frac{\alpha^2}{\beta^2}. \tag{41}$$

Snyder-de Sitter

The terms to be renormalized in $D = 4 - \epsilon$ dimensions are

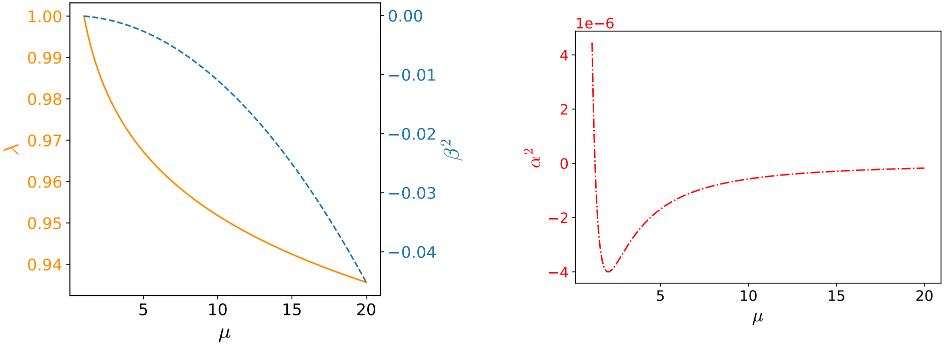
$$\Gamma^{(2)}_{\text{div}} = -\frac{\lambda}{192\pi^2\varepsilon}\int d^D x\,\phi\Bigg[-26\alpha_{\text{eff}}^2 + 6m^2 + 12m^4\beta^2 + \frac{6\alpha_{\text{eff}}^2 m^4}{\omega^2} + 48\beta^2\omega^2 + x^2(15\alpha_{\text{eff}}^2 m^2 + 6\omega^2 + 36m^2\beta^2\omega^2 - 4\beta^2\omega^2\partial^2) - 8\beta^2\omega^2 x_\mu x_\nu \partial^\mu\partial^\nu + x^4(9\alpha_{\text{eff}}^2\omega^2 + 24\beta^2\omega^4)\Bigg]\phi, \tag{42}$$

$$\Gamma^{(4)}_{\text{div}} = \frac{1}{4!}\frac{3\lambda^2}{16\pi^2\varepsilon}\int d^D x\,\phi^2\left[\frac{\alpha_{\text{eff}}^2\partial^2}{2\omega^2} - \frac{2m^2\left(\alpha_{\text{eff}}t^2 + 2\beta^2\omega^2\right) + \omega^2}{\omega^2} - \frac{x^2\left(5\alpha_{\text{eff}}^2 + 16\beta^2\omega^2\right)}{2}\right]\phi^2 - \beta^2\phi_{\star,(1)}, \tag{43}$$

Recall that in de Sitter

$$\sqrt{\det g} = \frac{1}{(1+\alpha^2 x^2)^{(D+1)/2}}. \tag{44}$$

SdS



Numerical solutions for the running of λ, β^2 and α^2.

What is WLF? To do QFT (or to compute HK) using QM in first quantization.

Consider a real scalar field φ with action S in a $d-$dimensional Euclidean space. The effective action is

$$\Gamma = S + \frac{\hbar}{2} \log \det \delta^2 S + \dots \tag{45}$$

For

$$S = \int dx \; \frac{1}{2}(\partial_x \varphi)^2 + \frac{m^2}{2}\varphi^2 + U(\varphi)$$

the quantum corrections can be written by using Schwinger proper time in the following form

$$\Gamma_{1-\mathrm{loop}} = -\frac{1}{2}\int_0^\infty \frac{d\beta}{\beta} e^{-\beta m^2} \underbrace{\mathrm{Tr}\left(e^{-\beta\left(-\partial^2 + U''(\phi)\right)}\right)}_{\text{Heat-Kernel trace}=\int dx\, K(x,x,\beta)} \tag{46}$$