Poisson gauge models and Seiberg-Witten maps

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Outline:

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Poisson gauge theories: motivation and definition.

- Non-commutative deformations of space-time may be characterised by the Kontsevich star product of functions on the space-time,

$$f \star g = f \cdot g + \frac{\mathrm{i}}{2}\{f, g\} + \dots ,$$

where the Poisson bracket is associated with the Poisson bivector Θ^{ij},

$$\{f, g\} = \Theta^{ij}(x)\, \partial_i f\, \partial_j g \, .$$

- Our aim is to construct the noncommutative gauge theory, requiring compatibility of the gauge algebra with space-time noncommutativity, namely infinitesimal gauge transformations should close the non-commutative algebra,

$$[\delta_f, \delta_g] A_j = \delta_{-\mathrm{i}[f,g]_\star} A_j, \qquad f, g \in \mathcal{C}^\infty(\mathcal{M}) \, .$$

The commutative limit has to reproduce standard abelian gauge transformations,

$$\lim_{\Theta \to 0} \delta_f A_j = \partial_j f \, .$$

Poisson gauge theories: motivation and definition.

- In my review I follow the lines of [Kupriyanov, Vitale' 2020]. For the sake of simplicity we work in the semi-classical approximation, therefore, star commutators

$$[f,g]_\star = f \star g - g \star f = \mathrm{i}\{f,g\} + \ldots\,,$$

will be replaced by imaginary unit times the Poisson brackets.

- In this approximation, the full non-commutative algebra of gauge transformations defines the Poisson gauge algebra,

$$[\delta_f, \delta_g] A_j = \delta_{\{f,g\}} A_j\,.$$

- If the Poisson bivector Θ^{ij} is constant, the required deformed gauge transformation can be easily constructed,

$$\delta_f^{can} A_j = \partial_j f + \{A_j, f\}_{can}\,.$$

General construction: gauge transformations.

- For non-constant Poisson bivectors Θ^{ij} the deformed gauge transformation can be defined in the following way

$$\delta_f A_j = \gamma_j^r(x,A)\,\partial_r f + \{A_j, f\}\,.$$

- In this formula the matrix γ satisfies the *first master equation,*

$$\gamma_i^j \partial_A^i \gamma_l^k - \gamma_i^k \partial_A^i \gamma_l^j + \Theta^{ji}\partial_i \gamma_l^k - \Theta^{ki}\partial_i \gamma_l^j - \gamma_l^i \partial_i \Theta^{jk} = 0\,,$$
$$\partial_A^l \equiv \frac{\partial}{\partial A_l}\,, \qquad \lim_{\Theta\to 0} \gamma_i^j = \delta_i^j.$$

- One can check by a direct calculation, that these deformed gauge transformations, indeed, close the required noncommutative Poisson gauge algebra and exhibit the correct commutative limit,

$$\begin{aligned}[\delta_f,\delta_g]A_j &= \delta_{\{f,g\}}A_j\,,\\ \lim_{\Theta\to 0}\delta_f A_j &= \partial_j f\,.\end{aligned}$$

General construction: field strength.

- The deformed field strength has to transform in a covariant way, and reproduce correctly the commutative limit,

$$\begin{aligned} \delta_f \mathcal{F}_{ab} &= \{\mathcal{F}_{ab}, f\}\,, \\ \lim_{\Theta\to 0} \mathcal{F}_{ab} &= \partial_a A_b - \partial_b A_a\,. \end{aligned}$$

- Such a field strength was constructed in [Kupriyanov'2021] in the following way,

$$\mathcal{F}_{ab} = \rho_a^c(x,A)\,\rho_b^d(x,A)\left(\gamma_c^l\,\partial_l A_d - \gamma_d^l\,\partial_l A_c + \{A_c\,, A_d\}\right).$$

- The matrix ρ, which appears in this formula, has to obey the second master equation,

$$\gamma_l^j \partial_A^l \rho_a^i + \rho_a^l \partial_A^i \gamma_l^j + \Theta^{jl} \partial_l \rho_a^i = 0\,, \qquad \partial_A^l \equiv \frac{\partial}{\partial A_l},$$

and it has to reduce to the identity matrix at the commutative limit,

$$\lim_{\Theta\to 0} \rho = \mathbb{I}\,.$$

General construction: covariant derivative and dynamics.

