A star product on N=1 chiral superspace

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1. Introduction

The Grassmannian of 2-planes in $\mathbb{C}^4$, $G(2,4)$ is the *complexified* conformal space in dimension 4: the complexified conformal group $\mathrm{SL}(4,\mathbb{C})$ acts on it and it is the conformal compactification of the complexifed Minkowski space M^4.
This is not true over the reals: the conformal group is $\mathrm{SU}(2,2)$ and the corresponding homogeneous space is not a Grassmannian. We will always talk about the complexified versions of our objects without stating it. Passing to the real picture is done by constructing a conjugation whose set of fixed points is the correct real form. We will not deal with this issue here.

Introduction

The space $\mathbb{C}^4$ in $G(2,4)$ is the twistor space. To convince ourselves, the subgroup of $\mathrm{SL}(4,\mathbb{C})$

$$\begin{pmatrix} x & 0 \\ Tx & y \end{pmatrix}, \qquad \det x \cdot \det y = 1$$

is the Poincaré group times dilations, with action on the big cell of $G(2,4)$

$$t \longrightarrow ytx^{-1} + T\,,$$

where

$$t = \sum_{\mu=0}^{3} z^\mu \sigma_\mu = \begin{pmatrix} z^0 + z^3 & z^1 - \mathrm{i}z^2 \\ z^1 + \mathrm{i}z^2 & z^0 + z^3 \end{pmatrix}.$$

Introduction

$$\begin{array}{ccc} \text{Conformal theory} & \xrightarrow{\text{restriction bigcell}} & \text{Broken symmetry theory} \\ \text{Conformal space} & \xleftarrow[\text{compactification}]{} & \text{Minkowski space}\,. \end{array}$$

Penrose belived at the beginning that through twistors one could introduce an indeterminacy principle in spacetime, since points are not fundamental quantities. This proved not to be the case. Twistor theory still describes classical spacetime and the indeterminacy must be introduced by hand.

Introduction

We will do that by substituting the conformal group by its quantum group counterpart $\mathrm{SL}_q(4, \mathbb{C})$ and quantizing the homogeneous space to a suitably defined $G_q(2, 4)$.

By restricting to the Minkowski space, we will be ble to give an explicit star product for this deformation. The star product corresponds to choose an ordering rule for the generators.

The process is non trivial, involves a lot of calculations and the result is not a 'nice' formula. But it is one of the few examples of explicit star product.

The Poisson bracket is quadratic in the generators, not like other deformations whose Poisson bracket is linear. This deformation has other properties that make it worthy to study. For example, it is better behaved under symmetries and it has a deformed Lorentz subgroup acting on it, property that is absent in the linear deformations.

Introduction

This picture carries over to the super setting. The superconformal group is then $\mathrm{SL}(4|1)$. We have two kinds of Minkowski superspaces:

- Chiral (antichiral) superspace, which corresponds to the super Grassmannian $G(2|0, 4|1)$. Superfields:

$$\Phi(x^\mu, \theta^\alpha) = \phi(x) + \theta\psi(x) + \theta\theta F(x)\,,$$

  $\alpha = 1, 2$ and $\mu = 0, \dots 4$. θ^α is a Weyl spinor, with definite chirality. This is an intrinsically complex super space, since θ^α does not have complex conjugate.

Introduction

- Real superspace, which corresponds to the super flag $Fl(2|0,2|1,4|1) \subset G(2|0,4|1) \times G(2|1,4|1)$. Superfields:

$$\begin{aligned}\Phi(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) = & \phi(x) + \theta\psi(x) + \bar{\theta}\bar{\psi}(x) + \\ & \theta\theta m(x) + \bar{\theta}\bar{\theta} n(x) + \theta\sigma^\mu\bar{\theta} A_\mu(x) + \\ & \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\psi(x) + \theta\theta\bar{\theta}\bar{\theta} F(x).\end{aligned}$$

with $\alpha, \dot{\alpha} = 1, 2,$ *and* with $\mu = 0, 1, 2, 3$
On this super space one can impose a reality condition. For larger number N of supersymmetries the number of component fields is too big and it is difficult to impose suitable covariant constraints.

