The Weak Gravity Conjecture with Scalar Fields in AdS

Marco Michel

work in collaboration with
Stefano Andriolo and Eran Palti

Ben-Gurion University of the Negev

Corfu 2022
Workshop on Holography and the Swampland

September 9, 2022

Table of Contents

Introduction: The Positive Binding Conjecture

The Weak Gravity Conjecture

- The Swampland program is motivated by the identification and verification of universal features of quantum gravity.
- A prominent example is the *Weak Gravity Conjecture* [Akani-Hamed, Motl, Nicolis, Vafa '06]

$$\sqrt{2} g q M_P \geq m$$

- An equivalent formulation is the *Repulsive-Force Conjecture* [Palti '17; Heidenreich, Reece, Rudelius '19], which requires a consistent gravitational theory, with a $U(1)$ gauge symmetry, must have a self-repulsive charged particle.
- The presents of an additional scalar field leads to a modification in the presence of additional massless scalar fields:

$$2g^2 q^2 M_p^2 \geq m^2 + \mu^2 M_p^2.$$

Positive Binding Conjecture

- A more appropriate formulation in AdS is the

Positive Binding Conjecture [Aharony, Palti '21]

For a (weakly coupled) gravitational theory with a $U(1)$ gauge field, there should exist at leat one charged particle in the theory, with charge of order one, to have a non-negative self-binding energy.

- Maps to certain convexity properties of charged operators in the holographically dual CFTs
- Demanding positive self-binding energy places constraints on the effective theory

Goal

We determine what those constraints are by calculating the self-binding energy for a charged particle in terms of its couplings.

Setup

- As a prototype model we consider following theory consisting of a charged scalar ϕ, a set of photons A_i, a graviton $h_{\mu\nu}$:

$$S[\phi, A^i, h] = \int \mathrm{d}^5x\sqrt{-g}\Bigg[\frac{1}{\kappa^2}\left(\frac{R}{2}+\frac{6}{L^2}\right) - \sum_{i=1}^{N}\frac{F_i^2}{4} - |D\phi|^2 - m^2|\phi|^2 - V(\phi)\Bigg]\,, \quad \text{with} \quad V = a|\phi|^4 + b|\phi|^2|D\phi|^2\,.$$

- We use global AdS coordinates to describe our background.

$$\mathrm{d}s^2 = \frac{L^2}{y^2}\left(-\mathrm{d}t^2 + \frac{\mathrm{d}y^2}{1-y^2} + (1-y^2)\mathrm{d}\Omega_3^2\right)\,.$$

- Under the assumption that

$$\frac{g_i}{L^{1/2}} \ll 1\,, \quad \frac{\kappa}{L^{3/2}} \ll 1\,, \quad \frac{a}{L} \ll 1\,, \quad \frac{b}{L^3} \ll 1\,,$$

we will find that $\mathcal{O}(\kappa^2), \mathcal{O}(g_i^2), \mathcal{O}(a), \mathcal{O}(b)$ are the leading contributions to the binding energy. In the context of AdS supergravity: $g_i \propto \frac{\kappa}{L}, a \propto \frac{\kappa^2}{L}, b \propto \kappa^2$, and thus this is ensured by $\frac{\kappa}{L^{3/2}} \ll 1$.

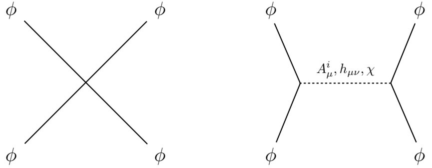
Binding energy from effective potentials

Binding energy

- The binding energy is defined as the difference between the energy of the bound state and twice the energy of the single (free) state $|\phi\rangle$:

$$\gamma \equiv E_{\phi\phi} - 2E_{\phi} = \langle \phi\phi | H | \phi\phi \rangle - 2 \langle \phi | H | \phi \rangle \ .$$

- States can be either $\gamma > 0$ self-repulsive or $\gamma < 0$ self-attractive. A state is BPS if in an interacting theory $\gamma = 0$.
- We are interested in computing the binding energy at tree level, in which case γ is merely determined by quartic interactions ϕ^4.
- In order to avoid going to second order perturbation theory we first integrating out classically all additional fields [Fitzpatrick, Shih '11].