- For any field $\psi(x)$, which transforms in a covariant way,

$$\delta_f \psi := \{f, \psi\} \,,$$

the covariant derivative has to transform properly,

$$\delta_f \left(\mathcal{D}_a \psi\right) = \{\mathcal{D}_a \psi, f\} \,,$$

and it has to exhibit a correct commutative limit,

$$\lim_{\Theta \to 0} \mathcal{D}_a \psi = \partial_a \psi \,.$$

- The expression for the deformed covariant derivative was presented in [Kupriyanov'2021], as follows,

$$\mathcal{D}_a \psi = \rho_a^i \left(\gamma_i^l \, \partial_l \psi + \{A_a, \psi\} \right) .$$

- The deformed covariant derivative and the deformed field strength allow to write down gauge-covariant equations of motion,

$$\mathcal{D}_a \mathcal{F}^{ab} = 0 \,, \qquad \delta_f \left(\mathcal{D}_a \mathcal{F}^{ab} \right) = \{\mathcal{D}_a \mathcal{F}^{ab}, f\}.$$

Symplectic geometric interpretation of the construction.

- Now I follow the logic of [Kupriyanov, Szabo' 2022]. Extending the Poisson structure from $\mathcal{M}$ to $T^*\mathcal{M}$,

$$\{x^i, x^j\} = \Theta^{ij}(x)\,, \qquad \{x^i, p_j\} = \gamma^i_j(x,p)\,, \qquad \{p_i, p_j\} = 0\,,$$

  we define the *symplectic embedding*.

- We also introduce a set of the *constraints* in this extended space,

$$\Phi_a(x,p) := p_a - A_a(x), \qquad a = 1, ..., n\,.$$

- The Jacobi identity on the extended symplectic space is equivalent to the first master equation,

$$\gamma^j_i \partial^i_p \gamma^k_l - \gamma^k_i \partial^i_p \gamma^j_l + \Theta^{ji}\partial_i \gamma^k_l - \Theta^{ki}\partial_i \gamma^j_l - \gamma^i_l \partial_i \Theta^{jk} = 0\,, \quad \partial^i_p \equiv \frac{\partial}{\partial p_i}.$$

  The deformed gauge transformation, introduced before, can be represented as a simple Poisson bracket on the extended symplectic space,

$$\delta_f A_a = \{f, \Phi_a\}_{\Phi=0}\,.$$

Symplectic geometric interpretation of the construction.

- Now I switch to [Kupriyanov, Kurkov, Vitale' 2022]. Performing a transformation in the basis of the constraints, we construct the new constraints,

$$\Phi_a(x,p) \;\to\; \Phi'_a(x,p) := \rho_a^m(x,p)\,\Phi_a(x,p)\,,$$

where ρ is a non-degenerate matrix. In particular,

$$\Phi'_a = 0\,, \quad \Leftrightarrow \quad \Phi_a = 0\,.$$

- The deformed field strength and the deformed covariant derivative can be represented as simple Poisson brackets in the extended symplectic space,

$$\mathcal{F}_{ab} = \{\Phi'_a, \Phi'_b\}_{\Phi'=0}\,, \qquad \mathcal{D}_a\psi = \{\psi, \Phi'_a\}_{\Phi'=0}\,.$$

- The gauge covariance condition,

$$\delta_f \mathcal{F}_{ab} = \{\mathcal{F}_{ab}, f\}\,,$$

is equivalent to the second master equation,

$$\gamma_l^j \partial_p^l \rho_a^i + \rho_a^l \partial_p^i \gamma_l^j + \Theta^{jl}\partial_l \rho_a^i = 0\,, \quad \partial_p^l \equiv \frac{\partial}{\partial p_l}\,.$$

Lie algebraic noncommutativities and the universal solution.