Introduction

Our goal is to give a deformation of $G(2|0,4|1)$ in line with the quantum supergroup $\mathrm{SL}_q(4|1)$ and choose an ordering rule to produce and explicit super star product.

2. The classical picture

We consider the supergroup [1]

$$\mathrm{SL}(4|1) = \left\{ \begin{pmatrix} g_{ij} & \gamma_{i5} \\ \gamma_{5j} & g_{55} \end{pmatrix}, \quad i, j = 1, \dots 4 \right\} .$$

An element of the supergrassmannian $G(2|0, 4|1)$ is a plane, that is, the span of two even vectors

$$\pi = (a, b) = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \\ \alpha_5 & \beta_5 \end{pmatrix}$$

that can be chosen up to the right action of $\mathrm{GL}(2, \mathbb{C})$.

[1] This can be understood better in terms of the *functor of points*.

2. The classical picture

$G(2|0,4|1)$ is a projective supervariety, embedded in $\mathbf{P}^{6|4} = \mathbf{P}\left(\bigwedge^2(\mathbb{C}^{4|1})\right)$ via the super Plücker embedding

$$\pi = (a,b) = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_4 & b_4 \\ \alpha_5 & \beta_5 \end{pmatrix} \to a \wedge b\,.$$

As in the non super case, the image of this map is a projective supervariety given in terms of generators and homogeneous relations

$$\begin{array}{ll} q_{12}q_{34} - q_{13}q_{24} + q_{14}q_{23} = 0, & \text{(classical Plücker relation)} \\ q_{ij}\lambda_k - q_{ik}\lambda_j + q_{jk}\lambda_i = 0, & 1 \le i < j < k \le 4 \\ \lambda_i\lambda_j = a_{55}q_{ij} & 1 \le i < j \le 4 \\ \lambda_i a_{55} = 0 & a_{55}^2 = 0\,. \end{array}$$

The classical picture

These relations are the super Plücker relations and the ideal generated by them in the (super)commutative free algebra is denoted by $\mathcal{I}_P$.

We consider now the first two rows of the supergroup element and construct the quantities

$$d_{ij} = g_{i1}g_{j2} - g_{i2}g_{j1}\,, \qquad \sigma_i = g_{1i}\gamma_{52} - g_{2i}\gamma_{51}\,, \qquad a = \gamma_{51}\gamma_{52}\,.$$

They satisfy the super Plücker relations and no other independent relations.

In this way one retrieves the algebra $\mathbb{C}[q_{ij}, \lambda_i, a_{55}]/\mathcal{I}_P$ of the super Plücker embedding as a subalgebra of $\mathrm{SL}(4|1)$.

The classical picture

We invert $d_{12} = \det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$, that is, we add an extra even generator, d_{12}^{-1}, satisfying the relation $d_{12}^{-1} d_{12} = 1$. Then we can reduce to the standard form

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ t_{31} & t_{32} \\ t_{41} & t_{42} \\ \tau_{51} & \tau_{52} \end{pmatrix},$$

with

$$\begin{pmatrix} t_{31} & t_{32} \\ t_{41} & t_{42} \\ \tau_{51} & \tau_{52} \end{pmatrix} = \begin{pmatrix} -d_{23} & d_{13} \\ -d_{24} & d_{14} \\ \sigma_1 & \sigma_2 \end{pmatrix} d_{12}^{-1}.$$

The classical picture

The (super)commutative algebra $\mathbb{C}[t_{ij}, \tau_i]$ is the coordinate algebra of the big cell, which is isomorphic to $\mathbb{C}^{4|2}$. Since it has the correct action of the super Poincaré subgroup,

$$\begin{pmatrix} x & 0 & 0 \\ Tx & y & y\eta \\ d\tau & 0 & d \end{pmatrix} \begin{pmatrix} I \\ t \\ \sigma \end{pmatrix} \approx \begin{pmatrix} I \\ y(t+\eta\sigma)x^{-1} + T \\ d(\sigma+\tau)x^{-1} \end{pmatrix}$$

it can be identified with the Minkowski superspace.