Basic scalar field in AdS

- Field decomposition in creation and annihilation operators:

$$\phi = \sum_{n,l,J} \left(a_{n,l,J} \psi^*_{n,l,J}(x) + b^\dagger_{n,l,J} \psi_{n,l,J}(x) \right),$$

- Free Hamiltonian is then given by

$$H_{free} = \sum_{n,l,J} E_{n,l} (a^\dagger_{n,l,J} a_{n,l,J} + b^\dagger_{n,l,J} b_{n,l,J}),$$

$$E_{n,l} = \Delta + 2n + l, \quad n,l = 0,1,2,\ldots \quad \Delta = 2 \pm \sqrt{4 + m^2 L^2}$$

- We are interested in the state with lowest energy $n = l = J = 0$,

$$|\phi\rangle = b_0^\dagger |0\rangle, \quad |\phi\phi\rangle = \frac{1}{\sqrt{2}} b_0^\dagger b_0^\dagger |0\rangle, \quad b_0 \equiv b_{0,0,0},$$

corresponding to the eigenfunction

$$\psi_0(x) \equiv \psi_{0,0,0}(x) = L^{-3/2} N_\Delta e^{it\Delta} y^\Delta, \quad N_\Delta = \sqrt{\frac{\Delta - 1}{2\pi^2}}.$$

Integrating out additional fields

- Expand

$$A_i = A_i^{(0)} + g_i A_i^{(1)} \quad g_{\mu\nu} = g_{\mu\nu}^{(0)} + \kappa h_{\mu\nu}$$

- Integrating out additional field (photon and graviton) classically by using their equations of motion:

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}F_i^{\mu\nu}) = g_i q_i J^\nu \,, \qquad \Delta_{\mu\nu}^{\rho\sigma} h_{\rho\sigma} = \frac{\kappa}{2} T_{\mu\nu} \,,$$

with sources

$$J_\mu[\phi,\phi^\dagger] = i\phi^\dagger \partial_\mu \phi + h.c. \,,$$
$$T_{\mu\nu}[\phi,\phi^\dagger] = g_{\mu\nu}\left(-|\partial\phi|^2) - m^2|\phi|^2\right) + \left(\partial_\mu\phi^\dagger\partial_\nu\phi + h.c.\right) \,.$$

- Resulting in an non-local effective potential that only depends on ϕ:

$$S_{\text{eff}} = \int \mathrm{d}^5 x \sqrt{-g}\left[-|\partial\phi|^2 - m^2\phi^2 - V_{\text{eff}}[\phi,\phi^\dagger]\right] \,,$$
$$V_{\text{eff}}[\phi,\phi^\dagger] = V[\phi,\phi^\dagger] + \frac{1}{2}\sum_i A_{i\mu}^{(1)}[\phi,\phi^\dagger] J_i^\mu[\phi,\phi^\dagger] - \frac{\kappa}{4} h_{\mu\nu}[\phi,\phi^\dagger] T^{\mu\nu}[\phi,\phi^\dagger]$$

Contributions to binding energy

- With the Hamiltonian

$$H = H_{\text{free}} + \delta H_{\text{eff}}\,, \qquad \delta H_{\text{eff}} = \int \mathrm{d}^4x\sqrt{-g}\; V_{\text{eff}}\,.$$

