Quantum Minkowski Superspace

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Plan of the talk:

1. Motivation
2. Ordinary Minkowski space as big cell
3. Minkowski superspace
4. Quantum Minkowski superspace
5. Quantum Riemannian geometry

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1. Motivation

Motivation

- **Quantum Groups**: are born to encode quantum symmetries. We treat physical geometric objects (Minkowski space) as *homogeneous spaces*. Up to now only **flat Minkowski** and **conformal (super)space** are successfully realized in their quantization together with their natural (super)symmetries:

  Conformal and Poincare' quantum (super)groups
- **Quantum Cartan Geometry**. We go towards a quantum theory of Cartan connections and (bi)covariant objects (e.g. covariant hamiltonian).
- **Non commutative (super)gravity**. The language of (quantum) differential forms is natural for any geometric theory like (super)gravity.

2. Ordinary Minkowski space

The ordinary (real) Minkowski space

Conformal Lie algebra

$$
\begin{aligned}
P_\mu &= \partial_\mu\,, && \text{(translations)},\\
D &= x^\mu \partial_\mu\,, && \text{(dilation)},\\
L_{\mu\nu} &= x_\nu \partial_\mu - x_\mu \partial_\nu\,, && \text{(Lorentz transformations)},\\
K_\mu &= 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu\,, && \text{(special conformal transformations)}.
\end{aligned}
$$

Conformal (finite) transformation on Minkowski space:

$$
\begin{aligned}
x'^\mu &= x^\mu + \epsilon a^\mu\,, && \text{(translations)},\\
x'^\mu &= \mathrm{e}^{\epsilon u/2} x^\mu\,, && \text{(dilation)},\\
x'^\mu &= \Lambda^\mu{}_\nu x^\nu\,, && \text{(Lorentz transformations)},\\
x'^\mu &= \frac{x^\mu + \epsilon b^\mu x^2}{1 + 2\epsilon b \cdot x + \epsilon^2 b^2 x^2}\,, && \text{(special conformal transformations)}.
\end{aligned}
$$

The ordinary complex Minkowski space as big cell in the Grassmannian $G(2,4)$

Penrose-Manin approach to complex conformal and Minkowski space:

$$M_{\mathbb{C}} \cong \mathbb{C}^4 \subset \mathrm{Gr}(2,4) = \mathrm{SL}_4(\mathbb{C})/Q, \qquad Q = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

The Poincarè symmetries are naturally encoded:

$$\begin{pmatrix} L & 0 \\ NL & R \end{pmatrix} \begin{pmatrix} I \\ A \end{pmatrix} Q = \begin{pmatrix} L \\ NL + RA \end{pmatrix} Q = \begin{pmatrix} I \\ N + RAL^{-1} \end{pmatrix} Q\,.$$

where

$$A = x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - \mathrm{i}x^2 \\ x^1 + \mathrm{i}x^2 & x^0 - x^3 \end{pmatrix}, \quad N = n^\mu \sigma_\mu = \begin{pmatrix} n^0 + n^3 & n^1 - \mathrm{i}n^2 \\ n^1 + \mathrm{i}n^2 & n^0 - n^3 \end{pmatrix}.$$

σ_μ: Pauli matrices

- $\mathrm{SL}_4(\mathbb{C})$: complex conformal group
- $\mathrm{SU}(2,2)$: real conformal group
- $\mathrm{Gr}(2,4) = \mathrm{SL}_4(\mathbb{C})/Q$: complex conformal space
- $P = \mathrm{SL}_2(\mathbb{C}) \otimes \mathbb{R}^4$: real Poincarè group ($\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{so}(1,3)$)

Action of Poincaré on $M = \mathbb{R}^{1,3}$ (adjoint action)

$$\begin{pmatrix} L & 0 \\ NL & {L^{\dagger}}^{-1} \end{pmatrix} \begin{pmatrix} I \\ A \end{pmatrix} Q = \begin{pmatrix} L \\ NL + {L^{\dagger}}^{-1} A \end{pmatrix} Q = \begin{pmatrix} I \\ N + {L^{\dagger}}^{-1} A L^{-1} \end{pmatrix} Q \, .$$