3. The deformation

Cervantes, Fioresi, Ll. 2011

Manin's relations.

In order to construct the quantum matrix superalgebra $M_q(r|s)$, we start with generators

$$\begin{pmatrix} z_{r\times r} & \xi_{r\times s} \\ \xi_{s\times r} & z_{s\times s} \end{pmatrix} . \tag{1}$$

It is convenient to have a common notation for even and odd variables.

$$a_{ij} = \begin{cases} z_{ij} & 1 \leq i,j \leq r, \ \text{ or } \quad r+1 \leq i,j \leq r+s, \\ \\ \xi_{ij} & 1 \leq i \leq r, \quad r+1 \leq j \leq r+s, \ \text{ or } \\ & r+1 \leq i \leq r+s, \quad 1 \leq j \leq r\,. \end{cases}$$

The deformation

Then, the ideal $\mathcal{I}_M$ of the free algebra over $\mathbb{C}_q := \mathbb{C}[q, q^{-1}]$ is generated by the commutation relations:

$$a_{ij}a_{il} = (-1)^{\pi(a_{ij})\pi(a_{il})} q^{(-1)^{p(i)+1}} a_{il}a_{ij}, \qquad \text{for } j < l$$

$$a_{ij}a_{kj} = (-1)^{\pi(a_{ij})\pi(a_{kj})} q^{(-1)^{p(j)+1}} a_{kj}a_{ij}, \qquad \text{for } i < k$$

$$a_{ij}a_{kl} = (-1)^{\pi(a_{ij})\pi(a_{kl})} a_{kl}a_{ij}, \qquad \begin{array}{l} \text{for } i < k,\ j > l \\ \text{or } i > k,\ j < l \end{array}$$

$$a_{ij}a_{kl} - (-1)^{\pi(a_{ij})\pi(a_{kl})} a_{kl}a_{ij} = (-1)^{\pi(a_{ij})\pi(a_{kl})} (q^{-1} - q) a_{kj}a_{il}, \qquad \text{for } i < k, j < l$$

Demanding that the quantum Berezinian is 1 we obtain the Hopf algebra $\mathrm{SL}_q(r|s)$.

Quantum twistor superspace

These commutation relations are required to have a coaction of the quantum supermatrices $M_q(r|s)$ on the affine quantum superspace $A_q(r|s)$ defined by generators x_i, $i = 1, \dots r + s$.

$$x_i x_j = q^{-1}(-1)^{p(i)p(j)} x_j x_i \qquad \text{for } i < j\,.$$

For $r = 4$ and $s = 1$ this is what we call the quantum twistor superspace.

The deformation

Definition

The quantum super Grassmannian $\mathrm{Gr}_q := Gr_q(2|0,4|1)$ is the subalgebra of $\mathrm{SL}_q(4|2)$ generated by the elements

$$D_{ij} := a_{i1}a_{j2} - q^{-1}a_{i2}a_{j1} \qquad D_{i5} := a_{i1}a_{52} - q^{-1}a_{i2}a_{51}$$
$$D_{55} := a_{51}a_{52}, \qquad \text{with } 1 \leq i < j \leq 4\,.$$

We want to give a presentation of this subalgebra in terms of generators and relations.

All we have to do is to compute the commutation relations among the generators above and find the deformation of the super Plücker relations (!).

The deformation

Commutation relations.