- Consider a class of Poisson bivectors, which are linear in coordinates,

$$\Theta^{ab} = f^{ab}_c\, x^c\,.$$

The constants f^{ab}_c satisfy the Jacobi identity,

$$f^{kl}_i f^{ja}_l + f^{jl}_i f^{ak}_l + f^{al}_i f^{kj}_l = 0\,,$$

therefore these constant can be seen as the structure constants of a Lie algebra.

- In [Kupriyanov, Szabo' 2022] a special solution of the first master equation was presented in terms of a single matrix-valued function, which is valid for *all* Poisson bivectors of the Lie algebraic type.

- Introducing the matrix,

$$[\hat{A}]^b_c = -\mathrm{i} f^{ab}_c A_a,$$

we can represent this universal solution as follows

$$\gamma(A) = G(\hat{A}), \qquad G(p) := \frac{\mathrm{i}\,p}{2} + \frac{p}{2}\cot\frac{p}{2}\,.$$

Lie algebraic noncommutativities and the universal solution.

- Also the second master equation exhibits the universal solution for ρ, expressed as a matrix valued function of the same matrix variable [Kupriyanov, Kurkov, Vitale'2022],

$$[\hat{A}]^b_c = -\mathrm{i} f^{ab}_c A_a \,.$$

- The explicit expression reads:

$$\rho(A) = F(\hat{A}), \quad F(p) = \frac{e^{\mathrm{i}p} - 1}{\mathrm{i}\, p} \,.$$

- The universal solutions for γ and ρ exhibit a simple connection,

$$\boxed{\rho^{-1} = \gamma - \mathrm{i}\, \hat{A} \,.}$$

Using the universal solutions for γ and ρ we can build the Poisson gauge model completely: we know both the gauge-covariant equations of motion, and the deformed gauge transformations, which close the Poisson gauge algebra.

Lie algebraic noncommutativities and the universal solution.

- The Poisson bivector, which corresponds to the (generalised) κ-Minkowski noncommutativity, is defined in the following way,

$$\Theta^{ij} = 2(\omega^i x^j - \omega^j x^i)\,.$$

The corresponding structure constants read

$$f^{ij}_k = 2(\omega^i \delta^j_k - \omega^j \delta^i_k)\,.$$

- We calculated the universal solutions [Kupriyanov, Kurkov, Vitale'2022]:

$$\begin{aligned}
\gamma^i_j(A) &= (\omega\cdot A)\,[1+\coth{(\omega\cdot A)}]\,\delta^i_j + \frac{1-(\omega\cdot A)-(\omega\cdot A)\coth{(\omega\cdot A)}}{\omega\cdot A}\,\omega^i A_j\,,\\
\rho^i_j(A) &= \frac{e^{2(\omega\cdot A)}-1}{2(\omega\cdot A)}\,\delta^i_j + \frac{1+2(\omega\cdot A)-e^{2(\omega\cdot A)}}{2(\omega\cdot A)^2}\,\omega^i A_j\,.
\end{aligned}$$

Arbitrariness and Seiberg-Witten maps.