Hence:

$$M = \mathbb{R}^{1,3} \cong A \mapsto N + {L^{\dagger}}^{-1} A L^{-1}$$

Real Minkowski metric

Metric is automatically preserved by Poincarè action:

$$A = x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - \mathrm{i}x^2 \\ x^1 + \mathrm{i}x^2 & x^0 - x^3 \end{pmatrix},$$

$$\text{metric} = det(A) = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

Poincarè action:

$$M = \mathbb{R}^{1,3} \cong A \mapsto N + {L^\dagger}^{-1} A L^{-1}$$

3. Minkowski Superspace

Wess-Zumino superalgebra: $N = 1$ SUSY

Wess-Zumino complex conformal superalgebra:

$$\mathfrak{sl}(4|1) = \left\{ \begin{pmatrix} p & \alpha \\ \beta & c \end{pmatrix} \;\middle|\; p \in \mathrm{M}_4(\mathbb{C}),\ \alpha, \beta^t \in \mathbb{C}^4,\ c \in \mathbb{C},\ \mathrm{tr}\, p = c \right\}$$

with

$$\mathfrak{sl}(4|1)_0 = \left\{ \begin{pmatrix} p & 0 \\ 0 & c \end{pmatrix} \right\} = \mathfrak{sl}_4(\mathbb{C}) \oplus \mathbb{C}\,, \quad \mathfrak{sl}(4|1)_1 = \left\{ \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \right\} = 4^{-1} \oplus \overline{4}^{+1}\,.$$

Observe:

- The even part $\mathfrak{sl}(4|1)$ contains, not only the conformal algebra, but also an *inevitable* extra factor $\mathbb{C}$.
- The odd part is a spinorial representation of $\mathfrak{sl}_4(\mathbb{C})$.

Real conformal superalgebra and supergroup: $N = 1$ SUSY

We need to look for real forms of the complex conformal superalgebra $\mathfrak{sl}(4|1)$: $\mathfrak{su}(2,2|1) = \mathfrak{sl}(4|1)^\sigma$ via the involution σ

$$\begin{array}{ccc} \sigma : \mathfrak{sl}(4|1) & \longrightarrow & \mathfrak{sl}(4|1) \\ \begin{pmatrix} p & \alpha \\ \beta & d \end{pmatrix} & \mapsto & \begin{pmatrix} -Fp^\dagger F & \mathrm{i}F\beta^\dagger \\ \mathrm{i}\alpha^\dagger F & -\bar{d} \end{pmatrix} \end{array}, \qquad F = \mathrm{i}\begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

Real conformal superalgebra:

$$\mathfrak{su}(2,2|1)_\pm = \left\{ \begin{pmatrix} p & \alpha \\ \beta & \mathrm{i}z \end{pmatrix} \ \middle| \ p \in \mathrm{M}_4(\mathbb{C}),\ \alpha, \beta^t \in \mathbb{C}^4,\ z \in \mathbb{R}; \right. \tag{1}$$

$$\left. Fp + p^\dagger F = 0,\ \mathrm{tr}\, p = \mathrm{i}z,\ \alpha = \pm \mathrm{i}F\beta^\dagger \right\} \tag{2}$$

Real conformal supergroup: $\mathrm{SU}(2,2|1) = \mathrm{SL}(4|1)^\Sigma$ are the fixed points of the involution:

$$\begin{pmatrix} X & \mu \\ \nu & x \end{pmatrix} \mapsto \begin{pmatrix} -FX^\dagger F & -jF\nu^\dagger \\ -j\mu^\dagger F & -\bar{x} \end{pmatrix}$$

Real Poincarè superalgebra and supergroup: $N = 1$ SUSY

The real Poincarè superalgebra and supergroups are retrieved as the fixed points of suitable involutions on $\mathfrak{sl}(4|1)$ and $\mathrm{SL}(4|1)$.

- **Real Poincarè superalgebra:**

$$\mathfrak{p} = \left\{ \begin{pmatrix} l & 0 & 0 \\ n & -l^\dagger & -\beta^\dagger \\ \beta & 0 & \mathrm{i}z \end{pmatrix} \right\}$$

- **Real Poincarè supergroup:**

$$P = \left\{ \begin{pmatrix} {L^\dagger}^{-1} & 0 & 0 \\ {L^\dagger}^{-1} M^\dagger {L^\dagger}^{-1} + {L^\dagger}^{-1} \chi^\dagger \varphi^\dagger & {L^\dagger}^{-1} & {L^\dagger}^{-1} - j\varphi \\ \bar{d}^{-1} \varphi^\dagger & 0 & \bar{d}^{-1} \end{pmatrix} \right\}$$

Minkowski Superspace: $N = 1$ SUSY

Complex Minkowski: big cell in *Flag Supermanifold* $\mathrm{SL}(4|1)/Q$.

