Noncommutative QFT and Curved Spacetimes

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Motivation

Does NC Geometry award us with a theory of quantum gravity?

QFT in Curved-Spacetimes is a zero order approximation to QG

$\Rightarrow$ Fuse NCG with QFT in Curved-Spacetimes to obtain first order approximation.

Intro, What

- **Generalize NQFT** in Minkowski to **curved spacetime** rigorously
- Prove that it complies with the **equivalence principle**

Intro, Why

- Physical (theoretical) Proof for NC (Quantum Spacetime)
- Supply a proof **connection to quantum gravity**

Intro QFTCST

QFTs are rigorously constructed for Globally Hyperbolic spacetimes

Advantages: Exist direction of a time, well-posed Cauchy problem

Disadvantages: No preferred State

(GNS) For a given state (positive linear functional) ω over the (unital) $*$-algebra $\mathscr{A}$, one obtains a quadruple $(\mathcal{H}_\omega, D_\omega, \pi_\omega, \Psi_\omega)$. Field operators are given by

$$\phi_\omega(F) = \pi_\omega(\phi(f)) : D_\omega \to \mathcal{H}_\omega$$

The he n-point function are given by

$$\omega_n(\phi(F_1)\cdots\phi(F_n)) = \langle\Psi_\omega|\,\pi_\omega(\phi(F_1))\cdots\pi_\omega(\phi(F_n))\,\Psi_\omega\rangle$$

Hadmard States

Preferred states: Hadamard States (resemble singularity structure Minkowski)

In a convex neighborhood C of (M, g) the Hadamard parametrix is

$$H_\epsilon(x,y) = \frac{u(x,y)}{\sigma_\epsilon^2(x,y)} + v(x,y)\log\left(\frac{\sigma_\epsilon^2(x,y)}{\lambda^2}\right)$$

where $\sigma^2(x,y)$ is the geodesic distance (the Synge function), T is any local time coordinate increasing towards the future, $\lambda > 0$ a length scale and

$$\sigma_\epsilon^2(x,y) \stackrel{\text{def}}{=} \sigma^2(x,y) + 2i\epsilon(T(x) - T(y)) + \epsilon^2\,,$$

NC Generalization

Goal: Generalize Star Product (or Rieffel product)

$$(f \times_\Theta g)(z) = \lim_{\epsilon \to 0} \iint \chi(\epsilon x, \epsilon y)\, f(z + \Theta x)\, g(z + y)\, e^{-i\, x \cdot y} d^4 x\, d^4 y,$$

to curved spacetimes in a sensible manner.

Technical problem: NO commuting translations that lead to associative product

Idea in Fröb, Much [JMP21] in case of de Sitter: Embed the spacetime in a higher dimensional Flat Minkowski where translations exist

This idea is **generalized to all globally hyperbolic spacetimes**, [AM, Make Deformation Quantization Physical Again 21]

Embedding in GHST

Theorem (Sanchez, Müller 11)

Let (M, g) be a GHM. Then, it admits an isometric embedding in $\mathbb{L}^N$

$\exists\ F : M \to \mathbb{L}^N$, with local coordinates $X^A = (X^\mu, X^a)$

$$\sum_{A=0}^{N} \frac{\partial X^A}{\partial x^\mu} \frac{\partial X_A}{\partial x^\nu} = g_{\mu\nu}.$$

Existence of $F \equiv$ existence of solutions for Diff. equations

$$X^A = X^A(x^\mu).$$

$(N-4)$ constraints on the coordinates X^a,

$$X^a = X^a(X^\mu).$$

Due to inverse function theorem

$$x^\mu = x^\mu(X^\nu).$$

Curved Star Product

Using the embedding coordinates we define an associative star product.

Definition

Let Z be the embedding point corresponding to z, then

$$
\begin{aligned}
(f \times_\Theta g)(z) &= \lim_{\epsilon \to 0} \iint \chi(\epsilon X, \epsilon Y)\, f(Z + \Theta X)\, g(Z + Y)\, e^{-i\, X \cdot Y} \\
&= \exp\big(i\, \Theta^{\rho\sigma}(x_1, x_2) \partial_{x_{\mathbf{1}}^\rho} \partial_{x_{\mathbf{2}}^\sigma}\big) f(x_1) g(x_2)|_{x_{\mathbf{1}} = x_{\mathbf{2}} = z},
\end{aligned}
$$

where the matrix $\Theta(x_1, x_2)$ is given by

$$
\begin{aligned}
\Theta^{\rho\sigma}(x_1, x_2) :&= \Theta^{\mu\nu} \frac{\partial x_1^\rho}{\partial X_1^\mu} \frac{\partial x_2^\sigma}{\partial X_2^\nu} \\
&= \Theta^{\mu\nu} J^\rho_{\ \mu}(x_1) J^\sigma_{\ \nu}(x_2),
\end{aligned}
$$

where J represents the Jacobian.