- Now we can use first order perturbation theory:

$$\begin{aligned} E_{\phi\phi} &\equiv \langle\phi\phi|\, H\,|\phi\phi\rangle = E^{\text{free}}_{\phi\phi} + \langle\phi\phi|\,\delta H_{\text{eff}}\,|\phi\phi\rangle \\ &= 2(E^{\text{free}}_{\phi} + \langle\phi|\,\delta H^{\text{quadratic}}_{\text{eff}}\,|\phi\rangle) + \langle\phi\phi|\,\delta H^{\text{quartic}}_{\text{eff}}\,|\phi\phi\rangle\ , \end{aligned}$$

- The binding energy is therefore

$$\gamma = \int \mathrm{d}^4x\sqrt{-g}\,\langle\phi\phi|\,V_{\mathsf{eff}}\,|\phi\phi\rangle = 2\int \mathrm{d}^4x\sqrt{-g}V_{\mathsf{eff}}[\phi(x)^{(\dagger)} = \psi_0(x)^{(\dagger)}]\,,$$

- Contribution to binding energy [Fitzpatrick, Shih '11] :

$$\gamma^{(a,b)} = \frac{\pi^2 N_\Delta^4\left(aL^2 - b(\Delta-2)\Delta\right)}{(\Delta-1)(2\Delta-1)L^3}\,,$$

$$\gamma_i^{(A_i)} = \frac{\pi^2 g_i^2 q_i^2 N_\Delta^4}{L(2\Delta-1)}\,, \quad \gamma^{(h)} = -\frac{2\pi^2(\Delta-2)\Delta^2\kappa^2 N_\Delta^4}{3(\Delta-1)(2\Delta-1)L^3}.$$

Contribution for scalar fields

Additional neutral scalar field

- Next we add an additional neutral scalar field χ.

$$S[\phi, A^i, h, \chi] = S[\phi, A, h] - \int \mathrm{d}^5x\sqrt{-g}\left[\frac{1}{2}(\partial\chi)^2 + \frac{M^2}{2}\chi^2 + Y\chi|\phi|^2\right] ,$$

- The equation of motion for χ is

$$\Box\chi - M^2\chi = Y|\phi|^2 \quad \text{with} \quad YL^{1/2} \ll 1$$

- Expand

$$\chi = \chi_0 + Y\chi_1$$

- χ contributes to the effective potential with

$$V^{(\chi)}[\phi, \phi^\dagger] = \frac{Y^2}{2}\chi_1[\phi, \phi^\dagger]|\phi|^2 ,$$

- Note, requiring boundary terms to vanish leads to restrictions of the solution. Alternatively, start with modified action [Klebanov, Witten '99].
- For ϕ in the ground state :

$$N_\Delta^2 y^{2\Delta} + M^2\chi_1 + y((y^2+3)\chi_1' + y(y^2-1)\chi_1'') = 0 ,$$

Massless case

- For $M = 0$, we can solve the eom analytically:

$$\chi_1 = \frac{N_\Delta^2}{4(\Delta-2)(\Delta-1)}\left[\frac{y^2}{y^2-1} - \log(1-y^2) + f_\Delta(y)\right],$$

where the integration constants are fixed by requiring that the solution is smooth at $y = 1$ and vanishes at $y = 0$.

$$f_\Delta(y) = \frac{\Delta-1}{\Delta}y^{2\Delta} + B_{y^2}(\Delta+1,-1) + (\Delta-1)B_{y^2}(\Delta+1,0),$$

$$B_x(a,b) \equiv \int_0^x \mathrm{d}t\, t^{a-1}(1-t)^{b-1},$$

where $B_x(a,b)$ are incomplete Beta functions.

- Contributing to the binding energy with

$$\gamma^{(\chi)} = \frac{Y^2\pi^2N_\Delta^4}{8(\Delta-2)^2(\Delta-1)^2}\left[1-\Delta+\frac{1}{\Delta}+\frac{4}{\Delta-1} + \frac{2}{2\Delta-1} + 4H_{\Delta-2} - 2H_{2\Delta}\right],$$

where H_n are the harmonic numbers.