$$\left(\begin{pmatrix} 1\!\!1 \\ A \\ \alpha \end{pmatrix}, \begin{pmatrix} 1\!\!1 & 0 \\ B & \beta \\ 0 & 1 \end{pmatrix} \right) \longrightarrow \left(\begin{pmatrix} 1\!\!1 \\ C + \frac{1}{2}\theta\bar{\theta}^t \\ \bar{\theta}^t \end{pmatrix}, \begin{pmatrix} 1\!\!1 & 0 \\ C - \frac{1}{2}\theta\bar{\theta}^t & \theta \\ 0 & 1 \end{pmatrix} \right)$$

Penrose twistor relation:

$$B = A - \theta\bar{\theta}^t$$

Note: we change coordinates from (A, B) to

$$C = \sum_{\mu=0}^{3} x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - \mathrm{i}x^2 \\ x^1 + \mathrm{i}x^2 & x^0 - x^3 \end{pmatrix}$$

Action of Super Poincaré: $N = 1$ SUSY

$$
\begin{aligned}
C &\longrightarrow R\big(C + \frac{1}{2}\varphi\bar{\theta}^t - \frac{1}{2}\theta\bar{\varphi}^t\big)L^{-1} + T, \\
\theta &\longrightarrow d^{-1}R(\theta + \varphi), \\
\bar{\theta} &\longrightarrow dL^{-1t}(\bar{\theta} + \bar{\varphi})\,.
\end{aligned}
\tag{3}
$$

Real Lorentz group is obtained by imposing

$$
L^{-1\dagger} = R, \quad \Leftrightarrow \quad L^{-1t} = \bar{R}\,,
$$

This treatment works also for $N = 2$ SUSY!

4. Quantum Minkowski Superspace

Complex Quantum Minkowski Superspace as Homogeneous space, $N = 1$, $(N = 2)$

Key Idea:

- Replace the geometric objects with their non commutative function algebras of *polynomials*:

$$M = \mathbb{C}^{4|N} \quad \Longrightarrow \quad \mathbb{C}_q[M] \quad \text{ring with gen. and rels.}$$

$$P = \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \times \mathbb{C}^4 \quad \Longrightarrow \quad \mathbb{C}_q[P]$$

- Replace actions with *coactions*.
  Action of a group on a space:

$$P \times M \longrightarrow M \quad \Longrightarrow \quad \mathbb{C}_q[M] \leftarrow \mathbb{C}_q[P] \otimes \mathbb{C}_q[M]$$

$\mathbb{C}_q := \mathbb{C}[q, q^{-1}]$ Laurent polynomials (think $q = e^{\hbar}$)

Quantum (chiral) Minkowski Superspace, $N = 1$, $(N = 2)$

Idea: Retrieve heuristically the quantum super ring of the Minkowski superspace *inside* the quantum comformal supergroup $\mathbb{C}_q[\mathrm{SL}(4|1)]$ given by Manin.

$$\begin{pmatrix} L & 0 & 0 \\ tL & R & 0 \\ \tau L & \nu & d \end{pmatrix} \begin{pmatrix} I_2 & s & \sigma \\ 0 & I_2 & \rho \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{14} & \alpha_{15} \\ \vdots & & \vdots & \vdots \\ a_{41} & \dots & a_{44} & \alpha_{45} \\ \alpha_{51} & \dots & \alpha_{54} & a_{55} \end{pmatrix}$$

Minkowski $\cong$ translation group in Poincaré (t, τ)

$$t = (t_{kj}) = \begin{pmatrix} -q^{-1} D_{23} {D_{12}}^{-1} & D_{13} {D_{12}}^{-1} \\ -q^{-1} D_{24} {D_{12}}^{-1} & D_{14} {D_{12}}^{-1} \end{pmatrix}$$

$$\tau = (\tau_{5j}) = \begin{pmatrix} -q^{-1} D_{25} {D_{12}}^{-1} & D_{15} {D_{12}}^{-1} \end{pmatrix}$$

D_{ij} quantum determinants.

