Charge and antipodal matching across spatial infinity

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Motivations

- Scattering theory [Strominger '13, He-Lysov-Mitra-Strominger '14]

  BMS charge conservation $\Leftrightarrow$ Weinberg's soft graviton theorem

Standard assumption

antipodal matching conditions on $m\,, N_A$ and C_{AB}

$\Rightarrow$ BMS charge conservation across i^0

- Asymptotic degrees of freedom

Open question

What is the *physical* asymptotic phase space?

polyhomogeneous in r? falloffs in u? ...

Advertisement: improved understanding of asymptotic phase space and BMS fluxes yields loop-corrected subleading soft graviton theorem *on the nose* [Donnay-KN-Ruzziconi '22, Pasterski '22]

Roadmap

Main message

Antipodal matching is a *property* of solutions satisfying specific u-falloffs at $\mathscr{I}^+_-$

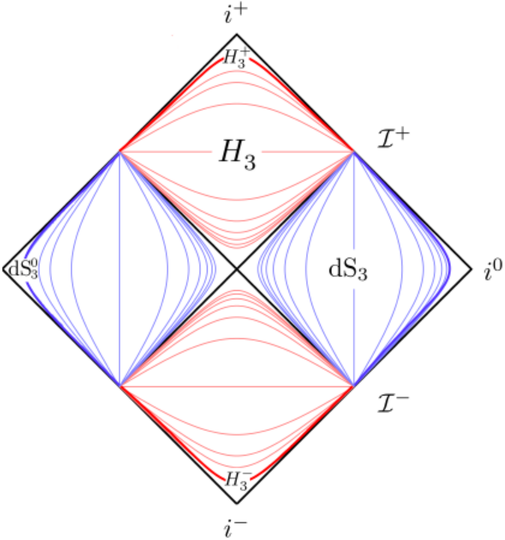
Strategy

- impose u-falloffs at $\mathscr{I}^+_-$ such that BMS charges are finite
- map data at $\mathscr{I}^+_-$ to data in the future of i^0
- use evolution equations to obtain data in the past of i^0
- map to data at $\mathscr{I}^-_+$
- assess antipodal matching

Novelties wrt. to previous works [Troessaert '17, Prabhu '19, Prabhu-Shehzad '21]

- systematic map between data at $\mathscr{I}$ and i^0
- treatment of nonlinearities $\longrightarrow$ antipodal matching of N_A
- matching of BMS charges at $\mathscr{I}$ and i^0

From $\mathscr{I}^+_-$ to $\mathscr{I}^-_+$ through i^0



[Ball et al. arXiv:1905.09809]

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Mapping data from $\mathscr{I}$ to i^0

Bondi gauge

$$ds^2 = g_{uu}\, du^2 + 2g_{ur}\, du\, dr + 2g_{uA}\, du\, dx^A + g_{AB}\, dx^A\, dx^B \,,$$

with

$$\begin{aligned}
g_{uu} &= -1 + \frac{2m}{r} + O(r^{-2})\,, \\
g_{ur} &= -1 + \frac{1}{16r^2}\, C_{AB}C^{AB} + O(r^{-3})\,, \\
g_{uA} &= \frac{1}{2}\nabla^B C_{AB} + \frac{2}{3r}\left(N_A + u\,\partial_A m - \frac{3}{32}\partial_A \left(C_{BC}C^{BC} \right) \right) + O(r^{-2})\,, \\
g_{AB} &= r^2\,\gamma_{AB} + r\, C_{AB} + O(r^0)\,.
\end{aligned}$$

Bondi gauge

assumptions on falloff rate to $\mathscr{I}^+_-$:

$$m = m^0 + u^{-1}\, m^1 + o(u^{-1})\,,$$
$$N_A = N_A^0 + o(u^0)\,,$$
$$C_{AB} = C_{AB}^0 + u^{-1}\, C_{AB}^1 + o(u^{-1})\,.$$

evolution/constraint equations:

$$\partial_u m = -\frac{1}{8} N_{AB} N^{AB} + \frac{1}{4}\nabla_A \nabla_B N^{AB} - 4\pi \lim_{r\to\infty} (r^2\, T_{uu})\,,$$
$$\partial_u N_A = -u\,\partial_A \partial_u m + \frac{1}{4}\partial_A \left(N_{BC} C^{BC}\right) - \frac{1}{4}\nabla_B \left(C^{BC} N_{CA}\right) + \frac{1}{2} C_{AB} \nabla_C N^{BC} - \frac{1}{4}\nabla_B \left(\nabla^B \nabla^C C_{AC} - \nabla_A \nabla_C C^{BC}\right) - 8\pi \lim_{r\to\infty} (r^2\, T_{uA})\,,$$

$$\Rightarrow \qquad C_{AB}^0 = -2\nabla_A \nabla_B C + \gamma_{AB} \nabla^2 C\,, \qquad m^1 = C_{AB}^1 = 0\,,$$