- i, j, k, l are *not* all distinct and D_{ij}, D_{kl} are not both odd:

$$D_{ij} D_{kl} = q^{-1} D_{kl} D_{ij}, \quad (i,j) < (k,l), \quad 1 \leq i,j,k,l \leq 5\,. \quad (2)$$

- i, j, k, l are all distinct and D_{ij}, D_{kl} are not both odd:

$$\begin{aligned}
D_{ij} D_{kl} &= q^{-2} D_{kl} D_{ij}, && 1 \leq i < j < k < l \leq 5 \\
D_{ij} D_{kl} &= q^{-2} D_{kl} D_{ij} - (q^{-1} - q) D_{ik} D_{jl}, && 1 \leq i < k < j < l \leq 5 \\
D_{ij} D_{kl} &= D_{kl} D_{ij}, && 1 \leq i < k < l < j \leq 5
\end{aligned}$$

- Commutations with D_{55} or involving two odd elements:

$$\begin{aligned}
& D_{ij} D_{55} = q^{-2} D_{55} D_{ij}, \\
& D_{i5} D_{j5} = -q^{-1} D_{j5} D_{i5} - (q^{-1} - q) D_{ij} D_{55} = -q D_{j5} D_{i5} \\
& D_{i5} D_{55} = D_{55} D_{i5} = 0.
\end{aligned}$$

The deformation

Quantum super Plücker relations.

$$\begin{aligned} &D_{12}D_{34} - q^{-1}D_{13}D_{24} + q^{-2}D_{14}D_{23} = 0, && \\ &D_{ij}D_{k5} - q^{-1}D_{ik}D_{j5} + q^{-2}D_{i5}D_{jk} = 0, && 1 \leq i < j < k \leq 4, \\ &D_{i5}D_{j5} = qD_{ij}D_{55}, && 1 \leq i < j \leq 4. \end{aligned}$$

We also prove that these are all the relations that they satisfy, so the free algebra in D_{ij}, D_{i5}, D_{55} modulo the ideal $\mathcal{I}_Q$ generated by the above commutation relations and the quantum super Plücker relations

$$\mathbb{C}\langle D_{ij}, D_{i5}, D_{55}\rangle / \mathcal{I}_Q \ ,$$

is a presentation of the quantum Grassmannian $G_q(2|0,4|1)$.
Also, the corresponding coaction of the quantum supergroup!

The deformation

As for the classical case, we can define a quantum super Poincaré group as the subgroup of $\mathrm{SL}_q(4|1)$ generated by the elements (formally as in the classical case)

$$\begin{pmatrix} x & 0 & 0 \\ Tx & y & y\eta \\ d\tau & 0 & d \end{pmatrix} = \begin{pmatrix} x & 0 & 0 \\ Tx & y & y\eta \\ \tilde{\tau}x & 0 & d \end{pmatrix} .$$

x, y and d are invertible.
It is convenient to make the change of variables $d\tau = \tilde{\tau}x$.
One can compute the commutation relations of these generators inside $\mathrm{SL}_q(4|1)$. What is crucial is to compute the commutation relations of T and $\tilde{\tau}$, which represent the (even and odd) translations. This will be our model for the quantum Minkowski superspace.

The deformation

Definition

The *complexified quantum Minkowski superspace* is the free algebra in six generators

$$\hat{t}_{41},\ \hat{t}_{42},\ \hat{t}_{31}\ \text{ and }\ \hat{t}_{32}, \qquad \text{(even)}$$
$$\hat{\tau}_{51}, \hat{\tau}_{52} \qquad \text{(odd)}$$

satisfying the commutation relations

$$\begin{aligned}
\hat{t}_{42}\hat{t}_{41} &= q^{-1}\hat{t}_{41}\hat{t}_{42}, \\
\hat{t}_{31}\hat{t}_{41} &= q^{-1}\hat{t}_{41}\hat{t}_{31}, \\
\hat{t}_{32}\hat{t}_{41} &= \hat{t}_{41}\hat{t}_{32} + (q^{-1} - q)\hat{t}_{42}\hat{t}_{31}, \\
\hat{t}_{31}\hat{t}_{42} &= \hat{t}_{42}\hat{t}_{31}, \\
\hat{t}_{32}\hat{t}_{42} &= q^{-1}\hat{t}_{42}\hat{t}_{32}, \\
\hat{t}_{32}\hat{t}_{31} &= q^{-1}\hat{t}_{31}\hat{t}_{32}\,,
\end{aligned}$$