Quantum (chiral) Super Minkowski as homogeneous space

The coaction of the super Poincaré is encoded naturally (restriction of the coaction of $\mathbb{C}_q[\mathrm{SL}(4|1)]$):

$$\begin{array}{ccc} \mathbb{C}_q[M] & \longrightarrow & \mathbb{C}_q[P] \otimes \mathbb{C}_q[M] \\ t_{ij} & \mapsto & t_{ij} \otimes 1 + \sum_{u,v} r_{iu} S(\ell_{vj}) \otimes t_{uv} + \sum_v \nu_{5i} S(\ell_{vj}) \otimes \tau_{5v} \\ \tau_{5j} & \mapsto & \nu_{5j} \otimes 1 + \sum_v d S(\ell_{vj}) \otimes \tau_{5v} \end{array}$$

where $R = (r_{ij})$, $L = (\ell_{kl})$.

Quantum chiral complex super Minkowski M for $N = 2$:

- M is the big cell in quantum super $\mathrm{Gr}(2|0; 4|2) = \mathrm{SL}(4|2)/Q$.
- The chiral superfields are defined as the spinor reps of the even part.
- The coaction of super Poincaré for $N = 2$ is more complicated.
- Problem with real forms! (in quantum groups: $*$ involutions)

Reference: F.-Lledo-Razzaq

$N = 2$ quantum chiral superfields and quantum super bundles
https://arxiv.org/abs/2204.01242

Quantum Super Minkowski: big cell in quantum flag supermanifold

Why not definining Quantum Super Minkowski as above?

Obstacles

- The supergrassmannian $\mathrm{Gr}(2|0,4|2)$ is projective, but the superflag $\mathrm{Fl}(2|0,2|1,4|2)$ is **not** a projective (super) manifold!
- Even when looking at $N=1$ (where both supergrassmannian and superflag are projective) the super ring is generated by quantum berezinians and localizations is impossible (computationally) to give in terms of generators and relations.

An indirect definition via the concept of *quantum section* is given in:

R. Fioresi, M.A. Lledo, E. Latini, Nadal, F. A,

The Segre embedding of the quantum conformal superspace, ATMP, Vol. 22, 8, 1939-2000, 2018.

Quantum Section

- G complex semisimple algebraic group.
- P a closed algebraic subgroup of G (P parabolic).

$\mathbb{C}_q(G)$: Hopf algebra quantization of the algebraic functions on G.
$\mathbb{C}_q(P)$: Hopf algebra quantization of the algebraic functions on P.
$d \in \mathcal{O}_q(G)$: *quantum section* that is:

$$\Delta(d) = d \otimes \pi(d)$$

where $\pi : \mathbb{C}_q[G] \longrightarrow \mathbb{C}_q[P] := \mathbb{C}_q[G]/I_q(P)$.
d controls the ring of algebraic functions on G/P: it gives a quantum deformation of the projective embedding of G/P.

$$\begin{aligned} \mathcal{O}_q(G/P) &:= \sum \mathcal{O}_q(G/P)_n, \quad \text{where} \\ \mathcal{O}_q(G/P)_n &:= \{ f \in \mathbb{C}_q[G] \,|\, (id \otimes \pi)\Delta(f) = f \otimes \pi\big(d^n\big) \}. \end{aligned}$$

5. Quantum Riemannian (super)geometry

Quantum Riemannian metric

Quantum Riemannian metric: quantum determinant!

$$det_q \begin{pmatrix} x^0 + x^3 & x^1 - \mathrm{i}x^2 \\ x^1 + \mathrm{i}x^2 & x^0 - x^3 \end{pmatrix} = (x^0 + x^3)(x^0 - x^3) - q^{-1}(x^1 - \mathrm{i}x^2)(x^0 - x^3)$$

Super Riemannian metric: the quantum Berezinian!

Next:

- Quantum differential calculus (Majid, Heckenberger-Kolb, ...)
- Quantum principal bundles (O'Buachalla, Krutov ...)
- Quantum Connections (Hajac,...)
- Quantum Curvature (Majid, ...)
- Quantum Gauge field theory (Landi, Aschieri, Pagani ...)

Conclusions

- We describe Super Conformal and Minkowski spaces for $N = 1$ and $N = 2$ SUSY:
  1. The Chiral Minkowski superspace is the big cell in the grasmannian $\mathrm{Gr}(2|0, 4|N)$
  2. The Minkowski superspace is the big cell in the flag supermanifold: $\mathrm{Fl}(2|0, 2|N; 4|N)$.
- The Chiral Minkowski space is successfully quantized for $N = 1$ and $N = 2$: it comes as a Ore (non commutative) *localization*.
  Warning: No (natural) real forms!
- Notion of Quantum Principal Bundle can be given and quantum differential calculus, vector fields and connections.
  Work in progress: Aschieri, F., Latini, Lledo, Razzaq, Weber.

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