Beig–Schmidt gauge

[Beig-Schmidt '82, Beig '84, Ashtekar-Hansen '78]

$$ds^2 = N^2\, d\rho^2 + H_{ab}\,(N^a\, d\rho + dx^a)\,(N^b\, d\rho + dx^b)\,,$$

where

$$\begin{aligned} N &= 1 + \frac{\sigma}{\rho}\,, \\ H_{ab}N^b &= o(\rho^{-1})\,, \\ H_{ab} &= \rho^2\left(h_{ab} + \rho^{-1}(k_{ab} - 2\sigma h_{ab}) + \rho^{-2} j_{ab} + o(\rho^{-2})\right)\,. \end{aligned}$$

Leading electric and magnetic Weyl tensors:

$$E_{ab} = -\left(D_a D_b + h_{ab}\right)\sigma\,, \qquad B_{ab} = \frac{1}{2}\epsilon_a{}^{cd} D_c k_{db}\,.$$

Beig–Schmidt gauge

Boundary conditions:

$$R\left[h\right]_{ab} = 2h_{ab}\,, \qquad k^a_a = 0\,.$$

Leading eoms:

$$(D^2+3)\sigma = 0\,, \qquad (D^2-3)k_{ab} = 0\,, \qquad D^a k_{ab} = 0\,,$$

Subleading eoms:

$$\left(D^2-2\right) j_{ab} = S_{ab}(\sigma, k)\,,$$

$$j^a_a = 12\sigma^2 + D_a\sigma\, D^a\sigma + \frac{1}{4}k^{ab}k_{ab} + k^{ab}D_aD_b\sigma\,,$$

$$D^b j_{ba} = \frac{1}{2}k^c_b\, D^b k_{ca} + D_a\left(8\sigma^2 + D_a\sigma\, D^a\sigma - \frac{1}{8}k^{cd}k_{cd} + k^{cd}D_cD_d\sigma\right).$$

Beig–Schmidt gauge

$$ds^2_{\mathcal{H}} = h_{ab}\, dx^a\, dx^b = -\, d\tau^2 + \cosh^2\tau\, \gamma_{AB}\, dx^A\, dx^B\, .$$

Large-τ expansions:

$$\sigma(\tau, x) = e^{\tau}\, \sigma^{(-1)} + e^{-\tau} \sigma^{(1)} + e^{-3\tau} \tau\, \tilde{\sigma} + e^{-3\tau}\, \sigma^{(3)} + \dots\, ,$$

$$k_{\tau\tau} = e^{-3\tau} \tau\, \tilde{k}_{\tau\tau} + e^{-3\tau}\, k^{(3)}_{\tau\tau} + \dots\, ,$$

$$k_{\tau A} = e^{-\tau} \tau\, \tilde{k}_{\tau A} + e^{-\tau}\, k^{(1)}_{\tau A} + \dots\, ,$$

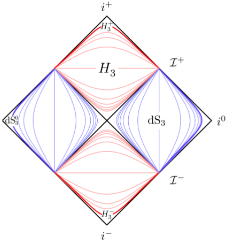
$$k_{AB} = e^{\tau} \tau\, \tilde{k}_{AB} + e^{\tau}\, k^{(-1)}_{AB} + \dots\, .$$

$$j_{\tau\tau} = e^{-2\tau} j^{(2)}_{\tau\tau} + e^{-4\tau} j^{(4)}_{\tau\tau} + \dots\, ,$$

$$j_{\tau A} = j^{(0)}_{\tau A} + e^{-2\tau} j^{(2)}_{\tau A} + \dots\, ,$$

$$j_{AB} = e^{2\tau} j^{(-2)}_{AB} + j^{(0)}_{AB} + \dots\, ,$$

From Bondi to Beig–Schmidt



Zeroth order transformation:

$$u = -\rho\, e^{-\tau}, \qquad r = \rho \cosh \tau,$$

such that

$$\frac{u}{r} = O(e^{-2\tau})\,.$$

Doubly asymptotic coordinate transformation

We work in the regime $r \gg |u| \gg 1$ and $\rho \gg e^{\tau} \gg 1$

From Bondi to Beig–Schmidt

We find the following map:

$$\sigma^{(-1)} = \tilde{k}_{AB} = j^{(0)}_{\tau A} = \dots = 0\,,$$

$$\sigma^{(3)} = 2m^0\,, \qquad k^{(-1)}_{AB} = \frac{1}{2}\,C^0_{AB}\,, \qquad j^{(2)}_{\tau A} = 4N^0_A + C^0_{AB}\nabla_C C^{BC}_0\,.$$