The deformation

which would be the commutation relations defining the non super quantum Minkowski space, together with

$$
\begin{aligned}
&\hat{\tau}_{51}\hat{\tau}_{52} = -q^{-1}\hat{\tau}_{52}\hat{\tau}_{51} && \\
&\hat{t}_{31}\hat{\tau}_{51} = q^{-1}\hat{\tau}_{51}\hat{t}_{31} && \hat{t}_{32}\hat{\tau}_{52} = q^{-1}\hat{\tau}_{52}\hat{t}_{32} \\
&\hat{t}_{41}\hat{\tau}_{51} = q^{-1}\hat{\tau}_{51}\hat{t}_{41}, && \hat{t}_{42}\hat{\tau}_{52} = q^{-1}\hat{\tau}_{52}\hat{t}_{42} \\
&\hat{t}_{31}\hat{\tau}_{52} = \hat{\tau}_{52}\hat{t}_{31} && \hat{t}_{41}\hat{\tau}_{52} = \hat{\tau}_{52}\hat{t}_{41} \\
&\hat{t}_{32}\hat{\tau}_{51} - \hat{\tau}_{51}\hat{t}_{32} = (q - q^{-1})\hat{t}_{31}\hat{\tau}_{52} && \hat{t}_{42}\hat{\tau}_{51} - \hat{\tau}_{51}\hat{t}_{42} = (q - q^{-1})\hat{t}_{41}\hat{\tau}_{52}
\end{aligned}
$$

These are a subalgebra of the quantum supermatrices $M_q(2|1)$. There is a well defined coaction of the quantum Poincaré supergroup on this algebra that reduces to the standard coaction of the non super case when $q = 1$.

4. The star product

Cervantes, Fioresi, Nadal, Ll. 2011

We first give the star product in the non super case (we put all the fermionic generators to 0). In order to do that we choose an ordering rule among the generators. We denote as $t_{41}, t_{42}, t_{31}, t_{32}$ the classical generators of the free commutative algebra $\mathbb{C}_q[t_{41}, t_{42}, t_{31}, t_{32}]$. One can prove that the map

$$\begin{array}{ccc} \mathbb{C}_q[t_{41}, t_{42}, t_{31}, t_{32}] & \xrightarrow{Q_0} & \mathbb{C}_q\langle \hat{t}_{41}, \hat{t}_{42}, t_{31}, \hat{t}_{32}\rangle/\mathcal{I}_{CR_0} \\ t_{41}^a t_{42}^b t_{31}^c t_{32}^d & \longrightarrow & \hat{t}_{41}^a \hat{t}_{42}^b \hat{t}_{31}^c \hat{t}_{32}^d \end{array} \tag{3}$$

$a, b, c, d = 0, 1, 2, 3, \ldots$ is a $\mathbb{C}_q$-module isomorphism (so it has an inverse). Then the star product is an associative, non commutative product on $\mathbb{C}_q[t_{41}, t_{42}, t_{31}, t_{32}]$:

$$f \star_0 g := Q_0^{-1}\left(Q_0(f)Q_0(g)\right), \qquad f, g \in \mathbb{C}_q[t_{41}, t_{42}, t_{31}, t_{32}])\,.$$

The star product

What is non trivial is to reorder $Q_0(f)Q_0(g)$ to compute Q_0^-1. We give here the result on two arbitrary monomials. First we define certain quantities by a recursion rule. For $k, m, n = 0, 1, 2, 3, \ldots$

$$F_k(q, m, n) = \beta_k(q, m) \prod_{l=0}^{k-1} F(q, n-l) \text{ with } F(q, n) = \left(\frac{1}{q^{2n-1}} - q \right)$$

and $\beta_k(q, m)$ is defined by the recursive relation

$$\begin{aligned} &\beta_0(q, m) = \beta_m(q, m) = 1, \quad \text{and} \\ &\beta_k(q, m+1) = \beta_{k-1}(q, m) + \beta_k(q, m) q^{-2k}. \end{aligned}$$

Moreover, $\beta_k(q, m) = 0$ if $k < 0$ or if $k > m$.