- The solutions for γ, presented above,

$$\gamma^i_j(A) = (\omega \cdot A)\,[1 + \coth(\omega \cdot A)]\,\delta^i_j + \frac{1 - (\omega \cdot A) - (\omega \cdot A)\coth(\omega \cdot A)}{\omega \cdot A}\,\omega^i A_j\,,$$

are different [Kupriyanov, Kurkov,Vitale'2021]

$$\tilde{\gamma}^k_a(A) = \left[\sqrt{1 + (\omega \cdot A)^2} + (\omega \cdot A)\right]\delta^k_a - \omega^k A_a\,.$$

- The solutions for ρ, presented above,

$$\rho^i_j(A) = \frac{e^{2(\omega \cdot A)} - 1}{2(\omega \cdot A)}\,\delta^i_j + \frac{1 + 2(\omega \cdot A) - e^{2(\omega \cdot A)}}{2(\omega \cdot A)^2}\,\omega^i A_j\,,$$

also differ from the ones, obtained in the previous studies,

$$\tilde{\rho}^k_a(A) = \left[\sqrt{1 + (\omega \cdot A)^2} + (\omega \cdot A)\right]\delta^k_a - \frac{\sqrt{1 + (\omega \cdot A)^2} + (\omega \cdot A)}{\sqrt{1 + (\omega \cdot A)^2}}\,\omega^k A_a\,.$$

Arbitrariness and Seiberg-Witten maps.

- For any invertible field redefinition $A \to \tilde{A}(A)$, the quantities

$$\tilde{\gamma}^i_j(\tilde{A}) = \left(\gamma^i_k(A) \cdot \frac{\partial \tilde{A}_j}{\partial A_k}\right)\Bigg|_{A=A(\tilde{A})}, \quad \tilde{\rho}^i_a(\tilde{A}) = \left(\frac{\partial A_s}{\partial \tilde{A}_i} \cdot \rho^s_a(A)\right)\Bigg|_{A=A(\tilde{A})},$$

  are again the solutions of the master equations. The matrices $\tilde{\gamma}$ and $\tilde{\rho}$ define one more Poisson model for the same Poisson bivector Θ.

- The infinitesimal gauge transformation of the new model read,

$$\tilde{\delta}_f \tilde{A}_a = \tilde{\gamma}^i_a(\tilde{A})\, \partial_i f + \{\tilde{A}_a, f\}\,.$$

  Upon the field redefinition the gauge orbits are mapped onto the gauge orbits,

$$\tilde{A}\left(A + \delta_f A\right) = \tilde{A}(A) + \tilde{\delta}_f \tilde{A}(A)\,.$$

  Therefore the invertible field redefinition can be seen as the Seiberg-Witten map between the two models.

Arbitrariness and Seiberg-Witten maps.

The universal solutions,

$$\gamma^i_j(A) = (\omega \cdot A)\,[1 + \coth(\omega \cdot A)]\,\delta^i_j + \frac{1 - (\omega \cdot A) - (\omega \cdot A)\coth(\omega \cdot A)}{\omega \cdot A}\,\omega^i A_j\,,$$

$$\rho^i_j(A) = \frac{e^{2(\omega \cdot A)} - 1}{2(\omega \cdot A)}\,\delta^i_j + \frac{1 + 2(\omega \cdot A) - e^{2(\omega \cdot A)}}{2(\omega \cdot A)^2}\,\omega^i A_j\,,$$

and the previous solutions,

$$\tilde{\gamma}^k_a(A) = \left[\sqrt{1 + (\omega \cdot A)^2} + (\omega \cdot A)\right]\delta^k_a - \omega^k A_a\,,$$

$$\tilde{\rho}^k_a(A) = \left[\sqrt{1 + (\omega \cdot A)^2} + (\omega \cdot A)\right]\delta^k_a - \frac{\sqrt{1 + (\omega \cdot A)^2} + (\omega \cdot A)}{\sqrt{1 + (\omega \cdot A)^2}}\,\omega^k A_a\,,$$

are connected through the following Seiberg-Witten map:

$$\tilde{A}_a = \frac{\sinh(\omega \cdot A)}{\omega \cdot A}\,A_a \quad \Leftrightarrow \quad A_a = \frac{\operatorname{arcsinh}(\omega \cdot \tilde{A})}{\omega \cdot \tilde{A}}\,\tilde{A}_a\,.$$

Summary.