Breitenlohner-Freedman bound saturation - $M^2 = -4$

- For $M = -4$, we can solve the eom analytically:

$$\chi_1 = \frac{N_\Delta^2}{4L(\Delta-1)^2}\left(\frac{-y^{2\Delta}+y^2}{y^2-1}\right).$$

- Note that here, in contrast to the $M = 0$ case, the second integration constant is fixed by requiring a sufficiently fast fall off towards the boundary.
- Computing the binding energy one finds

$$\gamma^{(\chi)} = -\frac{Y^2\pi^2 N_\Delta^4}{8(\Delta-1)^3}.$$

Numerical result - general case

- In case for general mass M and scaling dimension Δ an analytic solution is not available and one has to rely on numerical methods.

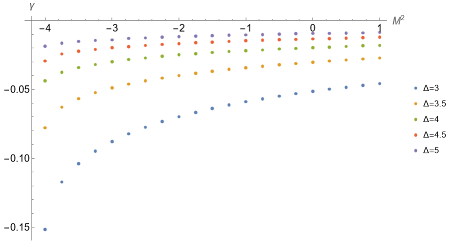


Figure: Binding energy for different scaling dimensions Δ and mass M^2.

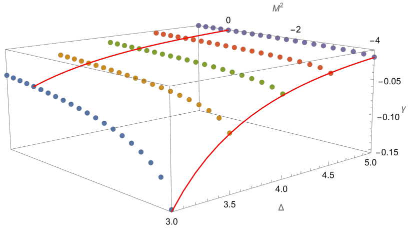


Figure: Binding energy for different scaling dimensions Δ and mass M^2.

Tests in special cases

Flat Space Limit and BPS State

- In the case of massless χ we can compare to the flat space case.
- Taking $L \to \infty, \Delta \to \infty$ while keeping $m = \frac{\sqrt{\Delta(\Delta-4)}}{L}$ fixed, reproduces the WGC bound:

$$\sum_i q_i^2 g_i^2 - \frac{2}{3} m^2 \kappa^2 - \frac{Y^2}{4m^2} + \frac{1}{L}\left(\frac{a}{m} - bm\right) \geq 0\,.$$

[1] Analysed in detail in [Ceresole, Dall'Agata, Kallosh, Van Proeyen '01]

Flat Space Limit and BPS State

- In the case of massless χ we can compare to the flat space case.
- Taking $L \to \infty, \Delta \to \infty$ while keeping $m = \frac{\sqrt{\Delta(\Delta-4)}}{L}$ fixed, reproduces the WGC bound:

$$\sum_i q_i^2 g_i^2 - \frac{2}{3} m^2 \kappa^2 - \frac{Y^2}{4m^2} + \frac{1}{L}\left(\frac{a}{m} - bm\right) \geq 0\,.$$

- We consider a $\mathcal{N} = 2$ 5d gauged supergravity [1] with a gravity multiplet, an hypermultiplet and a vector multiplet.
- The moduli space is given by $\frac{SU(2,1)}{SU(2)\times U(1)} \times O(1,1)$
- Using relations between the couplings:

$$\gamma^{(A_0)} + \gamma^{(A_1)} + \gamma^{(h)} + \gamma^{(a,b)} = -\gamma^{(\chi)}$$

and thus the total self-binding energy $\gamma_{z_1 z_1} = 0$ as expected for a BPS state.

[1]Analysed in detail in [Ceresole, Dall'Agata, Kallosh, Van Proeyen '01]

Summary

Summary

- Positive binding energy conjecture as a natural formulation of the WGC applicable to AdS.
- Requirement of the existence of at least one particle with non-negative binding energy

$$\gamma \geq 0$$

leads to constraints on couplings for effective theories.

- Our results contribute towards sharpening quantum gravity constraints on effective theories in Anti-de Sitter spacetime.
- Outlook: Extending analysis to AdS_4 and AdS_3, fermions, ...

Thank you!