The star product

Then we have for the star product of two monomials:

$$(t_{41}^a t_{42}^b t_{31}^c t_{32}^d) \star_0 (t_{41}^m t_{42}^n t_{31}^p t_{32}^r) = q^{-mc-mb-nd-dp} t_{41}^{a+m} t_{42}^{b+n} t_{31}^{c+p} t_{32}^{d+r} + \sum_{k=1}^{\mu=min(d,m)} q^{-(m-k)c-(m-k)b-n(d-k)-p(d-k)} F_k(q,d,m) \cdot t_{41}^{a+m-k} t_{42}^{b+k+n} t_{31}^{c+k+p} t_{32}^{d-k+r}\,.$$

The first order in h, $q = e^h$, antisymmetrized, is the Poisson bracket:

$$\begin{aligned}\{f,g\}_0 =& t_{41}t_{31}(\partial_{41}f\partial_{31}g - \partial_{41}g\partial_{31}f) + t_{42}t_{41}(\partial_{41}f\partial_{42}g - \partial_{41}g\partial_{42}f) + \\ & t_{32}t_{42}(\partial_{42}f\partial_{32}g - \partial_{42}g\partial_{32}f) + t_{32}t_{31}(\partial_{31}f\partial_{32}g - \partial_{31}g\partial_{32}f) + \\ & 2t_{42}t_{31}(\partial_{41}f\partial_{32}g - \partial_{41}g\partial_{32}f)\,.\end{aligned}$$

The star product

We have expressed the Poisson bracket in terms of derivatives in spite that the star product formula was given for polynomials. One can prove that at each order in h there exists a bidifferential operator, that reproduces the same result on the polynomials. But a differential operator is characterized by its action on polynomials, so the star product extends to C^∞ functions.

The star product

In terms of the standard variables of Minkowski space

$$z^0 = \frac{1}{2}(t_{31} + t_{42}), \quad z^1 = \frac{1}{2}(t_{32} + t_{41}),$$
$$z^2 = \frac{i}{2}(t_{32} - t_{41}), \quad z^3 = \frac{1}{2}(t_{31} - t_{42}),$$

the Poisson bracket becomes

$$\begin{aligned}\{f, g\}_0 =& \mathrm{i}\Big(\left((z^0)^2 - (z^3)^2\right) (\partial_1 f \partial_2 g - \partial_1 g \partial_2 f) \\ & z^0 z^1 (\partial_0 f \partial_2 g - \partial_0 g \partial_2 f) - z^0 z^2 (\partial_0 f \partial_1 g - \\ & \partial_0 g \partial_1 f) - z^1 z^3 (\partial_2 f \partial_3 g - \partial_2 g \partial_3 f) + \\ & z^2 z^3 (\partial_1 f \partial_3 g - \partial_1 g \partial_3 f)\Big) .\end{aligned}$$

5. The super star product

Fioresi, Ll. 2021

The super star product is a bit more involved because of the signs that appear when interchanging odd variables. Nevertheless it can be put in terms of the standard star product.

The ordering that we chose is, for $a, b, c, d = 0, 1, 2, \dots$ and $e, f = 0, 1$, the module isomorphism

$$\begin{array}{ccc} \mathbb{C}_q[\tau_{51}, \tau_{52}, t_{41}, t_{42}, t_{31}, t_{32}] & \xrightarrow{Q} & \mathbb{C}_q[\hat{\tau}_{51}, \hat{\tau}_{52}, \hat{t}_{41}, \hat{t}_{42}, \hat{t}_{31}, \hat{t}_{32}] \\ \tau_{51}^e \tau_{52}^f t_{41}^a t_{42}^b t_{31}^c t_{32}^d & \longrightarrow & \hat{\tau}_{51}^e \hat{\tau}_{52}^f \hat{t}_{41}^a \hat{t}_{42}^b \hat{t}_{31}^c \hat{t}_{32}^d \,. \end{array}$$

In order to simplify the notation we define

$$T(a, b, c, d) = t_{41}^a t_{42}^b t_{31}^c t_{32}^d \,,$$

so we will multiply monomials of the type $\tau_{51}^e \tau_{52}^f T(a, b, c, d)$. Then