Application to QFT in GHST

For a $*$-algebra $\mathscr{A} = \mathscr{A}(M, g)$ defined on a globally hyperbolic spacetime (M, g) generated by Klein-Gordon fields we have the following deformed 2-point function

$$\omega_2^{\Theta}(\phi(x_1)\phi(x_2)) = \lim_{\epsilon \to 0} \iint \chi(\epsilon X, \epsilon Y) \langle \Psi_\omega | \, \phi(X_1 + \Theta X)\phi(X_2 + Y) \, \Psi_\omega \rangle \, e^{-i X \cdot Y}$$

Furthermore, the deformed smeared two-point function

$$\omega_2^{\Theta}(\phi(F_1)\phi(F_2)) = \iint F_1(X_1) \, F_2(X_2) \, \omega_2^{\Theta}(\phi(x_1(X_1))\phi(x_2(X_2)))$$

for functions $F_1, F_2 \in C_0^\infty(\mathbb{R}^4)$ is well-defined.

Does the Deformation make Sense?

Let $\mathcal{N}$ be the bundle of nonzero null covectors on M:

$$\mathcal{N} = \{(x,\xi) \in T^*M : \xi \text{ a non-zero null at } p\}.$$
$$\mathcal{N}^{\pm} = \{(p,\xi) \in \mathcal{N} : \xi \text{ is future}(+)/\text{past}(-)\text{directed}\}.$$

Definition

A state ω obeys the **Microlocal Spectrum Condition** (μSC) if

$$WF(\omega_2) \subset \mathcal{N}^+ \times \mathcal{N}^-.$$

Theorem (Radzikowski 96)

*The μSC is equivalent to the **Hadamard condition**.*

Definition

If $u \in \mathcal{D}'(\mathbb{R}^n)$, a pair $(x, k) \in^n \times(^n \backslash \{0\})$ is a regular direction for u if $\exists$ constants C_N, $N \in$ so that

$$\left|\hat{\phi u}(k)\right| < \frac{C_N}{1+|k|^N}, \qquad \forall k \in V \phi \in C_0^\infty(\mathbb{R}^n)$$

Definition (Wave front set)

The *wavefront* set of u is defined to be

$WF(u) = \{(x, k) \in \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\}) : (x, k)$ is not *a regular direction for* u$\}$.

Useful properties

- If $f \in C_0^\infty$ it has an empty wavefront set $WF(f) = \emptyset$.
- If P is any differential operator with smooth coefficients, then

$$WF(Pu) \subset WF(u)$$

Micro-local condition in the deformed Setting

Theorem

Let the state ω obey the microlocal spectrum condition. Then, the deformed state ω_Θ obeys the microlocal spectrum condition as well, i.e.

$$WF(\omega_2^\Theta) \subset \mathcal{N}^+ \times \mathcal{N}^-.$$

Proof.

$$\begin{aligned} \omega_2^\Theta(x_1, x_2) &= \exp\left(i\Theta^{\rho\sigma}(x_1, x_2)\partial_{x_1^\rho}\partial_{x_2^\sigma}\right)\omega_2(x_1, x_2) \\ &= P\,\omega_2(x_1, x_2) \end{aligned}$$

where P is a differential operator with smooth coefficients.

$$WF(\omega_2^\Theta) = WF(P\,\omega_2) \subset WF(\omega_2) \subset \mathcal{N}^+ \times \mathcal{N}^-.$$

□

Corollary

The deformed two-point function ω_2^Θ is ***Hadamard****.*

$\Longrightarrow$ Deformations **physically meaningful** since they satisfy the **equivalence principle**

Conclusions and Outlook

- QFT in noncommutative (or quantized) curved spacetimes agrees with the equivalence principle
- Deformation can be extended to n-point functions
- Rigorous Framework to examine achievements in curved spacetime with NC component, e.g. Hawking effect, Quantum energy inequalities, Entropies (Joint work with H. Grosse, Rainer Verch), semi-classical effects
- Predict testible quantum gravitational effects