Vanishing of leading magnetic Weyl tensor

We demonstrate that k_{ab} admits the restricted form

$$k_{ab} = -\left(D_a D_b + h_{ab}\right)\Phi\,, \qquad (D^2+3)\Phi = 0\,,$$

such that

$$B_{ab} = 0\,.$$

Note that this was assumed in [KN-Salzer '21, Prabhu-Shehzad '21]

Antipodal matching relations

Antipodal matching relations

To show

$$m(x)\big|_{\mathscr{I}^+_-} = m(-x)\big|_{\mathscr{I}^-_+}\,,$$
$$C_{AB}(x)\big|_{\mathscr{I}^+_-} = -\,C_{AB}(-x)\big|_{\mathscr{I}^-_+}\,,$$
$$N_A(x)\big|_{\mathscr{I}^+_-} = N_A(-x)\big|_{\mathscr{I}^-_+}\,.$$

Dictionary:

$$\sigma^{(3)} = 2m^0\,, \qquad k^{(-1)}_{AB} = \frac{1}{2}\,C^0_{AB}\,, \qquad j^{(2)}_{\tau A} = 4N_A + C^0_{AB}\nabla_C C_0^{BC}\,.$$

Claim

The antipodal relations follow from parity properties of σ, k_{ab} and j_{ab} under $(\tau, x^A) \mapsto (-\tau, -x^A)$.

Example: mass aspect

eom:

$$\left[-\partial_\tau^2 - 2\tanh\tau\,\partial_\tau + \cosh^{-2}\tau\,\nabla^2 + 3\right]\sigma = 0\,.$$

harmonic decomposition:

$$\sigma(s, x^A) = \sqrt{1-s^2}\sum_{l,m}\sigma_{lm}(s)Y_l^m(x^A)\,, \quad s = \tanh\tau \in (-1,1)\,.$$

Legendre differential equation:

$$\left[(1-s^2)\partial_s^2 - 2s\partial_s + l(l+1) - \frac{4}{1-s^2}\right]\sigma_{lm}(s) = 0\,,$$

with solutions

$$P_l^2(s)\,,\; Q_l^2(s)\,.$$

Example: mass aspect

Remember that the Bondi data maps to $O(e^{-3\tau})$ solutions, which are given by

$$\sigma(s,x^A)=\sqrt{1-s^2}\sum_{l=2}^{\infty}\sum_{m=-l}^{l}a_{lm}\,P_l^2(s)Y_l^m(x^A)\,.$$

Since

$$Y_l^m(-x^A)=(-1)^l\,Y_l^m(x^A)\,,\qquad P_l^2(-s)=(-1)^lP_l^2(s)\,,$$

we infer the parity property [Herberthson-Ludvigsen '92, Troessaert '17]

$$\sigma(-s,-x^A)=\sigma(s,x^A)\,.$$

Result

$$m(x^A)\big|_{\mathscr{I}^+_-}=m(-x^A)\big|_{\mathscr{I}^-_+}\,.$$

Matching of BMS charges from $\mathscr{I}$ to i^0

Matching of charges from $\mathscr{I}$ to i^0

Various BMS charge proposals:

$$Q_{(\alpha,\beta)}[T,Y^A] = \frac{1}{8\pi G}\int_{\mathbb{S}^2} d\Omega \left(2T\,m + Y^A \hat{N}_A\right)\,,$$

$$\hat{N}_A \equiv N_A - \frac{\alpha}{16}\partial_A(C_{BC}C^{BC}) - \frac{\alpha}{4}C_{AB}\nabla_C C^{BC} + u\,\frac{\beta}{4}\nabla^B\left(\nabla_B\nabla^C C_{AC} - \nabla_A\nabla^C C_{BC}\right)\,.$$

We simply find

$$\lim_{u\to-\infty} Q_{(\alpha,\beta)}[T,Y^A] = \frac{1}{8\pi G}\int d\Omega\,(2T\,m^0 + Y^A N^0_A)\,,$$

where the term controlled by α is proportional to

$$\int d\Omega\, Y^A\left[\partial_A(C^0_{BC}C_0^{BC}) + 4C^0_{AB}\nabla_C C_0^{BC}\right] = 0\,.$$

Charge matching

We show that this agrees with conserved charges at i^0 [Compere-Dehouck '11]

Perspectives

Perspectives

- Investigation of more general u-falloff, including u^{-1} tail effects $\leftrightarrow \log\omega$ terms in soft expansion [Sen et al.]
- Extension of the asymptotic phase space: superrotations, non-polyhomogeneity, ... [work in progress]
- Mapping of subleading data from $\mathscr{I}$ to i^0
- Sub-subleading antipodal matching condition? [Freidel-Pranzetti-Raclariu '21]
- ...