The super star product

$$\tau_{51}^{e}\tau_{52}^{f}T(a,b,c,d)\star\tau_{51}^{u}\tau_{52}^{v}T(m,n,p,r)=$$

$$\delta_{u0}\delta_{v0}\tau_{51}^{e}\tau_{52}^{f}\;T(a,b,c,d)\star_0\;T(m,n,p,r)+$$
$$\delta_{u0}\delta_{v1}\left(q^{-(a+b+d)}\tau_{51}^{e}\tau_{52}^{f+1}\;T(a,b,c,d)\star_0\;T(m,n,p,r)\right)$$
$$\delta_{u1}\delta_{v0}\Big((-1)^f q^{f-a-c}\tau_{51}^{e+1}\tau_{52}^{f}\;T(a,b,c,d)\star_0\;T(m,n,p,r)+$$
$$q^{-c}(q-q^{-2b+1})\tau_{51}^{e}\tau_{52}^{f+1}\;T(a+1,b-1,c,d)\star_0\;T(m,n,p,r)+$$
$$q^{-b}(q-q^{-2d+1})\tau_{51}^{e}\tau_{52}^{f+1}\;T(a,b,c+1,d-1)\star_0\;T(m,n,p,r)\Big)+$$
$$\delta_{u1}\delta_{v1}\Big((-1)^f q^{-2a-b-c-d+f}\tau_{51}^{e+1}\tau_{52}^{f+1}\;T(a,b,c,d)\star_0\;T(m,n,p,r)+$$
$$q^{-(a+b+c+d)}(q-q^{-2b+1})\tau_{51}^{e}\tau_{52}^{f}\;T(a+1,b-1,c,d)\star_0\;T(m,n,p,r)+$$
$$q^{-a-2b-d+1}(q-q^{-2d+1})\tau_{51}^{e}\tau_{52}^{f}\;T(a,b,c+1,d-1)\star_0\;T(m,n,p,r)\Big)\;,$$

The super Poisson bracket

The term of order 1 in h in the above star product will give, upon antisymmetrization the super Poisson bracket. We denote by S_1 this term and by C_1 the same term in the non super star product. We also denote:

$$R_A = \tau_{52}^e \tau_{51}^f t_{41}^a t_{42}^b t_{31}^c t_{32}^d, \qquad R_M = \tau_{52}^u \tau_{51}^v t_{41}^m t_{42}^n t_{31}^r t_{32}^p \,.$$

Then

$$\begin{aligned} & S_1(R_A, R_M) = C_1(R_A, R_M) + \\ & (-t_{41}\partial_{t_{41}} - t_{42}\partial t_{42} - t_{32}\partial_{t_{32}}) R_A \cdot \tau_{52}\partial_{\tau_{52}} R_M + \\ & (\tau_{52}\partial_{\tau_{52}} - t_{41}\partial_{t_{41}} - t_{31}\partial_{t_{31}}) R_A \cdot \tau_{51}\partial_{\tau_{51}} R_M \,. \end{aligned}$$

As we can see, it can be put in terms of bidifferential operators. Presumably this happens at all orders in h.

6. Conclusions

- We have displayed a deformation of the $N = 1$ chiral conformal superspace that has a good behaviour with respect to symmetries like the quantum conformal supergroup and, in particular, the quantum Poincaré supergroup inside the conformal one. This deformation is given in terms of generators and relations and exploits the projectivity of the Grassmannian $G(2|0,4|1)$.
- One can define a big cell that, as in the classical case, plays the role of the quantum Minkowski superspace, with a quantum Poincaré supergroup acting on it.
- Restricting to the big cell it is possible to express this deformation in a more concrete way, in terms of a normal ordering and an induced star product.

Conclusions

- The Poisson bracket associated to this deformation is quadratic in the standard coordinates of the Minkowski superspace. This is different from other deformations of the Minkowski space, perhaps better known, that are linear.
- Because its symmetry properties, it will be worth to study, at least to first order in h the effect that this non commutativity could have in the description of field theories, even supersymmetric ones.