- For a given Poisson bivector Θ, defining the noncommutativity, the Poisson gauge model is completely determined by the matrices γ and ρ, which solve the two master equations.

- We completed the symplectic geometric description of the Poisson gauge models in terms of symplectic embeddings and constraints in the extended symplectic space.

- We obtained the universal solution of the second master equation, which is valid for any noncommutativity of the Lie algebraic type.

- Invertible field redefinitions give rise to new solutions of the master equations. All the outcoming Poisson gauge models are connected with each other through Seiberg-Witten maps.

Bonus #1: comments on the Lagrangian formulation.

- Using the deformed field strength $\mathcal{F}$, discussed before, we can construct the gauge-covariant Lagrangian density,

$$\mathcal{L} = -\frac{1}{4}\mathcal{F}_{ab}\,\mathcal{F}^{ab}\,, \qquad\qquad \delta_f \mathcal{L}_g = \{\mathcal{L}, f\}\,.$$

- Introducing the weigh-function $\mu(x)$, we get,

$$\mu(x)\delta_f\mathcal{L} = \text{total derivative}, \quad \Leftrightarrow \quad \partial_l\left(\mu(x)\,\Theta^{lk}(x)\right) = 0.$$

The gauge-invariant action reads,

$$S = \int \mathrm{d}^n x\,\mu(x)\,\mathcal{L}\,, \qquad\qquad \delta_f S = 0\,.$$

- The "natural" equations of motion, introduced before, are non-Lagrangian,

$$\mathcal{E}^b_N := \mathcal{D}_a\mathcal{F}^{ab}\,, \qquad \nexists \tilde{S}[A]: \quad \mathcal{E}^b_N = \frac{\delta\tilde{S}}{\delta A}, \qquad \frac{\delta\mathcal{E}^b_N}{\delta A_a} \neq \frac{\delta\mathcal{E}^a_N}{\delta A_b}.$$

However, if $\partial_l\Theta^{lk}(x) = 0$, the "natural" and the Euler-Lagrange equations are equivalent:

$$\mathcal{E}^a_{EL} := \frac{\delta S}{\delta A_a}, \qquad \mathcal{E}^b_{EL} = \rho^b_a\,\mathcal{E}^a_N.$$

Bonus #2: comments on the (deformed) Bianchi identity.

- The second pair of Maxwell equations, i.e. the Bianchi identity,

$$\partial_a F_{bc} + \text{cycl}(abc) = 0, \qquad F_{ab} = \partial_a A_b - \partial_b A_a,$$

exhibits its deformed version in the Poisson gauge theory.

- This deformed version is highly counterintuitive [Kupriyanov'2021],

$$\mathcal{D}_a\left(\mathcal{F}_{bc}\right) - \mathcal{F}_{ad}\,\mathcal{B}_b{}^{de}\,\mathcal{F}_{ec} - \left(\mathcal{K}_{ab}{}^{e} - \mathcal{K}_{ba}{}^{e}\right)\mathcal{F}_{ec} + \text{cycl}(abc) = 0\,,$$

$$\begin{aligned}
\mathcal{B}_b{}^{de}(A) &= \left(\rho^{-1}\right)^d_j \left(\partial^j_A \rho^m_b(A) - \partial^m_A \rho^j_b(A)\right)\left(\rho^{-1}\right)^e_m\,, \\
\mathcal{K}_{ab}{}^{e}(A) &= \rho^i_a(A)\,\gamma^m_i(A)\,\left(\partial_m \rho^j_b(x,p)\right)_{\Phi=0}\left(\rho^{-1}\right)^e_j\,.
\end{aligned}$$

- However, using the symplectic geometric construction, discussed before, this equation can be presented in the short form [Kupriyanov, Kurkov, Vitale'2022],

$$\{\Phi'_a, \{\Phi'_b, \Phi'_c\}\}_{\Phi'=0} + \text{cycl}(abc) = 0\,.$$